Generalized interpolative quantum statistics

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A generalized interpolative quantum statistics is presented by conjecturing a certain reordering of phase space due to the presence of possible exotic objects other than bosons and fermions. Such an interpolation achieved through a Bose-counting strategy predicts the existence of an infinite quantum Boltzmann-Gibbs statistics akin to the one discovered by Greenberg recently.

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The concept of fractional statistics has assumed importance in the context of what have been sometimes called "anyons" in the past few years [1,2]. Some investigators have also considered the possibility of a small violation of Fermi statistics by electrons and other particles and of Bose statistics by pions [3]. Recently Greenberg [4] proposed an infinite quantum Boltzmann statistics which arises from the algebra

$$a(k)a^{\dagger}(l)=\delta_{k,l} \tag{1}$$

with the Fock-like vacuum condition $a(k)|0\rangle=0$, and showed that the Hilbert space of the Fock-like representation is positive definite. He has further mentioned Polyakov's remark that this is a particular case of the more general "q-mutator" algebra $a(k)a^{\dagger}(l)-qa^{\dagger}(l)a(k)=\delta_{k,l}$ for q real, which interpolates between Fermi-Dirac and Bose-Einstein statistics. Fivel [5] and Zagier [5] have proved that the Fock-like representation of the q-mutator algebra has a positive-definite Hilbert space for $-1\le q\le 1$.

Many years ago I had done a work similar in spirit but employing a Bose strategy of counting and discovered a statistical distribution which interpolates (nontrivially) between the Fermi and Bose statistics through a family of statistical distributions corresponding to various values of a "q mutator" like group parameter q which lies in the range $-1\le q\le 1$, of which one happens to approximate the infinite quantum Boltzmann distribution discovered by Greenberg. The whole family of interpolating statistics is quantum by virtue of the indistinguishability of the particles incorporated in the present statistics.

This interpolation is achieved through a Bose-counting strategy through the following hypothesis: if z_j is the number of energy levels in the energy interval E_j to $E_j+\delta E_j$ for a system of N_j bosons in the same interval, then the corresponding degeneracy of the group of levels in the energy interval in the presence of N_j "exotic" objects must be $[z_j+\eta(N_j)-N_j]$, where $\eta(N_j)$ is a positive, linear function of N_j in order to make statistical counting possible. The further conjecture is that with this replacement for the degeneracy of the levels, we may count the number of ways N_j "exotic" objects in the energy interval E_j to $E_j+\delta E_j$ can be distributed in phase space by carrying out the Bose strategy of counting as though we are dealing with just a system of noninteracting bosons which are indistinguishable. The number of ways W_G, therefore, in which we can distribute N_j "exotic" particles in $[z_j+\eta(N_j)-N_j]$ cells is, by standard counting [6],

$$W_G=\prod_j \frac{[z_j+\eta(N_j)-1]!}{[z_j+\eta(N_j)-1-N_j]!N_j!}$$
$$\simeq\prod_j \frac{[z_j+\eta(N_j)]}{[z_j+\eta(N_j)-N_j]!N_j!}$$

with $z_j\sim N_j\gg 1$, when quantum features of the statistics are manifest. This will lead to a generalized quantum probability distribution through the entropy S defined by the standard Boltzmann ansatz

$$S=k\ln W_G\ . \tag{2}$$

A straightforward extremization of (2) under the total particle number and energy-conservation constraints,

$$N=\sum_j N_j \quad\text{and}\quad E=\sum_j N_jE_j\ , \tag{3}$$

leads to the relation

$$\left[\frac{z_j+\eta(N_j)}{N_j}-1\right]\Big/\{1-N_j/[z_j+\eta(N_j)]\}^{\partial\eta(N_j)/\partial N_j}=e^{\alpha+\beta E_j}\ , \tag{4}$$

where α and β are the undetermined multipliers. Putting $[z_j+\eta(N_j)]/N_j=x_j$ and $\partial\eta(N_j)/\partial N_j=\zeta(q)$ in (4) we have

$$x_j^{\zeta(q)}/(x_j-1)^{[\zeta(q)-1]}=e^{\alpha+\beta E_j}\ . \tag{5}$$

For the distribution to be realizable we require the exponents $\zeta(q)$ to be in the domain [0,1], as otherwise we will encounter the singularity at $x_j=1$ which is not permissible for real and finite α and β.

