Photon and graviton Green's functions on cosmic string space-times

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We present the Green's functions for photons and gravitons in the vicinity of an idealized cosmic string. We stress the importance of the Ward identities involved and the necessity for "smoothing" the curvature singularity in the space-time in order to carry out the calculations. The Green's functions are employed to determine the renormalized vacuum expectation value of the stress-energy tensor for scalar, electromagnetic, and linearized gravitational fields propagating in the neighborhood of an idealized cosmic string.

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I. INTRODUCTION

The space-time generated by an infinite, straight cosmic string has been studied in some detail [1]. In the most simple idealized case, where the string is assumed to have zero thickness, the classical curvature and stress-energy tensor have their support on a two-dimensional world sheet, and the space-time outside the string is flat. The only effect of the string is to generate a conical singularity in the curvature on the two-dimensional world sheet.

More realistically, cosmic strings are formed as topological defects of a gauge theory, and have internal structure, characterized by the *core radius* r_0 of the string. The radius r_0 is the Compton wavelength associated with the symmetry-breaking energy scale of the theory. Outside the core space-time is flat, just as in the idealized conical case. However, within the core the stress-energy tensor and curvature are nonzero.

Because string space-times are flat outside the string core, but globally curved, they provide interesting examples of curved space-time effects on quantum fields that can be calculated exactly [2-8]. In a previous paper [9], we studied the properties of quantized scalar fields on these space-time backgrounds. In particular, we examined the effects of $\xi R\varphi^2$ couplings between the curvature R and the scalar field φ. In order to make sense of the coupling term for the idealized string, the tip of the cone was "rounded off" to distribute the curvature $R_{\alpha\beta\gamma\delta}$ over a finite-sized region. We then examined the limit $r_0 \rightarrow 0$, in which the rounded-off cone approached the ideal conical case. We found that in this limit the Green's function $G_\xi(x,x')$ of a scalar field with arbitrary coupling ξ to the curvature of the rounded-off cone approached the usual Green's function $G_{\xi=0}(x,x')$ on the idealized space-time, provided the points x and x' were both outside the region of support of the curvature.

In this paper we first review the calculation of the scalar Green's function on the space-time generated by an idealized cosmic string and then proceed to perform the corresponding calculations of the electromagnetic and graviton Green's functions, along with their associated ghosts. To the best of our knowledge this is only the fourth example of a graviton Green's function that can be calculated in closed analytic form in a curved space-time (other examples are (anti−) de Sitter space [10] and certain Friedmann-Robertson-Walker (FRW) cosmological models [11]). The results are particularly interesting because both the Maxwell field A_μ and the linearized gravitational field $h_{\mu\nu}$ couple to the curvature. The presence of these coupling terms requires one to consider the rounded-off cone in order to perform the calculations.

Because the Maxwell field and linearized gravitational field have gauge symmetries, one must introduce symmetry-breaking terms into the action and also the compensating Faddeev-Popov ghost fields associated with those terms. The underlying symmetry of the theory requires that the fields and their associated ghosts satisfy a set of constraint equations known as Ward identities. At a practical level these identities determine the state of the ghost field entirely in terms of the physical state of the gauge field. As a check on our calculations we verify that the Ward identities are satisfied. The identities are particularly interesting in the gravitational case, where they include a nonlocal term.

It should be stressed that the technique of "rounding the cone" in these calculations is a necessary complication. If one were to proceed naively and attempt the calculation of the Green's functions for the graviton and its

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ghost entirely on the idealized conical space-time, by imposing the "natural" boundary conditions that the modes of which they are composed be well behaved at the origin $r=0$, then one would find that the resultant expressions fail to satisfy the Ward identities for the system. The reason for this failure becomes apparent when the procedure is repeated on the rounded-off cone. Here the Green's functions do satisfy the Ward identities appropriate to the space-time (which now contain nonlocal terms involving the curvature) and, furthermore, have sensible conical limits which satisfy the appropriate wave equations on the idealized cosmic string. However, while the conical limit of the graviton Green's function coincides with that obtained by the naive procedure described above, the same is not true of the ghost Green's function. In particular, the decomposition of the ghost Green's function on the rounded-off cone contains modes which in the conical limit are singular at $r=0$. This correct ghost Green's function on the ideal cone differs from the naively obtained expression by a solution of the homogeneous wave equation; the difference between the two may be viewed as a consequence of the fact that the "invisible" curvature couples with different signs to the wave equations for the graviton ghost and electromagnetic Green's functions on the ideal cone [cf., Eqs. (4.5) and (5.6)]. Thus, although the rounding of the cone may appear to be an unnecessary complication at first sight, it is, in fact, an essential element of the calculations and cannot be eliminated. As further evidence for this assertion, we note that, even with the correct ghost Green's function now in hand, the naive Ward identities on the idealized cosmic string still fail to be satisfied. This is because the Ward identities on the rounded-off cone, which contain nonlocal terms involving the curvature, do not have a well-defined conical limit. It is necessary to round the cone in order to verify the Ward identities.

In a sense, these issues all arise in connection with the boundary conditions on the various Green's functions. In order to determine the correct boundary conditions, we study a cone in which the space-time curvature is concentrated on a ring of radius r_0 (the "flower-pot" model of our previous paper [9]). In the interior region $r<r_0$, space-time is flat, and the boundary conditions for the "inner" mode functions $\Psi_<$ are those of flat space, namely, regularity at $r=0$. In this way, the "flower-pot" model of the rounded cone narrows the choice of possible boundary conditions to a unique choice. This ensures that our somewhat surprising results are not an artifact of an incorrect choice of these boundary conditions.

An important application for the Green's functions of a quantum field theory is in the calculation of the renormalized expectation value of the stress-energy tensor in some quantum state of that theory. We perform this calculation explicitly for each of the three theories considered here in their respective vacuum states, using the method of Hadamard regularization [12]. We close with some retrospective remarks concerning a scalar Green's function which contains a mode which is singular at the origin.

Throughout this paper we use units in which $\hbar=c=1$. Our sign conventions for the metric and curvature are those of Misner, Thorne, and Wheeler [13]. Properties and formulas for Bessel functions used in the text may be found in Gradshteyn and Ryzhik [14]; equation numbers prefixed by "GR" refer to formulas in that volume.

II. ROUNDED CONICAL METRICS

In this paper we consider the rounded conical metric studied by Allen and Ottewill [9]. This metric is useful because it enables us to disentangle the coordinate and curvature singularities that are conflated in the idealized conical model usually considered. In the rounded model the string is treated as static, straight, and infinitely long but has a finite radial size rather than being infinitesimally thin as in the idealized case. The line element for the rounded model is given in cylindrical coordinates (t,r,ϕ,z) by

$$ds^2=dt^2+P^2(r)dr^2+r^2d\phi^2+dz^2 , \tag{2.1}$$

where the range of the angular coordinate is $\phi\in[0,2\pi/\kappa)$, and $\kappa=(1-4\mu)^{-1}$, where μ is the mass per unit length of the string. For physically interesting cosmic strings $\mu\approx10^{-6}$ and so κ is slightly greater than 1. The function $P(r)$ has the properties

$$\lim_{r/r_0\to0} P(r)=1/\kappa \quad\text{and}\quad \lim_{r/r_0\to\infty} P(r)=1 . \tag{2.2}$$

The first condition states that there is no conical singularity at $r=0$, and the second condition means that, for large r, the cone has a deficit angle $8\pi\mu$. The function P should be a smooth monotonic function, and the condition that the curvature be concentrated in a region of radius r_0 about the string implies that all of the derivatives of $P(r)$ should be small outside that region.

Throughout most of this paper we work with the Euclidean space-time. Because the space-time is static, the corresponding Lorentzian space-time results may be obtained by a simple $t\to it$ Wick rotation. We note, however, that there are subtle issues regarding the choice of boundary conditions in the two cases; the correspondence is not one to one because the wave operators in one case are elliptic and in the other case they are hyperbolic.

One can show by direct calculation that the only nonvanishing Christoffel symbols associated with the line element (2.1) are

$$\Gamma^r_{rr}=\frac{P'}{P} , \quad \Gamma^r_{\phi\phi}=-\frac{r}{P^2} , \quad \Gamma^\phi_{r\phi}=\frac{1}{r} . \tag{2.3}$$

Furthermore, the curvature in this space-time is given by

$$R_{\alpha\beta\gamma\delta}=2R(\phi_{[\alpha}r_{\beta]}\phi_{[\gamma}r_{\delta]}) , \tag{2.4a}$$

$$R_{\alpha\beta}=\tfrac{1}{2}R(\phi_\alpha\phi_\beta+r_\alpha r_\beta) , \tag{2.4b}$$

$$R=\frac{2}{r}\frac{P'}{P^3} , \tag{2.4c}$$

where the prime denotes d/dr and we have introduced the obvious orthonormal tetrad $t_\alpha=\delta^0_\alpha$, $r_\alpha=P\delta^1_\alpha$, $\phi_\alpha=r\delta^2_\alpha$, $z_\alpha=\delta^3_\alpha$.

