Equivalence of convective and potential variational derivations of covariant superfluid dynamics

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The relativistically covariant theory of two-constituent superfluid dynamics that was derived by the convective variational approach (a specialization of the formalism developed for the covariant treatment of elastic media) is found to agree precisely, in the nondissipative limit, with the theory that was derived independently using a potential variational principle (a generalization of the classical Clebsch formalism).

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I. INTRODUCTION

The main purpose of this work is to show the complete equivalence, in the nondissipative limit, between two alternative kinds of approaches—using variation principles of respectively "convective" and "potential" type—to the covariant treatment of superfluid dynamics at a macroscopic level.

In particular, the relativistically covariant theory of two-constituent superfluid dynamics that was derived [1–3] by the convective variational approach (a specialization of the formalism developed [4,5] for the covariant treatment of elastic media) is found to agree *precisely* with the theory that was derived independently [6,7] using a potential variational principle (a generalization of the classical Clebsch formalism [8]). (For a specific application—such as the experimentally accessible example of liquid helium, or the theoretically predicted examples of highly compressed superfluids in neutron stars—the theory so obtained requires just the prescription of an appropriate equation of state giving the relevant action density as a single scalar function of three independent scalar variables.)

The convective variational approach [1–3] used a master function Λ that is given as a function of the three independent scalar variables $(n^\rho n_\rho,\ n^\rho s_\rho,\ s^\rho s_\rho)$ that can be constructed from an entropy current vector s^ρ, and a total-particle-number current vector n^ρ (which are both taken to be conserved in the nondissipative limit considered here) so that in a fixed background geometry its most general variation takes the form

$$d\Lambda=\mu_\rho dn^\rho+\Theta_\rho ds^\rho\ , \tag{1.1}$$

for coefficients μ_ρ and Θ_ρ that are interpretable as respectively particle and thermal four-momentum covectors. On the other hand, the potential variational approach [6,7] used a Lagrangian density Ψ that is given as a function of the three independent scalar variables $(I_1=\frac{1}{2}m^2v_\rho v^\rho,\ I_2=mkv_\rho w^\rho,\ I_3=\frac{1}{2}k^2w_\rho w^\rho)$ that can be constructed from superfluid and thermal momentum covectors v_μ and w_μ so that in a fixed geometrical background its most general variation is expressible as

$$d\Psi=j^\rho dv_\rho+s^\rho dw_\rho\ , \tag{1.2}$$

for coefficients j^ρ and s^ρ that are to be interpreted as representing rest mass and entropy currents.

As far as notation is concerned the translation between the two approaches is straightforward, the only significant difference being our use of opposite sign conventions for the specification of the momentum covectors and a difference of dimensionality between the particle-number current n^ρ and the corresponding mass current j^ρ whose specification implicitly incorporates a dimensional scale factor m that is interpretable as a fixed rest mass per particle. (This mass m might of course, without loss of physical generality, be set equal to unity, $m=1$, by choice of units. We shall, however, retain the particle rest mass m, together with the speed of light c, the Dirac Plank constant $\hbar$, and the Boltzmann's constant k, as freely adjustable parameters.) We obtain a translation table of the form

$$\begin{aligned}&mn^\rho\leftrightarrow j^\rho\ ,\quad ks^\rho\leftrightarrow s^\rho\ ,\\ &\mu_\rho\leftrightarrow -mv_\rho\ ,\quad \Theta_\rho\leftrightarrow -kw_\rho\\ &\mu\leftrightarrow m\mu\ ,\quad \Theta\leftrightarrow kT\ ,\quad \hbar\varphi\leftrightarrow m\alpha\ ,\end{aligned} \tag{1.3}$$

where the scalars on the last line repectively represent chemical potential, temperature, and velocity potential.

Although less trivial than a question of normalization, the relation between the two different kinds of Lagrangian is also straightforward, being given by a simple Legendre-type transformation whose respective versions are expressible by

$$-n^\rho\mu_\rho-s^\rho\Theta_\rho=\Psi-\Lambda=j^\rho v_\rho+s^\rho w_\rho\ . \tag{1.4}$$

However, the relation between the two kinds of variational principle involved is not quite so simple. Neither, of course, is a free variational principle (which would just give the trivial solution $n^\rho = s^\rho = 0$ in one approach and $v_\rho = w_\rho = 0$ in the other), and the nature of the constraints that need to be imposed is very different in the two cases. For the convective variational principle (which has the advantage of being easily generalizable [5] to treat cases such as that of superconductivity in an elastic solid) it is required that the allowed variations be constrained to be induced just by displacements of the current-flow world lines. On the other hand, for the potential variational principle (which has the advantage of being rather simpler to formulate explicitly) it is required that the momenta be constrained to be given in terms of freely variable dynamical potentials α, ζ, say, and auxiliary Clebsch-type potentials β, γ, say, according to a specification which in the present case takes the form

$$v_\rho = -\nabla_\rho \alpha \ , \quad w_\rho = -\nabla_\rho \zeta - \beta \nabla_\rho \gamma \ . \tag{1.5}$$

Clearly the requirement of invariance of the spacetime integral of Ψ under free variations of the potentials β, γ, α, and ζ leads directly to field equations given by

$$s^\rho \nabla_\rho \gamma = 0 \ , \quad \nabla_\rho (\beta s^\rho) = 0 \tag{1.6}$$

and

$$\nabla_\rho j^\rho = 0 \ , \quad \nabla_\rho s^\rho = 0 \ . \tag{1.7}$$

The latter set (1.7), giving just the usual conservation laws for particles and entropy, is physically satisfactory as it stands, while use of (1.5) to eliminate the potentials β, γ, α and ζ, in favor of the physically well-defined variables v_ρ and w_ρ, leads to the replacement of (1.6) by mainifestly gauge-invariant dynamic equations of the form

$$\nabla_{[\rho} v_{\sigma]} = 0 \ , \quad s^\rho \nabla_{[\rho} w_{\mu]} = 0 \tag{1.8}$$

where square brackets are used to indicate index antisymmetrization. This final result is exactly the same as is obtained by the more technically elaborate but more readily generalizable convective variational principle whose application will be described in the following sections. The final section briefly describes the way the present theory goes over exactly, in the nonrelativistic limit, to the standard Landau theory of superfluid dynamics in its original (nondissipative) version [9], though not quite in its original terminology. (It will be shown in a separate article [10] how the original Landau theory can be translated into terms that make its relationship to the present relativistic theory more evident.)

