Supersymmetric Chern-Simons vortex systems and fermion zero modes

Bum-Hoon Lee
Department of Physics, Hanyang University, Seoul 133-791, Korea

Choonkyu Lee
Department of Physics and the Center for Theoretical Physics, Seoul National University, Seoul, 151-742, Korea

Hyunsoo Min
Division of Liberal Arts, Seoul City University, Seoul, 130-743, Korea
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Fermion zero modes around a general multivortex background are analyzed in supersymmetrized self-dual (Maxwell–) Chern-Simons Higgs systems, using the index theorem and other means. In the models with an $N=2$ extended supersymmetry, a simple connection is established between all independent fermion zero modes and corresponding bosonic zero modes. We provide a supersymmetry-based explanation of the result.

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I. INTRODUCTION

Recently, there has been a surge of interest in $(2+1)$-dimensional gauge-field theories. This is partially due to the novel possibility involving the Chern-Simons term [1], which can significantly alter the long-distance behavior of the theory (e.g., the statistics of local excitations). The characteristics of allowed solitons are also affected by the presence of the Chern-Simons term. As is well known, the usual $(2+1)$-dimensional Abelian Higgs model supports only electrically neutral vortices as topologically stable soliton solutions [2]. But with the Chern-Simons term introduced into the theory, we can have electrically charged vortices [3] that are (extended) anyons [4]. It is conceivable that these anyonic vortices may turn out to have a significant dynamical role, say, in determining the phase structure of $(2+1)$-dimensional gauge theories.

With some special choice of the Higgs potential in $(2+1)$-dimensional gauge models, one can obtain interesting limiting theories in which the minimum-energy static-soliton solutions satisfy first-order differential equations, called the Bogomol'nyi or self-duality equations. This happens for a specific scalar quartic coupling in the usual Abelian Higgs model [5], while in the case of the "minimal" Chern-Simons Higgs model (i.e., without the Maxwell term in the action) a specific sixth-order potential form is required [6,7]. As described in Ref. [8], there are also more general self-dual systems in which both the Maxwell and Chern-Simons terms are simultaneously present. A remarkable feature with these self-dual systems is the existence of static multisoliton solutions which represent static configurations of several lumps, either superimposed at one point or separated in space. One can confirm this fact by counting independent zero modes to the boson fluctuation equations in the background field of a particular soliton solution, and the index theorem is useful for the purpose.

The appearance of self-dual structures for certain special Higgs potentials may be ascribed either to extended supersymmetry [9,10] or to suitable dimensional reductions of the 4D self-dual Yang-Mills system [11]. In the presence of the Chern-Simons term, the supersymmetry-based understanding (for the case of the minimal Chern-Simons Higgs model) was provided in Ref. [12], while that through dimensional reduction of the 4D Yang-Mills system is yet to be uncovered (but, see Ref. [13]). In the present paper, we shall elucidate the role of supersymmetry further by studying fermion zero modes around a multisoliton background in the context of the fully supersymmetrized versions of the above-mentioned $(2+1)$-dimensional self-dual systems. The index theorem and supersymmetry-based relationships will be two of our main theoretical tools. Note that fermion zero modes are very important in the quantum study of the models, representing the degeneracy of the soliton states (in contradistinction to bosonic zero modes which become collective coordinates [14]). In supersymmetric models in particular, they account for the soliton supermultiplet structure [15]; in this sense, our analysis also constitutes the first step toward a quantum theory of supersymmetric Chern-Simons vortices.

For the models under study it is found that *all* fermion zero modes around the general multivortex background are simply related to the corresponding bosonic zero modes (discussed earlier in Refs. [7] and [16]), independently of the fact that there ought to be a few fermion zero modes related to supersymmetry transformations of the background classical solution. Note that similar observations were made previously for other self-dual systems [17], but in our case the situation is more complex. We shall here provide a simple understanding for this fact by suitably extending the argument given by Zumino some time ago for the instanton background case [18]. The $N=2$ supersymmetry is crucial. Much of our discussion will be addressed to the self-dual Maxwell–Chern-

Simons system of Ref. [8], which contains other self-dual systems mentioned above as limiting cases.

Fermion zero modes of a Dirac operator around the rotationally symmetric vortex background have been analyzed in Ref. [19] using partial-wave analysis. For some specific cases their findings do serve as a useful check for our more formal considerations. But the models considered in Ref. [19] do not have built-in supersymmetry, and as it turns out, their analysis is insufficient to yield any definite conclusion about the number of fermion zero modes for the interesting case of the Dirac equations in $N=2$ supersymmetric Chern-Simons theories (see Sec. III).

The organization of this paper is as follows. In Sec. II we provide first the full content of the $N=2$ supersymmetric generalization (and also the $N=1$ model with fermion-number violation) of the Maxwell–Chern-Simons system of Ref. [8], and review briefly some pertinent facts on the structure of self-dual vortex solutions. In Sec. III, we study the fermion zero modes around the general multivortex background in supersymmetric models. In the $N=2$ models, we first present the index theorem analysis and then substantiate it by presenting the simple formulas for the fermion zero modes in terms of the corresponding bosonic zero modes (satisfying appropriate background gauge conditions). For the sake of comparison the fermion zero modes in the $N=1$ supersymmetric model are also discussed. Then, in Sec. IV, we clarify the role of supersymmetry in the relation between the fermion zero modes and the bosonic ones of the $N=2$ models. Section V contains a summary and discussion of our work. In the appendixes, we present the superfield formulation of the models considered by us and also give a proof of a certain fact which has been used in the main text.

II. SUPERSYMMETRIC MAXWELL–CHERN-SIMONS THEORY

In a (2+1)-dimensional Abelian Higgs model with both Maxwell and Chern-Simons terms in the gauge-field action, we can have static-vortex solutions satisfying the self-duality equations only when the scalar potential takes a very special form [8]. Just as in the case of self-dual Chern-Simons Higgs theory [12], requiring an $N=2$ supersymmetry uniquely fixes this special potential form. (See Appendix A for the proof.) There also exists an $N=1$ supersymmetric model which produces exactly the same bosonic part of the Lagrangian as that of the $N=2$ model. We will consider both possibilities here. Construction of these supersymmetric models is facilitated by the use of superfields. But this superfield formalism is relegated to Appendix A, and we shall below describe these models using component fields.

The Lagrangian for the $N=2$ model is given by

$$\mathcal{L}^{(2)}=\mathcal{L}_B+\mathcal{L}_F^{(2)} , \tag{2.1}$$

where

$$\mathcal{L}_B=-\tfrac{1}{4}F_{\mu\nu}F^{\mu\nu}+\tfrac{1}{4}\kappa\epsilon^{\mu\nu\lambda}F_{\mu\nu}A_\lambda-|D_\mu\phi|^2-\tfrac{1}{2}(\partial_\mu N)^2 -\tfrac{1}{2}(e|\phi|^2+\kappa N-ev^2)^2-e^2N^2|\phi|^2 \tag{2.2}$$

and

$$\mathcal{L}_F^{(2)}=i\bar\psi\gamma^\mu D_\mu\psi+i\bar\chi\gamma^\mu\partial_\mu\chi+\kappa\bar\chi\chi -i\sqrt{2}e(\bar\psi\chi\phi-\bar\chi\psi\phi^*)+eN\bar\psi\psi . \tag{2.3}$$

Here $D_\mu=\partial_\mu-ieA_\mu$ is the covariant derivative, N a real scalar, ϕ a complex charged scalar, and $\psi(\chi)$ a complex charged (neutral) two-component spinor. Our metric tensor $\eta^{\mu\nu}$ has the signature $(-,+,+)$, and the γ matrices satisfy the relation $\gamma^\mu\gamma^\nu=-\eta^{\mu\nu}-i\epsilon^{\mu\nu\lambda}\gamma_\lambda$. This theory possesses the supersymmetry

$$\begin{aligned}
&\delta_\eta A_\mu=i(\bar\eta\gamma_\mu\chi-\bar\chi\gamma_\mu\eta) ,\\
&\delta_\eta\phi=\sqrt{2}\bar\eta\psi,\quad \delta_\eta N=i(\bar\chi\eta-\bar\eta\chi) ,\\
&\delta_\eta\psi=-\sqrt{2}(i\gamma^\mu\eta D_\mu\phi-\eta eN\phi) ,\\
&\delta_\eta\chi=\gamma^\mu\eta(\partial_\mu N+\tfrac{1}{2}i\epsilon_{\mu\nu\lambda}F^{\nu\lambda})+i\eta(e|\phi|^2+\kappa N-ev^2) .
\end{aligned} \tag{2.4}$$

Here the spinor parameter η should be taken as being *complex* Grassmannian.

The Lagrangian with $N=1$ supersymmetry while possessing the same bosonic sector is

$$\mathcal{L}^{(1)}=\mathcal{L}_B+\mathcal{L}_F^{(1)} , \tag{2.5}$$

where

$$\mathcal{L}_F^{(1)}=i\bar\psi\gamma^\mu D_\mu\psi+i\bar\chi\gamma^\mu\partial_\mu\chi+\tfrac{1}{2}\kappa(\bar\chi\chi^c+\bar\chi^c\chi) -i\sqrt{2}e(\bar\psi\chi^c\phi-\bar\chi^c\psi\phi^*)-eN\bar\psi\psi . \tag{2.6}$$

The charge conjugate of χ is denoted χ^c and is obtained by complex conjugation in the Majorana basis. Note that fermion number is not preserved here, and this theory possesses an $N=1$ supersymmetry only. The Grassmannian transformation parameter η must be now taken as a real (i.e., Majorana) spinor, and the detailed $N=1$ transformation rules are that the formulas for $\delta_\eta A_\mu,\delta_\eta\phi$, and $\delta_\eta N$ assume the same form as those given in Eq. (2.4), while the rules for $\delta_\eta\psi$ and $\delta_\eta\chi$ should be changed to

$$\begin{aligned}
&\delta_\eta\psi=-\sqrt{2}(i\gamma^\mu\eta D_\mu\phi+\eta eN\phi) ,\\
&\delta_\eta\chi=\gamma^\mu\eta(\partial_\mu N+\tfrac{1}{2}i\epsilon_{\mu\nu\lambda}F^{\nu\lambda})-i\eta(e|\phi|^2+\kappa N-ev^2) .
\end{aligned} \tag{2.7}$$

See Appendix A for further explanations of the construction of the theory.

