Derivative expansions for affinely quantized field theories. II. Scale-invariant quantization

R. J. Rivers and C. C. Wong
Blackett Laboratory, Imperial College, London SW7 2BZ, United Kingdom

Carl M. Bender
Department of Physics, Washington University, St. Louis, Missouri 63130
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It is shown that a classically free scalar field quantized with a scale-invariant measure can plausibly become an interacting theory in $d \le 5$ dimensions.

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I. INTRODUCTION

In a previous work [1] (henceforth known as I), we used the tactics of derivative expansions [2] to examine the possible nontriviality of a classically *free* scalar field ϕ quantized with a *scale-covariant* measure. Such measures were proposed by Klauder [3] and examined by himself and others [4,5], motivated by questions of triviality [3], nonrenormalizability [6], and, less directly, by new ways to quantize gravity [7].

Scale-covariant measures arise naturally if, instead of the equal-time canonical commutation relations

$$[\phi(\mathbf{x},t),\Pi(\mathbf{y},t)]=i\delta(\mathbf{x}-\mathbf{y}) , \tag{1.1}$$

where $\Pi(x)$ is the field conjugate to $\phi(x)$, we adopt the equal-time affine commutation relations

$$[\phi(\mathbf{x},t),K(\mathbf{y},t)]=i\phi(\mathbf{x},t)\delta(\mathbf{x}-\mathbf{y}) . \tag{1.2}$$

In (1.2), $K=\frac{1}{2}(\Pi\phi+\phi\Pi)$ is the generator of scale transformations of the fields. The choice of (1.2) over (1.1) corresponds to implementing scale transformations unitarily, rather than field translations. Quantum mechanically, this choice is perfectly acceptable. For a more detailed discussion, see Ref. [8].

The Euclidean generating functional that we shall consider is

$$Z[j]=\int D'\phi \exp\left[-S[\phi]+\int j\phi\right] , \tag{1.3}$$

where $S[\phi]$ is the usual *free-field* action in d dimensions,

$$S[\phi]=\int d^dx[\tfrac{1}{2}(\nabla\phi)^2+\tfrac{1}{2}m_0^2\phi^2] . \tag{1.4}$$

It is the measure $D'\phi$ that reflects the unitary nature of field scaling; for positive functions $\Lambda(x)>0$, $D'\phi$ exhibits the scaling behavior

$$D'(\Lambda\phi)\propto D'\phi . \tag{1.5}$$

This is to be contrasted with the *canonical* measure $D\phi$, derived from (1.1), that preserves the translation invariance

$$D(\phi+\Lambda)=D\phi, \quad \forall\Lambda(x) . \tag{1.6}$$

One possible realization of $D'\phi$ is in terms of $D\phi$, as in [3],

$$D'\phi=\frac{D\phi}{\prod_x|\phi|^{1-2f}}, \quad f\ge 0 , \tag{1.7}$$

where the product is taken over all space-time points. A classically *free* theory quantized with the new measure $D'\phi$, as in (1.3), has been termed a *pseudofree* theory by Klauder. Our aim is to determine if there are any choices of f for which the pseudofree theory of (1.3) in an *interacting* theory. This will be sufficient to demonstrate that noncanonical quantization can change the equivalence class of a field theory, as anticipated in Refs. [3–8]. With one possible exception [9], this has not been confirmed outside special cases such as the large-N limit [10].

Despite earlier optimism [3], the results of Monte Carlo calculations [4,5], high-temperature series expansions [3,5], and the derivative expansion of I show that scale-covariant measures of the form (1.7) do not lead to interacting theories for nonzero f. However, the independent-value model (IVM) [11], in which the kinetic term $\frac{1}{2}(\nabla\phi)^2$ is omitted from $S[\phi]$ of (1.4), provides an example in which a nontrivial theory can be achieved provided $f\to 0$. In these circumstances $D'\phi$ is formally scale invariant, viz.,

$$D'(\Lambda\phi)=D\phi, \quad \Lambda(x)>0 . \tag{1.8}$$

We remark that we cannot simply set $f=0$ in (1.7) because of the logarithmic divergence at small ϕ. Thus the way that f vanishes is crucial. If the IVM is regularized by putting it on a cubic lattice having lattice spacing a, it can be shown [8,11,12] that

$$Z_{\rm IVM}[j]=\int D'\phi \exp\left[-\int \tfrac{1}{2}m_0^2\phi^2+\int j\phi\right] \tag{1.9}$$

is non-Gaussian in j if and only if

$$f=O(a^d) \quad \text{as } a\to 0 . \tag{1.10}$$

Specifically, if

$$f=ba^dm_0^d , \tag{1.11}$$

then, by direct construction [8],

$$Z_{\rm IVM}[j] = \exp\left[bm_0^d \int d^dx \int \frac{du}{|u|}(\cosh ju - 1)e^{-(1/2)\bar{m}^2u^2} \right] , \tag{1.12}$$

where $\bar{m}^2 = bm_0^{d+2}$.

The contrast between functionals such as (1.12) and their canonical counterparts is reflected in the operator formalism. Here the IVM scalar field takes a bilinear form in terms of annihilation and creation operators in a "translated" Fock space [11], with an unconventional operator-product expansion. This indicates how the commutation relations (1.2), based on operator products, can be inequivalent to their canonical counterparts (1.1).

There is reason to believe that the limit $f \to 0$ in the measure (1.7) is also important for the full noncanonical theory. A considerable effort [13] has been expended on attempting an analytical solution in this case, essentially without any success. In fact, calculations about non-Gaussian measures are most easily performed on the lattice, and we shall follow I in using a lattice to convert the Euclidean theory of (1.3) into a d-dimensional spin system. Treated as a continuous-spin ferromagnet, the single-site spin distribution acquires a factor $d\sigma/|\sigma|^{1-2f}$. As $f \to 0$, this distribution becomes so singular that the Lebowitz inequality, which bounds the dimensionless coupling constant γ_4 from below, no longer holds [3,4]. This bound is crucial in the proof of triviality, and its violation permits a change in universality class. Some caution is required because the continuum theory does not seem to exist for fixed $0 < 1-2f < 1$, for which the analogue spin system displays a *first-order* transition [4,5]. However, the limit $f \to 0$ can exist. This paper, in almost its entirety, will be devoted to examining the various ways in which f can be taken to zero in (1.3) and (1.7).