II. CONVECTIVE VARIATIONAL FORMALISM FOR IDEAL MULTICONSTITUENT FLUIDS

The particular example of the two-constituent superfluid theory is a special case within the more general multiple-constituent perfectly-conducting-fluid theory [2,3,5] for which the equations of motion are derivable just from a Lagrangian-type master function Λ, say, that apart from the background spacetime metric $g_{\rho\sigma}$ depends only on a set of (contravariant) current vectors n^ρ_X with "chemical" index label X. (In a typical application to an electrically conducting fluid, the relevant currents might be taken to be an entropy current $n^\rho_{(0)} = s^\rho$, say; a positive ion current $n^\rho_{(1)} = n^\rho_+$, say; and a negative electron current $n^\rho_{(2)} = n^\rho_-$, say.)

The primary role of the master function (not just in the strictly conservative case with which we are concerned here but also in the more general convective variational theory of resistive [1,3,5] and viscous [11,12] fluids) is to determine a set of generalized momenta and their associated force covectors, the latter being required to vanish in the strictly conservative case (but not in the dissipative generalizations). Specifically, in the present case, each independent current vector n^ρ_X will have as its dynamical conjugate what is interpretable as strictly a momentum-energy (not momentum-energy density) covector μ^X_ρ, which is defined as the corresponding partial derivative that may be read out from the general variation formula

$$d\Lambda = \sum_X (\mu^X{}_\rho dn_X{}^\rho + \mu_X{}^\rho n^{X\sigma} dg_{\rho\sigma}) \ , \tag{2.1}$$

the particular form of the coefficient of the fixed-point (Eulerian) variation $dg_{\rho\sigma}$ of the spacetime metric being derivable [2,5] as a Noether identity expressing general covariance. For each current vector there is also a corresponding naturally defined force density covector f^X_ρ, say, that is expressible as

$$f^X{}_\rho = \mu^X{}_\rho \nabla_\sigma n_X{}^\sigma + 2 n_X{}^\sigma \nabla_{[\sigma} \mu^X{}_{\rho]} \ , \tag{2.2}$$

and whose vanishing can be shown (as discussed in the next section) to be the necessary and sufficient condition for invariance of the spacetime integral of Λ with respect to convective variations, i.e., variations due just to displacements of the integral curves of the independent currents n^ρ_X.

Introducing the thermodynamic potential (or generalized pressure) function Ψ, say, that is obtained as the Legendre transform of the master function according to the specification

$$\Psi = \Lambda - \sum_X n_X{}^\rho \mu^X{}_\rho \ , \tag{2.3}$$

we can go on to construct the corresponding total stress-momentum-energy density tensor $T^{\rho\sigma}$ according to the specification

$$T^\rho{}_\sigma = \Psi g^\rho{}_\sigma + \sum_X n_X{}^\rho \mu^X{}_\sigma \ . \tag{2.4}$$

The covariant divergence of this tensor defines a total force density

$$f_\rho = \nabla_\sigma T^\sigma{}_\rho \tag{2.5}$$

which is decomposable as the sum of the separable constituent force densities (2.2) which, as a further Noether identity [5,16] will automatically satisfy

$$\sum_X f^X{}_\rho = f_\rho \ . \tag{2.6}$$

The foregoing formalism can be made more explicit by noting that the requirement that Λ be covariant means that it can depend only on the scalar products $n_X{}^\rho n_{Y\rho}$ of the currents so that the variation formula (2.1) can be written out equivalently in the form

$$d\Lambda = \frac{1}{2}\sum_{XY} \mathcal{K}^{XY} d(n_X{}^\rho n_{Y\rho}) , \tag{2.7}$$

in which, in view of the symmetry restriction $n_X{}^\rho n_{Y\rho} = n_Y{}^\rho n_{X\rho}$, there is no loss of generality in imposing that the partial-derivative coefficients $\mathcal{K}^{XY}$ should satisfy the corresponding symmetry condition

$$\mathcal{K}^{[XY]} = 0 . \tag{2.8}$$

We thereby obtain a symmetric "inertia metric" $\mathcal{K}^{XY}$ (that is uniquely defined provided the number of independent currents does not exceed the relevant spacetime dimension) in terms of which the momentum covectors will be given by the chemical-index-raising operation

$$\mu^X{}_\rho = \sum_Y \mathcal{K}^{XY} n_{Y\rho} . \tag{2.9}$$

It is thereby apparent that despite the asymmetry of the separate terms in (2.4) the total stress-momentum-energy density tensor will nevertheless be symmetric after all, i.e.,

$$T^{[\rho\sigma]} = 0 . \tag{2.10}$$

It as important to notice that the stress-energy-momentum tensor $T^\rho{}_\sigma$ as defined by (2.4), and hence also the total force f_ρ as defined by (2.5) or (2.6), are *invariant* with respect to a change of *chemical basis* of the form

$$n_X{}^\rho \mapsto n'_X{}^\rho , \quad n'_X{}^\rho = \sum_Y N_X{}^Y n_Y{}^\rho \tag{2.11}$$

whereby new currents are defined as linear combinations, with constant coefficients $N_X{}^Y$, of the original currents, which implies a corresponding contravariant transformation

$$\mu^X{}_\rho \mapsto \mu'^X{}_\rho , \quad \mu^X{}_\rho = \sum_Y \mu'^Y{}_\rho N_Y{}^X . \tag{2.12}$$