When the coupling strength κ for the Chern-Simons term becomes zero, both of the above Lagrangians are essentially equivalent and reduce to the $N=2$ supersymmetric Abelian Higgs model [10]. In another extreme limit of very large κ (with the ratio e^2/κ fixed), the neutral scalar field N (spinor field χ) can be represented in terms of the complex scalar field ϕ (spinor field ψ), since the kinetic term of the neutral scalar (spinor) can be neglected. For example, in the $N=2$ model, it becomes possible to write

$$N=-\frac{1}{\kappa}e(|\phi|^2-v^2),\quad \chi=-\frac{i}{\kappa}\sqrt{2}e\phi^*\psi , \tag{2.8}$$

and using these relations in Eqs. (2.2) and (2.3) yields the Lagrangian

$$\mathcal{L}_{\rm CS}^{(2)}=\frac{\kappa}{4}\epsilon^{\mu\nu\lambda}F_{\mu\nu}A_\lambda-|D_\mu\phi|^2-\frac{e^4}{\kappa^2}|\phi|^2(|\phi|^2-v^2)^2$$
$$+i\bar{\psi}\gamma^\mu D_\mu\psi-\frac{e^2}{\kappa}(3|\phi|^2-v^2)\bar{\psi}\psi\ . \quad (2.9)$$

The corresponding Lagrangian for the $N=1$ model is easily obtained in a similar way:

$$\mathcal{L}_{\rm CS}^{(1)}=\frac{\kappa}{4}\epsilon^{\mu\nu\lambda}F_{\mu\nu}A_\lambda-|D_\mu\phi|^2-\frac{e^4}{\kappa^2}|\phi|^2(|\phi|^2-v^2)^2$$
$$+i\bar{\psi}\gamma^\mu D_\mu\psi+\frac{e^2}{\kappa}(\phi^2\bar{\psi}\psi^c+\phi^{*2}\bar{\psi}^c\psi)$$
$$+\frac{e^2}{\kappa}(|\phi|^2-v^2)\bar{\psi}\psi\ . \quad (2.10)$$

These supersymmetric extensions of the minimal self-dual Chern-Simons Higgs model were given already in Ref. [12].

Now, let us briefly review the structure of self-dual vortices in the Maxwell–Chern-Simons theory (i.e., the theory defined by the bosonic Lagrangian $\mathcal{L}_B$ above). In this theory, there are two degenerate ground states, i.e., a symmetric one where $\phi=0$, $N=ev^2/\kappa$ and an asymmetric one where $|\phi|=v$, $N=0$. It is known that topological solitons exist in the asymmetric phase with the asymptotic behavior

$$N(\mathbf{r})\to 0,\quad |\phi(\mathbf{r})|\to v\quad \text{as } r\to\infty \quad (2.11)$$

and a quantized flux $\Phi=\pm(2\pi/e)n$ (n a positive integer). Nontopological solitons exist in the symmetric phase with the asymptotic behavior

$$N(\mathbf{r})\to\frac{ev^2}{\kappa}+\frac{\text{const}}{r^{2\alpha}},\quad |\phi(\mathbf{r})|\to\frac{\text{const}}{r^\alpha}\quad \text{as } r\to\infty \quad (2.12)$$

and a nonquantized flux $\Phi=\pm(2\pi/e)(n+\alpha)$ (here $n=0,1,2,\ldots$ and $\alpha\geq n+2$). All static solutions must satisfy the Gauss-law constraint

$$\partial_i F^{i0}+\kappa F_{12}+eJ^0=0\ , \quad (2.13)$$

with $J^0=-i(\phi^*D^0\phi-D^0\phi^*\phi)$. Integrating over the whole space then tells us that a configuration with the magnetic flux $\Phi=\int d^2x\, F_{12}$ carries the electric charge $Q\equiv\int d^2x\, J^0=-(\kappa/e)\Phi$. In this theory it has also been shown [8] that the energy of the configuration is bounded from below by the relation $E\geq ev^2|\Phi|$, and is saturated if the configurations satisfy the "self-duality" equations

$$(D_1\pm iD_2)\phi=0\ ,$$
$$F_{12}\pm(e|\phi|^2+\kappa N-ev^2)=0\ , \quad (2.14)$$
$$A^0\mp N=0\ ,$$

together with the Gauss law (2.13). The upper (lower) sign corresponds to a positive (negative) value of the magnetic flux Φ.

In the case of $\kappa=0$, we may consistently set $A^0=N=0$ in Eq. (2.14) and thereby find the equations for self-dual Landau-Ginzburg vortices [5]:

$$(D_1\pm iD_2)\phi=0\ ,$$
$$F_{12}\pm e^2(|\phi|^2-v^2)=0\ . \quad (2.15)$$

On the other hand, in the case of $\kappa\to\infty$, we have instead

$$A^0=\frac{\kappa}{2e^2}\frac{F_{12}}{|\phi|^2}\ , \quad (2.16)$$

and the self-duality equations reduce to those of Ref. [6], viz.,

$$(D_1\pm iD_2)\phi=0\ ,$$
$$F_{12}\pm\frac{2e^3}{\kappa^2}|\phi|^2(|\phi|^2-v^2)=0\ . \quad (2.17)$$

To understand the quantum aspects, we have to consider the fluctuations of fields around the background vortex solutions. Among these, we have zero-mode fluctuations. They play an important role in quantizing the theory and correspond to the collective coordinates associated with the vortices. The equations for the zero-mode fluctuations may be obtained by considering the variation of the self-duality equations (2.14) around the given classical vortex configuration. They read

$$(D_1+iD_2)\delta\phi-ie\phi(\delta A_1+i\delta A_2)=0\ ,$$
$$\partial_1\delta A_2-\partial_2\delta A_1+e(\phi^*\delta\phi+\phi\delta\phi^*)+\kappa\delta A^0=0\ , \quad (2.18)$$
$$(-\nabla^2+\kappa^2+2e^2|\phi|^2)\delta A^0$$
$$+e(\kappa+2eA^0)(\phi^*\delta\phi+\phi\delta\phi^*)=0\ ,$$

where the last equation follows from the Gauss law (2.13). Among the zero modes, those related to gauge transformations are of no interest and can be eliminated by imposing a gauge condition. One group of gauge choices is given by

$$\partial_1\delta A_1+\partial_2\delta A_2+\mathcal{G}(\delta\phi)=0\ . \quad (2.19)$$

If $\mathcal{G}(\delta\phi)=0$, it is the Coulomb gauge studied in Ref. [16]. We will here choose the form

$$\mathcal{G}(\delta\phi)=\kappa\mathcal{J}+ie(\phi^*\delta\phi-\phi\delta\phi^*)\ , \quad (2.20)$$

with $\mathcal{J}$ determined implicitly by the equation

$$(-\partial^2+\kappa^2+2e^2|\phi|^2)\mathcal{J}+ie(\kappa+2eA^0)(\phi^*\delta\phi-\phi\delta\phi^*)=0\ . \quad (2.21)$$

In the next section it will become clear that this particular gauge choice in fact stands out in the sense that the fermion zero modes are then directly related to the bosonic ones. If $\kappa=0$, this choice is not different from the usual background gauge. In the case of $\kappa\to\infty$, on the other hand, this reduces to

$$\partial^i\delta A_i+i\frac{2e^3}{\kappa^2}(2|\phi|^2-v^2)(\phi^*\delta\phi-\phi\delta\phi^*)=0\ . \quad (2.22)$$

Except for some scale difference for the scalar-field fluctuation term, this is the background-gauge choice adopted in Ref. [7].

The index theorem or its variants [20–24] can be used to count the number of bosonic zero modes satisfying Eq. (2.18) and the gauge-fixing condition (2.19). The bosonic zero modes for the $\kappa=0$ case were studied some time ago by Weinberg in the background gauge [23], and those for the $\kappa=\infty$ case in Ref. [7]. Recently the bosonic zero modes for a general value of κ were also studied in Ref. [16], with the Coulomb gauge condition adopted. Naturally the number of these bosonic zero modes is expected to be independent of the specific gauge conditions chosen. Accepting this, the results are as follows. In the background of a topological vortex configuration with vorticity n, there exist $2n$ bosonic zero modes. This is consistent with the interpretation of these zero modes as being related to translation of individual vortices. On the other hand, in the background of a nontopological soliton with vorticity n and asymptotic behavior (2.12) (which exists only when $\kappa\neq 0$), there are $2n+2\hat{\alpha}$ zero modes, $\hat{\alpha}$ being the greatest integer less than α. A physical interpretation of this number was offered in Ref. [7], but it is not completely understood yet.