The tuning of f will require some delicacy, and an analytic or semianalytic approach is necessary. The well-tabulated approach of using the high-temperature series is, however, predicated on the existence of second-order transitions and may be deceptive as $f \to 0$ through positive values. Instead, we shall continue with the approach of I and build upon the noncanonical success of the IVM by expanding about it in powers of field derivatives.

Formally,

$$Z[j] = \exp\left[-c \int \frac{1}{2} \left[\nabla \frac{\delta}{\delta j} \right]^2 \right] \times \int D'\phi \exp\left[-\int \frac{1}{2} m_0^2 \phi^2 + \int j\phi \right] \Bigg|_{c=1} . \tag{1.13}$$

The parameter c is a bookkeeping device. It will be set to unity after the partial series in c have been evaluated. Setting c to zero recovers the IVM of (1.9).

We should not assume that $D'\phi$ of (1.13) is necessarily the $D'\phi$ of the IVM, as given in (1.11). The vanishing of f as $a \to 0$ can be understood as a renormalization effect (with f an effective coupling) that will be modified by the presence of derivatives of the field ϕ, as will be shown.

In Sec. II we display the formal series in c, at $c=1$, for the dimensionless four-point coupling constant at zero momentum, γ_4. (For its definition see I. We merely note that, if the Lebowitz inequality is satisfied, then $\gamma_4 \geq 0$.) In Secs. III and IV we try to choose f so that $\gamma_4 \neq 0$ in the continuum limit; that is, we try to choose f so that we have a noninteracting theory. Despite the relative brevity of the series, we believe that we have the makings of a nontrivial pseudofree theory, as explained in Sec. V.

II. SERIES EXPANSION FOR γ_4

Our starting point is the series expansion in powers of c (at $c=1$) for the dimensionless four-point coupling constant γ_4. [This was given in I in Eq. (5.19) of that paper.] The relevant expansion parameter is

$$y = \frac{1}{a^2M^2} , \tag{2.1}$$

where a is the lattice spacing and M is the fixed renormalized mass of the ϕ particle (defined by the pole of the two-point function).

The reader is referred to I for the details of the series, which will not be recalculated here. For example, to order y^3

$$\gamma_4 = \frac{y^{-d/2}}{f^2}\{(2f-1)f + 4d(2f-1)fy + (2f-1)[4df^2 - d(10f-3)]y^2/f + O(y^3)\}_{c=1} . \tag{2.2}$$

We note that γ_4 vanishes when $2f=1$, as it must from (1.7), and we recover the canonical free theory.

The series (2.2) is the start of an infinite double series for γ_4 of the form

$$\gamma_4 = y^{-d/2}A(cy, 1/f)_{c=1} , \tag{2.3}$$

where

$$A(cy, 1/f) = \sum_{k,l \geq 0}^{N} c^k h_{kl} y^k (1/f)^l , \tag{2.4}$$

in which

$$h_{kl} = 0, \quad k > l > 1 . \tag{2.5}$$

The powers of c count the number of field derivatives that have been included. Higher powers require greater effort. It will be seen that relatively short series are sufficient to establish a pattern. We have calculated h_{kl} for $k, l \leq 5$. The results are given in Table I.

The IVM is obtained from the $k=0$ terms of (2.4) at $c=1$,

$$\gamma_4^{\rm IVM} = y^{-d/2}\left[2 - \frac{1}{f}\right] , \tag{2.6}$$

which corresponds to putting $cy=0$ in the double sum for A. In the continuum limit, $y \to \infty$ and $\gamma_4^{\rm IVM}$ vanishes at fixed f. However, provided $f \sim y^{-d/2}$ [i.e., $f = O(a^d)$], $\gamma_4^{\rm IVM}$ has a negative, finite continuum limit, as we had an-

TABLE I. Values of the coefficients h_{kl} of Eq. (2.9) in arbitrary dimension d for $k,l \le 5$.

k \ l	0	1	2	3	4	5
0	2	-1	0	0	0	0
1	$8d$	$-4d$	0	0	0	0
2	$8d^2-20d$	$16d-4d^2$	$-3d$	0	0	0
3	$32d$	$-112d/3$	$40d/3$	$-4d/3$	0	0
4	$60d-160d^2$	$(440d^2-68d)/3$	$(-98d^2-51d)/3$	$(21d-d^2)/3$	$-d/6$	0
5	$512d^3+208d^2$ $-384d$	$(-13\,120d^3-13\,920d^2$ $+14\,016d)/30$	$(2320d^3+8120d^2$ $-5024d)/30$	$(200d^3-1160d^2$ $+288d)/30$	$(-100d^2+72d)/30$	$4d/30$

ticipated earlier. [There are no difficulties with stability because the six-leg vertex $\gamma_6^{\rm IVM}=y^{-d}(2f-1)(3f-1)/f^2$ is positive in this limit.]

In I the calculations were performed at fixed f, but with our earlier comments in mind, we shall now explore the possibility that f vanishes in the continuum limit, not necessarily as $f\sim y^{-d/2}$, but as

$$f=\frac{1}{by^n}\ , \qquad (2.7)$$

for some $n>0$ (not necessarily integral) and fixed $b>0$. The series (2.4) then takes the form $(c=1)$

$$\gamma_4=y^{-d/2}\sum_{k>l}\sum_{l\ge 0} h_{kl}b^l y^{k+nl}\ . \qquad (2.8)$$

Let us assume that

$$\mathcal{A}(y;n,b)=A(y,by^n)=\sum_{k,l} h_{kl}b^l y^{k+nl} \qquad (2.9)$$

has the large-y (continuum) behavior

$$\mathcal{A}(y;n,b)\sim y^{\rho(n,b)}\ \text{ as } y\to\infty\ . \qquad (2.10)$$

A necessary condition that γ_4 remain finite and nonzero (i.e., that the theory is nontrivial) is

$$\rho(n,b)=d/2\ . \qquad (2.11)$$

The remainder of this paper is devoted to examining the circumstances under which (2.11) is likely to be true, given our knowledge of the partial series

$$\mathcal{A}_N(y;n,b)=\sum_{k,l=0}^{N} h_{kl}b^l y^{k+nl}\ , \qquad (2.12)$$

for $N\le 5$ as derived from Table I.