The "inertia matrix," which plays the role of a naturally-defined metric for the chemical vector space, undergos a transformation of the corresponding contravariant tensorial form

$$\mathcal{K}^{XY} \mapsto \mathcal{K}^{XY\prime} , \quad \mathcal{K}^{XY} = \sum_{ZW} \mathcal{K}^{ZW\prime} N_Z{}^X N_W{}^Y . \tag{2.13}$$

However, the chemical transformation properties of the individual force-density contributions are not quite so simple: They can be considered as the diagonal components

$$f^X{}_\rho = f_X{}^X{}_\rho \tag{2.14}$$

of a force-density matrix

$$f_X{}^Y{}_\rho = \mu^X{}_\rho \nabla_\nu n_Y{}^\nu + 2 n_Y{}^\nu \nabla_{[\nu} \mu^X{}_{\rho]} \tag{2.15}$$

whose chemical transformation law is of mixed (covariant and contravariant) tensorial type.

It is to be remarked that in the case of a constituent that is charged with electromagnetic coupling constant (per particle) e^X, say, the effect of an electromagnetic field with four-potential A_ρ can be allowed for [2,3,5,11] by a procedure of the usual kind whereby the physically well-defined momentum covector $\mu^X{}_\rho$ is replaced in the formalism by a corresponding gauge-dependent momentum covector $\pi^X{}_\rho = \mu^X{}_\rho + e^X A_\rho$.

III. CONVERSION FROM CONVECTIVE TO POTENTIAL VARIATIONAL FORMULATION

If the master function were used directly as an ordinary Lagrangian in a *free* variational principle, the corresponding variational equations would simply amount to the (chemically covariant) requirement that the momenta $\mu^X{}_\rho$ should all vanish, a condition which is so restrictive as to render the dynamics entirely trivial. To get variational models with nontrivial dynamics it is therefore necessary to specify constraints on the variation of the current fluxes. One of the simplest possibilities is that of a strictly conservative perfectly-conducting model as characterized by the (chemical-base dependent) condition that each of the forces $f^X{}_\rho$ should vanish, which is what results from a *convective* variational procedure of the kind introduced in the case of a simple perfect fluid by Taub [13], in which the variation of the current vectors is constrained to have the form naturally induced by infinitesimal displacements of the flow lines. Explicitly, if $\xi_X{}^\rho$ denotes the *convecting vector* field generating the infinitesimal displacements of the flow lines of the current $n_X{}^\rho$ then the corresponding Eulerian perturbation induced by the convection of the current will be given [4,2,5,11] by

$$dn_X{}^\rho = \xi_X{}^\sigma \nabla_\sigma n_X{}^\rho - n_X{}^\sigma \nabla_\sigma \xi_X{}^\rho + n_X{}^\rho \nabla_\sigma \xi_X{}^\sigma . \tag{3.1}$$

It is to be noticed that invariance of the integral of Λ with respect to the displacement generated by a vector field parallel to the corresponding flow, i.e., such that

$$\xi_X{}^{[\rho} n_X{}^{\sigma]} = 0 \tag{3.2}$$

requires only the vanishing of the corresponding tangential contraction of the force density, as given by the identity

$$n_X{}^\rho f^X{}_\rho = n_X{}^\rho \mu^X{}_\rho \nabla_\sigma n_X{}^\sigma . \tag{3.3}$$

It is apparent from the last identity that the (chemical-base-dependent) perfect-conductivity postulate to the effect that all the force densities vanish, i.e.,

$$f^X{}_\rho = 0 , \tag{3.4}$$

generically entails the (chemically covariant) condition that all the currents be conserved, i.e.,

$$\nabla_\rho n^\rho_X = 0 \tag{3.5}$$

and that when this last condition is satisfied the residual content of the perfect-conductivity condition (2.14) reduces to the standard form

$$n_X{}^\rho \nabla_{[\rho} \mu^X{}_{\sigma]} = 0 . \tag{3.6}$$

This (chemical-base dependent) condition of perfect conductivity includes as a special case the stricter condition of fully irrotational flow as characterized by the (chemically covariant, i.e., basis-independent) condition that all the momentum forms be closed, i.e., that each of the exterior derivatives $\nabla_{[\rho}\mu^X{}_{\sigma]}$ should vanish, which characterizes a class of fully superconducting superfluid models. The chemical covariance would be lost in a weaker specification requiring vanishing divergence and closure of the momentum form for only some but not all of the currents as, for example, would be appropriate for a model of a nonsuperfluid but superconducting liquid.

It is to be remarked that the formalism set above can be applied not only to perfectly-conducting fluids but also to the other nondissipative case of perfect insulation, in which some or all [14] of the currents are locked together, in which case the corresponding displacements in the variational principle should also be locked together, which means that only the sum (but not the separate values) of the corresponding forces is to be set to zero. It is also to be remarked that it is possible to get nondissipative models in which creation occurs, so that (3.5) is not satisfied, but where the tangential force contraction (3.3) is nevertheless made to vanish by instead requiring that the corresponding chemical affinity [5], namely $n_X^\rho \mu^X{}_\rho$, be set to zero.