III. ANALYSIS OF FERMION ZERO MODES

A. The $N=2$ model

We here consider the fermion sector around the self-dual soliton background in the $N=2$ supersymmetric model with the Lagrangian $\mathcal{L}^{(2)}$. Among all the fluctuation modes, we consider only the zero modes that satisfy the time-independent Dirac equations [see the fermion Lagrangian in Eq. (2.2)]

$$\begin{gathered}\gamma^i D_i \psi + ie(\gamma^0 A^0 - N)\psi - \sqrt{2}e\phi\chi = 0 ,\\ \gamma^i \partial_i \chi - i\kappa\chi + \sqrt{2}e\phi^*\psi = 0 .\end{gathered} \tag{3.1}$$

Here the background fields (A_μ,ϕ,N) may represent any particular solution to the self-duality equations (2.14). We will choose the γ-matrix basis where γ^0 is diagonal, specifically

$$\gamma^0=\sigma_3,\quad \gamma^1=i\sigma_2,\quad \gamma^2=i\sigma_1 . \tag{3.2}$$

Then, setting

$$\psi=\begin{bmatrix}\psi_\uparrow\\ \psi_\downarrow\end{bmatrix},\quad \chi=\begin{bmatrix}\chi_\uparrow\\ \chi_\downarrow\end{bmatrix}, \tag{3.3}$$

we have the above equations rewritten as (here we assume $\Phi=\int d^2\mathbf{r}\,F_{12}>0$)

$$(D_1+iD_2)\psi_\downarrow - \sqrt{2}e\phi\chi_\uparrow = 0 , \tag{3.4a}$$

$$(\partial_1 - i\partial_2)\chi_\uparrow + i\kappa\chi_\downarrow - \sqrt{2}e\phi^*\psi_\downarrow = 0 , \tag{3.4b}$$

$$(D_1 - iD_2)\psi_\uparrow + 2ieA^0\psi_\downarrow + \sqrt{2}e\phi\chi_\downarrow = 0 , \tag{3.4c}$$

$$(\partial_1 + i\partial_2)\chi_\downarrow - i\kappa\chi_\uparrow + \sqrt{2}e\phi^*\psi_\uparrow = 0 . \tag{3.4d}$$

One may apply the index theorem to learn about the fermion zero modes in the above model. The index of a Dirac-like operator $\mathcal{D}$ is defined by

$$\text{Index}(\mathcal{D}) = \dim(\text{kernel}\,\mathcal{D}) - \dim(\text{kernel}\,\mathcal{D}^\dagger) . \tag{3.5}$$

To evaluate this index, it is convenient to consider the quantity [22,23]

$$I(M^2) = \text{Tr}\left[\frac{M^2}{\mathcal{D}^\dagger\mathcal{D}+M^2}\right] - \text{Tr}\left[\frac{M^2}{\mathcal{D}\mathcal{D}^\dagger+M^2}\right] , \tag{3.6}$$

where M^2 is an arbitrary parameter. The index is recovered in the $M^2\to 0$ limit. In most cases, $I(M^2)$ is independent of M^2, and then one finds the index by evaluating this expression for $M^2\to\infty$, say, with the help of a large-mass expansion. (In the presence of a continuum spectrum extending to zero [24], or if there exist certain long-range fields [22,23], further correction terms can enter the index calculation.) Now what happens is that if one calculates the index for the Dirac-like operator directly associated with the set of equations (3.4a)–(3.4d), the index vanishes identically. Hence we do not get any information about the number of zero modes. This is because the adjoint operator $\mathcal{D}^\dagger$ has the same number of zero modes as $\mathcal{D}$.

Clearly, to get any useful information, some modification of strategy is needed. We shall discuss below the appropriate methods, first for the limiting cases of $\kappa=0$ and $\kappa\to\infty$, and then for the case of general κ.

1. The Landau-Ginzburg model (the $\kappa=0$ case)

With κ set to zero, the set of equations in (3.4) can be decoupled into two pieces, viz.,

$$(D_1+iD_2)\psi_\downarrow - \sqrt{2}e\phi\chi_\uparrow = 0 , \tag{3.7a}$$

$$(\partial_1 - i\partial_2)\chi_\uparrow - \sqrt{2}e\phi^*\psi_\downarrow = 0 \tag{3.7b}$$

and

$$(D_1 - iD_2)\psi_\uparrow + \sqrt{2}e\phi\chi_\downarrow = 0 , \tag{3.8a}$$

$$(\partial_1 + i\partial_2)\chi_\downarrow + \sqrt{2}e\phi^*\psi_\uparrow = 0 . \tag{3.8b}$$

Then we observe that Eqs. (3.8a) and (3.8b) allow no nontrivial solutions. To see this, apply $(\partial_1 - i\partial_2)$ to Eq. (3.8b) from the left and use Eq. (3.8a) and the self-duality equations. This yields

$$(-\nabla^2 + 2e^2|\phi|^2)\chi_\downarrow = 0 , \tag{3.9}$$

which has no nontrivial normalizable solution. Hence, it is enough to investigate the zero modes of Eqs. (3.7a) and (3.7b), and for this reduced system the index theorem becomes useful. A short calculation establishes that the index of the corresponding Dirac operator in a vortex background with vorticity n is equal to $2n$. [Here and henceforth, for an easy comparison with bosonic zero modes, the index (and the number of zero modes) in the case of Dirac equations will also be given for the corresponding *real* equations. This means that, when viewed as complex functions, the number is really half of the given value.] It is also a simple matter to show that the kernel of the adjoint operator is null. Actually the adjoint operator is nothing but the Dirac operator corresponding to Eqs. (3.8a) and (3.8b), which has no nontrivial solution. So we conclude that the number of fermion zero modes is equal to $2n$.

Note that the number of fermionic zero modes is equal to the number of bosonic ones. This is not an accident. Adopting the background-gauge condition in Eq. (2.20) with $\kappa=0$, the bosonic-zero-mode equations (2.18) reduce in the present case to (here note that $A^0=N=0$)

$$\begin{aligned}&(D_1+iD_2)\delta\phi-ie\phi(\delta A_1+i\delta A_2)=0 ,\\&(\partial_1-i\partial_2)(\delta A_1+i\delta A_2)+2ie\phi^*\delta\phi=0 .\end{aligned} \tag{3.10}$$

These become identical to Eqs. (3.7a) and (3.7b) with the identifications

$$\psi_\downarrow=\delta\phi,\quad \chi_\uparrow=\frac{i}{\sqrt{2}}(\delta A_1+i\delta A_2) . \tag{3.11}$$

This shows that each bosonic-zero-mode solution gives a fermionic zero mode and vice versa. This coincidence of bosonic-zero-mode equations and fermionic ones can be attributed to the underlying $N=2$ supersymmetry, as we shall show in Sec. IV.

2. The minimal Chern-Simons Higgs model (the $\kappa\to\infty$ case)

We now consider the $\kappa\to\infty$ case of our model or, more precisely, the theory defined by the Lagrangian (2.9). The equation for the fermion zero modes is easily obtained either from the Lagrangian (2.9) or from Eq. (3.4) with the relationships (2.8) assumed. It reads

$$\gamma^i D_i\psi+ie\left[\gamma^0 A^0+\frac{e}{\kappa}(3|\phi|^2-v^2)\right]\psi=0 , \tag{3.12}$$

with the background fields (A_μ,ϕ) now satisfying the self-duality equations (2.17) and (2.16). Again, in the γ-matrix basis given in Eq. (3.2), this can be written as

$$(D_1+iD_2)\psi_\downarrow+ie\left[A^0+\frac{e}{\kappa}(3|\phi|^2-v^2)\right]\psi_\uparrow=0 , \tag{3.13a}$$

$$(D_1-iD_2)\psi_\uparrow+ie\left[A^0-\frac{e}{\kappa}(3|\phi|^2-v^2)\right]\psi_\downarrow=0 . \tag{3.13b}$$

Then, considering $\phi^*\psi_\uparrow$ instead of $\psi_\downarrow$ and using the self-duality equations [and $A^0=(e/\kappa)(v^2-|\phi|^2)$], we can recast these equations into the form

$$\mathcal{D}\Psi=0,\quad \Psi=\begin{bmatrix}\psi_\downarrow\\ \phi^*\psi_\uparrow\end{bmatrix} , \tag{3.14}$$

with

$$\mathcal{D}=\begin{bmatrix}D_1+iD_2 & 2i(e^2/\kappa)\phi\\ 2i(e^2/\kappa)(v^2-2|\phi|^2)\phi^* & \partial_1-i\partial_2\end{bmatrix} . \tag{3.15}$$

For the above Dirac operator the $M^2\to\infty$ limit of the quantity $I(M^2)$ defined in Eq. (3.6) can easily be calculated, and it is equal to $(e/\pi)\Phi$. It is also not difficult to show that the corresponding adjoint operator has zero kernel. In a topological soliton background with vorticity n, we are then led to conclude [on the basis of the M^2 independence of $I(M^2)$] that $\dim(\text{kernel}\mathcal{D})=2n$, i.e., there exist $2n$ normalizable fermion zero modes (viewed as real functions). In the case of a nontopological soliton background, however, we have a continuous spectrum extending to zero, and this can give rise to a nonzero contribution to $\lim_{M^2\to 0}I(M^2)$. Therefore the correct number of normalizable zero modes is obtained only after we subtract this continuum contribution from the value $(e/\pi)\Phi=2(n+\alpha)$ [see Eq. (2.12)]. Note that the continuum contribution is sensitive only to the asymptotic behavior of $\mathcal{D}$. This implies that our system has the same continuum correction as the one defined solely by $(D_1+iD_2)\psi_\downarrow=0$. Now, using the well-known result [24] for the latter, we may conclude that the continuum contribution in our case is equal to $2(\alpha-\hat{\alpha})$. Thus the number of normalizable fermion zero modes in a nontopological soliton background is $2(n+\hat{\alpha})$.