Before going into detail, some general observations can be made. First, for $n>N$, $\mathcal{A}_N$ is independent of b, and the approximants for ρ are those obtained from the series

$$\mathcal{A}(y,0)=\sum_k b_{k0}y^k\ , \qquad (2.13)$$

in which b is set to zero in (2.9). That is, for $n>N$ we derive a b-independent ρ identical to that obtained from the limit $f\to\infty$ (for the same N), even though we are considering the case $f\to 0$. As we saw in I, the $f\to\infty$ limit of the pseudofree theory is identical to the strong-coupling limit of an *interacting* $\lambda\phi^4$ *canonical theory*, as calculated in [2]. This is independently confirmed by the calculations of [9] based on the high-temperature series, which show a change in universality class for large f from that of the Gaussian theory. Thus, for $n>N$, the model behaves nontrivially in $d=2$ and 3 dimensions and is otherwise trivial.

Of course, for fixed n we must take $N>n$, ultimately. At the opposite extreme, if we take $n=0$ in (2.9), we recover the partial series

$$\mathcal{A}_N(y;0,b)=\sum_{k,l=0}^{N} h_{kl}b^l y^k\ . \qquad (2.14)$$

This is just the partial sum for $A(y,b)$, the series for fixed $1/f=b$. In I we showed that, for $b>2$, the series did not permit a simple power behavior, corresponding to the absence of a continuum limit (anticipated on other grounds [4,5]). However, for $b<2$, ρ is calculable. Thus, if there were continuity in $\rho(n,b)$ as n varies from 0 to N, the results of I would fix the extreme behavior. In reality the situation is much more complicated, but the results of I remain a useful touchstone.

III. LOOKING FOR NONTRIVIALITY

Let us return to the IVM of (2.6), rewritten as

$$\gamma_4^{\rm IVM}=\frac{y^{-d/2}}{f}(2f-1)\ . \qquad (3.1)$$

Consider paths in the plane with axes labeled by $-\ln f$ and $-\ln y$ (Fig. 1). The physical region $(y\to\infty)$ as $f\to 0$ lies in the top left quadrant. The choice (2.7),

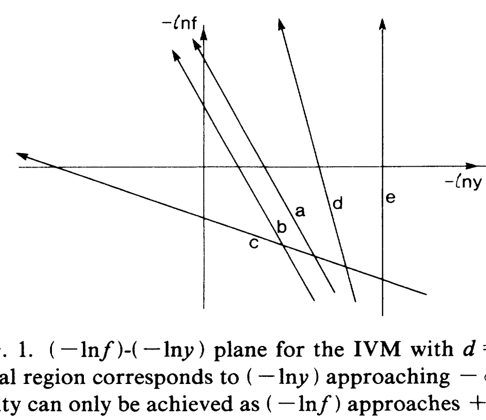


FIG. 1. $(-\ln f)$-$(-\ln y)$ plane for the IVM with $d=4$. The physical region corresponds to $(-\ln y)$ approaching $-\infty$. Nontriviality can only be achieved as $(-\ln f)$ approaches $+\infty$ along lines such as a and b with gradient -2. The model is trivial along line c and singular along lines d and e.

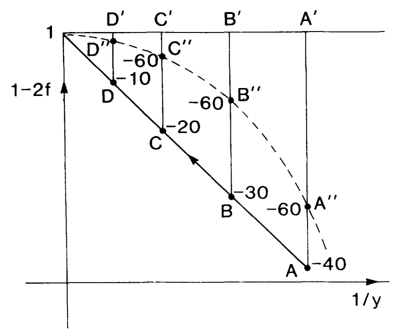


FIG. 2. $(1-2f)$-y^{-1} plane for the IVM in $d=4$ dimensions. The top left-hand corner ($f=0$, $y=\infty$) gives the continuum interacting limit. Approached along the line ABCD, the model is trivial. Along the line A'B'C'D' the model is singular. For a given value of γ_4, it is possible to find a curve (e.g., A''B''C''D'') in which a nontrivial continuum limit can be achieved. The numbers displayed against the points are the values of γ_4 chosen in the example.

$$-\ln f=\ln b+n\ln y \tag{3.2}$$

describes a line, slope $-n<0$, with intercept $\ln b$.

To be specific we take $d=4$. Then only for lines with gradient $n=d/2=2$ is there a nontrivial limit in the second quadrant. Lines for which the slope is shallower give vanishing γ_4 in the limit, lines for which the slope is steeper give a divergent γ_4 in the limit.

In particular, a vertical line (fixed y, vanishing f) gives a singular γ_4. We observe that, had we not known in advance to choose $n=1$, the existence of lines along which γ_4 vanishes, and adjacent lines along which γ_4 diverges would have strongly implied lines along which γ_4 achieved a nonzero limit.

This can be justified from a different viewpoint. Figure 1 provides the best way to discriminate between different n and b, with large (small) b corresponding to lines with large positive (negative) intercepts. However, consider the alternative provided by Fig. 2. Suppose, not knowing that we needed curves with $n=2$ for nontriviality, we had chosen a straight-line path with $n=1$, for which the continuum limit is trivial. Then, as we go toward the top left corner ($f=0$, $y=\infty$), γ_4 vanishes. Specifically, consider the points A, B, C, and D (given schematically in Fig. 2) at which γ_4 takes values -40, -30, -20, and -10, say. At each of the four points, take the vertical lines of constant y^{-1} (AA',BB',CC',DD'). Suppose, for example, now that we wished to find a path on which $\gamma_4=-60$. Assuming continuity in γ_4, along the vertical lines, each possesses a point (A'',B'',C'',D'') for which $\gamma_4=-60$. Joining these points gives a path on which the nontrivial continuum limit can be recovered as $y\rightarrow\infty$. Different choices of γ_4 naturally require different paths, all parabolic near the top left corner of Fig. 2.

Before attempting to replicate these results for the full theory, there is a further calculation to be performed. It was trivial, from (3.1), to see that γ_4^{IVM} diverges as $f\rightarrow 0$ for fixed y. In general, we assume that, for *fixed y*,

$$\gamma_4 \sim y^{-d/2}(1/f)^{\tilde{\rho}}\ , \tag{3.3}$$

as $f\rightarrow 0$. Let $\tilde{\rho}_N(y)$ denote the Nth approximant to $\tilde{\rho}(y)$, as calculated by the method of [2]. We find that

$$\begin{aligned}
\tilde{\rho}_3(y)&=\frac{24.3}{[1631.2+47.19y^{-1}-7.67y^{-2}+O(y^{-3})]^{1/3}}\ ,\\
\tilde{\rho}_4(y)&=\frac{7.68}{[235.4+12.41y^{-1}-1.53y^{-2}+O(y^{-3})]^{1/4}}\ ,\\
\tilde{\rho}_5(y)&=\frac{18.75}{[101\,431.8+7728.2y^{-1}-806.4y^{-2}+O(y^{-3})]^{1/5}}\ .
\end{aligned} \tag{3.4}$$

In each case $\tilde{\rho}_N(y)\geq 0$, for large y (Fig. 3), showing that γ_4 diverges (negatively) as $f\rightarrow 0$ for fixed y at least as severely as for the IVM.