To see how the generic system of perfectly-conducting-fluid equations of motion (3.5) and (3.6) can be converted into potential variational form, the first step (following lines that are well established in the nonconducting case [8,14]) is to introduce a (gauge-dependent) set of scalar fields, α^X, say, and a corresponding set of flow-transported covectors, $\kappa^X{}_\rho$, according to the prescription

$$\mu^X{}_\rho = \kappa^X{}_\rho + \nabla_\rho \alpha^X \ , \quad n_X{}^\rho \nabla_\rho \alpha^X = n_X{}^\rho \mu^X{}_\rho \ , \tag{3.7}$$

which, by (3.6), automatically entails the conditions

$$n_X{}^\rho \nabla_{[\rho}\kappa^X{}_{\sigma]} = 0 \ , \quad n_X{}^\rho \kappa^X{}_\rho = 0 \ , \tag{3.8}$$

which are well known to ensure that the covector $\kappa^X{}_\rho$ is invariant under transport by the flow congruence of the corresponding current $n_X{}^\rho$ (in the strong sense of having zero Lie derivative with respect to any variable multiple of the current vector). Under these conditions it follows without loss of generality (making use of the gauge freedom in the choice of α^X and on the understanding, of course, that we are not concerned with generalizations to more than four dimensions) from the theorem of Pfaff [8] that we can take $\kappa^X{}_\rho$ to have the form

$$\kappa^X{}_\rho = \beta^X \nabla_\rho \gamma^X \ , \tag{3.9}$$

in terms of generalized Clebsch potentials β^X and γ^X that are themselves constant along the flow world lines, i.e.,

$$n_X{}^\rho \nabla_\rho \beta^X = 0 \ , \quad n_X{}^\rho \nabla_\rho \gamma^X = 0 \ . \tag{3.10}$$

Proceeding conversely, it can be seen that when taken in conjunction with the combined expression

$$\mu^X{}_\rho = \nabla_\rho \alpha^X + \beta^X \nabla_\rho \gamma^X \ , \tag{3.11}$$

as obtained from (3.7) and (3.9), the Clebsch-type equations of motion (3.10) are sufficient by themselves to give back the original gauge-independent form (3.6) of the dynamic equations, the evolution equations (3.6) for the dynamic potentials α^X being merely an automatic consequence.

Having gotten to this point, it is now straightforward to verify that the complete set of dynamical equations, which in this potential reformulation consist just of (3.5) and (3.10), is obtainable from the general variation formula given for the generalized pressure function Ψ by (2.1) and (2.3), i.e.,

$$d\Psi = \sum_X \left(-n_X^\rho d\mu^X{}_\rho + \mu_X^\rho n^{X\sigma} dg_{\rho\sigma} \right) \ , \tag{3.12}$$

by directly substituting the expression (3.11) and imposing the requirement that the spacetime integral of Ψ be invariant with respect to infinitesimal variations of the independent dynamical potentials α^X and of the independent Clebsch potentials β^X and γ^X. Within this system the special case of an *irrotational* flow, $\nabla_{[\rho}\mu^X{}_{\sigma]}=0$, as required for superfluidity of superconductivity, is obtainable directly at the level of the variational principle simply by omitting the auxiliary variables, i.e., dropping the final Clebsch term in the expression (3.12) for the relevant momentum covector (which is equivalent to restricting β^X or γ^X to be uniform).

IV. SINGLE-CONSTITUENT FLUID

As a preparation for discussing the two-constituent case that is of greatest interest, it is worthwhile to briefly recapitulate how the convective variational formalism works out for the familiar case of a single-constituent fluid, including the special case of a zero-temperature superfluid, with only one independent timelike current vector n^μ, say, so that the master function Λ depends only on a single scalar n, say, defined as the magnitude of the current by

$$n^\mu = n u^\mu \tag{4.1}$$

where u^μ is a unit flow tangent vector as characterized by the normalization condition

$$u^\mu u_\mu = -c^2 \ . \tag{4.2}$$

According to the principle expressed by (2.1) the corresponding momentum covector works out as

$$\mu_\rho = \frac{\mu}{c^2} u_\rho \tag{4.3}$$

with

$$\mu = -\frac{d\Lambda}{dn} \ . \tag{4.4}$$

Under these circumstances the Legendre-transformed potential Ψ will be given by

$$\Psi = \Lambda + n\mu \ , \tag{4.5}$$

while the stress-momentum-energy density tensor is obtained in the familiar perfect-fluid form

$$T^{\rho}{}_{\sigma}=\left[\frac{\mathscr{E}+P}{c^2}\right]u^{\rho}u_{\sigma}+Pg^{\rho}{}_{\sigma} \quad (4.6)$$

where the energy-mass density $\mathscr{E}$ and pressure P are given in this case simply by

$$\mathscr{E}=-\Lambda\ ,\quad P=\Psi\ . \quad (4.7)$$

It is to be noted that in this simple "barytropic" case the "inertia metric" defined by (2.7) has only a single component that will be given as $\mathscr{K}=(\mathscr{E}+P)/n^2$.