We can again relate the fermion-zero-mode equations to those for bosonic ones. The zero-mode fluctuations of Bose fields should satisfy the equations

$$\begin{aligned}&(D_1+iD_2)\delta\phi-ie\phi(\delta A_1+i\delta A_2)=0 ,\\&\partial_1\delta A_2-\partial_2\delta A_1+i\frac{2e^3}{\kappa^2}(2|\phi|^2-v^2)(\phi^*\delta\phi-\phi\delta\phi^*)=0 .\end{aligned} \tag{3.16}$$

With the gauge-fixing equation (2.22) for the bosonic zero modes, these equations are in fact equivalent to the above Dirac equation with the identifications

$$\psi_\downarrow=\delta\phi,\quad \phi^*\psi_\uparrow=-\frac{\kappa}{2e}(\delta A_1+i\delta A_2) . \tag{3.17}$$

Indeed, the number of fermion zero modes is identical to the number of bosonic zero modes (as determined in Ref. [7]). Moreover, using our connection formulas (3.17), all fermion zero modes may be obtained immediately from corresponding bosonic zero modes and vice versa. But there is one tricky point, which we address below.

Note that the fermion zero modes we got are, strictly speaking, those of $\psi_\downarrow$ and $\phi^*\psi_\uparrow$ (rather than $\psi_\uparrow$). Hence we must make sure that the number of zero modes is unchanged by having the multiplicative factor ϕ^*. This is not a trivial matter due to the zeros in the function ϕ^*. For this consideration it is convenient to reorganize Eqs. (3.13a) and (3.13b) into the equivalent set of equations

$$\left[\nabla^2+\frac{4e^4}{\kappa^2}(v^2-2|\phi|^2)|\phi|^2\right]\left[\frac{1}{\phi}\psi_\downarrow\right]=0 , \tag{3.18a}$$

$$\phi^*\psi_\uparrow=i\frac{\kappa}{2e^2}(\partial_1+i\partial_2)\left[\frac{1}{\phi}\psi_\downarrow\right] . \tag{3.18b}$$

Equation (3.18a) does not admit any normalizable solution for $(1/\phi)\psi_\downarrow$ (as asserted in Ref. [7]). But, in spite of this, it will be argued below that we can have normalizable zero modes for $\psi_\downarrow$. This is possible because some non-normalizable solutions for $(1/\phi)\psi_\downarrow$ can transform into normalizable ones for $\psi_\downarrow$ once the prefactor $1/\phi$ is removed.

For a better understanding of the situation, consider Eq. (3.18a) for a rotationally symmetric topological vortex background with vorticity n. We then have

$$\phi(\mathbf{r})=f(r)e^{in\theta} , \tag{3.19}$$

with the function $f(r)$ being $O(r^n)$ as $r \to 0$ and approaching v for large r. In this background, the general solution to Eq. (3.18a) can be written as

$$\frac{1}{\phi}\psi_\downarrow(r,\theta)=h_0^{(+)}(r)+\sum_{l=1}^{\infty}[h_l^{(+)}(r)e^{il\theta}+h_l^{(-)}(r)e^{-il\theta}] , \tag{3.20}$$

where the radial functions $h_l^{(\pm)}(r)$ satisfy

$$\frac{1}{r}\frac{d}{dr}\left[r\frac{d}{dr}h_l^{(\pm)}\right] - \left[\frac{l^2}{r^2}-\frac{4e^4}{\kappa^2}(v^2-2|\phi|^2)|\phi|^2\right]h_l^{(\pm)}(r)=0 . \tag{3.21}$$

We here want both $\psi_\downarrow$ and $\psi_\uparrow$ [determined through Eq. (3.18b)] to be regular at the origin. This implies that $(1/\phi)\psi_\downarrow$ can be singular at $r=0$ if (i) the singularity is not worse than $(1/r^n)e^{-in\theta}$, and (ii) one has $(\partial_1+i\partial_2)[(1/\phi)\psi_\downarrow]\sim O(r^n)$ as r tends to zero. Since $(\partial_1+i\partial_2)(r^le^{il\theta})=0$ for any integer l, the appropriate boundary condition for $(1/\phi)\psi_\downarrow$ may thus read, in addition to asymptotic square normalizability, there should exist some constants $(C_{-n},C_{-n+1},\dots,C_n)$ so that one has

$$r\to 0: \quad \left|\frac{1}{\phi}\psi_\downarrow(r,\theta)-\sum_{l=-n}^{n}C_lr^le^{il\theta}\right|\sim O(r^{n+1}) . \tag{3.22}$$

From this boundary condition and the very fact that Eq. (3.18a) admits no everywhere-regular normalizable solution for $(1/\phi)\psi_\downarrow$, a strong restriction follows as regards what terms on the right-hand side of Eq. (3.20) might be kept for acceptable solutions. One finds only

$$\frac{1}{\phi}\psi_\downarrow(r,\theta)=\sum_{l=1}^{n}h_l^{(-)}(r)e^{-il\theta} , \tag{3.23}$$

with the functions $h_l^{(-)}(r)$ satisfying the following properties: $h_l^{(-)}(r)=(\text{const}/r^l)+O(r^{n+1})$ as $r\to 0$, and $h_l^{(-)}(r)$ vanishes (exponentially) for $r\to\infty$. It is a nontrivial mathematical problem to prove the existence of solutions involving the functions $h_l^{(-)}(r)$ $(l=1,2,\dots,n)$ with these properties. Such a solution is not to be expected generically. But we conjecture that, for the very equation being considered, such a solution does exist for each value of $l=1,2,\dots,n$; these would then account for n acceptable complex solutions (i.e., $2n$ real solutions) for $(1/\phi)\psi_\downarrow$. We here remark that the analysis of Ref. [19] is for the generic case, and hence, not particularly useful for our problem.

To support our conjecture, we consider specifically the fermion zero modes directly related to the supersymmetry transformations of the given soliton background. Those modes have the form

$$\psi=-\sqrt{2}\left[i(D_\mu\phi)\gamma^\mu\eta-\frac{e^2}{\kappa}(v^2-|\phi|^2)\phi\eta\right] = \begin{bmatrix} 2\sqrt{2}(e^2/\kappa)(v^2-|\phi|^2)\phi\eta_\uparrow \\ i\sqrt{2}(D_1-iD_2)\phi\eta_\uparrow \end{bmatrix} , \tag{3.24}$$

where, to obtain the last expression, we have used the fact that $(D_1+iD_2)\phi=0$. This provides us with the c-number zero-energy solutions

$$\psi_\uparrow=2\frac{e^2}{\kappa}(v^2-|\phi|^2)\phi, \quad \psi_\downarrow=i(D_1-iD_2)\phi , \tag{3.25}$$

and correspondingly

$$\frac{1}{\phi}\psi_\downarrow=\frac{i}{\phi}(D_1-iD_2)\phi=i(\partial_1-i\partial_2)\ln|\phi|^2 . \tag{3.26}$$

It is evident that $\psi_\uparrow$ and $\psi_\downarrow$ given in Eq. (3.25) are regular everywhere. Moreover, for the rotationally symmetric self-dual vortex of vorticity n, it is easily shown that

$$r\to 0: \quad |\phi|\sim r^n[1+O(r^{2n+2})] . \tag{3.27}$$

Based on this, we can identify the solution (3.26) with the $l=1$ mode on the right-hand side of Eq. (3.23), and then we have

$$h_{l=1}^{(-)}(r)=i\frac{d}{dr}\ln|\phi|^2 \underset{r\to 0}{\sim} i\frac{2n}{r}+O(r^{2n+1}) , \tag{3.28}$$

which is fully consistent with our boundary condition. For $l=1$, we have now verified our conjecture. Similarly we expect appropriate modes with $l=2,\dots,n$ to exist also, although we have not been able to verify this yet.

We believe that, in a general topological or nontopological self-dual vortex background, the space of fermion zero modes has the same dimension as that of bosonic zero modes (after eliminating pure gauge modes), and our formula (3.17) gives a simple explicit map between the two. For this to be true, the multiplicative factor ϕ^* in the second relation of Eq. (3.17) should not lead to a singular expression for $\psi_\uparrow$, i.e., δA_1 and δA_2 should vanish where ϕ^* vanishes. (If this happens not to be the case by any chance, there would be fewer fermion zero modes than corresponding bosonic zero modes, and quite likely this would result in some sort of supersymmetry breaking in the quantum theory.)