IV. RESULTS IN $d\geq 2$ DIMENSIONS FOR INTEGER n

The results of I were very sensitive to the value of the dimension d, and we expect that to be the case here also. We shall consider the dimensions in ascending order.

A. $d=2$ dimensions

A necessary condition that the theory be nontrivial in $d=2$ dimensions is, from (2.11), that

$$\rho(n,b)=1\ . \tag{4.1}$$

Taking the IVM (with $n=d/2$) as a guide, we first consider this possibility for integer values of $n\geq 1$, for which

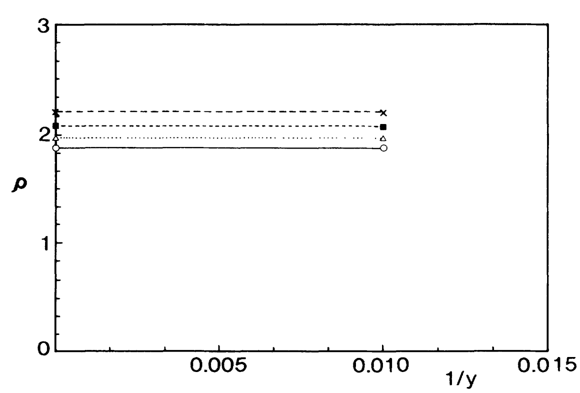


FIG. 3. Approximants for the critical index $\tilde{\rho}$ of (3.3) for the partial series of (2.4) at $k=5$ with $N=3$, 4, and 5, and denoted by squares, triangles, and circles, respectively. The approximant for $\tilde{\rho}$ with partial series (2.4) at $k=4$ is also given by the crosses. For all values of N and large y, $\tilde{\rho}_N>0$. $\tilde{\rho}$ is approximately constant over the range shown.

the series (2.9) can be used directly.

For $n=1$, the series $\mathcal{A}(y;1,b)$ takes the form

$$\mathcal{A}(y;1,b)=h_{00}+(h_{10}+h_{01}b)y+(h_{20}+h_{11}b)y^2 +(h_{30}+h_{21}b)y^3+(h_{10}+h_{31}b+h_{22}b^2)y^4 +(h_{50}+h_{41}b+h_{32}b^2)y^5+O(y^6)\ . \tag{4.2}$$

Adopting the method of I, we use the series (4.2) to calculate the extrapolants for $\rho_N(1,b)$ for $\rho(1,b)$ when $N=2,3,4,5$. Here N denotes the highest power of y taken in the partial series $\mathcal{A}_N(y;1,b)$. Details of the method can be found there. For the case in hand,

$$\begin{aligned}\rho_2(1,b)&=\frac{(b-16)^2}{(b^2-2b+320)}\ ,\\ \rho_3(1,b)&=\frac{4(b-16)^2}{(16b^4+496b^3+32\,516b^2+101\,760b+1\,172\,480)^{1/2}}\ ,\\ \rho_4(1,b)&=9(b-16)^2(729b^6+69\,498b^5+5\,598\,828b^4+78\,619\,672b^3+1\,338\,961\,920b^2\\ &\quad+2\,776\,958\,976b+12\,350\,062\,592)^{-1/3}\ ,\\ \rho_5(1,b)&=16(b-16)^2(65\,536b^8+12\,550\,144b^7+1\,433\,409\,536b^6+42\,803\,614\,539b^5+1\,075\,917\,665\,637b^4\\ &\quad+9\,076\,399\,564\,459b^3+73\,835\,378\,488\,661b^2+88\,741\,717\,823\,853\,b+291\,339\,511\,005\,117)^{-1/4}\ .\end{aligned} \tag{4.3}$$

We first note that ρ_N is very sensitive to the value b. However, for all calculated orders, we observe that

$$\rho_N(1,b)\to 1 \quad \text{as } b\to\infty\ . \tag{4.4}$$

At the other extreme, we see that

$$\rho_N(1,b)\simeq 1 \quad \text{as } b\to 0\ , \tag{4.5}$$

for $N>2$. Specifically, we have, for $b=0$,

$$\begin{aligned}\rho_3&=0.946\ ,\\ \rho_4&=0.997\ ,\\ \rho_5&=0.991\ .\end{aligned} \tag{4.6}$$

The series (4.6) must converge to $\rho=1$ for $N\to\infty$ because, as we observed earlier, it is the critical index for a strong-coupling canonical theory, and such a theory is nontrivial in $d=2$ dimensions. The sequence (4.6) provides a positive check on the accuracy of our scheme in low order. In Fig. 4 we plot $\rho_N(1,b)$ for $N=2,3,4,5$ for $0<b<\infty$. We see that these functions agree quite well over the whole range of b. (Even the $N=2$ approximant is in good agreement for $b>16$.)

For $n>1$, the agreement between low-N approximants becomes worse as n gets larger, as one might expect from our earlier comments.

For example, for $d=2$ and $n=4$,

$$\mathcal{A}(y;4,b)=h_{00}+h_{10}y+h_{20}y^2+h_{30}y^3+(h_{40}+h_{01}b)y^4 +(h_{50}+h_{11}b)y^5+O(y^6)\ . \tag{4.7}$$

It follows that ρ_2 and ρ_3 are independent of b, whereas ρ_4 and ρ_5 depend significantly upon b. Thus ρ_2,ρ_3 can only give a good estimate when $b\ll 1$. As b becomes large, only extrapolants with $N\geq 4$ will reflect the true scaling character of the large-N series.

In Fig. 5 we display the extrapolants for $d=2$, $n=4$ at various values of b. We have observed that $\rho(n,0)$ is independent of n. Therefore we are not surprised that, for $n=4$,

$$\rho_N(4,0)\simeq 1\ , \tag{4.8}$$

just as for $n=1$. However, we now have

$$\rho_N(4,b)\to 0 \quad \text{as } b\to\infty\ , \tag{4.9}$$

unlike the case for $n=1$.

