Improved $SU(2)_k$ Wess-Zumino-Witten model

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The $SU(2)_k$ Wess-Zumino-Witten model with the improved energy-momentum tensor is studied in terms of parafermion fields. It is shown that the parafermion fields are variables which are quite convenient to describe this model. Using the Coulomb gas formalism we show that some correlation functions are easily calculated. A possible application to the selection rules for nonvanishing correlation functions in $N=2$ minimal models is also discussed. These are quite useful in studying couplings of massless fields in models compactified by $N=2$ minimal models.

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I. INTRODUCTION

Among the recent developments in conformal field theories (CFT's), one of the quite important subjects is the free-field realization of the affine Kac-Moody current algebra. The currents for $SU(2)_k$ algebra were first constructed by Wakimoto [1], and then such a construction was generalized to $SU(3)_k$, $SU(n)_k$, and some other groups [2–5].

Apparently, there are a number of advantages of expressing currents in terms of free fields. It is already known that many of the existing conformal models can be constructed as coset models. The energy-momentum tensors are obtained by the Sugawara method from the currents, and the free-field representation makes such construction much easier. For example, given an $SU(2)_k$ current algebra, the minimal model can be obtained as the coset $SU(2)_k \times SU(2)_1/SU(2)_{k+1}$ [6]. This can be easily generalized to the coset of $SU(n)_k$, and the energy-momentum tensor of W_n algebra is constructed [7]. Another useful application of free-field realization of currents is the building of generalized parafermion theories based on G_k [8], which may be regarded as a more fundamental building block. This related to the fact that the currents for Cartan subalgebra take a very simple form in this free-field construction, and therefore removing the U(1) fields associated with the Cartan subalgebra is rather straightforward in this approach. The simplest is the well-known Z_k parafermions, obtained as $SU(2)_k/U(1)$, which is familiar in the application to the representations of $N=2$ minimal super CFT. It has already been discussed in detail how the $N=2$ minimal series are very effective in achieving string compactifications providing massless spectra with the spacetime supersymmetry, the so-called (2,2) compactifications [9]. Through the use of the Z_k parafermion fields, it was possible in such a compactification scheme to calculate three-point functions and even to derive some conditions for the nonvanishing correlation functions [10].

One of the related topics is the Wess-Zumino-Witten (WZW) model based on the group G_k obtained by improving (or deforming) the energy-momentum tensor $T(z)$ as [2]

$$\hat{T}(z)=T(z)-\rho\cdot\partial H(z) , \qquad (1.1)$$

where ρ is called the deformation parameter and is half the sum of the positive roots of the group G and $H(z)$ is the current for Cartan subalgebra (see below). If one considers $G_k=SU(n)_k$, for example, the central charge $\hat{c}$ is given by

$$\hat{c}=n^4-1-n(n^2-1)\left[\frac{1}{k+n}+k+n\right] , \qquad (1.2)$$

and apparently we are dealing with a fractional level WZW model [11] as can be seen by putting $p'/p=k+n$, with p and p' being comprime to each other. A coset construction of the unitary and nonunitary minimal models of the (super) conformal models is studied using such fractional level representations [12]. The fractional level representations of algebras are also getting some attention from the point of view of constructing new $N=2$ superconformal models.

The purpose of this paper is to consider an improved $SU(2)_k$ WZW model which may be regarded as a building block for further applications. In the following section, we show that expressing $SU(2)_k$ free fields in terms of Z_k parafermion fields and a U(1) field is quite convenient in the present case. We also discuss the conjugate primary states in the vertex operator representation so that the Coulomb gas formalism can be applicable for the computation of correlation functions. The charge-neutrality conditions are derived. In Sec. III we consider the three-point function and some four-point functions as an application. From the fact that this model is formulated in terms of Z_k parafermions, there exists a direct application to $N=2$ minimal super CFT. We derive the conditions for having nonvanishing correlation functions in $N=2$ models in Sec. IV. Section V is devoted to a discussion.

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II. IMPROVED SU(2)$_2$ WZW MODEL AND PARAFERMIONS

The affine SU(2)$_k$ currents for an arbitrary level k can be expressed in terms of the following free fields: $\varphi(z)$ and a pair of the ghosts $\beta(z)$ and $\gamma(z)$. They are given by

$$J_- = \beta(z) , \tag{2.1a}$$

$$J_+ = -\gamma^2(z)\beta(z) + k\,\partial\gamma(z) + i\alpha_+\alpha\,\partial\varphi(z) , \tag{2.1b}$$

$$H = \alpha\beta(z)\gamma(z) - i\alpha_+\partial\varphi(z) , \tag{2.1c}$$

where $\alpha + = \sqrt{k+2}$ with k being a positive integer and $\alpha = \sqrt{2}$ the simple root of SU(2). $J_-(z)$ and $J_+(z)$ are the currents associated with the negative and positive roots, respectively, and $H(z)$ is the one associated with the Cartan subalgebra. The field $\varphi(z)$ and commuting ghosts $\beta(z)$ and $\gamma(z)$ have the correlations

$$\varphi(z)\varphi(w) \sim -\ln(z-w) , \tag{2.2a}$$

$$\beta(z)\gamma(w) \sim -\gamma(z)\beta(w) \sim \frac{1}{z-w} . \tag{2.2b}$$

With these one can easily show that the currents satisfy the operator-product expansions (OPE's) as required:

$$J_+(z)J_-(w) \sim \frac{k}{(z-w)^2} + \frac{\alpha H(w)}{z-w} , \tag{2.3a}$$

$$H(z)J_\pm(w) \sim \frac{\pm\alpha J_\pm(w)}{z-w} . \tag{2.3b}$$

The ghost fields can be bosonized by the scalar fields $\phi(z)$ and $\chi(z)$ as

$$\beta(z) = i\partial\chi(z)e^{-\phi(z)-i\chi(z)} , \tag{2.4a}$$

$$\gamma(z) = e^{\phi(z)+i\chi(z)} , \tag{2.4b}$$

where $\phi(z)$ and $\chi(z)$ satisfy

$$\phi(z)\phi(w) \sim \chi(z)\chi(w) \sim -\ln(z-w) . \tag{2.5}$$

This bosonization procedure is necessary before introducing the parafermion fields. The energy-momentum tensor is obtained through Sugawara method as

$$\begin{aligned} T(z) &= -\frac{1}{2}(\partial\varphi)^2 - i\frac{\rho}{\alpha_+}\partial^2\varphi - \beta\,\partial\gamma \\ &= -\frac{1}{2}(\partial\varphi)^2 - i\frac{\rho}{\alpha_+}\partial^2\varphi \\ &\quad -\frac{1}{2}[(\partial\phi)^2 + (\partial\chi)^2 + \partial^2\phi + i\partial^2\chi] , \end{aligned} \tag{2.6}$$

where $\rho = \frac{1}{2}\sum_{\alpha>0}\alpha$ and is $1/\sqrt{2}$ for SU(2). The field $\phi(z)$ has the imaginary background charge, as can be expected from the bosonization formulas.