3. The Maxwell–Chern-Simons Higgs model

Let us now discuss the coupled Dirac equations (3.1) [or equivalently Eqs. (3.4a)–(3.4d)] for an arbitrary finite value of κ. Equations (3.42a)–(3.4d) cannot be separated into two decoupled ones. Here we may regard Eq. (3.4d) as the equation determining $\phi^*\psi_\uparrow$ in terms of χ's, viz.,

$$\phi^*\psi_\uparrow=\frac{1}{\sqrt{2}e}[i\kappa\chi_\uparrow-(\partial_1+i\partial_2)\chi_\downarrow] , \tag{3.29}$$

and rewrite Eq. (3.4c) as

$$(\partial_1-i\partial_2)\phi^*\psi_\uparrow+2ieA^0\phi^*\psi_\downarrow+\sqrt{2}e|\phi|^2\chi_\downarrow=0 . \tag{3.30}$$

These two, together with Eqs. (3.4a) and (3.4b), can then be put into the single matrix equation

$$\mathcal{D}\Psi=0, \quad \Psi=\begin{bmatrix}\psi_\downarrow\\ \chi_\uparrow\\ \phi^*\psi_\uparrow\\ \chi_\downarrow\end{bmatrix} , \tag{3.31}$$

with

$$\mathcal{D}=\begin{pmatrix} D_1+iD_2 & -\sqrt{2}e\phi & 0 & 0 \\ -\sqrt{2}e\phi^* & \partial_1-i\partial_2 & 0 & i\kappa \\ 2ieA^0\phi^* & 0 & \partial_1-i\partial_2 & \sqrt{2}|\phi|^2 \\ 0 & -i\kappa & \sqrt{2}e & \partial_1+i\partial_2 \end{pmatrix} . \tag{3.32}$$

The index for this operator can be determined by the same procedure as used for the operator (3.15), and we again find the value $2n$ (or $2n+2\hat{\alpha}$) in the topological soliton (nontopological soliton) background of vorticity n. Also, it is not difficult to show that the adjoint operator $D^\dagger$ has no normalizable zero mode: this means $2n$ (or $2n+2\hat{\alpha}$) independent zero modes for Ψ. This is equal to the number of corresponding bosonic zero modes [16], and as we shall demonstrate below, there again exist simple connection formulas relating these two sets of zero modes.

The bosonic zero modes in this model should satisfy the three equations in (2.18), and additionally, we will subject them to the (nonlocal) gauge condition which is specified by Eqs. (2.19)–(2.21). For the sake of comparison, it will also be convenient to have the quantity $\phi^*\psi_\uparrow$ eliminated from Eq. (3.30) by using Eq. (3.29). We then find

$$(-\nabla^2+\kappa^2+2e^2|\phi|^2)\chi_\downarrow+\sqrt{2}ie(\kappa+2eA^0)\phi^*\psi_\downarrow=0 , \tag{3.33}$$

which may well be separated into two equations, one for $\mathrm{Re}\chi_\downarrow$ and the other for $\mathrm{Im}\chi_\downarrow$. Now we notice that Eqs. (3.4a), (3.4b), and (3.33) for $(\psi_\downarrow,\chi_\uparrow,\mathrm{Re}\chi_\downarrow,\mathrm{Im}\chi_\downarrow)$ are completely equivalent to Eqs. (2.18) and (2.19) for $(\delta\phi,\delta A_1,\delta A_2,\delta A^0)$ once we make the identifications

$$\begin{aligned} &\psi_\downarrow=\delta\phi, \quad \mathrm{Im}\chi_\downarrow=\frac{1}{\sqrt{2}}\delta A^0 , \\ &\chi_\uparrow=\mathrm{Re}\chi_\uparrow+i\,\mathrm{Im}\chi_\uparrow=\frac{i}{\sqrt{2}}(\delta A_1+i\delta A_2) . \end{aligned} \tag{3.34}$$

[Here note that no separate formula has been given for $\psi_\uparrow$ because of Eq. (3.29), and $\sqrt{2}\,\mathrm{Re}\chi_\downarrow$ goes to the quantity $\mathcal{F}$ appearing in our background-gauge condition.] In the appropriate limits, this connection formula correctly reproduces the earlier ones given in Eqs. (3.11) and (3.17). As we will show in Sec. IV, the $N=2$ supersymmetry is responsible for this formula.

For every known fermion zero mode, Eq. (3.34) tells us that there is a corresponding bosonic zero mode. But its converse does not follow necessarily. The point is that for every known bosonic zero mode, Eq. (3.34) provides us with a zero mode for Ψ, but this zero mode for Ψ can correspond to a singular fermion mode (because Ψ involves $\phi^*\psi_\uparrow$ rather than $\psi_\uparrow$ itself). A complete matching between bosonic and fermionic zero modes is secured only when there is no wrongdoing due to the multiplicative function ϕ^* (in $\phi^*\psi_\uparrow$ of Ψ) in getting the fermion zero modes. This is the same problem which we encountered already in the minimal Chern-Simons Higgs model. Here we conjecture again that all $2n$ (or $2n+2\hat{\alpha}$) zero modes for Ψ lead to nonsingular fermion zero modes. This has the implication that if $\psi_\uparrow$ is found using Eq. (3.29), it should come out to be regular at zero of ϕ^*. As one can check readily, this is certainly the case for the fermion zero modes directly related to the supersymmetry transformation of the given soliton background.

B. The $N=1$ model

The fermionic part of the $N=1$ supersymmetric Lagrangian is given in Eq. (2.6). In the background of the self-dual vortex solutions satisfying Eq. (2.14), the equations for the fermionic zero modes now read

$$\begin{aligned} &i\gamma^i D_i\psi-e(\gamma^0A^0+N)\psi-\sqrt{2}ie\phi\chi^c=0 , \\ &i\gamma^i\partial_i\chi+\kappa\chi^c-\sqrt{2}ie\phi\psi^c=0 . \end{aligned} \tag{3.35}$$

In the γ-matrix basis given in Eq. (3.2), Eq. (3.35) gives rise to the component equations

$$(D_1+iD_2)\psi_\downarrow+2ieA^0\psi_\uparrow+\sqrt{2}\phi\chi_\downarrow^*=0 , \tag{3.36a}$$

$$(D_1-iD_2)\psi_\uparrow-\sqrt{2}e\phi\chi_\uparrow^*=0 , \tag{3.36b}$$

$$(\partial_1+i\partial_2)\chi_\downarrow+i\kappa\chi_\downarrow^*+\sqrt{2}e\phi\psi_\downarrow^*=0 , \tag{3.36c}$$

$$(\partial-i\partial_2)\chi_\uparrow-i\kappa\chi_\uparrow^*-\sqrt{2}e\phi\psi_\uparrow^*=0 . \tag{3.36d}$$

Note that we here have

$$\chi^c=i\gamma^2\chi^*=-\begin{pmatrix}\chi_\downarrow^*\\ \chi_\uparrow^*\end{pmatrix}$$

for

$$\chi=\begin{pmatrix}\chi_\uparrow\\ \chi_\downarrow\end{pmatrix} .$$

We then note that Eqs.(3.36b) and (3.36d) have a trivial solution only, i.e., $\chi_\uparrow=\psi_\uparrow=0$. This follows since, as we eliminate $\psi_\uparrow$ from the two equations and use the self-duality equations for the background fields, we end up with

$$(\nabla^2-2e^2|\phi|^2)\chi_\uparrow^*+i\kappa(\partial_1-i\partial_2)\chi_\uparrow=0 , \tag{3.37}$$

which has a trivial solution only. Thus the nontrivial part of the fermion-zero-mode equations becomes just

$$\begin{aligned} &(D_1+iD_2)\psi_\downarrow+\sqrt{2}e\phi\chi_\downarrow^*=0 , \\ &(\partial_1+i\partial_2)\chi_\downarrow+i\kappa\chi_\downarrow^*+\sqrt{2}e\phi\psi_\downarrow^*=0 . \end{aligned} \tag{3.38}$$

Zero-mode equations analogous to Eq. (3.38) were studied previously in Ref. [25], and one can calculate the index for the system (3.38) by a parallel procedure. Indeed, employing the real-field basis, we have the Dirac operator relevant to the system (3.38) in the form

$$\mathcal{D}=\begin{pmatrix} I(\partial_1+eA_2)-i\sigma_2(\partial_2-eA_1) & -\sqrt{2}e\phi_1 I+i\sqrt{2}e\phi_2\sigma_2 \\ -\sqrt{2}e\phi_1 I-i(\kappa+\sqrt{2}e\phi_2)\sigma_2 & I\partial_1+i\sigma_2\partial_2 \end{pmatrix} , \tag{3.39}$$

where I denotes the 2×2 unit matrix. For this operator form the index calculation is not much different from the cases already considered by us, and so we state the results only. The number of fermion zero modes is again given by the value $2n$ in a topological soliton background, and by $2(n+\hat{\alpha})$ in a nontopological soliton background. In either case, we thus see that the number of fermion zero modes matches precisely the number of bosonic ones. But no simple relationship connecting these two sets of zero modes seems to exist, in contrast to the case of the $N=2$ model. In the next section we shall see that there is a simple reason for this.

In the large-κ limit, the theory is described by the Lagrangian (2.10). Here, for fermion zero modes around a self-dual vortex background, we have just one Dirac equation

$$i\gamma^i D_i\psi - e\left[\gamma^0 A^0 - \frac{e}{\kappa}(|\phi|^2-v^2)\right]\psi + 2\frac{e^2}{\kappa}\phi^2\psi^c = 0 , \tag{3.40}$$

or in our γ-matrix basis, the component equations

$$(D_1+iD_2)\psi_\downarrow - 2i\frac{e^2}{\kappa}(|\phi|^2-v^2)\psi_\uparrow + 2i\frac{e^2}{\kappa}\psi_\downarrow^* = 0 , \tag{3.41a}$$

$$(D_1-iD_2)\psi_\uparrow - 2i\frac{e^2}{\kappa}\phi^2\psi_\uparrow^* = 0 . \tag{3.41b}$$

Then observe that, thanks to the self-duality equations (2.14), Eq. (3.41b) can be recast as

$$(\partial_1 - i\partial_2)\phi^*\psi_\uparrow - 2i\frac{e^2}{\kappa}|\phi|^2(\phi^*\psi_\uparrow)^* = 0 . \tag{3.42}$$

This immediately leads to the conclusion

$$2i\frac{e^2}{\kappa}\int d^2x\,|\phi|^2|(\phi^*\psi_\uparrow)|^2 = \frac{1}{2}\int d^2x(\partial - i\partial_2)(\phi^*\psi_\uparrow)^2 = 0 , \tag{3.43}$$

assuming that $\phi^*\psi_\uparrow$ vanishes sufficiently fast at spatial infinity. Hence, we are allowed to set $\psi_\uparrow\equiv0$, and the remaining equation (3.41a) is now simplified to

$$(D_1+iD_2)\psi_\downarrow + 2i\frac{e^2}{\kappa}\phi^2\psi_\downarrow^* = 0 . \tag{3.44}$$

It is straightforward to determine the number of independent normalizable solutions to Eq. (3.44) with the help of the index theorem or by explicit mode analysis (in a rotationally symmetric background) [25,19]. For the case of either the topological or nontopological self-dual vortex background, the answer agrees with the result we found for the more complicated Dirac equations in Eq. (3.35). Again, in this $N=1$ model, we have not found any simple formula connecting the fermion zero modes to the bosonic ones.