The four vectorial components of the vanishing force equation

$$f_{\rho}=0 \quad (4.8)$$

that is obtained by direct application of the original Taub variational principle [13] (the prototype for the general convective variational formalism used here) can be decomposed into two parts. The first is a scalar part which, in the usual algebraically unconstrained version of the theory, is satisfied by the imposition of the current conservation law which in this case can be written out in terms of differentiation with respect to proper time τ along the flow as

$$\frac{dn}{d\tau}+n\nabla_{\rho}u^{\rho}=0\ . \quad (4.9)$$

The remaining part takes what is an ordinary *Hamiltonian* form of the *uniformly* canonical type [15]. What this means is that the equations for the flow lines are given by a single particle-type Hamiltonian in which covariance is obtained by using proper time τ as a fifth independent variable, the ordinary time variable x^0 being treated on the same footing as the space variables x^i, $i=1,2,3$, while similarly the mass-energy component μ, say, is combined with the ordinary space momentum components p_i, say, to form the covector $\mu_{\rho}\leftrightarrow\{\mu,p_i\}$. The superfluous degree of freedom that would otherwise be introduced in such a covariant formalism must be removed by the imposition of an appropriate constraint fixing the initial value of the Hamiltonian, which is of course conserved along each world line. The statement that the flow is of *uniform* canonical type means that it has the same form for each world line, which implies without loss of generality that it can be arranged to have the standard form

$$H=0\ . \quad (4.10)$$

The Hamiltonian equations themselves have a form that is just the obvious $(4+1)$-dimensional analogue of the usual $(3+1)$-dimensional form, i.e.,

$$u^{\mu}=\frac{dx^{\mu}}{d\tau}=\frac{\partial H}{\partial\mu_{\rho}} \quad (4.11)$$

and

$$\frac{d\mu_{\rho}}{d\tau}=-\frac{\partial H}{\partial x^{\rho}} \quad (4.12)$$

with x^{ρ} replacing x^i and μ_{ρ} replacing p_i.

These equations are, as they stand, covariant with respect to ordinary Lorentz transformations if we are using Minkowski coordinates in a flat background, but unlike (4.11) the momentum equation (4.12) has the disadvantage of not being covariant under nonlinear transformations such as are needed even in flat space if one wishes to use spherical or cylindrical coordinates. If one were dealing with just an individual particle world line, the best one could do to get general covariance would be to add in an appropriate term involving the Christoffel connection so as to convert (4.11) to the form $u^{\nu}\nabla_{\nu}\mu_{\rho}=-(\partial H/\partial x^{\rho}+u^{\nu}\mu_{\sigma}\Gamma_{\nu}{}^{\sigma}{}_{\rho})$. However, in a fluid where one has a whole congruence of trajectories there is a neater alternative, which is to use the fact that H will be determined as a field over spacetime whose gradient

$$\nabla_{\rho}H=\partial H/\partial x^{\rho}+u^{\sigma}\partial\mu_{\sigma}/\partial x^{\rho}$$

is covariant even though its separate terms are not. Under such conditions it is apparent that the general Hamiltonian equation (4.12) can be rewritten in the *canonical form*

$$2u^{\sigma}\nabla_{[\sigma}\mu_{\rho]}=-\nabla_{\rho}H\ , \quad (4.13)$$

whose right-hand side drops out in the uniform case constrained by (4.10), and which in any case is not only generally covariant but has the convenient feature that, due to the antisymmetrization on the left, its evaluation does not involve the Christoffel connection.

In the simple perfect-fluid case with which we are concerned, the relevant point-particle-type Hamiltonian is given as a function of the position coordinates x^{μ} and the four-momentum components μ_{ρ}, by

$$H=\frac{c^2}{2\mu}g^{\rho\sigma}\mu_{\rho}\mu_{\sigma}+\frac{\mu}{2}\ . \quad (4.14)$$

This confirms the interpretation of the covector μ_{ρ} as an ordinary four-momentum in the usual technical sense. [It is to be observed here that μ/c^2 plays the role of an effective mass in the kinetic term of (4.14)].

Subject to (4.10), the canonical equation of motion (4.13) is interpretable as meaning that the *vorticity two-form*

$$w_{\rho\sigma}=2\nabla_{[\rho}\mu_{\sigma]} \quad (4.15)$$

is conserved in the strong sense [15] (meaning that it is dragged along by the flow, i.e., its Lie derivative with respect to an arbitrary multiple of the flow vector is zero) so that in particular its (positive-indefinite) magnitude $w_{\rho\sigma}w^{\rho\sigma}$ is constant along each flow world line. Under these conditions the vorticity form is expressible in terms of the ordinary kinematic rotation vector

$$\omega^{\mu}=\frac{1}{2c}\varepsilon^{\mu\nu\rho\sigma}u_{\nu}\nabla_{\rho}u_{\sigma} \quad (4.16)$$

by the relation

$$w_{\mu\nu}=\frac{2\mu}{c}\varepsilon_{\mu\nu\rho\sigma}\omega^{\rho}u^{\sigma}\ . \quad (4.17)$$

As a corollary one obtains [16] an ordinary divergence-type conservation law for the relativistic *helicity flux* $\mu^2\omega^{\rho}$, i.e.,

$$\nabla_\rho(\mu^2\omega^\rho)=0 \ . \qquad (4.18)$$

If it were not already obvious from the form of the momentum transport equation (4.13), Eq. (4.18) makes it manifestly evident that any solution that is initially irrotational in the sense of having vanishing vorticity $w_{\rho\sigma}$, or (equivalently in this case) vanishing rotation ω^ρ, will remain irrotational in its subsequent evolution. By the Poincaré lemma such irrotational flow is evidently characterizable in terms of a flow potential.

$$w_{\rho\sigma}=0 \iff \mu_\rho=\hbar\nabla_\rho\varphi \ , \qquad (4.19)$$

the inclusion of the Dirac Plank constant having the effect of rendering the potential φ conveniently dimensionless. It is this special class of irrotational solutions that is relevant to the superfluid case, in which the flow potential is interpretable as being proportional to the phase of an underlying quantum wave function, so that one gets what is obviously [16] the simplest and most natural relativistic generalization of the single-constituent London theory for a zero-temperature superfluid with no "normal" component. In this theory the global periodicity of the phase gives rise to the familiar topological quantization condition on the circulation. When the number current of marker particles is normalized to agree with that of the superfluid bosons then the potential φ will presumably be directly identifiable with an ordinary phase angle so that its periodicity should have the standard value 2π. It is to be remarked that this canonically normalized potential φ differs [see Eqs. (1.3)] from the potential α introduced directly by the classical formula $\nabla_\rho\alpha=-v_\rho$ [as in (1.5)] by a dimensional factor: $\varphi/\alpha=m/\hbar$.