Furthermore, the screening currents are also known for the affine SU(2)$_k$ currents. Screening currents are the operators of conformal dimension 1, and all the OPE's with the affine currents are either regular or a total derivative (for simplicity called "commute" hereafter). The following three such currents are known for SU(2)$_k$ [5,13,14] (sometimes the ghosts fields $\beta(z)$ and $\gamma(z)$ are used just for the simplicity of the notations with the understanding of the bosonization [Eqs. (2.4a) and (2.4b)]):

$$S_-(z) = \beta e^{-i\alpha\varphi(z)/\alpha_+} , \tag{2.7a}$$

$$S_+(z) = e^{(k+2)\phi(z)+(k+1)i\chi(z)+i\alpha_+\alpha\varphi(z)} , \tag{2.7b}$$

$$\eta(z) = e^{i\chi(z)} . \tag{2.7c}$$

Let us introduce a primary state. A primary state for SU(2)$_k$ is given by the vertex operator

$$V_\Lambda = e^{i\Lambda\varphi/\alpha_+} , \tag{2.8}$$

where Λ is a weight vector and the conformal dimension of V_Λ is given by

$$h = \frac{\Lambda(\Lambda+2\rho)}{2(k+2)} . \tag{2.9}$$

This vertex operator has a regular OPE with the raising operator $J_-(z)$, and the OPE with a lowering operator $J_+(z)$ has a $1/z$ singularity and produces another state. One can show that generally such descendent states are given by multiplying some power of $\gamma(z)$ to the primary vertex operator as $[\gamma(z)]^n V_\Lambda(z)$ (the integer n being determined by the integrability condition).

Let us now introduce the Z_k parafermion fields and discuss rewriting the affine currents. First, we define the U(1) fields $\Phi(z)$ from the current of Cartan subalgebra $H(z)$:

$$\begin{aligned} H(z) &= -\alpha\,\partial\phi(z) - i\alpha_+\partial\varphi(z) \\ &\equiv i\sqrt{k}\;\partial\Phi(z) . \end{aligned} \tag{2.10}$$

Then define φ, ϕ, and χ in terms of the new fields in the following way[1] [15]:

$$\varphi(z) = \Psi + \frac{\alpha_+}{\sqrt{k}}(-\Phi + i\Phi) , \tag{2.11a}$$

$$\phi(z) = \frac{i\alpha\Phi}{\sqrt{k}} + \frac{k+g}{g}\frac{\alpha\tilde{\Phi}}{\sqrt{k}} - i\frac{\alpha_+}{g}\alpha\Psi , \tag{2.11b}$$

$$i\chi(z) = -\frac{\sqrt{k}}{g}\alpha\tilde{\Phi} + i\frac{\alpha_+}{g}\alpha\Psi , \tag{2.11c}$$

where $g=2$ is the dual Coxeter number of SU(2). These new fields are orthonormal to each other, namely,

$$\Psi(z)\Psi(w) \sim \tilde{\Phi}(z)\tilde{\Phi}(w) \sim \Phi(z)\Phi(w) \sim -\ln(z-w) , \tag{2.12}$$

and all the other correlations are regular. By this set of fields, the energy-momentum tensor in Eq. (2.6) is given as

[1]Another approach to define generalized parafermions is discussed in [16]; however, the equivalence of it to the one employed here is proved in [15].

$$T(z)=T_{\rm PF}(z)+T_{\rm U(1)}(z)$$
$$=-\frac{1}{2}(\partial\Psi)^2-\frac{i\rho}{\alpha_+}\partial^2\Psi-\frac{1}{2}(\partial\tilde{\Phi})^2-\frac{1}{2}(\partial\Phi)^2 . \qquad (2.13)$$

So, compared with the expression in Eq. (2.6), one can see that the transformations introduced above are the rotation which leaves the background charge for $\Psi(z)$ the same as $\varphi(z)$ and makes $\Phi(z)$ and $\tilde{\Phi}(z)$ be zero.

From this discussion it is rather clear that the improvement of the energy-momentum tensor in Eq. (1.1) is nothing but changing the background charge of the U(1) field from zero to some value fixed by the deformation parameter

$$\hat{T}(z)=T_{\rm PF}(z)-\frac{1}{2}(\partial\Phi)^2-i\left[\frac{k}{2}\right]^{1/2}\partial^2\Phi . \qquad (2.14)$$

Again, the point is that although the representations of $SU(2)_k$ can be given by either set of fields, the net effect of the deformation can be very simply described by the U(1) field Φ, as can be seen from Eq. (2.14). When the original variables ϕ and φ are used, the background charges of both of these fields are changed and the vertex operator representations for such a choice would be more complicated, while the representation of the parafermions is well understood [14] and is fully utilized in the following discussion.

Let us make some additional remarks about parafermions. The currents $J_\pm(z)$ can be easily expressed in terms of these new fields, and one finds that the U(1) field $\Phi(z)$ appears just as the exponential factor. Therefore the procedure of removing U(1) from $SU(2)_k$ can be achieved by simply dropping this exponential factor of the U(1) field.

Second, the primary field of this parafermion model is given by

$$\phi_{l,m}=\exp\left[i\frac{l}{\sqrt{2(k+2)}}\Psi-\frac{m}{\sqrt{2k}}\tilde{\Phi}\right], \qquad (2.15)$$

where l and m are the integers such that $0\le l\le k$ and $-l\le m\le l$ for a given integer level k. The conformal weight of this operator is given by

$$h=\frac{l(l+2)}{4(k+2)}-\frac{m^2}{4k} . \qquad (2.16)$$

Finally, we mention the screening currents for the parafermion currents. One can show that when the screening currents in Eq. (2.7) are rewritten by the new fields, they do not depend on the U(1) field $\Phi(z)$. Hence there are no changes of the conformal dimensions and the screening currents of $SU(2)_k$ current algebra are indeed the screening currents of Z_k parafermion currents. They are given as

$$S_-(z)=\left[i\left[\frac{k+2}{2}\right]^{1/2}\partial\Psi(z)-\left[\frac{k}{2}\right]^{1/2}\partial\tilde{\Phi}(z)\right]$$
$$\times\exp\left[-i\frac{2}{\sqrt{2(k+2)}}\Psi(z)\right], \qquad (2.17a)$$

$$\eta(z)=\exp\left[\frac{i}{\sqrt{2}}(k+2)\Psi(z)-\frac{k}{\sqrt{2}}\tilde{\Phi}(z)\right], \qquad (2.17b)$$

$$S_+(z)=\exp\left[\frac{i}{\sqrt{2}}(k+2)\Psi(z)+\frac{k}{\sqrt{2}}\tilde{\Phi}(z)\right]. \qquad (2.17c)$$

It should be noticed that the first one does not have the $\tilde{\Phi}$-field dependence in the exponential and is actually very useful in the subsequent discussion of correlation functions. This is deeply related to the fact that the two fields $\tilde{\Phi}$ and Φ have the same quantum number m in the vertex operators of primary states [see Eq. (2.19) below]. The screening charges are defined as the integrals of those currents around some closed contours:

$$Q_-=\int_c dt\,\partial\tilde{\Phi}(t)\exp\left[-i\frac{2}{\sqrt{2(k+2)}}\Psi(t)\right], \qquad (2.18a)$$

$$Q_0=\int_c dt\,\exp\left[\frac{i}{\sqrt{2}}(k+2)\Psi(t)-\frac{k}{\sqrt{2}}\tilde{\Phi}(t)\right], \qquad (2.18b)$$

$$Q_+=\int_c dt\,\exp\left[\frac{i}{\sqrt{2}}(k+2)\Psi(t)+\frac{k}{\sqrt{2}}\tilde{\Phi}(t)\right], \qquad (2.18c)$$

where in defining Q_- the total derivative part has been dropped and the overall factor has been changed.