In fact, the $n=4$ case is typical of the range $1<n\leq N$, $N\leq 5$. In Fig. 6 we show ρ_3,ρ_4,ρ_5 as functions of b for $n=2,3,5$. In each case $\rho_N(n,b)$ decreases from $\rho_N\approx 1$ at

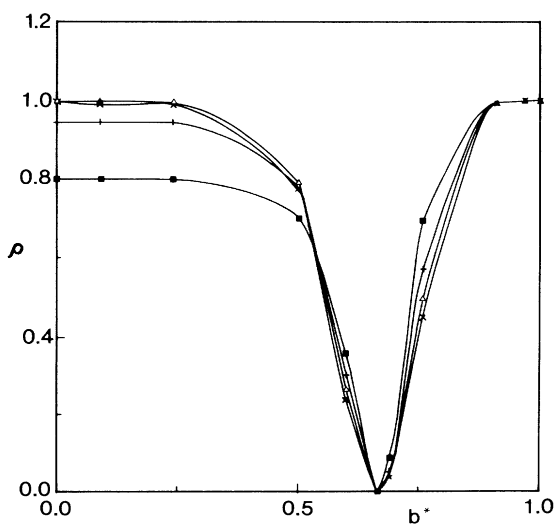


FIG. 4. Plot of ρ as a function of $b^*=b^{1/4}/(1+b^{1/4})$ in $d=2$ dimensions for $n=1$, where the squares, straight crosses, triangles, and slant crosses denote the two-, three-, four-, and five-line extrapolants, respectively.

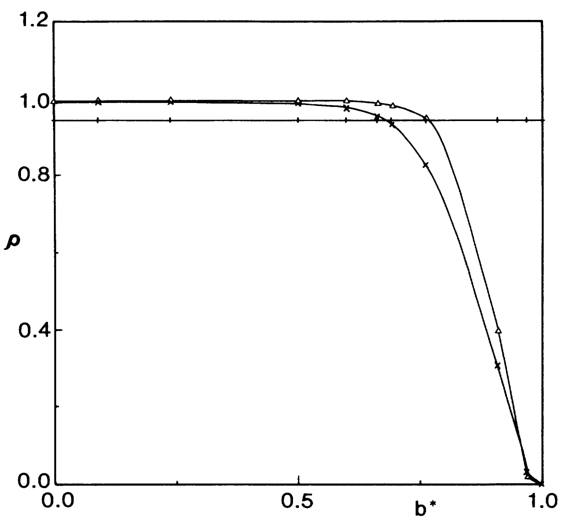


FIG. 5. Plot of ρ as a function of $b^* = b^{1/4}/(1+b^{1/4})$ in $d=2$ dimensions for $n=4$, using the notation of Fig. 1.

$b=0$ to $\rho_N=0$. These additional results give nothing new, as is seen when we plot ρ_5 for different n over a wide range of b (Fig. 7). We shall return to this later.

B. $d=3$ dimensions

The calculations for $d>2$ dimensions are much less complicated than for $d=2$ dimensions. A necessary condition for nontriviality in $d=3$ dimensions is that $\rho=1.5$. For integer n we plot, in Fig. 8, the behavior of $\rho(n,b)$ for different b. We observe that, qualitatively, the picture is very similar to that for $d=2$ dimensions.

In particular, for $n=1-5$ the theory will be trivial apart from the $b \to 0$ limit when we recover the nontriviality of the canonical strong-coupling theory. This enables us to estimate the errors in low-order perturbation theory. For example, for $n=2$,

$$\rho_3 = 1.247 ,$$
$$\rho_4 = 1.369 , \quad (4.10)$$
$$\rho_5 = 1.385 ,$$

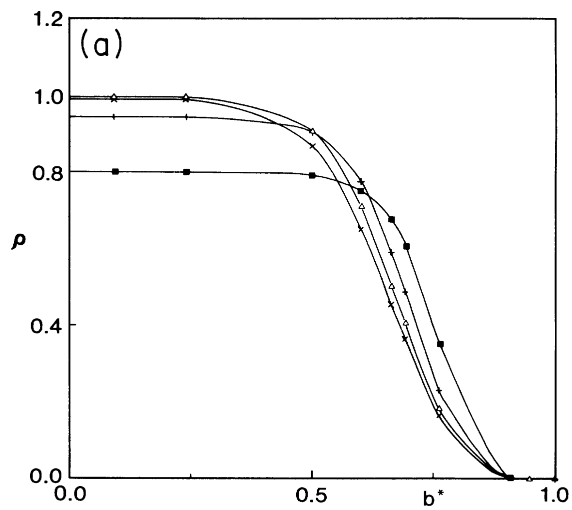
not yet achieving 1.5. On the other hand, the $b \to \infty$ limit at $n=1$ still gives $\rho_N(1,\infty)=1$, for all N, now not a desirable goal.

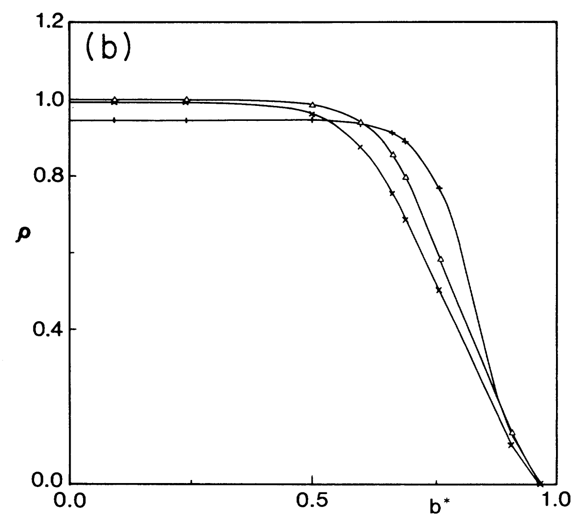
C. $d \geq 4$ dimensions

Beginning with $d=4$ dimensions, it is necessary that $\rho=2$ for there to be a chance of a nontrivial theory. Our results are given in Fig. 9. There is no indication of nontriviality, but we note that ρ_N increases with n for fixed $b>0$. Two points are worth making. First, for the small-b limit,