The simplest form for the function $\eta(N_j)$ which represents the occupancy-induced change in the degeneracy of energy levels is $\eta(N_j)=\zeta(q)N_j$, where $\zeta(q)$ is a function of a real number in the range $1\geq\zeta(q)\geq 0$ with $q\in R$, a real number. The demand that $\eta(N_j)$ be an integer function imposes the restriction that only those values of N_j are allowed for a given q which make $\eta(N_j)$ an integer. Since we are dealing with the statistics of indistinguishable particles, the statistics is necessarily quantum and therefore we identify q as the quantum group parameter with real values in the range $-1\leq q\leq 1$ and also choose the simplest polynomial form for $\zeta(q)$ as $\zeta(q)=\mu(q)^2+\nu q+\delta$ where μ, ν, and δ are real numbers. The boundary conditions to be satisfied by $\zeta(q)$ are obviously $\zeta(+1)=1$ and $\zeta(-1)=0$ leading to the values $\mu=\frac{1}{2}-\delta$ and $\nu=\frac{1}{2}$.

For $q=-1$, the distribution (5) yields the Fermi-Dirac distribution

$$N_j=z_j/(e^{\alpha+\beta E_j}+1)$$

and, for $q=+1$, the distribution (5) yields the Bose-Einstein distribution

$$N_j=z_j/(e^{\alpha+\beta E_j}-1)\ .$$

For the third important case of $q=0$, the distribution (5) leads approximately to the infinite quantum Boltzmann statistics [7] (see Fig. 1)

$$N_j=z_j/e^{\alpha+\beta E_j}\quad \text{for } \delta\approx 0.565\ .$$

This agrees approximately with Greenberg's infinite quantum Boltzmann statistics [4], obtained from a very different premise. All the intermediate q values lead to "exotic" statistics which may possibly correspond to new exotic statistics and all the statistics are quantum because indistinguishability is assumed in the counting of states and it leads to the conventional quantum statistics as particular cases. One more value of $\zeta(q)$ for which (5) is exactly solvable is $\zeta(q)=\frac{1}{2}$ or $q\simeq-\frac{2}{9}$ when the distribution (5) yields

$$N_j=z_j/(e^{2\alpha+2\beta E_j}+\tfrac{1}{4})^{1/2} \tag{6}$$

which is more Fermi-like than Bose-like but has many intermediary features of both.

With all these striking similarities with Greenberg's "quon" statistics, their exact relationship, if any, needs further examination in the near future.

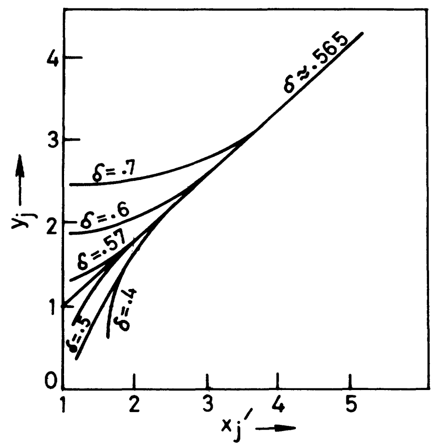

FIG. 1. Linearization value of δ for which the quantum Boltzmann distribution occurs for $q=0$.

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[7] The numerical solution of (5) for $q=0$ with $x_j'=(x_j-\delta)$ and $y_j=x_j^{\delta}/(x_j-1)^{(\delta-1)}$ can be seen from Fig. 1, which gives the plot for y_j as a function of x_j'. The plot becomes approximately linear for $\delta\simeq 0.565$, for even the strong quantum case of $x_j'=1$, the allowed range being $\infty\geq x'\geq 1$. Linearity is attained for all values as $x_j'\rightarrow\infty$, which is the classical limit.