The representation of vertex operator $V_{l,m}$ in this improved $SU(2)_k$ WZW model can be obtained by multiplying the vertex operators for the U(1) field $\Phi(z)$ and the parafermion model:

$$V_{l,m}(z)=\exp\left[\frac{il}{\sqrt{2(k+2)}}\Psi(z)+\frac{m}{\sqrt{2k}}[-\tilde{\Phi}(z)+i\Phi(z)]\right]. \qquad (2.19)$$

In this way one can think of the representations of (improved) $SU(2)_k$ as the combination of parafermions and the U(1) field. Hence the conformal dimension Δ of the vertex operator $V_{l,m}(z)$ is given as the sum of h for the parafermion part and that of Φ. The latter (for $e^{(im/\sqrt{2k})\Phi}$) is calculated as

$$h'=\frac{m(m+2k)}{4k} \qquad (2.20)$$

and one can easily see that this h' partly cancels with the contribution of $\tilde{\Phi}$ to h, when they are added together, leaving the contribution from the background charge of Φ, and Δ becomes

$$\Delta=h+h'=\frac{l(l+2)}{4(k+2)}+\frac{m}{2} . \qquad (2.21)$$

In applying the Coulomb gas formalism to this model, we first have to construct the primary state which is conjugate to $V_{l,m}$, namely, the state having the same conformal dimension. Actually, such a conjugate state can be obtained without difficulty as in the case of the minimal model [17]. Let us remember very briefly that in the background charge approach to the minimal model, the energy-momentum tensor is modified by adding the background charge α_0,

$$T_M(z)=-\tfrac{1}{2}(\partial\varphi)^2-i\alpha_0\partial^2\varphi \ , \tag{2.22}$$

and the correct central charge for the model is reproduced by choosing α_0 appropriately. When a primary state $V_\beta=e^{i\beta\varphi}$ is considered, its conformal weight is given by $h=\frac{1}{2}\beta^2-\alpha_0\beta$. The state conjugate to this has the same conformal weight and is obtained as $V_{2\alpha_0-\beta}$. Note that in this conjugate operator, $V_{2\alpha_0}$ is nothing but the conjugate identity operator which has the conformal dimension zero. In the improved $SU(2)_k$ WZW model, the fields $\Psi(z)$ and $\Phi(z)$ have nonzero background charges and a similar construction of conjugate vertex operators can be done.

Let us consider the original vertex operator $\phi_{l,m}$ for the parafermion model as the product of the two pieces for Ψ and $\tilde{\Phi}$, respectively (these are indicated by the indices l and m, respectively, as shown below); then one can easily check that the vertex operator

$$\tilde{\phi}_l(z)=\exp\left[-i\frac{l+2}{\sqrt{2(k+2)}}\Psi(z)\right] \tag{2.23}$$

has the same conformal weight as ϕ_l. And similarly, for the Φ field (which also has nonzero background charge), we get

$$\tilde{V}_m(z)=\exp\left[-i\frac{m+2k}{\sqrt{2k}}\Phi(z)\right] , \tag{2.24}$$

for $V_m=e^{im/\sqrt{2k}}$; namely, these are the conjugate operators. For the $\tilde{\Phi}$ field which has a vanishing background charge, the conformal weight of the vertex operator $\phi_m=e^{-(m/\sqrt{2k})\tilde{\Phi}}$ is given by $h=m^2/4k$, which is invariant under the change $m\rightarrow -m$. It is then convenient to define the "conjugate" as $\tilde{\phi}m=e^{(m/\sqrt{2k})\tilde{\Phi}}$ by flipping the sign of m in accordance with the sign change of m in the definition of the conjugate vertex operator for the U(1) field Φ. Hence the product of these three operators, namely,

$$\tilde{V}_{l,m}=\exp\left[-i\frac{l+2}{\sqrt{2(k+2)}}\Psi+\frac{m}{\sqrt{2k}}\tilde{\Phi}-i\frac{m+2k}{\sqrt{2k}}\Phi\right] ,$$

is the conjugate operator to $V_{l,m}$ in the improved WZW model.

With this preparation let us now turn to the charge neutrality conditions. As was discussed in the original paper by Dotsenko and Fateev [17], the charge-neutrality conditions are derived by considering a new ensemble defined by (consider the minimal model again)

$$\langle V_{\alpha_1}(z_1)\cdots V_{\alpha_n}(z_n)\rangle_c =\lim_{z\to\infty}\frac{\langle e^{-2i\alpha_0\varphi(z)}V_{\alpha_1}(z_1)\cdots V_{\alpha_n}(z_n)\rangle}{\langle e^{-2i\alpha_0\varphi(z)}e^{2i\alpha_0\varphi(0)}\rangle} , \tag{2.25}$$

where the charge of the additionally inserted operator has been chosen as the minus of that of the conjugate identity operator $\tilde{I}(z)=V_{2\alpha_0}(z)=e^{2i\alpha_0\varphi(z)}$. So, before discussing the neutrality conditions, let us briefly mention about the conjugate identity operators of the relevant fields. Since the identity operators are the ones which have the vanishing conformal dimension, such objects are easily constructed by requiring $h=\frac{1}{2}\beta(\beta-2\alpha_0)=0$ for the vertex operator $e^{i\beta\varphi}$ in the minimal model. Namely, $\beta=0$ gives the identity 1, while the other choice $\beta=2\alpha_0$ leads to the conjugate identity (as is now apparent, a conjugate identity operator is implicitly obtained in the course of constructing conjugate primary operator). In our model the relevant conjugate identity operators can be constructed in the same way and are obtained as $e^{-i[\alpha/\sqrt{(k+2)}]\Psi(z)}$ and $e^{-i\sqrt{2k}\,\Phi(z)}$, respectively.

We are now in a position to derive the charge-neutrality conditions for the improved WZW model. Apparently, it is the condition that the total charge of the vertex operators in the expectation value of the right-hand side (RHS) of Eq. (2.25) is zero. The similar operators to $V_{-2\alpha_0}$ to be inserted to define a new ensemble can be read off from the conjugate identity operators, which were just discussed. Therefore, taking this point into consideration, for the string of n vertex operators $V_{l,m}$ and N screening charges Q_- in Eq. (2.25), the condition for the Ψ field that the sum of the charges be zero becomes

$$\sum_{i=1}^{n-1} l_i-(l_n+2)-2N=-2 \tag{2.26}$$

or

$$\sum_{i=1}^{n-1} l_i-l_n=2N , \tag{2.27}$$

where the conjugate operator $\tilde{V}_{l,m}$ is introduced for nth state and the charge -2 in the RHS is the contribution from the analogue of $V_{-2\alpha_0}$ inserted in the new ensemble. Similarly, from the part of the n vertex operator of the U(1) field Φ, we have

$$\sum_{i-1}^{n-1} m_i-(m_n-2k)=2k \quad\text{or}\quad \sum_{i=1}^{n-1} m_i-m_n=0 . \tag{2.28}$$

As is obvious, this condition also neutralizes the total charges of the $\tilde{\Phi}$ field. Therefore these two are the conditions which have to be satisfied for any nonvanishing correlation functions.

From this discussion, for example, we see that the two-point function in parafermion model can be reproduced immediately [18] using Eq. (2.12):

$$\langle \phi_{l,m}(z)\tilde{\phi}_{l,m'}(w)\rangle_c=\delta_{m,m'}\frac{1}{(z-w)^{2h}} , \tag{2.29}$$

where the meaning of the conjugate operator $\tilde{\phi}_{l,m}$ would be obvious from the above and h is given in Eq. (2.16).