$$\rho(n,0) \sim 1.65 < 2 , \quad (4.11)$$

for $n=1,2,3,4,5$. This concurs with the result of I and





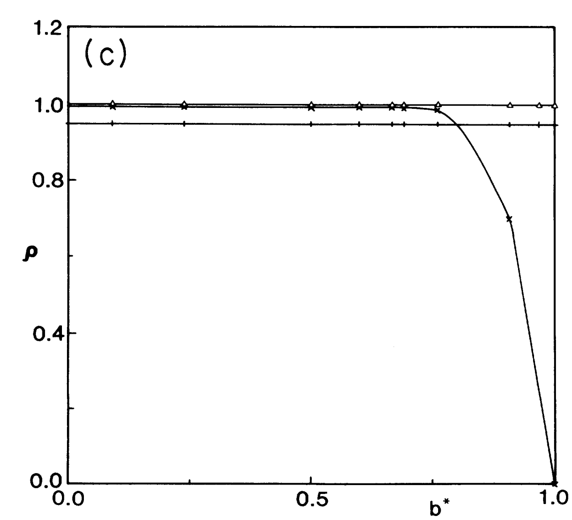


FIG. 6. Same plot as Fig. 4 for (a) $n=2$, (b) $n=3$, and (c) $n=5$.

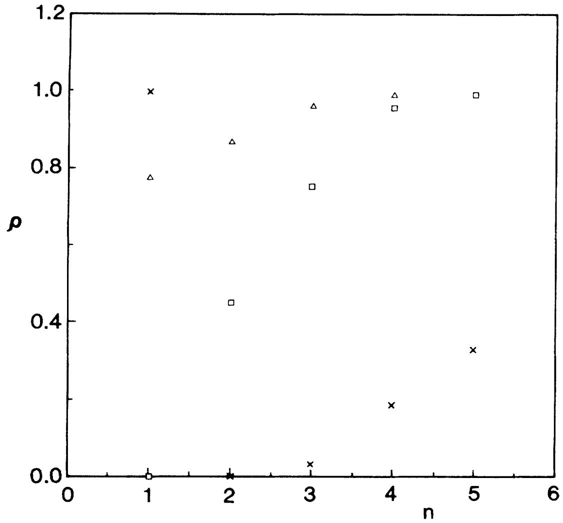


FIG. 7. Plot of ρ ($d=2$ dimensions) as a function of n for $n=1$, 2, 3, 4, and 5, where the triangles, boxes, and crosses denote $b=1$, 15, and 10^5, respectively.

[2] that the canonical $\lambda\phi^4$ theory is trivial in the strong-coupling limit. (At $n=1$ the $b\to\infty$ limit gives $\rho=1$ as before.) Second, for $n=0$ and $b<2$,

$$\rho(0,b)\sim 1.75<\frac{d}{2}=2 \;. \tag{4.12}$$

This result was also obtained in I. Thus small b, small n is not a likely region for finding nontriviality.

For $d\geq 5$ dimensions we believe the theory to be trivial for all constant $f>0$ (from I). Just as for $d=4$, there seems to be no hope of nontriviality when $n=2,3,4,5$, but we shall not display the results here.

V. PRELIMINARY ANALYSIS

We begin with $d\geq 4$ dimensions, plotting the results of the previous section in Fig. 10 in the style of Fig. 1.

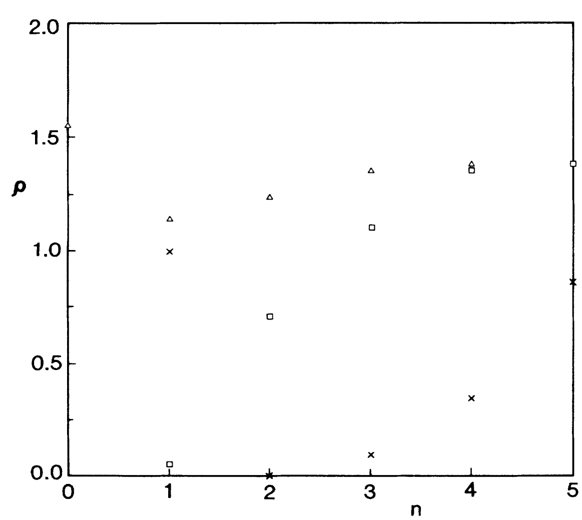


FIG. 8. Same plot as Fig. 7, but for $d=3$ dimensions.

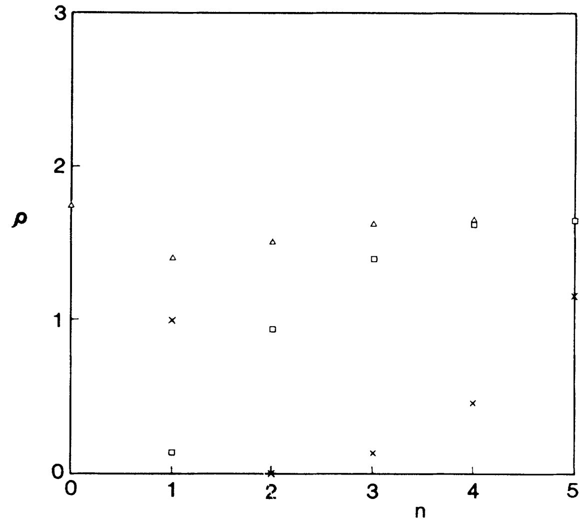


FIG. 9. Same plot as Fig. 7, but for $d=4$ dimensions.

Lines a,b,c,d,e,f correspond to $n=0,1,2,3,4,5$. As we move from a to b to c to d to e to f, the number of usable approximants decreases and our conclusion becomes more tentative. However, there is no reason to expect anything but triviality in any of these directions. Nonetheless, as the slope of the lines increases, the intercept (and hence b) also increases, potentially driving us closer to nontriviality. From Sec. III we know the vertical line g to be leading to a singular theory. There is therefore the strong possibility of finding a direction between f and g, for large n, in which a nontrivial theory is recovered.

The greatest complications occur for small n. The line a is expected to lead to a trivial theory only when the intercept $\ln b<\ln 2$. This is well understood numerically for $d=4$ in Monte Carlo calculations for which the continuum limit ($y\to\infty$) is inaccessible [4,5] when $b>2$ because of a first-order phase transition. The inaccessible region is represented schematically in Fig. 10 by the shaded area.