IV. ZERO MODES AND $N=2$ SUPERSYMMETRY

We have seen in Sec. III that, in the $N=2$ models, there exist simple transformation formulas which produce all fermionic zero modes around the given self-dual vortex background from the corresponding bosonic ones. The observation was based on the direct comparison of the equations of motion for the respective zero modes. One may naturally suspect whether a certain symmetry is responsible for this. The answer is in the affirmative, and as we will see below, it is the extended supersymmetry which plays an important role.

We start from the *static* version of the field equations for the $N=2$ supersymmetric Maxwell–Chern-Simons system. They are easily found from the Lagrangian in Eqs. (2.1)–(2.3). The static field equations for Bose fields read

$$\partial_i F^{ij} + \kappa\epsilon^{ij}\partial_i A^0 - ie[\phi^* D^j\phi - (D^j\phi)^*\phi] = -e\bar{\psi}\gamma^j\psi , \tag{4.1a}$$

$$\nabla^2 A^0 + \frac{\kappa}{2}\epsilon^{ij}F_{ij} - 2e^2 A^0|\phi|^2 = -e\bar{\psi}\gamma^0\psi , \tag{4.1b}$$

$$D^2\phi - e\phi(e|\phi|^2+\kappa N - ev^2) + e^2[(A^0)^2 - N^2]\phi = -\sqrt{2}ie\bar{\chi}\psi , \tag{4.1c}$$

$$\nabla^2 N - \kappa(e|\phi|^2+\kappa N - ev^2) - 2e^2|\phi|^2 N = -e\bar{\psi}\psi , \tag{4.1d}$$

while those for fermion fields are in fact just the fermion-zero-mode equations in Eq. (3.1). Self-dual vortex configurations or solutions of Eq. (2.14) satisfy the static field equations of pure bosonic theory, which coincide with Eq. (4.1) but for the bilinear fermion source terms on the right-hand sides. Also note that, to be able to discuss supersymmetry of the classical field equations in a consistent way, one must now view the fermion fields ψ,χ appearing above as Grassmannian objects.

Our next observation is that static classical solutions to the full field equations—Eq. (4.1) together with Eq. (3.1)—can be constructed with the help of static bosonic configurations satisfying the self-duality equations (2.14) and fermion zero-mode functions [or "c-number solutions" to the Dirac equations (3.1)] in a given self-dual background. Let the (in general complex-valued) spinor functions (ψ_0,χ_0) represent any particular c-number solution to Eq. (3.1) in some given self-dual background (A_μ,ϕ,N). Then we have the solution to our full field equations in

$$\Psi_0(x) = (A_\mu(x),\phi(x),N(x) , \psi(x)=\epsilon\psi_0(x),\chi(x)=\epsilon\chi_0(x)) , \tag{4.2}$$

where ϵ is a constant *real* Grassmann number by which all individual components of spinor zero-mode functions get multiplied. Since the square of a real Grassmann number is identically zero, all bilinear fermion source terms in Eq. (4.1) in fact vanish for our configuration (4.2), and therefore that $\Psi_0(x)$ solves the full field equations is guaranteed. (We here remark that bilinear fermion source terms do not vanish if one inserts c-number solutions for ψ and χ. In such a situation, our simple trick of attaching a real Grassmann number does not appear to have been utilized before in the literature.) Now, remembering that there are $2n$ (or $2n+2\hat{\alpha}$) independent fermion-zero-mode functions in a given topological (nontopological) self-dual vortex background, we see that the construction in Eq. (4.2) allows us to obtain $2n$ ($2n+2\hat{\alpha}$) independent solutions to the full field equations of the supersymmetric theory from any given purely bosonic solution $\Psi_0^{(b)}(x)=(A_\mu(x),\phi(x),N(x),\psi=0,\chi=0)$.

A supersymmetry transformation of any given solution of the form (4.2) should also be a solution to the full field equations. This has the implication that the supersymmetry-related infinitesimal change $\delta_\eta\Psi(x)=(\delta_\eta A_\mu(x),\delta_\eta\phi(x),\delta_\eta N(x),\delta_\eta\psi(x),\delta_\eta\chi(x))$ should satisfy the appropriate small-fluctuation equations which are readily derived from the field equations. We shall now make use of this fact in deriving the interrelationship between the bosonic and fermionic zero modes around the purely bosonic solution $\Psi_0^{(b)}$. For that, we again go to the γ-matrix basis given in Eq. (3.2) and write the supersymmetry transformation parameter η as

$$\eta=\begin{bmatrix}\eta_\uparrow\\ \eta_\downarrow\end{bmatrix} . \tag{4.3}$$

Here, $\eta_\uparrow$ and $\eta_\downarrow$ are mutually independent complex Grassmann numbers which together generate $N=2$ supersymmetry. Notice that the mode functions of $\delta_\eta\Psi(x)$, obtained through supersymmetry transformations of the configuration (4.2), will not necessarily satisfy the bosonic- and fermionic-zero-mode equations in the purely bosonic background $\Psi_0^{(b)}$. Let us assume that $\Psi_0^{(b)}$ describes the self-dual vortex background with $\Phi=\int d^2\mathbf{r}\,F_{12}>0$. Now, if one considers a supersymmetry transformation involving $\eta_\uparrow$ only (in short, $\delta_{\eta_\uparrow}=\delta_\uparrow$), it is not difficult to show that $\delta_\uparrow\Psi$ gives zero modes in the bosonic background $\Psi_0^{(b)}$ only when $\delta_\uparrow\Psi$ is a supersymmetry transformation of $\Psi_0^{(b)}$ [i.e., the configuration (4.2) with $\psi=\chi=0$]. For the latter, we find $\delta_\uparrow A_\mu=\delta_\uparrow\phi=\delta_\uparrow N\equiv 0$ and ($\delta_\uparrow\psi$, $\delta_\uparrow\chi$) gives familiar supersymmetry-related fermion zero modes. The situation for the $\eta_\downarrow$ transformation is more interesting, and is discussed below.

With the restriction

$$\eta=\begin{bmatrix}0\\ \eta_\downarrow\end{bmatrix} ,$$

the supersymmetry transformation (2.4) can be expressed as

$$\begin{aligned}
&\delta_\downarrow A_1+i\delta_\downarrow A_2=-2i\eta_\downarrow^*\chi_\uparrow ,\\
&\delta_\downarrow\phi=+\sqrt{2}\eta_\downarrow^*\psi_\downarrow ,\\
&\delta_\downarrow A^0=\delta_\downarrow N=-i(\eta_\downarrow^*\chi_\downarrow-\chi_\downarrow^*\eta_\downarrow) ,\\
&\delta_\downarrow\psi_\uparrow=-\sqrt{2}i\eta_\downarrow(D_1+iD_2)\phi ,\\
&\delta_\downarrow\psi_\downarrow=-\sqrt{2}e\phi(A^0-N)\eta_\downarrow ,\\
&\delta_\downarrow\chi_\uparrow=(\partial_1+i\partial_2)(N-A^0)\eta_\downarrow ,\\
&\delta\chi_\downarrow=i(F_{12}+e|\phi|^2+\kappa N-ev^2)\eta_\downarrow .
\end{aligned} \tag{4.4}$$

We then immediately notice that, in the self-dual vortex background,

$$\delta_\downarrow\psi_\uparrow=\delta_\downarrow\psi_\downarrow=\delta_\downarrow\chi_\uparrow=\delta_\downarrow\chi_\downarrow=0 \tag{4.5}$$

thanks to the self-duality equations (2.4); consequently, all bilinear fermion sources appearing in Eq. (4.1) give rise to vanishing fluctuations, i.e.,

$$\delta_\downarrow(\bar\psi\gamma^i\psi)=\delta_\downarrow(\bar\chi\psi)=0 . \tag{4.6}$$

Now, considering the above transformation with the form (4.2) as the original configuration, we are able to assert that the bosonic mode determined by

$$\begin{aligned}
&\delta A_1(x)+i\delta A_2(x)=-2i(\eta_\downarrow^*\epsilon)\chi_{0\uparrow}(x) ,\\
&\delta\phi(x)=\sqrt{2}(\eta_\downarrow^*\epsilon)\psi_{0\downarrow}(x) ,\\
&\delta A^0(x)=\delta N(x)\\
&\qquad=-i[(\eta_\downarrow^*\epsilon)\chi_{0\downarrow}(x)-(\epsilon\eta_\downarrow)\chi_{0\downarrow}^*(x)]
\end{aligned} \tag{4.7}$$

should solve the second-order bosonic fluctuation equations that are derived from the field equations (4.1), but without the fermion source terms. The relation (4.7) is in fact completely equivalent to our connection formula (3.34), since every fermion zero mode ($\psi_0(x),\chi_0(x)$) is defined only up to an arbitrary complex constant (which multiplies ψ_0 and χ_0 simultaneously). As for the contents of the formulas (4.7) and (3.34), there is one apparent difference: the bosonic modes entering Eq. (4.7) are supposed to solve the second-order bosonic fluctuation equations, while those entering Eq. (3.34) satisfy the bosonic-zero-mode equations (2.18). But the two sets of equations are really equivalent (see Appendix B), and so we are now entitled to say that the origin of our connection formula (3.34) lies in supersymmetry. [The supersymmetry argument does not tell us what sort of gauge condition the bosonic zero modes given by Eq. (4.7) are going to satisfy, although our equation-by-equation comparison in Sec. III provides this information also.]