III. THREE- AND FOUR-POINT FUNCTIONS

In this section we apply the formalism developed above to study correlation functions. Let us start with the three-point function

$$G(z_1,z_2,z_3)=\langle V_{l_1,m_1}(z_1)V_{l_2,m_2}(z_2)\tilde{V}_{l_3,m_3}(z_3)\rangle_c \ , \quad (3.1)$$

where the conjugate state is introduced for the last operator. Since each of the vertex operators consists of three fields, the charge-neutrality conditions have to be satisfied separately for each sector. However, as mentioned before, there are actually two such conditions from the fact that the primary state is specified by the two quantum numbers. The charge-neutrality conditions to be satisfied by the vertex operators in Eq. (3.1) are given by

$$l_1+l_2-l_3=0 \ , \quad (3.2)$$

$$m_1+m_2-m_3=0 \ . \quad (3.3)$$

We now start with the evaluation of the parafermion part of $G(z_1,z_2,z_3)$ denoted by $J_{\rm PF}(z_1,z_2,z_3)$, which is calculated to be

$$J_{\rm PF}(z_1,z_2,z_3)=\frac{1}{(z_1-z_2)^{h_{12}}(z_2-z_3)^{h_{23}}(z_1-z_3)^{h_{13}}} \ , \quad (3.4)$$

where, with the use of Eq. (3.2),

$$h_{ij}=\frac{l_i l_j}{k+2}=h_l-h_i-h_j \ , \quad (3.5)$$

with h_i being given in Eq. (2.16). Next, the contribution of the Φ field J in Eq. (3.1) is

$$\begin{aligned} J(z_1,z_2,z_3)&=\left\langle \exp\left[i\frac{m_1}{\sqrt{2k}}\Phi(z_1)\right]\exp\left[i\frac{m_2}{\sqrt{2k}}\Phi(z_2)\right]\exp\left[-i\frac{m_3+2k}{\sqrt{2k}}\Phi(z_3)\right]\right\rangle_c \\ &=(z_1-z_2)^{h'_{12}}(z_2-z_3)^{h'_{23}}(z_1-z_3)^{h'_{13}} \ , \end{aligned} \quad (3.6)$$

where, with the help of Eq. (3.3), we get

$$h'_{12}=h'_3-h'_1-h'_2 \ , \quad (3.7)$$

with h'_i being defined in Eq. (2.20). Hence, by taking the product of these parts, the three-point function $G(z_1,z_2,z_3)$ is obtained as

$$G(z_1,z_2,z_3)=\frac{C_{123}}{(z_1-z_2)^{\Delta_{12}}(z_2-z_3)^{\Delta_{23}}(z_1-z_3)^{\Delta_{13}}} \ , \quad (3.8)$$

where

$$\Delta_{ij}=\Delta_l-\Delta_i-\Delta_j \ , \quad (3.9)$$

with Δ_i in Eq. (2.21). C_{123} is an operator-product coefficient and is not determined by the discussion of the three-point function, and one needs to consider four-point functions to calculate it [19].

A general form of four-point functions with the insertion of N screening charges Q_- is given as

$$G(z_1,z_2,z_3,z_4)=\langle V_{l_1,m_1}(z_1)V_{l_2,m_2}(z_2)V_{l_3,m_3}(z_3) \times\tilde{V}_{l_4,m_4}(z_4)Q_-\cdots Q_-\rangle_c \ , \quad (3.10)$$

where Q_- is given in Eq. (2.18a). However, from the fact that the field $\tilde{\Phi}$ does not have the exponential dependence in this screening charge, the evaluation of the expectation value of the relevant operators with an arbitrary number of Q_- becomes rather cumbersome. Therefore, in this paper, we consider some cases where a lower number of Q_- is used to neutralize the charges of the vertex operators. Below we study the cases with $N=0$, 1, and 2.

When $N=0$ it is rather straightforward, and after fixing the SL$(2,C)$ gauge in the standard way to be $z_1=\infty$, $z_2=1$, $z_3=z$, and $z_4=0$, we get, from Eq. (3.10),

$$G_0(z)=(1-z)^{l_2l_3/2(k+2)}z^{-l_3(l_4+2)/2(k+2)} \ , \quad (3.11)$$

where, from the neutrality conditions, the quantum numbers have to satisfy

$$l_1+l_2+l_3-l_4=2 \quad (3.12)$$

and

$$m_1+m_2+m_3-m_4=0 \ . \quad (3.13)$$

Next, consider the case when only one screening charge is being inserted: $N=1$. It is convenient to express the correlator as the product of each sector:

$$\begin{aligned} G_1(z)&=\int_c dt\,\langle V_{l_1,m_1}(\infty)V_{l_2,m_2}(1)V_{l_3,m_3}(z)\tilde{V}_{l_4,m_4}(0)S_-(t)\rangle_c \\ &=\int_c dt\, J_\Psi(z,t)\tilde{J}(z,t)J(z,t) \ , \end{aligned} \quad (3.14)$$

where

$$
\begin{aligned}
J\Psi(z,t) &= \left\langle \exp\left[i\frac{l_1}{\sqrt{2(k+2)}}\Psi(\infty)\right] \exp\left[i\frac{l_2}{\sqrt{2(k+2)}}\Psi(1)\right] \exp\left[i\frac{l_3}{\sqrt{2(k+2)}}\Psi(z)\right] \right.\\
&\quad \left. \times \exp\left[-i\frac{l_4+2}{\sqrt{2(k+2)}}\Psi(0)\right] \exp\left[-i\frac{2}{\sqrt{2(k+2)}}\Psi(t)\right] \right\rangle_c \\
&= (-1)^{(l_4+2)/\sqrt{2(k+2)}} G_0(z) t^{(l_4+2)/2(k+2)} (1-t)^{-l_2/2(k+2)} (z-t)^{-l_3/2(k+2)}
\end{aligned}
\tag{3.15}
$$

and

$$
\begin{aligned}
J(z,t) &= \left\langle \exp\left[i\frac{m_1}{\sqrt{2k}}\Phi(\infty)\right] \exp\left[i\frac{m_2}{\sqrt{2k}}\Phi(1)\right] \exp\left[i\frac{m_3}{\sqrt{2k}}\Phi(z)\right] \exp\left[-i\frac{2k+m_4}{\sqrt{2k}}\Phi(0)\right] \right\rangle_c \\
&= (1-z)^{m_2 m_3/2k} z^{-\frac{m_3(2k+m_4)}{2k}} ;
\end{aligned}
\tag{3.16}
$$

namely, $J(z,t)$ is the same as in the previous case $N=0$. The remaining part, the $\tilde{\Phi}$ contribution, is calculated as follows. Note that in this case the evaluation of the expectation value is easy, but in general when N screening charges are inserted, since we have to take all possible contractions of the fields, the result contains a large number of terms. One way to evaluate in a systematic manner is to use the following trick:

$$
\partial\tilde{\Phi} = \lim_{\beta\to 0} \frac{1}{\beta}(\partial e^{\beta\tilde{\Phi}}) ,
\tag{3.17}
$$

where the limit $\beta \to 0$ should be taken after the differentiation is performed. Then the expectation value of the exponential operators can be computed as in the previous case, and the final answer is obtained by taking the derivatives and limiting procedure $\beta \to 0$. Thus $\tilde{J}(z,t)$ is computed as

$$
\begin{aligned}
\tilde{J}(z,t) &= \left\langle \exp\left[-\frac{m_1}{\sqrt{2k}}\tilde{\Phi}(\infty)\right] \exp\left[-\frac{m_2}{\sqrt{2k}}\tilde{\Phi}(1)\right] \exp\left[-\frac{m_3}{\sqrt{2k}}\tilde{\Phi}(z)\right] \exp\left[\frac{m_4}{\sqrt{2k}}\tilde{\Phi}(0)\right] \partial\tilde{\Phi}(t) \right\rangle_c \\
&= \frac{1}{\sqrt{2k}} \left[\frac{m_2}{1-t} + \frac{m_3}{z-t} + \frac{m_4}{t} \right] (1-z)^{-m_2 m_3/2k} z^{m_3 m_4/2k} .
\end{aligned}
\tag{3.18}
$$