Of particular interest to us is a simple model, called the flower-pot model in [9], in which the curvature of space-time is concentrated on a ring of radius r_0. This corresponds to the choice of P,

$$P(r)=\begin{cases}\frac{1}{\kappa}\ , & r<r_0-\epsilon\ ,\\ 1\ , & r>r_0+\epsilon\ ,\end{cases} \tag{2.5}$$

for ϵ/r_0 infinitesimal. The function $P(r)$ is assumed to vary smoothly in the region $|r-r_0|<\epsilon$. In the limit as $\epsilon\to 0$, the scalar curvature R approaches $[2(\kappa-1)/rP]\delta(r-r_0)$.

III. THE SCALAR FIELD

In this section, we briefly recall the results of Ref. [9] to establish our method and notation. The Green's function

$$G(x,x')=i\langle 0|T[\varphi(x)\varphi(x')]|0\rangle$$

for a massless scalar field with arbitrary coupling to the scalar curvature R satisfies the equation

$$(\Box-\xi R)G(x,x')=\left[\frac{\partial^2}{\partial t^2}+\frac{\partial^2}{\partial z^2}+\frac{1}{r^2}\frac{\partial^2}{\partial\phi^2}+\frac{1}{rP}\frac{\partial}{\partial r}\left[\frac{r}{P}\frac{\partial}{\partial r}\right]-\frac{2\xi}{r}\frac{P'}{P^3}\right]G(x,x')=-\delta^4(x,x')\ . \tag{3.1}$$

Here the covariant δ function is

$$\delta^4(x,x')=P^{-1}r^{-1}\delta(\Delta t)\delta(\Delta r)\delta(\Delta\phi)\delta(\Delta z)\ , \tag{3.2}$$

with coordinate differences $\Delta t=t-t'$, $\Delta r=r-r'$, and so on.

The cylindrical and temporal symmetries of the cosmic string background suggest that we seek a solution of the form

$$G(x,x')=\frac{\kappa}{(2\pi)^3}\int d\mathbf{k}\, e^{i\mathbf{k}\cdot\Delta\mathbf{x}}\sum_{n=-\infty}^{\infty}e^{in\kappa\Delta\phi}g_{n,\mathbf{k}}(r,r')\ , \tag{3.3}$$

where $\Delta\mathbf{x}=(\Delta t,\Delta z)$, and $\mathbf{k}\in\mathbb{R}^2$. The differential equation satisfied by $g_{n,\mathbf{k}}(r,r')$ is found by substituting the above trial solution into Eq. (3.1); we obtain

$$\left[\frac{\partial}{\partial r}\left[\frac{r}{P}\frac{\partial}{\partial r}\right]-s^2rP-\frac{(n\kappa)^2P}{r}-2\xi\frac{P'}{P^2}\right]g_{n,\mathbf{k}}(r,r')=-\delta(r-r')\ , \tag{3.4}$$

where $s\equiv|\mathbf{k}|$.

In the standard way we can now write

$$g_{n,\mathbf{k}}(r,r')=\Psi_<(r_<)\Psi_>(r_>)\ , \tag{3.5}$$

where $r_<=\min(r,r')$ and $r_>=\max(r,r')$. Here the functions $\Psi_<$ and $\Psi_>$ satisfy the homogeneous version of Eq. (3.4). $\Psi_<$ is taken to be regular as $r\to 0$ and $\Psi_>$ to fall off as $r\to\infty$. Equation (3.4) gives the normalization condition

$$\Psi'_>(r)\Psi_<(r)-\Psi'_<(r)\Psi_>(r)=\frac{-P(r)}{r}\ . \tag{3.6}$$

The function $\Psi_<$ is determined by choosing the solution of Eq. (3.4) which is well behaved at $r=0$ and integrating it out. The resulting function is, of course, continuous. In the case of the flower pot, the ξR term in the equation of motion is nonzero only in the infinitesimal region $|r-r_0|<\epsilon$. In the limit as ϵ vanishes, the effect of this ξR term can be seen by integrating Eq. (3.4) through the point $r=r_0$. One obtains the relation

$$\lim_{r\to r_0+}\left[\frac{d}{dr}\Psi_<(r)\right]-\kappa\lim_{r\to r_0-}\left[\frac{d}{dr}\Psi_<(r)\right]=2\xi\frac{\kappa-1}{r_0}\Psi_<(r_0)\ , \tag{3.7}$$

which implies that the mode function $\Psi_<$ is continuous at $r=r_0$ but has a discontinuity in its slope at $r=r_0$.

The homogeneous solutions of Eq. (3.4) are Bessel functions. The regularity condition as $r\to 0$ gives, as solutions for $\Psi_<(r)$,

$$\Psi_<(r)=\begin{cases}I_{|n|}(sr/\kappa) & \text{for } r<r_0\ ,\\ A_0I_{\kappa|n|}(sr)+B_0K_{\kappa|n|}(sr) & \text{for } r>r_0\ .\end{cases} \tag{3.8}$$

The ratio of the constants $C_0\equiv B_0/A_0$ is determined by the jump condition (3.7) and continuity of $\Psi_<$ at $r=r_0$ to be $C_0(sr_0,n,\xi)$, where

$$C_0(x,n,\xi)=\frac{xI'_{\kappa|n|}(x)I_{|n|}(x/\kappa)-xI_{\kappa|n|}(x)I'_{|n|}(x/\kappa)-2\xi(\kappa-1)I_{\kappa|n|}(x)I_{|n|}(x/\kappa)}{xK_{\kappa|n|}(x)I'_{|n|}(x/\kappa)-xK'_{\kappa|n|}(x)I_{|n|}(x/\kappa)+2\xi(\kappa-1)K_{\kappa|n|}(x)I_{|n|}(x/\kappa)}\ . \tag{3.9}$$

The solutions for $\Psi_>(r)$ are determined by the condition that they fall off when $r\to\infty$. Together with the normalization condition (3.6) this yields

$$\Psi_>(r)=\frac{1}{A_0}K_{\kappa|n|}(sr)\quad\text{for } r>r_0\ . \tag{3.10}$$

The Green's function on the four-dimensional flower pot is now given by the expression

$$G(x,x')=\frac{\kappa}{(2\pi)^3}\sum_{n=-\infty}^{\infty} e^{in\kappa\Delta\phi}\int d\mathbf{k}\, e^{i\mathbf{k}\cdot\Delta\mathbf{x}}K_{\kappa|n|}(sr_>)[I_{\kappa|n|}(sr_<)+C_0(sr_0,n,\xi)K_{\kappa|n|}(sr_<)]$$

$$=\frac{\kappa}{(2\pi)^2}\sum_{n=-\infty}^{\infty} e^{in\kappa\Delta\phi}\int_0^\infty ds\, s\, J_0(s|\Delta\mathbf{x}|)K_{\kappa|n|}(sr_>)[I_{\kappa|n|}(sr_<)+C_0(sr_0,n,\xi)K_{\kappa|n|}(sr_<)] . \quad (3.11)$$

In this expression we have assumed that both r and r' are greater than r_0. The only dependence upon ξ and r_0 here is through the function C_0.