The superfluid subcase obtained, as just described, by supplementing the general equations of motion by (4.14) with the initial-value restraint (4.19) can be gotten directly [without any need to consider the more general unrestrained solutions of (4.14)] using a less severely constrained variational principle [16]. Instead of requiring that the allowed variations be convective, one allows the current vector to vary freely apart from the requirement that is is required to preserve the conservation law (4.9), which in this case is imposed in advance as a restraint whose effect may be taken into account in a free variation principle with an augmented Lagrangian of the form $\Lambda+\hbar\varphi\nabla_\sigma n^\sigma$ where φ is a locally defined dimensionless Lagrange multiplier. This leads immediately to the restricted irrotational dynamical equations (4.19) with the Lagrange multiplier itself taking the role of the flow potential. It is to be remarked that by subtracting off the total divergence $\nabla_\rho(\hbar\varphi n^\rho)$ one obtains an equivalent but more convenient potential gauge-invariant Lagrangian of the form $\Lambda-\hbar n^\rho\nabla_\rho\varphi$ whose value after substitution of the ensuing field equation (4.19) is just that of the ordinary pressure P.

V. THE TWO-CONSTITUENT FLUID

We now come to the most important application, namely the case of a two-constituent superfluid. According to the principles described in the introductory sections, the two-fluid theory is to be constructed in terms of a Lagrangian scalar Λ depending on a pair of currents s^ρ and n^ρ, say, for which there will be associated momentum covectors Θ_μ and μ_ρ, say, that are determined by a (fixed-background) variation formula of the form

$$d\Lambda=\Theta_\rho ds^\rho+\mu_\rho dn^\rho \ , \qquad (5.1)$$

so that the corresponding force densities will be expressible as

$$f^{(0)}_\rho=\Theta_\rho\nabla_\sigma s^\sigma+2s^\sigma\nabla_{[\sigma}\Theta_{\rho]} \ , \qquad (5.2)$$

and

$$f^{(1)}_\rho=\mu_\rho\nabla_\sigma n^\sigma+2n^\sigma\nabla_{[\sigma}\mu_{\rho]} \ . \qquad (5.3)$$

The thermodynamic potential, as given by the Legendre-type transformation (2.3), will take the form

$$\Psi=\Lambda-s^\rho\Theta_\rho-n^\rho\mu_\rho \qquad (5.4)$$

and the stress-momentum-energy density tensor (2.4) will be expressible as

$$T^\rho{}_\sigma=\Psi g^\rho{}_\sigma+s^\rho\Theta_\sigma+n^\rho\mu_\sigma \ . \qquad (5.5)$$

In ordinary applications to heat-transporting fluids one would wish to interpret n^ρ as a current of conserved marker particles which (except in exotic circumstances) might appropriately be taken to be baryons, or (more conveniently for superfluid applications) bosonic combinations of baryons (corresponding to neutron Cooper pairs, or α-particle quartets for example) while s^μ would be identified with a current of entropy, which would also be conserved in nondissipative applications such as those with which we are concerned in the present work (but which would not be conserved in more general applications such as have been considered elsewhere [6,7,11]). Such an interpretation is particularly convenient in the case of superfluids in view of the comparative ease with it is possible, from a phenomenological or experimental point of view, to identify not only the baryon current but even the entropy current. (Although the most appropriate identification of the latter may still be subject to some ambiguity, it is clearer in superfluid applications than in most other fluid contexts. The simplest case is that of the low-temperature limit, in which proton contributions can be neglected so that the entropy current can be recognized [17] as consisting essentially just of a current of phonons.) With this identification, μ_ρ is to be interpreted as an effective momentum per marker particle (e.g., per baryon or per bosonic combination of baryons) and Θ_ρ as an effective momentum per unit entropy.

To obtain the original version of what from a convective variational point of view is the simplest imaginable two-constituent superfluid theory, we apply the variation principle with respect to strictly convective variations of the entropy current s^ρ while allowing free variations of the current n^ρ but ensuring its conservation, in the manner described for the single-constituent case in the previous section, by adding an appropriate Lagrange multiplier term $\hbar_\varphi\nabla_\rho n^\rho$ and then subtracting the divergence $\nabla_\rho(\hbar\varphi n^\rho)$ so as to obtain a corresponding modification $\Lambda \to X$, say, of the Lagrangian function which will take the form

$$X = \Lambda - \hbar n^\rho \nabla_\rho \varphi \ . \tag{5.6}$$

Applying the variational principle with respect to the multiplier φ now gives the particle conservation law

$$\nabla_\rho n^\rho = 0 \ , \tag{5.7}$$

while allowing the entropy current s^ρ to vary in accordance with the more restrictive convective ansatz of the form (3.1) implies that the corresponding thermal force density (5.2) must vanish. Achieving this latter requirement by the usual vanishing-divergence condition of the form (3.5) so as to obtain a nondissipative model in which entropy is also conserved,

$$\nabla_\rho s^\rho = 0 \ , \tag{5.8}$$

one finds that the vanishing of the remaining part of the thermal force density simply gives an evolution equation of the standard form

$$s^\rho \nabla_{[\rho} \Theta_{\sigma]} = 0 \tag{5.9}$$

for the thermal momentum. Finally, applying the variational principle with respect to free variations of n^ρ in (5.6), one finds [consistently with the vanishing of the other force density (5.3)] that the evolution of the particle momentum will be determined by the stricter irrotationality condition

$$\nabla_{[\rho} \mu_{\sigma]} = 0 \tag{5.10}$$

that follows from the potential flow condition

$$\mu_\rho = \hbar \nabla_\rho \varphi \tag{5.11}$$

obtained from (5.6). Thus (as already remarked for the zero-temperature case) the Lagrange multiplier φ becomes what may be interpreted a representing a superfluid phase angle, with periodicity 2π if the particle current is normalized in such a way that there is just one marker particle per bosonic unit (whether it is an α particle in ordinary helium or a baryonic Cooper pair in a neutron star) of the type whose condensation is supposed to underly the superfluidity phenomenon.