Putting it all together, $G_1(z)$ is expressed as $G_0(z)$ multiplied by the integral, which can be written as the hypergeometric functions

$$
\begin{aligned}
G_1(z) &= \frac{1}{\sqrt{2k}}(-1)^{(l_4+2)/2(k+2)} G_0(z) \int_c dt \left[\frac{m_2}{1-t} + \frac{m_3}{z-t} + \frac{m_4}{t} \right] t^{(l_4+2)/2(k+2)} (1-t)^{-l_2/2(k+2)} (z-t)^{-l_3/2(k+2)} \\
&= \frac{1}{\sqrt{2k}}(-1)^{(l_4+2)/2(k+2)} G_0(z) \left[\sum_{J=2,3,4} m_J I_i^J(z,t) \right] .
\end{aligned}
\tag{3.19}
$$

Depending on the choice of contours (indicated by the index i), $I_i^J(z)$ is defined as follows and is expressed in terms of the hypergeometric function $F(a,b,c;z)$:

$$
\begin{aligned}
I_1^J(z) &= \int_1^\infty dt\, t^{a_J}(t-1)^{b_J}(z-t)^{c_J} \\
&= \frac{\Gamma(-a_J-b_J-c_J-1)\Gamma(b_J+1)}{\Gamma(-a_J-c_J)} F(-c_J, -a_J-b_J-c_J-1, -a_J-c_J; z)
\end{aligned}
\tag{3.20}
$$

and

$$
\begin{aligned}
I_2^J(z) &= \int_0^z dt\, t^{a_J}(1-t)^{b_J}(z-t)^{c_J} \\
&= z^{1+a_J+c_J} \int_0^1 dt\, t^{a_J}(1-t)^{c_J}(1-zt)^{b_J} \\
&= z^{1+a_J+b_J} \frac{\Gamma(a_J+1)\Gamma(c_J+1)}{\Gamma(a_J+c_J+2)} F(-b_J, a_J+1, a_J+c_J+2, z) ,
\end{aligned}
\tag{3.21}
$$

where, for simplicity, by defining $a=l_2+2/2(k+2)$, $b=-l_2/2(k+2)$, and $c=-l_3/2(k+2)$, a_J, b_J, and c_J are given by

$$a_2=a_3=a,\quad a_4=a-1\ ,$$
$$b_2=b-1,\quad b_3=b_4=b\ , \tag{3.22}$$
$$c_2=c_4=c,\quad c_3=c-1\ .$$

These two solutions (specified by the contours) correspond to two solutions of second-order differential equations [17].

In order to study the structure of the conformal blocks, we have to examine the monodrony property of this amplitude. Fortunately, for each $F(a_J,b_J,c_J;z)$ of the constituents of $G_1(z)$, the monodromy has been studied in the case of the minimal model. In our case the monodromy of

$$I_i(z)=\sum_{J=2}^{4} m_J I_i^J(z) \tag{3.23}$$

can be obtained as the sum of $I_i^J(z)$ over J; therefore, it is enough to know the monodromy of $I_i^J(z)$ for each J. Denoting the contribution of the antiholomorphic sector by an overbar, we get [17]

$$I=I(z)\bar{I}(\bar{z})=\sum_{J=2}^{4}\left[\sum_{i=1,2} X_i^J I_i^J(z)\bar{I}_i^J(\bar{z})\right]\ ; \tag{3.24}$$

namely, the monodromy can be found for each J through the relation

$$I_i^J(z)=\sum_j \alpha_{ij}^J \tilde{I}_j^J(1-z)\ . \tag{3.25}$$

The α_{ij}^J matrix can be calculated by changing the contours as was discussed in detail in [17]:

$$\alpha_{11}^J=\frac{s(a_J)}{s(b_J+c_J)},\quad \alpha_{12}^J=-\frac{s(c_J)}{s(b_J+c_J)}\ ,$$
$$\alpha_{21}^J=-\frac{s(a_J+b_J+c_J)}{s(b_J+_J)},\quad \alpha_{22}^J=-\frac{s(b_J)}{s(b_J+c_J)}\ , \tag{3.26}$$

where $s(a)=\sin(\pi a)$. And from the condition

$$\sum_{k,l} X_i\alpha_{ik}\alpha_{il}=0,\quad k\neq l\ , \tag{3.27}$$

we get

$$\frac{X_1^J}{X_2^J}=-\frac{\alpha_{21}^J\alpha_{22}^J}{\alpha_{11}^J\alpha_{22}^J}=\frac{s(a_J+b_J+c_J)s(b_J)}{s(a_J)s(c_J)}\ . \tag{3.28}$$

The invariant function $G(z,\bar{z})$ is found by multiplying the extra z-dependent part (and $\bar{z}$) from Eq. (3.19).

For the special case where the quantum numbers m_J are $m_1=m_4$ and $m_2=m_3=0$ such that they are consistent with the condition Eq. (3.13), the correlation function $G_1(z)$ has a simple structure analogous to an example in the minimal model.

Let us now consider the example with two screening charges, $N=2$, in Eq. (3.10):

$$G_2(z)=\int_c dt_1\int_c dt_2\langle V_{l_1,m_1}(\infty)V_{l_2,m_2}(1)V_{l_3,m_3}(z)\tilde{V}_{l_4,m_4}(0)S_-(t_1)S_-(t_2)\rangle_c\ , \tag{3.29}$$

with the restrictions on the quantum number l from Eqs. (2.26),

$$l_1+l_2+l_3-l_4=4\ , \tag{3.30}$$

and Eq. (3.13) for m_J. Just as in the previous case, $G_2(z)$ is expressed as the product of the following pieces:

$$\begin{aligned}J_\Psi^{(2)}(z,t)=&\left\langle\exp\left[i\frac{l_1}{\sqrt{2(k+2)}}\Psi(\infty)\right]\exp\left[i\frac{l_2}{\sqrt{2(k+2)}}\Psi(1)\right]\exp\left[i\frac{l_3}{\sqrt{2(k+2)}}\Psi(z)\right]\right.\\&\left.\times\exp\left[-i\frac{l_4+2}{\sqrt{2(k+2)}}\Psi(0)\right]\exp\left[-i\frac{2}{\sqrt{2(k+2)}}\Psi(t_1)\right]\exp\left[-i\frac{2}{\sqrt{2(k+2)}}\Psi(t_2)\right]\right\rangle_c\\=&\,t_1^a(1-t_1)^b(z-t_1)^c t_2^a(1-t_2)^b(z-t_2)^c(t_1-t_2)^g\ ,\end{aligned} \tag{3.31}$$

where $g=1/2(k+2)$ and a,b,c are as defined before, $J^{(2)}(z,t)$, which is the same as given in Eq. (3.16) for any number of insertion of the screening charges, and

$$\begin{aligned}\tilde{J}^{(2)}(z,t)=&\left\langle\exp\left[-\frac{m_1}{\sqrt{2k}}\tilde{\Phi}(\infty)\right]\exp\left[-\frac{m_2}{\sqrt{2k}}\tilde{\Phi}(1)\right]\exp\left[-\frac{m_3}{\sqrt{2k}}\tilde{\Phi}(z)\right]\exp\left[\frac{m_4+2k}{\sqrt{2k}}\tilde{\Phi}(0)\right]\partial\tilde{\Phi}(t_1)\partial\tilde{\Phi}(t_2)\right\rangle\\=&\left[\frac{1}{(t_1-t_2)^2}+\frac{1}{2k}\sum_{J,L=2}^{4}\frac{M_JM_L}{(z_J-t_1)(z_L-t_2)}\right](1-z)^{-m_2m_3/2k}z^{m_3m_4/2k}\ ,\end{aligned} \tag{3.32}$$

where we defined $M_J=m_J$ for $J=2,3$ and $M_4=-m_4$. Equation (3.32) implies that each conformal block contains ten terms which are specified by the products of the quantum number m_J.