We may now consider the limit to the idealized cone which may be obtained for fixed r and r' by taking the limit $r_0/r_< \to 0$ or, loosely speaking, $r_0 \to 0$. From (3.9) one can show that, in the limit as $x \to 0$, for $\xi \neq 0$ and $\kappa > 1$,

$$C_0(x,n,\xi)\sim\begin{cases}\dfrac{2\xi(\kappa-1)}{2\xi(\kappa-1)[\ln(x/2)+\mathcal{C}]-1} & \text{for } n=0 ,\\[2ex] \dfrac{-2}{\Gamma(\kappa|n|+1)\Gamma(\kappa|n|)}\left[\dfrac{\xi(\kappa-1)}{\kappa|n|+\xi(\kappa-1)}\right]\left[\dfrac{x}{2}\right]^{2\kappa|n|} & \text{for } n\neq 0 ,\end{cases} \quad (3.12a)$$

where $\mathcal{C}$ is Euler's constant. When $\xi=0$ (and $\kappa>1$) one can show that

$$C_0(x,n,\xi=0) \sim\frac{2}{\Gamma(\kappa|n|+2)\Gamma(\kappa|n|+1)}\frac{\kappa-1}{\kappa(|n|+1)}\left[\frac{x}{2}\right]^{2\kappa|n|+2} \quad (3.12b)$$

in the same limit. Thus, in all cases $C_0 \to 0$, although for $\xi\neq 0$ the $n=0$ term does so logarithmically slowly. Denoting quantities in the ideal cone limit by the subscript C, we therefore have

$$G_C(x,x')=\frac{\kappa}{(2\pi)^2}\sum_{n=-\infty}^{\infty} e^{in\kappa\Delta\phi}\int_0^\infty ds\, s\, J_0(s|\Delta\mathbf{x}|)\times K_{\kappa|n|}(sr_>)\times I_{\kappa|n|}(sr_<) . \quad (3.13)$$

This expression can be written in closed form. First the integral is computed using the identity (cf., GR 6.578.11, GR 8.754.4)

$$\int_0^\infty ds\, s\, J_0(as)K_\mu(bs)I_\mu(cs)=\frac{e^{-\mu\gamma}}{2bc\,\sinh\gamma} ,$$

$$\mathrm{Re}(\mu)>-1 ,\ a>0 ,\ \mathrm{Re}(b)>|\mathrm{Re}(c)| , \quad (3.14)$$

where γ is defined to be the positive solution of $2bc\cosh\gamma=a^2+b^2+c^2$, to give

$$G_C(x,x')=\frac{\kappa}{8\pi^2}\frac{1}{rr'\sinh\eta}\sum_{n=-\infty}^{\infty} e^{in\kappa\Delta\phi-\kappa|n|\eta} \quad (3.15)$$

with η defined to be the positive solution of

$$\cosh\eta=\frac{(\Delta t)^2+(\Delta z)^2+r^2+r'^2}{2rr'} . \quad (3.16)$$

By splitting this into two sums, one from $-\infty$ to -1 and the other from 0 to ∞, GR 1.461 can be employed to yield the closed form

$$G_C(x,x')=\frac{\kappa}{8\pi^2}\frac{1}{rr'\sinh\eta}\frac{\sinh\kappa\eta}{\cosh\kappa\eta-\cos\kappa\Delta\phi} \quad (3.17)$$

which is independent of the value of ξ. In Lorentzian space-time, one thus obtains the following expression for the scalar Green's function on an idealized cosmic string:

$$G_C(x,x')=i\langle 0|T[\varphi(x)\varphi(x')]|0\rangle =\frac{i\kappa}{8\pi^2}\frac{1}{rr'\sinh\eta}\frac{\sinh\kappa\eta}{\cosh\kappa\eta-\cos\kappa\Delta\phi} , \quad (3.18a)$$

where η is defined to be the positive solution of the equation

$$\cosh\eta=\frac{-(\Delta t)^2+(\Delta z)^2+r^2+r'^2}{2rr'} . \quad (3.18b)$$

The conclusion is that the coupling term $\xi R\varphi^2$ in the action has no effect on the Green's function for fixed r, r' in the ideal conical limit $r_0 \to 0$.

Given the Green's function (3.18) it is a straightforward matter to determine the renormalized vacuum expectation value of the stress-energy tensor for the scalar field in the vicinity of an idealized cosmic string, using the Hadamard regularization scheme outlined in Ref. [12]. The first step is to isolate the regular part $W(x,x')$ of the Green's function, which, in this case, is given by the formula

$$W(x,x')=-8\pi^2 i\{G_C(x,x')-[G_C(x,x')]_{\kappa=1}\} \quad (3.19)$$

(since the regular part of the Green's function for a massless theory in Minkowski space is known to be zero). Next we insert (3.18) for $G_C(x,x')$ into (3.19) and expand the result to second order in powers of the coordinate differences $\Delta x^\mu=(x-x')^\mu$; the expansion

$$\eta^2=r^{-2}[-(\Delta t)^2+(\Delta r)^2+(\Delta z)^2]+O((\Delta x^\mu)^3) \quad (3.20)$$

is useful in this regard. We obtain

$$W(x,x')=\frac{\kappa^2-1}{6r^2}\left[1+\frac{\Delta r}{r}+\frac{\kappa^2+11}{60r^2}[(\Delta t)^2-(\Delta z)^2] -\frac{\kappa^2-49}{60r^2}(\Delta r)^2+\frac{\kappa^2+1}{20}(\Delta\phi)^2\right] +O((\Delta x^\mu)^3) . \quad (3.21)$$

The Taylor coefficients of $W(x,x')$ required for the computation of the renormalized stress-energy tensor may now be determined by substituting (3.21) into the formulas

$$w(x)=\lim_{x\to x'}[W(x,x')]$$
$$w_\mu(x)=\lim_{x\to x'}[-g_\mu{}^{\rho'}(x,x')W(x,x')_{;\rho'}] ,$$
$$w_{\mu\nu}(x)=\lim_{x\to x'}[g_\mu{}^{\rho'}(x,x')g_\nu{}^{\tau'}(x,x')W(x,x')_{;\rho'\tau'}] . \tag{3.22}$$

Here $g_\mu{}^{\nu'}(x,x')$ is the bivector of parallel transport, which, for the idealized cosmic string space-time, is given by

$$g^\mu{}_{\nu'}=\begin{bmatrix}1 & 0 & 0 & 0\\ 0 & \cos\Delta\phi & r'\sin\Delta\phi & 0\\ 0 & -r^{-1}\sin\Delta\phi & r'r^{-1}\cos\Delta\phi & 0\\ 0 & 0 & 0 & 1\end{bmatrix} . \tag{3.23}$$

We find that the only nonvanishing coefficients are

$$w=\frac{\kappa^2-1}{6r^2} ,$$
$$w_r=\frac{\kappa^2-1}{6r^3} ,$$
$$w_{tt}=-w_{zz}=\frac{(\kappa^2-1)(\kappa^2+11)}{180r^4} , \tag{3.24}$$
$$w_{rr}=-\frac{(\kappa^2-1)(\kappa^2-49)}{180r^4} ,$$
$$w_{\phi\phi}=\frac{(\kappa^2-1)(\kappa^2-9)}{60r^2} .$$

It is reassuring to check that these coefficients satisfy the identities [12]

$$w_\mu{}^\mu=0 ,$$
$$w_{\mu\nu}{}^{;\nu}=\tfrac{1}{4}(\Box w)_{;\mu} , \tag{3.25}$$
$$w_\mu=-\tfrac{1}{2}w_{;\mu}$$

[the first two of which follow from substituting the Hadamard form of the Green's function into the wave equation (3.1) and the last from the symmetry constraint $G_C(x,x')=G_C(x',x)$]. Finally, substituting expressions (3.24) into the formula in [12] which gives the renormalized vacuum expectation value of the stress-energy tensor for a scalar field on an arbitrary background directly in terms of Taylor coefficients, we find that

$$\langle 0|T_\mu{}^\nu|0\rangle_R=\frac{\kappa^2-1}{1440\pi^2r^4}\times[(\kappa^2+1)\ \mathrm{diag}\ (1,1,-3,1)_\mu{}^\nu+10(6\xi-1)\ \mathrm{diag}\ (2,-1,3,2)_\mu{}^\nu] \tag{3.26}$$

in precise agreement with the result of Frolov and Serebriany [5] obtained previously using a more traditional renormalization procedure.

IV. THE ELECTROMAGNETIC FIELD

In terms of the vector potential, A_μ, the Maxwell action is

$$S_M=-\tfrac{1}{4}\int d^4x\sqrt{|g|}F_{\mu\nu}F^{\mu\nu} , \tag{4.1}$$

where $g\equiv\det[g_{\mu\nu}]$ and $F_{\mu\nu}\equiv 2\nabla_{[\mu}A_{\nu]}$. The field strength $F_{\mu\nu}$ is invariant under the gauge transformation $A_\mu\to A_\mu+\nabla_\mu\Lambda$ for an arbitrary scalar field Λ. To quantize the theory, one must introduce a gauge-breaking (GB) term into the action; the standard term is

$$S_{\rm GB}=-\tfrac{1}{2}\int d^4x\sqrt{|g|}(A_\mu{}^{;\mu})^2 . \tag{4.2}$$

The ghost action needed to compensate for this choice of gauge-breaking term is

$$S_{\rm gh}=\int d^4x\sqrt{|g|}\bar{c}\Box c , \tag{4.3}$$

where c and $\bar{c}$ are the (scalar) ghost and antighost field, respectively. For the purposes of quantization, the total action is then $S_M+S_{\rm GB}+S_{\rm gh}$. The action gives the following equation of motion for A_ν:

$$(g_\mu{}^\nu\Box-R_\mu{}^\nu)A_\nu=0 . \tag{4.4}$$

The remaining equations of motion are $\Box c=0$ and $\Box\bar{c}=0$ for the ghost and antighost fields.