A parallel analysis can be given for the $N=2$ supersymmetric minimal Chern-Simon Higgs theory, and it should suffice to say that the connection formula (3.17) is a consequence of supersymmetry involving the parameter $\eta_\downarrow$. In the 4D Yang-Mills instanton problem we also have this mode-by-mode matching between bosonic and fermionic zero modes [17], and the $N=2$ supersymmetry-based explanation for it was given by Zumino [18]. It would be also interesting to look at the case of our $N=1$ supersymmetry model. Using the fermion zero modes, it will be possible to construct the solutions of the form (4.2) to the full field equations even in this case. But, here, the supersymmetry transformation parameter η is a Majorana spinor, and so $\eta_\downarrow$ is actually identical to $-\eta_\uparrow^*$. Hence, it becomes impossible to consider the restricted symmetry transformations involving $\eta_\downarrow$ only [as in Eq. (4.4)]. This in turn implies that, in the $N=1$ model, the supersymmetry transformations of the fermion bilinear source terms never vanish, and so we do not obtain bosonic modes satisfying the purely bosonic-fluctuation equations by considering the supersymmetry transformations with a configuration like the one in Eq. (4.2). This explains the lack of a simple formula connecting bosonic and fermionic zero modes in the $N=1$ model.

V. SUMMARY AND DISCUSSION

Self-dual structures appearing in field theory have a deep connection with extended supersymmetry, and probably the *quantum* self-dual system makes sense only in the context of a suitably supersymmetrized theory [9]. In this paper, we have given the full contents of the $N=2$ (and $N=1$) supersymmetric Maxwell–Chern-Simons theory, which reduces in an appropriate limit to the $N=2$ supersymmetric Abelian Higgs model [10] or to the $N=2$ (and $N=1$) supersymmetric Chern-Simons Higgs model of Ref. [12]. Within this theory we have investigated the fermion zero modes around the topological or nontopological self-dual vortex background by making effective use of the index theorem and also by discovering a suitable formula (for the $N=2$ model) that converts every fermion zero mode into a corresponding bosonic zero mode. The simple relationship existing between the bosonic and fermionic zero modes is shown to be a consequence of extended supersymmetry.

The natural direction to go from here will be to study how the fermion zero modes we found become responsible for the vortex supermultiplet structures. Also it should be interesting to see whether one can develop the supercollective-coordinate formalism in which bosonic and fermionic zero modes are treated on an equal footing. (Note that our findings in the $N=2$ model strongly suggest this possibility.) It is not inconceivable that *slow* dynamics concerning vortex supermultiplets may find a simple description using such supercollective coordinates, say within the theoretical framework given in Ref. [26]. Quantum aspects of solitons can also be studied, but this requires a knowledge of not only zero modes but also nonzero modes (or, if you wish, propagators defined in the soliton background). An interesting issue here, in the $N=2$ supersymmetric models, is whether or not the Bogomol'nyi bound remains saturated when loop corrections are included. Another curious possibility in our $N=2$ models is that there might exist a very simple formula which relates the fermion propagator in the soliton background to the corresponding boson propagator, just as in the case of Yang-Mills instanton background [27]. These issues are under investigation.

Very recently, one-loop quantum and thermal corrections to the effective potential in the $N=2$ supersymmetric (Maxwell–)Chern-Simons theory (in the trivial vacuum sector) have been calculated by Ipekoglu, Leblanc and Thomaz [28]. They find some interesting results, and interested readers should consult their paper.

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APPENDIX A

Here we shall introduce the $N=1$ and $N=2$ supersymmetric Maxwell–Chern-Simons Lagrangians using the superfield language. We use the $N=1$ superfield notations following Ref. [29] closely. (Recently, Ivanov [30] has given an $N=2$ superfield formulation for the supersymmetric Chern-Simons Higgs model of Ref. [12].) A real scalar superfield $S(x,\theta)$ consists of a real scalar $N(x)$, a Majorana fermion $\xi(x)$, and a real auxiliary field $G(x)$:

$$S(x,\theta)=N(x)+\xi(x)^\alpha\theta_\alpha-G(x)\theta^2 \ . \tag{A1}$$

Here, $\theta^2\equiv\frac{1}{2}\theta^\alpha\theta_\alpha$ (θ is real Grassmannian) and α ($=1,2$) is an SL$(2,R)$ spinor index. Charged-matter fields are described by two real scalar superfields or, equivalently one complex superfield

$$\Phi(x,\theta)=\phi(x)+\psi^\alpha(x)\theta_\alpha-F(x)\theta^2 \ , \tag{A2}$$

with all the component fields now taken to be complex. In addition, we require a real spinor gauge superfield $\Gamma^\alpha(x,\theta)$, which in the Wess-Zumino gauge contains a real photon field $A_\mu(x)$ and a Majorana spinor "photino" field $\lambda^\alpha(x)$. The field-strength superfield W_α is then given by $W_\alpha=\frac{1}{2}D^\beta D_\alpha\Gamma_\beta$, where $D_\alpha=\partial/\partial\theta^\alpha+i\theta^\beta\partial_{\beta\alpha}$.

A general $N=1$ supersymmetric gauge theory with the Chern-Simons term, which one can construct by using these superfields, is described by the superspace action

$$S=\int d^3x\, d^2\theta[\tfrac{1}{4}W^\alpha W_\alpha-\tfrac{1}{4}\kappa\Gamma^\alpha W_\alpha-\tfrac{1}{4}(D^\alpha S)(D_\alpha D)$$
$$-\tfrac{1}{2}(\nabla^\alpha\Phi)^*(\nabla_\alpha\Phi)+f(S,\Phi^*\Phi)] \ , \tag{A3}$$

where $\nabla_\alpha\equiv D_\alpha-ie\Gamma_\alpha$ is the supercovariant derivative, and the superpotential $f(S,\Phi^*,\Phi)$ can be arbitrary. (The real scalar superfield S needs to be introduced for $N=2$ supersymmetry. See below.) This action is invariant under the $N=1$ supersymmetry transformation

$$\delta_{\eta_1}S=-\eta_1^\alpha Q_\alpha S, \quad \delta_{\eta_1}\Gamma=-\eta_1^\alpha Q_\alpha\Gamma, \quad \delta_{\eta_1}\Phi=-\eta_1^\alpha Q_\alpha\Phi \ , \tag{A4}$$

where η_1^α are (Grassmannian) Majorana spinor parameters and $Q_\alpha\equiv\partial/\partial\theta^\alpha-i\theta^\beta\partial_{\beta\alpha}$. The form of the superpotential $f(S,\Phi^*\Phi)$ becomes severely restricted if the theory is required to have $N=2$ supersymmetry. Let us impose this requirement also. For $N=2$ supersymmetry, the theory must be invariant under another supersymmetry transformation which reads

$$\delta_{\eta_2}S=2\eta_2^\alpha W_\alpha, \quad \delta_{\eta_2}\Gamma_\alpha=-\eta_{2\alpha}S, \quad \delta_{\eta_2}\Phi=-i\eta_2^\alpha\nabla_\alpha\Phi \ . \tag{A5}$$

Then the only allowed form for the superpotential is

$$f(S,\Phi^*\Phi)=-\tfrac{1}{2}\kappa S^2-e\Phi^*\Phi S+ev^2S \ , \tag{A6}$$

with an arbitrary positive constant v^2. Inserting this superpotential form into the expression (A3) gives the action for the $N=2$ supersymmetric Maxwell–Chern-Simons theory. For $\kappa=0$, this reduces to the theory discussed already in Ref. [10]. For very large κ, on the other hand, one may express the real superfield S (by using its field equation) as

$$S=-\frac{e}{\kappa}(\Phi^*\Phi-v^2)\ , \tag{A7}$$

and, using this with Eq. (A3), we obtain (after dropping less dominant terms) the action for the $N=2$ supersymmetric minimal Chern-Simons Higgs model,

$$S_{\rm CS}^{(2)}=\int d^3x\, d^2\theta\left[-\frac{\kappa}{4}\Gamma^\alpha W_\alpha-\frac{1}{2}(\nabla^\alpha\Phi)^*(\nabla_\alpha\Phi)+\frac{e^2}{2\kappa}(\Phi^*\Phi-v^2)^2\right]. \tag{A8}$$

With the above superspace action one can use the standard procedure [29] to obtain the component-field action in the Wess-Zumino gauge. Auxiliary fields F and G are eliminated by using their equations of motion, viz.,

$$F=e\phi N,\quad G=e|\phi|^2+\kappa N-ev^2\ , \tag{A9}$$

and we combine the Majorana field ξ^α with another Majorana (photino) field λ^α to form a Dirac field

$$\chi=\frac{1}{\sqrt{2}}(\lambda+i\xi)\ . \tag{A10}$$

The resulting component Lagrangian for the $N=2$ supersymmetric Maxwell–Chern-Simons theory is then precisely the expression in Eqs. (2.1)–(2.3). The component Lagrangian (2.9), which becomes relevant in the large-κ limit, is obtained as a result of the condition (A7) which translates for the component fields into Eq. (2.8). The supersymmetry transformations in Eqs. (A4) and (A5) do not preserve the Wess-Zumino gauge condition, and to fix this problem one must supplement the transformations by suitable supergauge transformations. Only after that, one finds the $N=2$ component supersymmetry transformation rules in Eq. (2.4). The parameter η in Eq. (2.4) is just the combination of the two real Grassmannian spinors η_1 and η_2, i.e., $\eta=(1/\sqrt{2})(\eta_1+i\eta_2)$.