Again, by combining these expressions, the amplitude is obtained as

$$G_2(z)=(-1)^{(l_4+2)/(k+2)}G_0(z)\int_c dt_1\int_c dt_2 t_1^a(1-t_1)^b(z-t_1)^c t_2^a(1-t_2)^b(z-t_2)^c(t_1-t_2)^g$$
$$\times\left[\frac{1}{(t_1-t_2)^2}+\frac{1}{2k}\sum_{J,L=2}^{4}\frac{M_M M_L}{(z_J-t_1)(z_L-t_2)}\right]. \quad (3.33)$$

We consider a particular case as mentioned before where the m quantum numbers are $m_1=m_4$ and $m_2=m_3=0$. Then the expression in square brackets in the above equation simplifies to

$$\frac{1}{(t_1-t_2)^2}+\frac{1}{2k}\frac{m_4^2}{t_1t_2}. \quad (3.34)$$

In this situation the integrand is symmetric in t_1 and t_2 (apart from a phase factor) and the analysis of the monodromy becomes simpler.

As the $N=1$ case, let us define

$$J_1(a,b,c,g:z)=2c(\tfrac{1}{2}g)I_1(a,b,c,g:z)$$
$$=2c(\tfrac{1}{2}g)\int_1^\infty dt_1\int_1^\infty dt_2 t_1^a(t_1-1)^b(t_1-z)^c t_2^a(t_2-1)^b(t_2-z)^c(t_1-t_2)^g, \quad (3.35)$$
$$J_2(a,b,c,g:z)=I_2(a,b,c,g:z)$$
$$=z^{a+c+1}\int_1^\infty dt_1\int_0^1 dt_2 t_1^a(t_1-1)^b(t_1-z)^c t_2^a(1-t_2)^b(1-zt_2)^c(t_1-t_2)^g, \quad (3.36)$$
$$J_3(a,b,c,g:z)=2c(\tfrac{1}{2}g)I_3(a,b,c,g:z)$$
$$=z^{2(a+c+1)+g}\int_0^1 dt_1\int_0^1 dt_2 t_1^a(1-t_1)^b(1-zt_1)^c t_2^a(1-t_2)^b(1-zt_2)^c(t_1-t_2)^g, \quad (3.37)$$

where $c(a)=\cos(\pi a)$.

Now, corresponding to the two terms in Eq. (3.34), we again use the index J to distinguish them. By repeating a similar procedure to the previous case, the invariant amplitude can be found. The α_{ij}^J matrix is defined as Eq. (3.25) (now it is 3×3) and is obtained as (only the matrix elements which are needed to determine the ratio X_i^J/X_j^J are given below) [17]

$$\alpha_{13}^J\frac{s(c_J)s(c_J+\frac{1}{2}g_J)}{s(b_J+c_J+\frac{1}{2}g_J)s(b_J+c_J+g_J)},$$
$$\alpha_{31}^J=\frac{s(a_J+b_J+c_J+\frac{1}{2}g_J)s(a_J+b_J+c_J+g_J)}{s(b_J+c_J)s(b_J+c_J+\frac{1}{2}g_J)},$$
$$\alpha_{32}^J=\frac{s(a_J+b_J+c_J+g_J)s(b_J)}{s(b_J+c_J)s(b_J+c_J+g_J)},$$
$$\alpha_{33}^J=\frac{s(b_J)s(b_J+\frac{1}{2}g_J)}{s(b_J+c_J+\frac{1}{2}g_J)s(b_J+c_J+g_J)}. \quad (3.38)$$

And the ratios are calculated as

$$\frac{X_1^J}{X_3^J}=\frac{s(a_J+b_J+c_J+g_J)s(a_J+b_J+c_J+\frac{1}{2}g_J)s(b_J)s(b_J+\frac{1}{2}g_J)s(a_J+c_J+g_J)}{s(a_J)s(a+\frac{1}{2}g_J)s(c_J)s(c_J+\frac{1}{2}g_J)s(a_J+c_J)},$$
$$\frac{X_2^J}{X_3^J}=\frac{s(a_J+b_J+c_J+g_J)s(a_J+c_J+\frac{1}{2}g_J)s(b_J)}{s(c_J+\frac{1}{2}g_J)s(a_J+\frac{1}{2}g_J)s(a_J+c_J)2c(\frac{1}{2}g_J)}. \quad (3.39)$$

The invariant function is expressed as

$$G_2(z,\bar z)\sim\sum_{J=1,2}C_J(X_1^JI_1\bar I_1+X_2^JI_2\bar I_2+X_3^JI_3\bar I_3), \quad (3.40)$$

where C_J are the coefficients which give the relative weight of two terms in Eq. (3.34) and are defined as $C_1=1$ and $C_2=m_4^2/2k$.

When one considers a general case without any restriction on m_J, there is no longer the symmetry between the integration variables t_1 and t_2. Such a situation did not occur in the minimal model, but one can see that it always occurs, for example, in the study of correlation functions in the $SU(n)_k$ WZW model. A detailed examination will be given separately.

As one can see from these examples, one complication in evaluating the correlation functions with an arbitrary number of screening charges Q_- being inserted is that conformal blocks contain a large number of terms coming from the $\tilde{\Phi}$ field. An effective way will be needed to compute systematically with any number of Q_-. On the contrary, it may not be so difficult to compute five- or even higher-point functions with a small number of Q_-.

IV. SELECTION RULES IN $N=2$ MINIMAL MODELS

In the improved $SU(2)_k$ WZW model [also our discussion obviously applies to the $SU(2)_k$ WZW model], the conditions by which any nonzero correlation functions exist are the ones of charge neutrality in Eqs. (2.23) and (2.28). It turns out that from these conditions, which may be called "selection rules," one is able to get some more useful information when they are applied to $N=2$ minimal models.

$N=2$ minimal models with a central charge $c=3k/(k+2)$ are known to be quite useful for achieving string compactification. By taking the tensor product of minimal models such that $c_{\rm tot}=9$, one can construct four-dimensional string models with massless spectra having $N=1$ spacetime supersymmetry. One advantage of this compactification scheme is that one can also study three- and even higher-point correlation functions among those massless states. In the three-generation model, the so-called 1×16^3 model, all the three-point functions of massless states have been calculated, and in order to study possible higher point functions, the selection rules were derived [10].

In the previous study of 1×16^3 model, the calculability of correlation functions is due to the fact that the primary fields of the $N=2$ minimal model are related through the Z_k parafermion to those of the $SU(2)_k$ WZW model where it is known how correlation functions can be calculated [18]. However, although correlation functions are obtainable in general, one has to solve differential equations, which is usually not easy. On the other hand, it is easy to derive a set of conditions which have to be satisfied by any correlation functions having nonvanishing behavior. However, the criteria derived there are only the necessary conditions and not the sufficient conditions. Now, with this Coulomb gas formalism for Z_k parafermions, we can derive the exact conditions for nontrivial correlation functions.