The Lorentzian Feynman functions of the vector potential and ghost field are defined by the time-ordered expectation values

$$G^{\mu\nu'}=i\langle 0|T[A^\mu(x)A^{\nu'}(x')]|0\rangle$$

and

$$\tilde{G}=i\langle 0|T[\bar{c}(x)c(x')]|0\rangle ,$$

respectively. In Euclidean space-time, the corresponding Green's functions satisfy

$$(g_{\mu\nu}\Box-R_{\mu\nu})G^{\nu\rho'}(x,x')=-g_\mu{}^{\rho'}\delta^4(x,x') \tag{4.5}$$

and

$$\Box\tilde{G}(x,x')=-\delta^4(x,x') . \tag{4.6}$$

Comparing (4.6) with (3.1), we see that the ghost Green's function is equal to the scalar Green's function with $\xi=0$: $\tilde{G}=G_{\xi=0}$.

An important check on our results is given by the Ward identity which may be derived from the Becchi-Rouet-Stora (BRS) invariance of the total action under the perturbations $\delta A_\mu=(\nabla_\mu c)\delta\zeta$, $\delta\bar{c}=(A_\nu{}^{;\nu})\delta\zeta$, $\delta c=0$, where ζ is the infinitesimal anticommuting scalar BRS parameter. This invariance implies the Ward identity

$$G^{\mu\nu'}(x,x')_{;\mu}+\tilde{G}(x,x')^{;\nu'}=0 . \tag{4.7}$$

It is this identity which ensures us that we are dealing with electromagnetism rather than an uncoupled vector-scalar theory.

The most efficient way to determine the above Green's functions on a cosmic-string background is to express everything in terms of a null complex tetrad for the corresponding Euclideanized background. A natural choice is given by

$$\begin{aligned} e^{\mu}_{(1)} &= \frac{1}{\sqrt{2}}(it^{\mu}+z^{\mu}) = \frac{1}{\sqrt{2}}(i,0,0,1)^{\mu} , \\ e^{\mu}_{(2)} &= \frac{1}{\sqrt{2}}(it^{\mu}-z^{\mu}) = \frac{1}{\sqrt{2}}(i,0,0,-1)^{\mu} , \\ e^{\mu}_{(3)} &= \frac{1}{\sqrt{2}}(r^{\mu}+i\phi^{\mu}) = \frac{1}{\sqrt{2}}(0,1/P,i/r,0)^{\mu} , \\ e^{\mu}_{(4)} &= \frac{1}{\sqrt{2}}(r^{\mu}-i\phi^{\mu}) = \frac{1}{\sqrt{2}}(0,1/P,-i/r,0)^{\mu} . \end{aligned} \tag{4.8}$$

For later convenience we note that the only nonvanishing $e^{\mu}_{(a);\nu}$ are

$$e^{\mu}_{(3);\phi} = -\frac{i}{P}e^{\mu}_{(3)} , \quad e^{\mu}_{(4);\phi} = \frac{i}{P}e^{\mu}_{(4)} \tag{4.9}$$

and that the only nonvanishing $\Box e^{\mu}_{(a)}$ are

$$\Box e^{\mu}_{(3)} = -\frac{1}{r^2P^2}e^{\mu}_{(3)} , \quad \Box e^{\mu}_{(4)} = -\frac{1}{r^2P^2}e^{\mu}_{(4)} . \tag{4.10}$$

Because $e^{\mu}_{(3)}$ and $e^{\mu}_{(4)}$ are interchanged by complex conjugation, the tetrad components of any real tensor are complex conjugated under a complete $(3)\leftrightarrow(4)$ replacement. This symmetry reduces the number of components of the Euclidean Green's functions that need to be determined. Substituting

$$G^{\mu\nu'} = e^{\mu}_{(a)}e^{\nu'}_{(b')}G^{(a)(b')} \tag{4.11}$$

into the Green's function equation (4.5) and using Eqs. (4.9) and (4.10), we find that the tetrad components $G^{(a)(b')}$ of the Euclideanized photon Green's function satisfy the differential equations

$$\Box G^{(1)(1')} = 0 , \tag{4.12a}$$

with similar equations for $G^{(1)(3')}$, $G^{(2)(2')}$, and $G^{(2)(3')}$ (and hence by symmetry for $G^{(1)(4')*}$, and $G^{(2)(4')}$),

$$\Box G^{(1)(2')} = \delta^4(x,x') \tag{4.12b}$$

with a similar equation for $G^{(2)(1')}$,

$$\left[\Box - \frac{2i}{r^2P}\frac{\partial}{\partial\phi} - \frac{1}{r^2P^2} - \frac{P'}{rP^3}\right]G^{(3)(1')} = 0 , \tag{4.12c}$$

with similar equations for $G^{(3)(2')}$, and $G^{(3)(3')}$ (and hence for $G^{(4)(1')*}$, $G^{(4)(2')*}$, and $G^{(4)(4')*}$), and

$$\left[\Box - \frac{2i}{r^2P}\frac{\partial}{\partial\phi} - \frac{1}{r^2P^2} - \frac{P'}{rP^3}\right]G^{(3)(4')} = -\delta^4(x,x') \tag{4.12d}$$

(with a similar equation for $G^{(4)(3')*}$). In Eqs. (4.12), $\Box$ is the scalar wave operator.

The only regular solution of the homogeneous equations (4.12a) and (4.12c) is zero. Furthermore, up to a sign, Eq. (4.12b) is just the scalar equation (3.1) with $\xi=0$, and so the appropriate solution is $G^{(1)(2')} = -G_{\xi=0}$. It remains to find the solution of Eq. (4.12d). As before, we seek a solution of the form

$$G^{(a)(b')}(x,x') = \frac{\kappa}{(2\pi)^3}\int d\mathbf{k}\, e^{i\mathbf{k}\cdot\Delta\mathbf{x}} \sum_{n=-\infty}^{\infty} e^{in\kappa\Delta\phi} g^{(a)(b')}_{n,\mathbf{k}}(r,r') \tag{4.13}$$

and find that $g^{(3)(4')}_{n,\mathbf{k}}(r,r')$ must satisfy

$$\left[\frac{\partial}{\partial r}\left[\frac{r}{P}\frac{\partial}{\partial r}\right] - s^2 rP - \frac{1}{rP}(n\kappa P-1)^2 - \frac{P'}{P^2}\right]g^{(3)(4')}_{n,\mathbf{k}} = -\delta(r-r') . \tag{4.14}$$

At this point we digress briefly to examine the form of the Ward identity (4.7) on the cosmic-string background. Substituting the tetrad form (4.11), we find that the tetrad components of the photon Green's function are related to the ghost Green's function according to

$$\left[i\frac{\partial}{\partial t} \pm \frac{\partial}{\partial z}\right]G^{(1)(2')} = \left[i\frac{\partial}{\partial t'} \pm \frac{\partial}{\partial z'}\right]\tilde{G} \tag{4.15a}$$

and

$$\left[\frac{1}{P(r)}\frac{\partial}{\partial r} + \frac{i}{r}\frac{\partial}{\partial\phi} + \frac{1}{rP(r)}\right]G^{(3)(4')} = -\left[\frac{1}{P(r')}\frac{\partial}{\partial r'} + \frac{i}{r'}\frac{\partial}{\partial\phi'}\right]\tilde{G} . \tag{4.15b}$$

Introducing the obvious representation for the ghost Green's function, these become

$$g^{(1)(2')}_{n,\mathbf{k}}(r,r') = -\tilde{g}_{n,\mathbf{k}}(r,r') , \tag{4.16a}$$

$$\left[\frac{1}{P(r)}\frac{\partial}{\partial r} - \frac{n\kappa P(r)-1}{rP(r)}\right]g^{(3)(4')}_{n,\mathbf{k}}(r,r') = -\left[\frac{1}{P(r')}\frac{\partial}{\partial r'} + \frac{n\kappa}{r'}\right]\tilde{g}_{n,\mathbf{k}}(r,r') . \tag{4.16b}$$

We immediately see that Eq. (4.16a) is in agreement with our previous choices $G^{(1)(2')} = -G_{\xi=0}$ and $\tilde{G} = G_{\xi=0}$.