The bosonic sector of the Lagrangian is self-dual only because of a very special potential form. In the above analysis we have shown that requiring $N=2$ supersymmetry automatically provides it. We then notice that, keeping the bosonic sector of the theory unchanged, we can construct another theory with only $N=1$ supersymmetry. Such a theory is obtained by choosing the superpotential $f(S,\Phi^*\Phi)$ to have the same form as that in Eq. (A6) but for the overall minus sign. This theory loses the second supersymmetry but has the same bosonic sector. If one expresses this theory in terms of component fields (in the Wess-Zumino gauge), the fermion-number-violating Lagrangian in Eq. (2.5) follows.

APPENDIX B

We will show that the second-order bosonic fluctuation equations, which can be deduced from Eqs. (4.1a)–(4.1d) with the fermion source terms dropped, are in fact equivalent to our bosonic-zero-mode equations in Eq. (2.18). The background fields we start with are assumed to satisfy Eq. (2.14), say, with the upper sign chosen. Then we observe that the second-order fluctuation equations derived from Eqs. (4.1a) and (4.1c) can be expressed in the form

$$(\partial_1+i\partial_2)[V+\kappa(\delta A^0-\delta N)]-2e\phi^*U=0\ , \tag{B1}$$

$$(\nabla^2-2e^2|\phi|^2)\delta A^0+\kappa V-(\kappa+2eA^0)e(\phi^*\delta\phi+\phi\delta\phi^*)-\kappa^2\delta N=0\ , \tag{B2}$$

$$(\partial_1-i\partial_2)\phi^*U-e|\phi|^2V+2e^2|\phi|^2A^0(\delta A^0-\delta N)=0\ , \tag{B3}$$

$$(-\nabla^2+\kappa^2+2e^2|\phi|^2)\delta N+e(\kappa+2eA^0)(\phi^*\delta\phi+\phi\delta\phi^*)=0\ , \tag{B4}$$

where we have defined

$$\begin{aligned}U&=(D_1+iD_2)\delta\phi-ie\phi(\delta A_1+i\delta A_2)\ ,\\ V&=\partial_1\delta A_2-\partial_2\delta A_1+e(\phi^*\delta\phi+\phi\delta\phi^*)+\kappa\delta N\ .\end{aligned} \tag{B5}$$

Eliminating ϕ^*U from Eqs. (B1) and (B3) gives

$$(-\nabla^2+2e^2|\phi|^2)[V+\kappa(\delta A^0-\delta N)]+2e^2(\kappa+2eA^0)|\phi|^2(\delta N-\delta A^0)=0\ , \tag{B6}$$

while we also have from Eqs. (B2) and (B4)

$$(-\nabla^2+2e^2|\phi|^2+\kappa^2)(\delta N-\delta A^0)+\kappa[V+\kappa(\delta A^0-\delta N)]-0\ . \tag{B7}$$

The pair of equations is in fact identical to Eqs. (3.16b) and (3.16e) of Ref. [16], and it was asserted there that no nontrivial solution exists. Therefore we must set

$$\delta A^0=\delta N,\quad V=0\ , \tag{B8}$$

and then, because of Eq. (B1), $U=0$ also. These conditions and Eq. (B4) obviously account for our zero-mode equations in Eq. (2.18).

[1] R. Jackiw and S. Templeton, Phys. Rev. D **23**, 2291 (1981); J. Schonfeld, Nucl. Phys. **B185**, 157 (1981); S. Deser, R. Jackiw, and S. Templeton, Phys. Rev. Lett. **48**, 975 (1982); Ann. Phys. (N.Y.) **140**, 372 (1982).
[2] H. B. Nielsen and P. Olesen, Nucl. Phys. **B61**, 45 (1973).
[3] S. Paul and A. Khare, Phys. Lett. B **174**, 420 (1986); H. de Vega and F. Schaposnik, Phys. Rev. Lett. **56**, 2564 (1986); L. Jacobs, A. Khare, C. Kumar, and S. Paul, Int. J. Mod. Phys. A **6**, 3441 (1991).
[4] J. M. Leinaas and J. Myrheim, Nuovo Cimento B **37**, 1 (1977); G. Goldin, R. Menikoff, and D. Sharp, J. Math. Phys. **21**, 650 (1980); F. Wilczek, Phys. Rev. Lett. **48**, 1144 (1982); **49**, 957 (1982).
[5] E. B. Bogomol'nyi, Yad. Fiz. **24**, 861 (1976) [Sov. J. Nucl.

Phys. **24**, 449 (1976)].

[6] J. Hong, Y. Kim, and P. Y. Pac, Phys. Rev. Lett. **64**, 2230 (1990); R. Jackiw and E. J. Weinberg, *ibid.* **64**, 2234 (1990).

[7] R. Jackiw, K. Lee, and E. J. Weinberg, Phys. Rev. D **42**, 3488 (1990).

[8] C. Lee, K. Lee, and H. Min, Phys. Lett. B **252**, 79 (1990).

[9] E. Witten and D. Olive, Phys. Lett. **78B**, 97 (1978).

[10] P. di Vecchia and S. Ferrara, Nucl. Phys. **B130**, 93 (1977).

[11] C. Taubes, Commun. Math. Phys. **75**, 207 (1980); E. Witten, Phys. Rev. Lett. **38**, 121 (1977); P. Forgacs and N. S. Manton, Commun. Math. Phys. **72**, 15 (1980); R. S. Ward, Philos. Trans. R. Soc. London **A315**, 451 (1985).

[12] C. Lee, K. Lee, and E. Weinberg, Phys. Lett. B **243**, 105 (1990).

[13] J. Lee and S. Nam, Phys. Lett. B **261**, 437 (1991).

[14] R. Jackiw and C. Rebbi, Phys. Rev. D **13**, 2298 (1976).

[15] For related discussions in the case of supersymmetric self-dual monopoles, see H. Osborn, Phys. Lett. **83B**, 321 (1979); P. Rossi, Phys. Rep. **86**, 317 (1982); B.-H. Lee, Ph. D. thesis, Columbia University, 1989.

[16] C. Lee, H. Min, and C. Rim, Phys. Rev. D **43**, 4100 (1991).

[17] L. S. Brown, R. D. Carlitz, and C. Lee, Phys. Rev. D **16**, 417 (1977); P. Rossi, Phys. Rep. **86**, 317 (1982).

[18] B. Zumino, Phys. Lett. **69B**, 369 (1977).

[19] G. Grignani and G. Nardelli, Phys. Rev. D **43**, 1919 (1991).

[20] L. S. Brown, R. D. Carlitz, and C. Lee, Phys. Rev. D **16**, 417 (1977); A. S. Schwartz, Phys. Lett. **67B**, 172 (1977); M. Atiyah, N. Hitchin, and I. Singer, Proc. Natl. Acad. Sci. U.S.A. **74**, 2662 (1977); R. Jackiw and C. Rebbi, Phys. Lett. **67B**, 189 (1977).

[21] M. Atiyah and I. Singer, Ann. Math. **87**, 484 (1968); M. Atiyah, R. Bott, and V. Patodi, Invent. Math. **19**, 279 (1973).

[22] C. Callias, Commun. Math. Phys. **62**, 213 (1978).

[23] E. J. Weinberg, Phys. Rev. D **19**, 3008 (1979); **24**, 2669 (1981).

[24] J. Kiskis, Phys. Rev. D **15**, 2329 (1977); D. Boyanovsky and R. Blankenbecler, *ibid.* **31**, 3234 (1985); T. Jaroszewicz, *ibid.* **34**, 3128 (1986); R. Musto, L. O'Raifeartaigh, and A. Wipf, Phys. Lett. B **175**, 433 (1986); M. Ansourian, Phys. Lett. **70B**, 301 (1977).

[25] R. Jackiw and P. Rossi, Nucl. Phys. **B190**, 681 (1981); E. J. Weinberg, Phys. Rev. D **24**, 2669 (1981).

[26] N. S. Manton, Phys. Lett. **110B**, 54 (1982); **154B**, 397 (1985); P. J. Ruback, Nucl. Phys. **B296**, 669 (1988).

[27] L. S. Brown, R. D. Carlitz, D. Creamer, and C. Lee, Phys. Rev. D **17**, 1583 (1978).

[28] Y. Ipekoglu, M. Leblanc, and M. T. Thomaz, Ann. Phys. (N.Y.) **214**, 160 (1992); M. Leblanc and M. T. Thomaz, MIT Report No. CTP#2031, 1991 (unpublished); Phys. Rev. D (to be published).

[29] S. J. Gates, M. T. Grisaru, M. Roček, and W. Siegel, *Superspace or One Thousand and One Lessons in Supersymmetry* (Benjamin-Cummings, Reading, MA, 1983), Chap. 2.

[30] E. A. Ivanov, Max-Planck-Institute Report No. MPI-PAE/PTh 43/91 (unpublished).