Let us remember that a primary field of $N=2$ minimal models is written as

$$\Phi^{l;\bar{l}}_{q,s;\overline{qs}}(z,\bar{z})=\phi^{l;\bar{l}}_{q-s;\bar{q}-\bar{s}}(z,\bar{z})e^{i\alpha_{qs}\varphi'(z)+i\alpha_{\overline{qs}}\varphi'(\bar{z})}\ , \qquad (4.1)$$

where the holomorphic part $\phi^{l}_{q-s}(z)$ is defined in Eq. (2.15) and $\varphi'(z)$ is another boson, with the charge α_{qs} being given by

$$\alpha_{qs}=\frac{1}{\sqrt{k(k+2)}}\left[-q+\frac{1}{2}(k+2)s\right]\ , \qquad (4.2)$$

and similarly for the antiholomorphic sector. For a given level, k, l, q, and s are integers such that $0\le l\le k$, $s=0,\pm1,2$ (defined mod 4), and $|q-s|\le l$, while the primary field for the $SU(2)_k$ WZW model is given in Eq. (2.19). Hence the vertex operators for these two models are related through the Z_k parafermion field. Therefore, by translating Eqs. (2.26) and (2.28), we get the selection rules for correlation functions in the $N=2$ minimal model.

In four-dimensional string models compactified by tensor products of $N=2$ minimal models, the L-point function which we want to study takes the form

$$\langle V_{-1}(z_1,\bar{z}_1)V_{-1/2}(z_2,\bar{z}_3)V_{-1/2}(z_3,\bar{z}_3)\times V_0(z_4,\bar{z}_4)V_0(z_5,\bar{z}_5)\cdots V_0(z_L,\bar{z}_L)\rangle\ , \qquad (4.3)$$

where V_{-1} is a vertex operator for a spacetime boson in the -1 picture and V_0 is the picture-changed version of the operator V_{-1}, while $V_{-1/2}$ is a spacetime fermion vertex operator. For example,

$$V_{-1}(z,\bar{z})=e^{-\xi}e^{ik\cdot X(z,\bar{z})}O(z,\bar{z})\tau^a\ , \qquad (4.4)$$

where $O(z,\bar{z})$ is the internal part which is built by a tensor product of the $N=2$ minimal model

$$O(z,\bar{z})=\otimes_{i=1}^{r}\Phi^{l_i;\bar{l}_i}_{q_i,s_i;\bar{q}_i,\bar{s}_i}(z,\bar{z})\ , \qquad (4.5)$$

ξ is a ghost field [20], and τ^a is the generator of SO(10) representation. V_0 and $V_{-1/2}$ are obtained from V_{-1} by the picture-changing and supersymmetry operators, respectively.

In this string compactification, the holomorphic (right-moving) sector gives the spacetime property of spectra, while the antiholomorphic (left-moving) sector is responsible for the structure of gauge groups of string models. The internal part of the massless spectra is represented by a set of integers $(l_i,q_i,s_i{:}\bar{l}_i,\bar{q}_i,\bar{s}_i)$, $i=1,\ldots,r$, with r being the number of $N=2$ minimal models in the tensor product.

In studying spectra in four-dimensional string models in this compactification scheme, the massless states in the representation of the gauge group E_6 are our main interest. It turns out that the representations of E_6 can be given by those of $SO(10)\times U(1)$, and it is convenient to specify the quantum numbers $(l_i,q_i,s_i{:}\bar{l}_i,\bar{q}_i,\bar{s}_i)$ for scalars of SO(10). Then $(l_i,q_i,s_i:\bar{l}_i,\bar{q}_i,\bar{s}_i)$ for other states can be obtained by using the supersymmetry and gauge representation operators. Let us first discuss the holomorphic sector. For a given $(l_i,q_i,s_i{:}\bar{l}_i,\bar{q}_i,\bar{s}_i)$ of V_{-1}, the quantum numbers of the internal part of $V_{-1/2}$ and V_0 are obtained as (for example, see [10,21] for details)

$$V_{-1/2}\sim\bigotimes_{i=1}^{r}(l_i,q_i+1,s_i+1{:}\bar{l}_i,\bar{q}_i,\bar{s}_i)\ , \qquad (4.6)$$

$$V_0\sim\sum_{j=1}^{r}\Big[\bigotimes_{i=1}^{j-1}(l_i,q_i,s_i{:}\bar{l}_i,\bar{q}_i,\bar{s}_i)\otimes(l_j,q_j,s_j+2{:}\bar{l}_j,\bar{q}_j,\bar{s}_j)\bigotimes_{k=j+1}^{r}(l_k,q_k,s_k{:}\bar{l}_k,\bar{q}_k,\bar{s}_k)\Big]\ , \qquad (4.7)$$

respectively, as the consequences of the supersymmetry and picture-changing operators.

We now turn to the antiholomorphic sector. As mentioned, the representation **27** of E_6 is decomposed under $SO(10)\times U(1)$ as $\mathbf{10}_{-2}+\mathbf{16}_{1/2}+\mathbf{1}_1$, where the subscripts represent the U(1) charges of the SO(10) states, and these different representations can be related by the gauge representation transformation. Starting with $\mathbf{10}_{-2}$, one can get $\mathbf{16}_{-1/2}$ and $\mathbf{1}_2$ by successively performing the shifts $\bar{q}\to\bar{q}+1$ and $\bar{s}\to\bar{s}+1$.

Now we are focusing on the correlation function of the internal part of the vertex operators; hence, the relevant part of Eq. (4.3) is given by (here only one of the sectors in the tensor product of the $N=2$ minimal model is considered for simplicity)

$$\langle \Phi^{L_i}_{q_i,s_i}(z_1)\Phi^{l_2}_{q_2,s_2}(z_2)\cdots\rangle=\langle \phi^{l_1}_{q_1-s_1}(z_1)\exp[i\alpha_{q_1s_1}\varphi'(z_1)]\phi^{l_2}_{q_2-s_2}(z_2)\exp[i\alpha_{q_2+1s_2+1}\varphi'(z_2)]\phi^{l_3}_{q_3-s_3}(z_3)$$
$$\times\exp[i\alpha_{q_3+1s_3+1}\varphi'(z_3)]\cdots\phi^{l_L}_{q_L-s_L}(z_L)\exp[i\alpha_{q_Ls_L}\varphi'(z_L)]\rangle\ . \tag{4.8}$$

In our Coulomb gas formulation, this correlation function is understood as

$$\langle \phi^{l_1}_{q_1-s_1}(z_1)\exp[i\alpha_{q_1s_1}\varphi'(z_1)]\phi^{l_2}_{q_2-s_2}(z_2)\exp[i\alpha_{q_2+1s_2+1}\varphi'(z_2)]\phi^{l_3}_{q_3-s_3}(z_3)$$
$$\times\exp[i\alpha_{q_3+1s_3+1}\varphi'(z_3)]\cdots\bar{\phi}^{l_L}_{q_L-s_L}(z_L)\exp[\bar{\alpha}_{q_Ls_L}\varphi'(z_L)]QQ\cdots Q\rangle_c\ , \tag{4.9}$$

where the conjugate is introduced for the Lth vertex operator and N screening charges are inserted. It is also understood that, according to Eq. (4.7), the picture-changing operation has been done for the last $L-3$ vertex operators.