To proceed further we need to make a specific choice for P and choose to work with the flower-pot model. In that case we can immediately solve Eq. (4.14) to yield the interior solution which is regular at $r=0$ as

$$\Psi^{(3)(4')}_{<}(r) = \begin{cases} I_{|n-1|}(sr/\kappa) & \text{for } r<r_0 , \\ A_1 I_{|n\kappa-1|}(sr) + B_1 K_{|n\kappa-1|}(sr) & \text{for } r>r_0 , \end{cases} \tag{4.17}$$

where

$$\lim_{r\to r_0+}\left[\frac{d}{dr}\Psi^{(3)(4')}_{<}(r_0)\right] - \kappa \lim_{r\to r_0-}\left[\frac{d}{dr}\Psi^{(3)(4')}_{<}(r_0)\right] = \frac{\kappa-1}{r_0}\Psi^{(3)(4')}_{<}(r_0) \tag{4.18}$$

and the properly normalized exterior solution which dies at infinity as

$$\Psi_{>}^{(3)(4')}(r)=\frac{1}{A_1}K_{|n\kappa-1|}(sr) \quad \text{for } r>r_0 \ . \tag{4.19}$$

As in the scalar case, the jump condition (4.18) and continuity of $\Psi_{<}^{(3)(4')}$ at $r=r_0$ determine the ratio $C_1(sr_0,n)=B_1/A_1$. We obtain

$$C_1(x,n)=\frac{xI'_{|n\kappa-1|}(x)I_{|n-1|}(x/\kappa)-xI_{|n\kappa-1|}(x)I'_{|n-1|}(x/\kappa)-(\kappa-1)I_{|n\kappa-1|}(x)I_{|n-1|}(x/\kappa)}{xK_{|n\kappa-1|}(x)I'_{|n-1|}(x/\kappa)-xK'_{|n\kappa-1|}(x)I_{|n-1|}(x/\kappa)+(\kappa-1)K_{|n\kappa-1|}(x)I_{|n-1|}(x/\kappa)} \ . \tag{4.20}$$

The required solution of (4.14) is

$$\begin{aligned} g_{n,k}^{(3)(4')}&=\Psi_{<}^{(3)(4')}(r_<)\Psi_{>}^{(3)(4')}(r_>)\\ &=K_{|n\kappa-1|}(sr_>)\\ &\quad\times[I_{|n\kappa-1|}(sr_<)+C_1(sr_0,n)K_{|n\kappa-1|}(sr_<)] \end{aligned} \tag{4.21}$$

for r, $r'>r_0$. The Ward identity (4.16b) now reduces to

$$C_1(x,n)=-C_0(x,n,\xi=0) \tag{4.22}$$

when $\kappa>1$, which is readily verified using (4.20) and (3.9).

At this point one can take the limit of an ideal cone, corresponding to $x=sr_0\to 0$. One finds from (4.20) that, as $x\to 0$,

$$C_1(x,n)\sim\frac{1}{\Gamma(\kappa|n|+2)\Gamma(\kappa|n|+1)}\frac{2(1-\kappa)}{\kappa(|n|+1)}\left[\frac{x}{2}\right]^{2\kappa|n|+2} \tag{4.23}$$

provided $\kappa>1$. [Note that this result is consistent with (4.22) and (3.12b).] It is then clear that $C_1\to 0$ as $x\to 0$. (The case $0<\kappa<1$, though unphysical, is interesting from a mathematical viewpoint and we shall return to it presently.) In the ideal conical limit, one therefore obtains

$$g_{n,k}^{(3)(4')}(r,r')=I_{|n\kappa-1|}(sr_<)K_{|n\kappa-1|}(sr_>) \ .$$

Substituting this expression into the representation (4.13) and computing the integral and sum as for the scalar case, we obtain first

$$\begin{aligned} G_C^{(3)(4')}&=\frac{\kappa}{8\pi^2}\frac{1}{rr'\sinh\eta}\sum_{n=-\infty}^{\infty}e^{in\kappa\Delta\phi-|n\kappa-1|\eta}\\ &=\frac{\kappa}{8\pi^2}\frac{1}{rr'\sinh\eta}\left[\sum_{n=-\infty}^{0}e^{in\kappa\Delta\phi+(n\kappa-1)\eta}+\sum_{n=1}^{\infty}e^{in\kappa\Delta\phi-(n\kappa-1)\eta}\right] \end{aligned} \tag{4.24}$$

and then

$$G_C^{(3)(4')}=\frac{\kappa}{8\pi^2}\frac{1}{rr'\sinh\eta}\left[\frac{e^{i\kappa\Delta\phi}\sinh\eta+\sinh(\kappa-1)\eta}{\cosh\kappa\eta-\cos\kappa\Delta\phi}\right] . \tag{4.25}$$

Finally, the tensor components of the electromagnetic Green's function

$$G_C^{\mu\nu'}=i\langle 0|T[A^\mu(x)A^{\nu'}(x')]|0\rangle$$

in the Feynman gauge on the Lorentzian ideal conical space-time are found to be

$$\begin{aligned} G_C^{tt'}&=-G_C^{zz'}\\ &=-G_C\\ &=-\frac{i\kappa}{8\pi^2}\frac{1}{rr'\sinh\eta}\left[\frac{\sinh\kappa\eta}{\cosh\kappa\eta-\cos\kappa\Delta\phi}\right] ,\\ G_C^{rr'}&=rr'G_C^{\phi\phi'}\\ &=\frac{i\kappa}{8\pi^2}\frac{1}{rr'\sinh\eta}\\ &\quad\times\left[\frac{\sinh\eta\cos\kappa\Delta\phi+\sinh(\kappa-1)\eta}{\cosh\kappa\eta-\cos\kappa\Delta\phi}\right] ,\\ r'G_C^{r\phi'}&=-rG_C^{\phi r'}\\ &=\frac{i\kappa}{8\pi^2}\frac{1}{rr'\sinh\eta}\left[\frac{\sinh\eta\sin\kappa\Delta\phi}{\cosh\kappa\eta-\cos\kappa\Delta\phi}\right] , \end{aligned} \tag{4.26}$$

where G_C is the scalar Green's function (3.18) on the ideal cone. To the best of our knowledge this is the first time that the electromagnetic propagator on an idealized cosmic string has been determined in closed form.

We now return to the case $0<\kappa<1$. Let the positive integer N be defined so that $0<1/(N+1)\le\kappa<1/N\le 1$. Provided either $n<1$ or $n>N$, one can take the $x\to 0$ limit of (4.20) just as before and obtain (4.23); but, when $1\le n\le N$, the lowest-order term in the denominator of (4.20) vanishes and the resulting behavior of C_1 for small x differs from (4.23) in this case. In fact, we find that

$$C_1(x,n)\to\begin{cases}0 & \text{for } n<1 \text{or } n>N \ ,\\ 2\dfrac{\sin(1-n\kappa)\pi}{\pi} & \text{for } 1\le n\le N\end{cases} \tag{4.27}$$

as $x\to 0$. This limit could also be derived from the Ward identity which now takes the form

$$C_1(x,n)=\begin{cases}-C_0(x,n,\xi=0) & \text{for } n<1 \text{ or } n>N \ ,\\ -C_0(x,n,\xi=0)+2\dfrac{\sin(1-n\kappa)\pi}{\pi} & \\ \qquad\text{for } 1\le n\le N \ . & \end{cases} \tag{4.28}$$

It follows that, in the ideal cone limit, for $1\le n\le N$,

$$\begin{aligned} \Psi_{<}^{(3)(4')}(r)&=A_1\left[I_{|n\kappa-1|}(sr)+2\frac{\sin(1-n\kappa)\pi}{\pi}K_{|n\kappa-1|}(sr)\right]\\ &=A_1I_{-|n\kappa-1|}(sr) \ . \end{aligned} \tag{4.29}$$

Hence, $\Psi_{<}^{(3)(4')}(r)$ diverges at the origin, although only ever as $r^{-1+\epsilon}$. Equation (4.29) is precisely what is required to ensure that the summed form for the Green's function in the ideal cone limit is unchanged. Thus, even for $0<\kappa<1$, the closed forms (4.25) and (4.26) given above are correct.