A first-order transition is predicted by mean-field

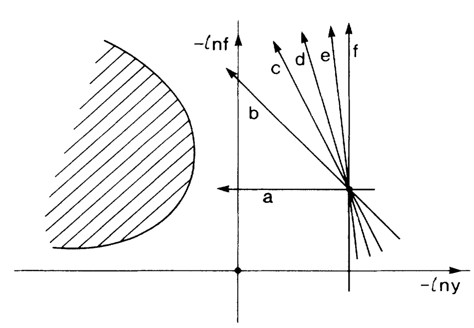


FIG. 10. $(-\ln f)$-$(-\ln y)$ plane. Lines b, c, d, and e lead to triviality in all dimensions, whereas line f gives a singular γ_4. The inaccessible (shaded) region is best understood for $d=4$ dimensions.

theory, and although it had not been checked explicitly, we expect such a region to exist for $d>4$ dimensions. Equally, pseudofree theories for fixed $f<\frac{1}{2}$ do not seem to exist in $d=2$ dimensions [5]. As a working hypothesis, we assume such a region for all d.

For $d=2$ dimensions and $d=2$ alone, there is an additional effect in that the line with $n=1$ (line b in Fig. 10) takes us to nontriviality as the intercept ln$b$ increases, but otherwise the situation is effectively the same, with hope for nontriviality for large n.

From the observations above, we need a better understanding of the situation for large and small n. Some progress can be made on each front.

A. Speculations for large n

Suppose, in trying to approach the $n\to\infty$ limit in Fig. 10, we substitute y for f in A of (2.4) by

$$y=\frac{1}{\bar{b}}\left[\frac{1}{f}\right]^{1/n}, \tag{5.1}$$

rather than (2.7). That is, $\bar{b}=b^n$ is taken to be n independent. Then $A(y,1/f)$ becomes

$$A\left[y,\frac{1}{f}\right]=\sum_{k,l\geq 0}h_{kl}\frac{1}{\bar{b}^k}\left[\frac{1}{f}\right]^{l+k/n}. \tag{5.2}$$

The sum obtained by taking the limit $n\to\infty$ (for which y is fixed, as in line f of Fig. 10) is

$$\bar{A}=\sum_{l\geq 0}H_l\left[\frac{1}{f}\right]^l, \tag{5.3}$$

where

$$H_l=\sum h_{kl}\frac{1}{\bar{b}^k}. \tag{5.4}$$

H_l is the weighted sum of the lth column of Table I to all values of k, of which we know nothing. However, let us suppose that, for large $\bar{b}$, H_l is dominated by the first few terms of the series. If, for small f,

$$\bar{A}\sim\left[\frac{1}{f}\right]^{\sigma}, \tag{5.5}$$

then $\rho\sim n\sigma$. Thus, if $\sigma>0$, for large n, $\rho>d/2$ and we have a singular theory. We note that, as $\bar{b}\to\infty$, we recover the IVM for which $\sigma=1$.

Approximants σ_N to σ are calculated as before. For example, at $\bar{b}=10^4$ in $d=4$ dimensions, σ_N takes the values

$$\begin{aligned}\sigma_2&=1.999\,,\\ \sigma_3&=1.286\,,\\ \sigma_4&=1.143\,,\\ \sigma_5&=1.087\,.\end{aligned} \tag{5.6}$$

The sequence (5.6) is plausibly converging to a positive limit near unity, indicating the existence of a singular γ_4 in the continuum limit for n sufficiently large. The sequence (5.6) is approximately d independent. This reinforces our prejudices that, with fine tuning, nontrivial pseudofree theories can be found in all dimensions.

B. Speculations for small n

Without any particular expectations for achieving nontriviality for small $n<1$, the presence of an inaccessible region should make itself felt in the approximants.

For $n<1$ it is only possible to construct approximants on the lines of our earlier calculations when n^{-1} is a positive integer. To see this consider the first case of $n=\frac{1}{2}$, writing

$$\frac{1}{f}=bz,\quad y=z^2\,. \tag{5.7}$$

We now have a 15-term series

$$\begin{aligned}\mathcal{A}(z;n,b)=&h_{00}+bh_{01}z+h_{10}z^2+h_{11}bz^3\\ &+h_{20}z^4+h_{21}bz^5+(h_{22}b^2+h_{30})z^6\\ &+h_{31}bz^7+(h_{32}b^2+h_{40})z^8+\cdots\,.\end{aligned} \tag{5.8}$$

If we treat this series in the same way as the series in y, we expect that for $b\gg 1$ the series will give sensible answers up to the 14th extrapolant. However, in the small-b region, the even powers of z, $z^{2p}(p=1,2,3,\ldots)$ dominate. Thus, when spanning the whole range in b, we restrict ourselves to approximants $\rho_{2p}(\frac{1}{2},b)$, $p=1,2,\ldots,7$. [In fact, the odd approximants $\rho_{2p+1}(\frac{1}{2},b)$ agree with the even approximants for large b, but behave irregularly for small b.]

The results are given in Fig. 11 for $d=2$ dimensions. We find that the even approximants only converge for small b or for large b. For intermediate values of b, the

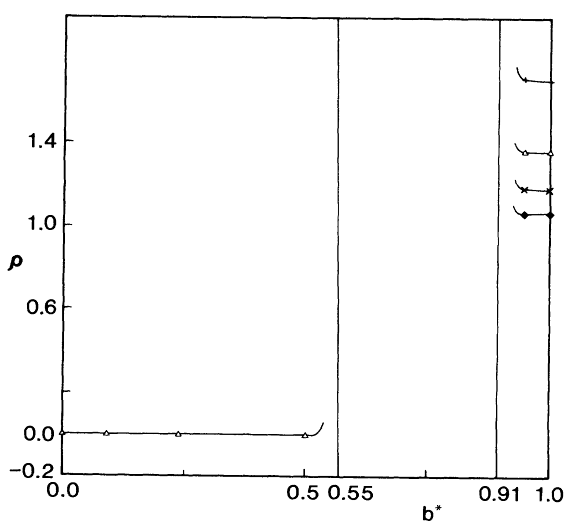


FIG. 11. Same plot as Fig. 1 for $n=\frac{1}{2}$, where the central block denotes the range of b for which the extrapolants become complex. For $b<\frac{1}{2}$ the extrapolants are numerically indistinguishable.

approximants become *complex*. From our previous comments we interpret this as a sign of an inaccessible part of the f-y^{-1} plane. However, some care is needed. Although this region does not make its effect felt at large b (as we would have anticipated), small b is puzzling. It may be that, since small b mitigates the effect of large y in (2.7), such calculations are less reliable.