With this form it is rather clear that, from the charge neutrality Eqs. (2.27) and (2.28), the following conditions have to be satisfied for the parafermion part ϕ^l_{q-s}:

$$\sum_{J=1}^{L-1} l_j-l_L=2N \tag{4.10}$$

and

$$\sum_{J=1}^{L-1}(q_J-s_J)-2\left[\sum_{J=4}^{L-1}d_J-d_L\right]-(q_L-s_L)=0\ , \tag{4.11}$$

where $d_J=0$ or 1 represents the effect of the picture-changing operation. Obviously, Eq. (4.10) is the condition to determine the number of screening charges for the given l_J's. Now, as for the restriction on the charge α_{qs} of the U(1) field φ', from the previous discussion of defining a conjugate operator, we need to flip the sign as $\alpha_{qs}\to-\alpha_{qs}=\bar{\alpha}_{qs}$, which is simply achieved by $(q,s)\to(-q,-s)$ in Eq. (4.2). Thus we get

$$\alpha_{q_1s_1}+\alpha_{q_2+1s_2+1}+\alpha_{q_3+1s_3+1}+\alpha_{q_4s_4}+\cdots+\alpha_{q_{L-1}s_{L-1}}-\alpha_{q_Ls_L}$$
$$=\sum_{J=1}^{L-1}\alpha_{q_Js_J}-\alpha_{q_Ls_L}-\left[\frac{k+2}{k}\right]^{1/2}\left[\sum_{J=4}^{L-1}d_J-d_L+\frac{k}{k+2}\right]$$
$$=0\ . \tag{4.12}$$

From Eqs. (4.11) and (4.12), one may derive a condition involving only the quantum number q_J:

$$\sum_{J=1}^{L-1}q_J-q_L+2=0\ . \tag{4.13}$$

By comparison we see that this condition corresponds to the one for q_J obtained in [10] and it is the advantage of this Coulomb gas approach that we can derive an explicit condition for the quantum number l. One may also derive a condition for s as has been done in [10], but it contains d_J and the condition in Eq. (4.13) is more useful.

Equations (4.10), (4.11), and (4.13) are the basic form of the selection rules. The previous selection rules were derived from the requirement that the operator product of parafermion fields in correlators be proportional to unity; hence, a stronger condition was not obtained for l.

Let us now discuss the antiholomorphic sector. As higher-point functions, one may be interested in the forms $(\mathbf{27}^3)^N$, $(\overline{\mathbf{27}}^3)^N$, or $(\mathbf{27}\times\overline{\mathbf{27}})^L$. However, as mentioned before, some representations of SO(10) are actually chosen in these correlation functions. Since **10** of SO(10) has been chosen as the canonical state in calculating massless spectra in this compactification scheme, let us first consider the simple case $(\mathbf{10}\times\overline{\mathbf{10}})^L$. The selection rules for such cases can be derived similarly as in Eqs. (4.11)–(4.13):

$$\sum_{i=J}^{2L-1}\bar{l}_J-\bar{l}_{2L}=2N\ , \tag{4.14}$$

$$\sum_{J=1}^{2L-1} \bar{q}_J - \bar{q}_{2L} = 0 \; . \tag{4.15}$$

Generally speaking, a correlation function $(\mathbf{27}\times\overline{\mathbf{27}})^L$ may contain some numbers of representations **10**, **16**, and **1** and the selection rules for such a case are easily obtained from those derived above by performing the shifts of $\bar{q}$ and $\bar{s}$ in an appropriate way.

In a practical application of these selection rules to correlation functions in a three-generation string model, obviously one is also interested in the couplings of the moduli (and **27** singlets) to **27** and $\overline{\mathbf{27}}$. Since quite a large number of singlets are found in this model, a separate study will be needed.

V. DISCUSSION

In this paper we showed how Z_k parafermions and U(1) fields can be used to describe the improved $SU(2)_k$ WZW model. In terms of the free-field representation of the parafermions, we also calculated some correlation functions. For some values of l and m, four-point functions are shown to be very similar to those in the minimal model.

When improving the energy-momentum tensor of $SU(2)_k$, we have especially chosen the deformation parameter to be ρ in connection with a fractional level of an algebra. It is obvious, however, that our approach can be employed for any choice of the parameter. It is also possible to choose the deformation parameter such that the energy-momentum tensor has a vanishing central charge. Such a model is shown to be related to a topological conformal theory [22]. Generally, a different parameter simply corresponds to a different value of the background charge of the U(1) field $\Phi(z)$. Hence it is easy to see how a change of parameter results, for example, in scaling behaviors of correlation functions.

For other groups it is also known how generalized parafermions can be introduced. The improvement of the energy-momentum tensor is still given by Eq. (1.1). Let us consider a group G_k where a set of positive roots α is denoted as Δ_+. In this case the U(1) field is still defined from the Cartan subalgebra

$$\begin{aligned} H(z) &= \sum_{\alpha\in\Delta_+} \alpha\beta_\alpha\partial\gamma_\alpha - i\alpha + \partial\varphi \\ &= \sum_{\alpha\in\Delta_+} \alpha\phi_\alpha(z) - i\alpha + \partial\varphi \\ &\equiv i\sqrt{k}\;\partial\Phi(z) \; , \end{aligned} \tag{5.1}$$

where a pair of ghosts $(\beta_\alpha,\gamma_\alpha)$ is introduced for each positive root α and the bosonization is carried out in terms of the set of scalars $(\phi_\alpha,\chi_\alpha)$. Φ is now a vector whose number of components is given by the rank of G_k. With these fields the rotation to the generalized parafermion fields is obtained by the following redefinition of the fields [14] [corresponding to Eq. (5.1), Ψ and $\tilde{\Phi}$ are also vectors and these new fields are orthogonal as in Eq. (2.12)]:

$$\varphi = \Psi + \frac{\alpha_+}{\sqrt{k}}(-\Phi + i\tilde{\Phi}) \; , \tag{5.2}$$

$$\phi_\alpha = i\frac{\alpha\cdot\Phi}{\sqrt{k}} + \frac{k+g}{g}\,\frac{\alpha\cdot\Phi}{\sqrt{k}} - i\frac{\alpha_+}{g}\alpha\cdot\Psi + f_\alpha \; , \tag{5.3}$$

$$i\chi_\alpha = -\frac{\sqrt{k}}{g}\alpha\cdot\tilde{\Phi} + i\frac{\alpha_+}{g}\alpha\cdot\Psi + ig_\alpha \; , \tag{5.4}$$

which are the extension of those for the $SU(2)_k$ case in Eqs. (2.11). f_α and g_α are the fields with the constraints

$$\sum_{\alpha\in\Delta_+} f_\alpha = 0, \quad \sum_{\alpha\in\Delta_+} g_\alpha = 0 \; ; \tag{5.5}$$

hence, not all of them are independent. Actually, one can show that the correlations of f_α and g_α are given by

$$\begin{aligned} f_\alpha(z)f_\beta(w) &= g_\alpha(z)g_\beta(w) \\ &= -\left[\delta_{\alpha\beta} - \frac{\alpha\cdot\beta}{g}\right]\ln(z-w) \; . \end{aligned} \tag{5.6}$$

Because f_α and g_α are constrained, one can show that this set of fields can reproduce the same central charge as the G_k WZW model. Needless to say, for $SU(2)_k$ there is only one root and the constraints are satisfied only for $f_\alpha = g_\alpha = 0$, leading to Eq. (2.11).

For the improved G_k WZW model with Eq. (1.1), it may still be possible to use these two sets of constraint fields with Eq. (5.6). However, for lower-rank groups, for example $SU(3)_k$, it is easy to solve the constraints and redefine independent orthonormal fields. Such a study will be pursued in a future publication.

Another possible direction may be an explicit evaluation of five- or even higher-functions in this formalism.

Finally, we comment on the connection to the work by Distler and Qiu [23], who constructed the irreducible representation of the $SU(2)_k$ WZW model from the corresponding parafermions. Our improved model has been constructed by changing only the Cartan current. Therefore the parafermions can be derived from J_+ in Eqs. (2.1a) and (2.1b) in the same manner as Ref. [23]. This means that the irreducible representation of the improved $SU(2)_k$ model can be obtained along the line of Distler and Qiu [23].

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