It is a valuable check on the above calculation to determine the renormalized stress tensor from this Green's function according to the Hadamard renormalization scheme of Brown and Ottewill [12]. One starts by finding the regular part of the Lorentzian Green's function given above. It still proves most economical to work with the tetrad components; i.e., we will determine

$$W^{(a)(b)}(x,x')\equiv g^{(b)}{}_{(b')}(x,x')W^{(a)(b')}(x,x') , \qquad (4.30)$$

where $g^{(b)}{}_{(b')}$ are the tetrad components of the bivector of parallel transport, given by

$$g^{(b)}{}_{(b')}(x,x')= \text{diag}\,(1,1,\exp(i\Delta\phi),\exp(-i\Delta\phi))^{(b)}{}_{(b')} . \qquad (4.31)$$

$W^{(a)(b)}$ is obtained by substituting the tetrad components of $G_C^{(a)(b')}$ into the formula

$$W^{(a)(b)}(x,x')=-8\pi^2 i g^{(b)}{}_{(b')}(x,x') \times\{G_C^{(a)(b')}(x,x')-[G_C^{(a)(b')}(x,x')]_{\kappa=1}\} \qquad (4.32)$$

and expanding the result as a power series in the coordinate differences Δx^μ. A straightforward though tedious computation produces the result

$$W^{(1)(2)}=W^{(2)(1)}=-W , \qquad (4.33a)$$

where W is the scalar expansion (3.21), and

$$W^{(3)(4)}=W^{(4)(3)*}=\frac{\kappa-1}{6r^2}\left\{(\kappa-5)+(\kappa-5)\frac{\Delta r}{r}+\frac{(\kappa+1)(\kappa^2-19)}{60r^2}[(\Delta t)^2-(\Delta z)^2] -\frac{1}{60}(\kappa^3+\kappa^2-79\kappa+281)\frac{(\Delta r)^2}{r^2}+\frac{1}{20}(\kappa^3+\kappa^2-29\kappa+31)(\Delta\phi)^2\right\} -\frac{i(\kappa-1)(\kappa-2)}{3r^2}\left[\Delta\phi+\frac{\Delta r\Delta\phi}{r}\right]+O((\Delta x^\mu)^3) \qquad (4.33b)$$

with all other components vanishing. The next step is to determine the first three coefficients occurring in the covariant Taylor expansion of $W^{(a)(b)}$ by substituting (4.33) into the formulas

$$\begin{aligned} w^{(a)(b)}(x)&=\lim_{x\to x'}[W^{(a)(b)}(x,x')] ,\\ w^{(a)(b)}{}_{\mu}(x)&=\lim_{x\to x'}[-g_\mu{}^{\rho'}(x,x')W^{(a)(b)}(x,x')_{;\rho'}] ,\\ w^{(a)(b)}{}_{\mu\nu}(x)&=\lim_{x\to x'}[g_\mu{}^{\rho'}(x,x')g_\nu{}^{\tau'}(x,x')W^{(a)(b)}(x,x')_{;\rho'\tau'}] . \end{aligned} \qquad (4.34)$$

We obtain

$$w^{(1)(2)}{}_{\alpha\cdots\beta}=w^{(2)(1)}{}_{\alpha\cdots\beta}=-w_{\alpha\cdots\beta} , \qquad (4.35a)$$

where $w_{\alpha\cdots\beta}$ are the scalar coefficients (3.24), and

$$\begin{aligned} w^{(3)(4)}&=\frac{1}{6r^2}(\kappa-1)(\kappa-5) ,\\ w^{(3)(4)}{}_r&=\frac{1}{6r^3}(\kappa-1)(\kappa-5) ,\\ w^{(3)(4)}{}_\phi&=-\frac{i}{3r^2}(\kappa-1)(\kappa-2) ,\\ w^{(3)(4)}{}_{tt}&=-w^{(3)(4)}{}_{zz}\\ &=\frac{1}{180r^4}(\kappa^2-1)(\kappa^2-19) , \qquad (4.35b)\\ w^{(3)(4)}{}_{rr}&=-\frac{1}{180r^4}(\kappa-1)(\kappa^3+\kappa^2-79\kappa+281) ,\\ w^{(3)(4)}{}_{r\phi}&=w^{(3)(4)}{}_{\phi r}=-\frac{2i}{3r^3}(\kappa-1)(\kappa-2) ,\\ w^{(3)(4)}{}_{\phi\phi}&=\frac{1}{60r^2}(\kappa-1)(\kappa^3+\kappa^2-39\kappa+81) \end{aligned}$$

with the only other nonvanishing components being $w^{(4)(3)}{}_{\alpha\cdots\beta}=w^{(3)(4)}{}_{\alpha\cdots\beta}{}^*$. At this point we can simultaneously revert to tensor indices and compute the symmetric and antisymmetric coefficients required for calculating the stress tensor using

$$\begin{aligned} s^{\mu\nu}{}_{\rho\cdots\tau}&=e^{(\mu}_{(a)}e^{\nu)}_{(b)}w^{(a)(b)}{}_{\rho\cdots\tau} ,\\ a^{\mu\nu}{}_{\rho\cdots\tau}&=e^{[\mu}_{(a)}e^{\nu]}_{(b)}w^{(a)(b)}{}_{\rho\cdots\tau} . \end{aligned} \qquad (4.36)$$

One may now verify that the coefficients above satisfy the identities [12]

$$\begin{aligned} a^{\mu\nu\rho}{}_{;\rho}&=0 ,\\ s^{\mu\nu\rho}{}_{\rho}&=0 ,\\ s^{\mu\nu\rho\tau}{}_{;\tau}&=\tfrac{1}{4}\Box(s^{\mu\nu;\rho}) , \end{aligned} \qquad (4.37)$$

which arise from the wave equation (4.5). Finally, inserting our expressions for $s^{\mu\nu}{}_{\rho\cdots\tau}$, $a^{\mu\nu}{}_{\rho\cdots\tau}$ into the relevant formulas of Ref. [12], we find that the vacuum expectation value of the renormalized stress-energy tensor for the electromagnetic field in the vicinity of an idealized cosmic string is

$$\langle 0|T_\mu{}^\nu|0\rangle_R=\frac{(\kappa^2-1)(\kappa^2+11)}{720\pi^2r^4}\,\text{diag}\,(1,1,-3,1)_\mu{}^\nu \qquad (4.38)$$

in agreement with Refs. [5] and [15].

V. THE LINEARIZED GRAVITATIONAL FIELD

We write the metric as $g_{\mu\nu}+h_{\mu\nu}$, where $g_{\mu\nu}$ is the classical background metric and $h_{\mu\nu}$ will be treated as a small perturbation. We expand the Einstein-Hilbert action (without cosmological constant) in powers of $h_{\mu\nu}$ and assume that the cosmic string is made of a classical matter field, whose classical stress-energy tensor cancels the first-order term in the expansion [16]. Then keeping only terms quadratic in $h_{\mu\nu}$, our action is given by

$$S_2=\frac{1}{32\pi G}\int\sqrt{|g|}d^4x[\tfrac{1}{2}h^{\mu\nu}\Box h_{\mu\nu}-\tfrac{1}{4}h\Box h+(\nabla^\mu h_{\mu\nu}-\tfrac{1}{2}\nabla_\nu h)^2 +h^{\mu\nu}R_{\mu\rho\nu\tau}h^{\rho\tau}+h^{\mu}{}_{\nu}R^{\nu\rho}h_{\mu\rho}-hh^{\mu\nu}R_{\mu\nu}-\tfrac{1}{2}Rh^{\mu\nu}h_{\mu\nu}+\tfrac{1}{4}Rh^2]\ , \tag{5.1}$$

where $h\equiv g^{\mu\nu}h_{\mu\nu}$ is the trace of $h_{\mu\nu}$.

To quantize the theory we must add a gauge-breaking term to the action and compensating complex, anticommuting vector ghost and antighost fields c^μ and $\bar{c}^\mu$:

$$S_{\rm GB}=-\frac{1}{32\pi G}\int\sqrt{|g|}d^4x(\nabla^\mu h_{\mu\nu}-\tfrac{1}{2}\nabla_\nu h)^2\ , \tag{5.2}$$

$$S_{\rm gh}=\frac{1}{32\pi G}\int\sqrt{|g|}d^4x\bar{c}_\mu(g^{\mu\nu}\Box+R^{\mu\nu})c_\nu\ . \tag{5.3}$$

The total action is $S=S_2+S_{\rm GB}+S_{\rm gh}$.