Even for large b the situation is complicated, as can be seen by developing ρ_N as a series in b^{-2}. For example, the later approximants for $b \gg 1$ are

$$\rho_{12}(\tfrac{1}{2},b)=\frac{0.5470}{[9.70\times10^{-5}-4.476\times10^{4}b^{-2}+4.518\times10^{8}b^{-4}+O(b^{-6})]^{1/12}}\ ,$$
$$\rho_{14}(\tfrac{1}{2},b)=\frac{0.5913}{[2.36\times10^{-4}-2.32\times10^{5}b^{-2}+3.67\times10^{9}b^{-4}+O(b^{-6})]^{1/14}}\ . \tag{5.9}$$

[All previous approximants have been calculated and higher orders in b^{-2} in these and (5.9) are also known.]

As $b\to\infty$, we find that

$$\rho_8(\tfrac{1}{2},\infty)=1.706\ ,$$
$$\rho_{10}(\tfrac{1}{2},\infty)=1.360\ ,$$
$$\rho_{12}(\tfrac{1}{2},\infty)=1.183\ ,$$
$$\rho_{14}(\tfrac{1}{2},\infty)=1.074\ . \tag{5.10}$$

The sequence appears converging, but it is not possible to decide whether, as $N\to\infty$, ρ_N is greater, or less, than unity. What is more relevant is that $\rho(\frac{1}{2},b)\to\rho(\frac{1}{2},\infty)$ from above. Specifically, ρ_N diverges above the upper boundary of the region in b for which ρ is complex.

Accepting these caveats, the approximants show that, for $b\sim10^9$,

$$\rho(\tfrac{1}{2},b)\gg1\ . \tag{5.11}$$

Thus, if $\rho(\frac{1}{2},\infty)<1$, there is a value of b, b_c, for which $\rho(\frac{1}{2},b_c)=1$.

A similar situation occurs for $n=\frac{1}{4}$, for which $\rho_{20}(\frac{1}{4},b)$, say, is complex for $O(1)<b<O(10^3)$, approaching $\rho_{20}(\frac{1}{4},b)\simeq\frac{1}{2}$ from above. However, for $n=\frac{1}{3}$, $\rho_{3N}(\frac{1}{3},b)$ is real, bounded above by $\rho_{3N}(\frac{1}{3},\infty)$, all of which are less than unity for $3N$ large. This suggests caution in identifying the complexity of b with an inaccessible region in the parameter space. Nevertheless, this intermittent behavior is a signal that something is amiss.

The pattern of complex $\rho_N(\frac{1}{2},b)$, $\rho_N(\frac{1}{4},b)$ [and real $\rho_N(\frac{1}{3},b)$] persists in all dimensions. For example, for $d=3$ we have

$$\rho_{14}(\tfrac{1}{2},b)=\frac{1.8708}{[2.36\times10^{3}-3.48\times10^{12}b^{-2}+8.01\times10^{16}b^{-4}+O(b^{-6})]^{1/14}}\ , \tag{5.12}$$

complex for intermediate values of b, greater than 1.5 for carefully chosen $b\gg1$. Similar expressions to (5.12) can be written in $d=4$ and 5 dimensions.

VI. CONCLUSIONS

In the search for nontrivial noncanonical scalar theories, the most hopeful sign has come from the independent-value model of (1.9) onward. Different ways of taking $f\to0$ in the measure (1.7) enables us to recover trivial, singular, or *nontrivial* continuum limits for this derivative-deficient "heavy-mass" limit. We have attempted to build upon this simple model by including more and more field derivatives, as in our previous paper I. The parametrization (2.7) gives us a systematic way in which to let f vanish (whereon we recover a scale-invariant measure). Our conclusions are the following.

(i) In all dimensions d there are directions in the f-y^{-1} plane [or the $(-\ln f)$-$(-\ln y)$ plane] in which the theory becomes trivial and adjacent directions in which it becomes singular. This strongly suggests the existence of the paths for large n (and large b) along which a nontrivial theory can be attained in the continuum limit. This is our main result, giving support to the program proposed by Klauder and developed in the earlier references. As such, it goes some way to repair the deficiencies of earlier attempts to build upon the IVM [14].

(ii) For small n $(n=\frac{1}{4},\frac{1}{2})$, the approximants show pathological behavior in becoming complex for intermediate values of b, although they become real again for large b. This behavior is consistent with a known inaccessible region in parameter space for which the continuum limit cannot be achieved. Such directions are to be avoided.

(iii) In $d=2$ dimensions there is a further limiting case $(n=1,\ b\to\infty)$ in which nontriviality may be obtained. The fact that b becomes infinite should not necessarily cause concern. In the language of Fig. 2, taking $b\to\infty$ for $n=1$ corresponds to folding the line ABCD (along which γ_4 still vanishes) against the singular line A′B′C′D′. A different parametrization to (2.7) might enable us to come in on a nontrivial path.

All this has been achieved with diagrams with $N\le5$ internal lines, using MACSYMA to evaluate the approximants analytically. This is a high-order approximation in comparison to some series that have been developed recently (e.g., the δ expansion recently developed by Bender *et al.* [15]). However, the order of perturbation theory is low in comparison to current high-temperature series expansions in the bookkeeping parameter K [9,16] for which even much older calculations by Baker and Kincaid gave series to eleventh order [17]. (We note that our tactics are similar in spirit to earlier work by Baker and Johnson [18].)

Would we be more predictive if we were to extend our series by a few terms? We doubt it. The patterns already present for $N \leq 5$ would persist, but would not permit us to be more accurate in charting nontriviality. It would take very much longer series to enable us to perform a useful multivariate approximant analysis on γ_4, with its trivial or singular behavior.

Suppose we had found a path along which a nontrivial pseudofree field theory was recovered. In all likelihood it would be sufficiently complicated that the resulting field theory would not be easily calculable. (Even the IVM was difficult enough.) Our assumption is that the path-integral functional $Z[j]$ may not necessarily be the most useful starting point. Rather, we could use the existence of the nontrivial path integral to attempt, yet again, an operator approach to the theory. (For example, the incorporation of time derivatives of the scalar field in the ultralocal model [2] has an operator realization superficially similar to that of the IVM, but its relationship to the singular measures in the path integral is unclear.) Despite the distance yet to go to construct noncanonical theories, we consider our results presented here to provide a modest step along the way.

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