It is to be observed that (5.11) can be used *post facto* for the evaluation of the augmented Lagrangian X, which thus works out to be given by

$$X = \Lambda - n^\rho \mu_\rho = \Psi + s^\rho \Theta_\rho \ . \tag{5.12}$$

The general variation (in a fixed background) of this hybrid between Λ and Ψ will evidently take the form

$$dX = \Theta_\rho ds^\rho - n^\rho d\mu_\rho \ . \tag{5.13}$$

It is apparent that this hybid function can be used as the Lagrangian for a hybrid variational formulation that is of the convective type only for the variable s^ρ, while being of potential (Clebsch) type for the other independent field variable which in this version is μ_ρ, whose constraint is simply that it should have the form (5.11). Explicitly this means that the most general allowed variation is expressible in terms of an infinitesimal convection vector ζ^μ and a potential variation $d\varphi$ by

$$ds^\rho = \zeta^\sigma \nabla_\sigma s^\rho - s^\sigma \nabla_\sigma \zeta^\rho + s^\rho \nabla_\sigma \zeta^\sigma \ , \quad d\mu_\rho = \hbar \nabla_\rho (d\varphi) \ . \tag{5.14}$$

Thus having first introduced the scalar φ with the status of a Lagrange multiplier in (5.6) we now find that in this hybrid formulation it has taken on the role of a fully fledged dynamic flow potential from the outset.

VI. THE NONRELATIVISTIC LIMIT

In the absence of any adequate microscopic theory of the nonlinear regime (in which not only phonons but also more complicated roton-type excitations are present) the phsyical justification for the theory described in the preceding section is simply that it is the mathematically simplest possibility that is characterized not only by relativistic covariance but also by formal agreement with the standard Landau theory in the appropriate nonrelativistic limit.

To see how the relativistic formalism described in the main part of this article can be translated in the nonrelativistic limit into the Newtonian formalism of the (rather inelegant) traditional kind, it is necessary at this point to take account of the long tradition of describing superfluids not in terms of readily identifiable marker particle (e.g., baryon or superfluid boson) and entropy currents but in terms of various "mass currents" whose empirical identification is rather less clear cut, being motivated by Newtonian considerations that lose their relevance in a relativistic context. The one whose definition is least ambiguous is the so called "total-rest-mass current"

$$j^\rho = m n^\rho \ , \tag{6.1}$$

where m is some proportionality constant having the dimensions of a mass. The choice of this "rest mass per particle" is quite arbitrary as far as the fully relativistic formalism is concerned, its use for a condensed medium in a highly relativistic context being just a hangover from a prerelativistic conceptual framework or from the theory of dilute gases, in which the interaction between the particles is so weak that their separate rest masses remain well defined. The notion of a fixed "rest mass per particle" does, however, acquire a certain degree of physical significance in the context of a nonrelativistic limit approximation, whose characterization requires not only that relative velocities in the fluid be small compared with the speed of light (by a factor that can be thought of as defining what is meant by "of order $1/c$") but it also requires conditions of the form

$$\frac{\mu}{mc^2} = 1 + O(1/c^2) \ , \quad \frac{\Theta}{mc^2} = O(1/c^2) \tag{6.2}$$

as $1/c \to 0$, in terms of a mass parameter m whose value is thereby fixed to within small, $O(c^{-2})$, adjustments whose affect will be unimportant unless one is so unreasonable as to want an extremely high degree of accuracy without having to go to the trouble of calculating corrections to a correspondingly high order. In practice, the freedom of adjustment that can be tolerated to accommodate personal tastes and preferences will usually be sufficiently generous that, for example, if the particles under consideration are atoms of helium-4, it will seldom matter whether one takes the "rest mass" to be four times the mass of a proton (which might be the most obvious choice for a high-energy physicist), or $\frac{1}{4}$ of the mass of an isolated atom of oxygen (which might be the most obvious choice for a chemist), or $\frac{1}{14}$ of the mass of an iron nucleus (which might be most obvious for a nuclear physicist, who might also settle for that of just one α particle) rather than the mass of a single isolated helium-4 atom in its ground state (which is probably what a low-temperature physicist would choose).

However the problem of its precise normalization may have been resolved, the rescaled current vector j^ρ will have a dynamical conjugate v_ρ, say, that can be read out of the fixed-background ($dg_{\rho\sigma}=0$) variation formula

$$d\Lambda=\Theta_\rho ds^\rho+v_\rho dj^\rho \ . \tag{6.3}$$

This covector v_ρ will therefore be given in terms of the true momentum per particle, μ_ρ, by the formula

$$\mu_\rho=-mv_\rho \ , \tag{6.4}$$

which differs from the superficially analogous relation (4.3) in that m (unlike μ/c^2) is a constant, while $-v_\rho$ (unlike u_ρ) has a generically variable scalar magnitude. This latter one-form v_ρ is what, in the traditional jargon, would be referred to as the superfluid "velocity", a terminology which is rather misleading: the sleight of hand whereby it has been scaled so as to acquire the right dimensions for a velocity does not alter the fact that [unlike the unit covector u_ρ in (4.3)] v_ρ still has the dynamical role not of a velocity but of a momentum.