In Lorentzian space-time, the graviton and ghost Green's functions are defined by the time-ordered expectation values

$$G_{\mu\nu\rho'\tau'}(x,x')=\frac{i}{32\pi G}\langle 0|T[h_{\mu\nu}(x)h_{\rho'\tau'}(x')]|0\rangle\ ,$$
$$\tilde{G}_{\nu\rho'}(x,x')=\frac{i}{32\pi G}\langle 0|T[\bar{c}_\nu(x)c_{\rho'}(x')]|0\rangle\ . \tag{5.4}$$

As before, we will calculate these quantities in Euclidean space-time, and then Wick rotate back to the Lorentzian result. The Euclidean graviton Green's function satisfies

$$(g_{\mu\rho}g_{\nu\tau}\Box+2R_{\mu\rho\nu\tau}+2R_{\mu(\rho}g_{\tau)\nu}-R_{\mu\nu}g_{\rho\tau}-R_{\rho\tau}g_{\mu\nu}-Rg_{\mu\rho}g_{\nu\tau}+\tfrac{1}{2}Rg_{\mu\nu}g_{\rho\tau})G^{\mu\nu\rho'\tau'}(x,x') =-\tfrac{1}{2}(g_\rho{}^{\rho'}g_\tau{}^{\tau'}+g_\tau{}^{\rho'}g_\rho{}^{\tau'}-g_{\rho\tau}g^{\rho'\tau'})\delta^4(x,x') \tag{5.5}$$

and by definition possesses the symmetries $G^{\mu\nu\rho'\tau'}=G^{(\mu\nu)(\rho'\tau')}$. The ghost Green's function satisfies the equation

$$(g_{\mu\nu}\Box+R_{\mu\nu})\tilde{G}^{\nu\rho'}(x,x')=-g_\mu{}^{\rho'}\delta^4(x,x')\ , \tag{5.6}$$

which is similar to the electromagnetic equation (4.5) except for a crucial change of sign in the Ricci tensor term.

As for electromagnetism, the gauge and ghost fields are connected by a set of Ward identities. The situation is more complicated in this case, however. The total action $(S_2+S_{\rm GB}+S_{\rm gh})$ for the linearized gravitational field is invariant under the infinitesimal BRS transformations

$$\delta h_{\mu\nu}=(c^\lambda\partial_\lambda g_{\mu\nu}+g_{\mu\lambda}\partial_\nu c^\lambda+g_{\nu\lambda}\partial_\mu c^\lambda)\delta\zeta\ ,$$
$$\delta c^\mu=c^\lambda\partial_\lambda c^\mu\delta\zeta\ ,$$
$$\delta\bar{c}^\mu=(\nabla^\nu h^\mu{}_\nu-\tfrac{1}{2}\nabla^\mu h)\delta\zeta\ . \tag{5.7}$$

Here ζ is an infinitesimal anticommuting scalar parameter. From this invariance one may derive Ward identities for the Green's functions of the linearized theory. However, owing to the fact that S_2 is invariant under infinitesimal gauge transformations $h_{\mu\nu}\to h_{\mu\nu}+\xi_{(\mu;\nu)}$ only when the Einstein tensor of the background space-time vanishes, the Ward identities are nonlocal when $R^{\mu\nu}-\frac{1}{2}Rg^{\mu\nu}\neq 0$. (Note that the two-dimensional Gauss-Bonnet theorem requires the existence of curvature in any cosmic-string space-time.) The Ward identities on a general background are given by [17]

$$G^{\mu\nu\rho'\tau'}{}_{;\mu}(x,x')-\tfrac{1}{2}G_\mu{}^{\mu\rho'\tau';\nu}(x,x')+\tilde{G}^{\nu(\rho';\tau')}(x,x') =\int\sqrt{|g''|}d^4x''\tilde{G}^{\nu\alpha''}(x,x'')[R^{\beta''\gamma''}(x'')-\tfrac{1}{2}R(x'')g^{\beta''\gamma''}](\delta^{\delta''}_{\beta''}\nabla_{\alpha''}-2\delta^{\delta''}_{\alpha''}\nabla_{\beta''})G_{\delta''\gamma''}{}^{\rho'\tau'}(x'',x')\ . \tag{5.8}$$

As in the electromagnetic case, we shall obtain the gravitational Green's functions by expressing the equations in terms of the null complex tetrad (4.8). We find that the only nonzero tetrad components of the graviton Green's function are $G^{(1)(1)(2')(2')}=G^{(2)(2)(1')(1')}$, $G^{(1)(2)(3')(4')}=G^{(3)(4)(1')(2')}$, $G^{(1)(3)(2')(4')}=G^{(2)(3)(1')(4')}$ and $G^{(3)(3)(4')(4')}$ [plus all others obtainable from these by the symmetries $G^{(a)(b)(c')(d')}=G^{((a)(b))((c')(d'))}$ or by interchange of (3) and (4) under complex conjugation]. These are solutions of the equations

$$\left[\Box-\frac{2P'}{rP^3}\right]G^{(1)(1)(2')(2')}=-\delta^4(x,x')\ , \tag{5.9a}$$

$$\Box G^{(1)(2)(3')(4')}=-\tfrac{1}{2}\delta^4(x,x')\ , \tag{5.9b}$$

$$\left[\Box-\frac{2i}{r^2P}\frac{\partial}{\partial\phi}-\frac{1}{r^2P^2}-\frac{P'}{rP^3}\right]G^{(1)(3)(2')(4')}=\tfrac{1}{2}\delta^4(x,x')\ , \tag{5.9c}$$

$$\left[\Box-\frac{4i}{r^2P}\frac{\partial}{\partial\phi}-\frac{4}{r^2P^2}-\frac{2P'}{rP^3}\right]G^{(3)(3)(4')(4')}=-\delta^4(x,x')\ . \tag{5.9d}$$

We conclude that certain components of the graviton Green's function are equal to scalar or electromagnetic components previously determined:

$$G^{(1)(1)(2')(2')}=G_{\xi=1}\ , \tag{5.10}$$

$$G^{(1)(2)(3')(4')}=\tfrac{1}{2}G_{\xi=0}\ , \tag{5.11}$$

$$G^{(1)(3)(2')(4')}=-\tfrac{1}{2}G^{(3)(4')}\ . \tag{5.12}$$

It remains to find the solution of (5.9d). Substituting

$$G^{(3)(3)(4')(4')}(x,x')=\frac{\kappa}{(2\pi)^3}\int d\mathbf{k}\, e^{i\mathbf{k}\cdot\Delta\mathbf{x}}\sum_{n=-\infty}^{\infty}e^{in\kappa\Delta\phi}g_{n,\mathbf{k}}^{(3)(3)(4')(4')}(r,r') \tag{5.13}$$

for $G^{(3)(3)(4')(4')}$ in (5.9d) results in the radial equation

$$\left[\frac{\partial}{\partial r}\left[\frac{r}{P}\frac{\partial}{\partial r}\right]-s^2rP-\frac{1}{rP}(m\kappa P-2)^2-\frac{2P'}{P^2}\right]g_{n,\mathbf{k}}^{(3)(3)(4')(4')}=-\delta(r-r')\ . \tag{5.14}$$

Similarly, from the ghost wave equation we find that certain components are given in terms of the scalar Green's function

$$\tilde{G}^{(1)(2')}=\tilde{G}^{(2)(1')}=-G_{\xi=0}\ , \tag{5.15}$$

while the radial part of the remaining nonzero component $\tilde{G}^{(3)(4')}=\tilde{G}^{(4)(3')*}$ satisfies the equation

$$\left[\frac{\partial}{\partial r}\left[\frac{r}{P}\frac{\partial}{\partial r}\right]-s^2rP-\frac{1}{rP}(n\kappa P-1)^2+\frac{P'}{P^2}\right]\tilde{g}_{n,\mathbf{k}}^{(3)(4')}=-\delta(r-r')\ , \tag{5.16}$$

which differs in the sign of the final (curvature) term from the corresponding electromagnetic equation (4.14).