The inverse of the relevant application of the momentum to current relationship (2.9) is obtainable directly from the generalized pressure function Ψ whose variation formula is given by the Legendre transformed analogue (1.2) of (6.3). Explicitly, the relation between the pair of independent current vectors n^ρ ($=m^{-1}j^\rho$) and s^ρ and the corresponding conjugate pair of momentum covectors μ_ρ ($=-mv_\rho$) and Θ_ρ ($=-kw_\rho$) is obtainable in terms of the three independent scalars

$$I_1=\tfrac{1}{2}g^{\rho\sigma}\mu_\rho\mu_\sigma \ , \quad I_2=g^{\rho\sigma}\mu_\rho\Theta_\sigma \ , \quad I_3=\tfrac{1}{2}g^{\rho\sigma}\Theta_\rho\Theta_\sigma \tag{6.5}$$

in the form

$$n^\rho=-g^{\rho\sigma}\left[\frac{\partial\Psi}{\partial I_1}\mu_\sigma+\frac{\partial\Psi}{\partial I_2}\Theta_\sigma\right] \ ,$$
$$s^\rho=-g^{\rho\sigma}\left[\frac{\partial\Psi}{\partial I_2}\mu_\sigma+\frac{\partial\Psi}{\partial I_3}\Theta_\sigma\right] \ . \tag{6.6}$$

To see how the nonrelativistic limit is obtained from the generally covariant theory set up above, the essential step is the introduction of a "3+1" decomposition with respect to some inertial reference system in what, form this point on, is to be understood as a flat background with standard Minkowski coordinates, $\{x^\mu\}=\{x^0,x^i\}$ with $i=1,2,3$. With respect to any inertial frame there will be a well-defined "normal" and total current three-velocity v_N^i and v^i with corresponding density (three-) scalars given by

$$j^\mu\leftrightarrow\{\rho,j^i\}=mn\{1,v^1\} \ , \quad s^\mu\leftrightarrow s\{1,v_N{}^i\} \ . \tag{6.7}$$

In the corresponding decomposition of the conjugate four- momenta it is convenient to introduce an effective *chemical potential* μ and an effective *temperature* Θ (preferring this symbol to the use of T in this context because of the possibility of confusing the latter with the trace of T^ρ_μ) that are given modulo a Lorentz factor by the thermal (i.e., "normal") frame component of the corresponding momentum covector, according to the specification

$$s^\rho\mu_\rho=-s\mu \ , \quad s^\mu\Theta_\mu=-s\Theta \ . \tag{6.8}$$

This leads to a decomposition taking the form

$$-m^{-1}\mu_\rho=v_\rho\leftrightarrow\{m^{-1}\mu+v_N{}^j v_{Sj},-v_{Si}\} \ ,$$
$$-\Theta_\mu=kw_\mu\leftrightarrow\{\Theta+v_N{}^j\Theta_j,-\Theta_i\} \tag{6.9}$$

(which of course simplifies somewhat in units such that $k=m=1$). The covector (as opposed to vector, a distinction that would still matter even in three-dimensional space if nonflat, e.g., spherical, coordinates are used) v_{Si} that makes its appearance here is interpretable—in the nonrelativistic limit—as the much discussed quantity that is commonly but rather misleadingly described as the superfluid "velocity" vector. Its analogously defined thermal partner Θ_i, on the other hand, is something that is hardly ever discussed explicitly, more importance being traditionally attached to the combined total momentum density which is given by

$$\rho_i=pv_{Si}+s\Theta_i \ . \tag{6.10}$$

In terms of these quantities the variation of the generalized pressure function takes the form

$$d\Psi=n\,d\mu+s\,d\Theta+\rho_i dv_N{}^i-(j^i-\rho v_N{}^i)dv_{Si} \ . \tag{6.11}$$

In the nonrelativistic limit, as characterized by (6.2) one obtains the simplification

$$\rho_i=j_i+O(1/c^2) \tag{6.12}$$

whose substitution finally reduces (6.11) to a form that is well known as a possible starting point for the development [17] of the standard classical theory [9].

Having derived this limit as a check on consistency with previously established results, it is to be remarked that while a fully relativistic treatment is of course necessary for astrophysical applications such as the treatment of neutron star interiors (not the mention more exotic contexts such as cosmic strings), the covariant treatment also has some conceptual advantages even in the terrestrial-laboratory context of ordinary liquid helium, for which a Newtonian approximation is an extremely good approximation but for which, in practice, the traditional use of inappropriate noncovariant concepts remains a source of unnecessary complication and consequent misunderstanding, so that a fresh covariant approach can provide useful clarification. This implies that it is worthwhile to develop a covariant description within a Newtonian framework in order to translate the original formulation [9] of the Landau theory into a form [10] that gives a better match to the more elegant relativistic version described in the preceding sections. In particular, the covariant approach makes it clearer that what has traditionally be referred to as superfluid "velocity" should more correctly be called superfluid *momentum*: whereas the "normal" velocity transforms *contravariantly* under Lorentz boosts as a genuine velocity should do, on the other hand the so-called superfluid "velocity" transform *covariantly*, i.e., like a differential one-form, which is the behavior characteristic not of a velocity but of a momentum. This can also be understood from a quantum-mechanical point of view according to which a genuine (necessarily subluminal) velocity represents the bicharacteristic direction of propagation of a wave packet, whereas a momentum corresponds to the phase gradient of a wave front (which without violating causality can exceed the speed of light).

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