Next we examine the tetrad form of the Ward identities (5.8). We begin by noting that the Einstein tensor may be expressed as

$$R^{\alpha\beta}-\tfrac{1}{2}Rg^{\alpha\beta}=\tfrac{1}{2}R(e^{\alpha}_{(1)}e^{\beta}_{(2)}+e^{\alpha}_{(2)}e^{\beta}_{(1)})\ .$$

The tetrad components of (5.8) are then found to be

$$\left[i\frac{\partial}{\partial t}\pm\frac{\partial}{\partial z}\right]G^{(1)(2)(3')(4')}=\frac{1}{2}\left[i\frac{\partial}{\partial t'}\pm\frac{\partial}{\partial z'}\right]\tilde{G}^{(1)(2')}\ , \tag{5.17a}$$

$$\left[\frac{1}{P(r)}\frac{\partial}{\partial r}+\frac{i}{r}\frac{\partial}{\partial\phi}+\frac{1}{rP(r)}\right]G^{(1)(3)(2')(4')}=-\frac{1}{2}\left[\frac{1}{P(r')}\frac{\partial}{\partial r'}+\frac{i}{r'}\frac{\partial}{\partial\phi'}\right]\tilde{G}^{(1)(2')}\ , \tag{5.17b}$$

$$\left[\frac{1}{P(r)}\frac{\partial}{\partial r}+\frac{i}{r}\frac{\partial}{\partial\phi}+\frac{2}{rP(r)}\right]G^{(3)(3)(4')(4')}=-\left[\frac{1}{P(r')}\frac{\partial}{\partial r'}+\frac{i}{r'}\frac{\partial}{\partial\phi'}-\frac{1}{r'P(r')}\right]\tilde{G}^{(3)(4')}\ , \tag{5.17c}$$

$$\left[i\frac{\partial}{\partial t}\pm\frac{\partial}{\partial z}\right]G^{(1)(1)(2')(2')}=\left[i\frac{\partial}{\partial t'}\pm\frac{\partial}{\partial z'}\right]\left[\tilde{G}^{(1)(2')}-\int d^4x''\sqrt{g''}\,\tilde{G}^{(1)(2'')}R(x'')G^{(1'')(1'')(2')(2')}\right]\ , \tag{5.17d}$$

$$\left[i\frac{\partial}{\partial t}\pm\frac{\partial}{\partial z}\right]G^{(1)(3)(2')(4')}=\left[i\frac{\partial}{\partial t'}\pm\frac{\partial}{\partial z'}\right]\left[\frac{1}{2}\tilde{G}^{(3)(4')}+\int d^4x''\sqrt{g''}\,\tilde{G}^{(3)(4'')}R(x'')G^{(1'')(3'')(2')(4')}\right]\ , \tag{5.17e}$$

$$\left[\frac{1}{P(r)}\frac{\partial}{\partial r}-\frac{i}{r}\frac{\partial}{\partial\phi}\right]G^{(1)(2)(3')(4')}=-\frac{1}{2}\left[\frac{1}{P(r')}\frac{\partial}{\partial r'}-\frac{i}{r'}\frac{\partial}{\partial\phi'}+\frac{1}{r'P(r')}\right]\tilde{G}^{(3)(4')}$$
$$-\int d^4x''\sqrt{|g''|}\tilde{G}^{(3)(4'')}R(x'')\left[\frac{1}{P(r'')}\frac{\partial}{\partial r''}-\frac{i}{r''}\frac{\partial}{\partial\phi''}\right]G^{(1'')(2'')(3')(4')} . \quad (5.17f)$$

Owing to the symmetry of the space-time and vacuum state, all Green's functions depend only on the differences $(z-z')$ and $(t-t')$ so that

$$e^{\mu}_{(a)}\frac{\partial}{\partial x^{\mu}}G^{\bullet}=-e^{\mu'}_{(a)}\frac{\partial}{\partial x^{\mu'}}G^{\bullet} , \quad a=1,2 , \quad (5.18)$$

where $\bullet$ indicates zero, two, or four indices. It follows that Eqs. (5.17a), (5.17d), and (5.17e) reduce to

$$G^{(1)(2)(3')(4')}=-\tfrac{1}{2}\tilde{G}^{(1)(2')} , \quad (5.19a)$$

$$G^{(1)(1)(2')(2')}=-\tilde{G}^{(1)(2')}+\int d^4x''\sqrt{|g''|}\tilde{G}^{(1)(2'')}R(x'')G^{(1'')(1'')(2')(2')} , \quad (5.19b)$$

$$G^{(1)(3)(2')(4')}=-\tfrac{1}{2}\tilde{G}^{(3)(4')}-\int d^4x''\sqrt{|g''|}\tilde{G}^{(3)(4'')}R(x'')G^{(1'')(3'')(2')(4')} . \quad (5.19c)$$

It is immediately clear that Eq. (5.19a) is satisfied, because of Eqs. (5.11) and (5.15). It is also reassuring to see that Eqs. (5.19b) and (5.19c) are consistent with Eqs. (5.9a) and (5.9c), respectively. For example, operating on Eq. (5.19b) with the wave operator at x and using (5.15), (3.1) we recover Eq. (5.9a):

$$\Box G^{(1)(1)(2')(2')}=-\delta^4(x,x')+RG^{(1)(1)(2')(2')} . \quad (5.20)$$

This consistency depends crucially on the nonlocal term in the Ward identity. Introducing our standard representations for the Green's functions, the Ward identities (5.17b), (5.17c), (5.17f), (5.19b), and (5.19c) take the form

$$\left[\frac{1}{P(r)}\frac{\partial}{\partial r}-\frac{n\kappa}{r}+\frac{1}{rP(r)}\right]g^{(1)(3)(2')(4')}=-\frac{1}{2}\left[\frac{1}{P(r')}\frac{\partial}{\partial r'}+\frac{n\kappa}{r'}\right]\tilde{g}^{(1)(2')} , \quad (5.21a)$$

$$\left[\frac{1}{P(r)}\frac{\partial}{\partial r}-\frac{n\kappa}{r}+\frac{2}{rP(r)}\right]g^{(3)(3)(4')(4')}=-\left[\frac{1}{P(r')}\frac{\partial}{\partial r'}+\frac{n\kappa}{r'}-\frac{1}{r'P(r')}\right]\tilde{g}^{(3)(4')} , \quad (5.21b)$$

$$\left[\frac{1}{P(r)}\frac{\partial}{\partial r}+\frac{n\kappa}{r}\right]g^{(1)(2)(3')(4')}=-\frac{1}{2}\left[\frac{1}{P(r')}\frac{\partial}{\partial r'}-\frac{n\kappa}{r'}+\frac{1}{r'P(r')}\right]\tilde{g}^{(3)(4')}$$
$$-\int dr''r''P(r'')R(r'')\tilde{g}^{(3)(4'')}\left[\frac{1}{P(r'')}\frac{\partial}{\partial r''}+\frac{n\kappa}{r''}\right]g^{(1'')(2'')(3')(4')} . \quad (5.21c)$$

$$g^{(1)(1)(2')(2')}=-\tilde{g}^{(1)(2')}+\int dr''r''P(r'')R(r'')\tilde{g}^{(1)(2'')}g^{(1'')(1'')(2')(2')} , \quad (5.21d)$$

$$g^{(1)(3)(2')(4')}=-\tfrac{1}{2}\tilde{g}^{(3)(4')}-\int dr''r''P(r'')R(r'')\tilde{g}^{(3)(4'')}g^{(1'')(3'')(2')(4')} , \quad (5.21e)$$

Again to proceed further we need to make a specific choice for P and choose to work with the flower-pot model. For simplicity, we shall henceforth restrict ourselves to the range $1<\kappa<2$, which includes the realistic case of a cosmic string for which $\kappa\approx 1+4\times 10^{-6}$. We can then solve Eq. (5.14) to yield the interior solution which is regular at $r=0$ as

$$\Psi^{(3)(3)(4')(4')}_{<}(r)=\begin{cases} I_{|n-2|}(sr/\kappa) & \text{for } r<r_0 , \\ A_2 I_{|n\kappa-2|}(sr)+B_2 K_{|n\kappa-2|}(sr) & \text{for } r>r_0 , \end{cases} \quad (5.22)$$

where the jump condition at radius r_0 is

$$\lim_{r\to r_0+}\left[\frac{d}{dr}\Psi^{(3)(3)(4')(4')}_{<}(r)\right]-\kappa\lim_{r\to r_0-}\left[\frac{d}{dr}\Psi^{(3)(3)(4')(4')}_{<}(r)\right]=2\frac{\kappa-1}{r_0}\Psi^{(3)(3)(4')(4')}_{<}(r_0) . \quad (5.23)$$

The exterior solution which dies at infinity is

$$\Psi^{(3)(3)(4')(4')}_{>}(r)=\frac{1}{A_2}K_{|n\kappa-2|}(sr) \quad \text{for } r>r_0 . \quad (5.24)$$

Thus, in the region $r, r'>r_0$,

$$g_{n,\mathbf{k}}^{(3)(3)(4')(4')}=\Psi_<^{(3)(3)(4')(4')}(r_<)\Psi_>^{(3)(3)(4')(4')}(r_>)$$

$$=K_{|n\kappa-2|}(sr_>)[I_{|n\kappa-2|}(sr_<)+C_2(sr_0,n)K_{|n\kappa-2|}(sr_<)] , \tag{5.25}$$

where, as before, the ratio $C_2(sr_0,n)=B_2/A_2$ is determined by the jump condition (5.23) and continuity of $\Psi_<^{(3)(3)(4')(4')}$ at $r=r_0$ to be

$$C_2(x,n)=\frac{xI'_{|n\kappa-2|}(x)I_{|n-2|}(x/\kappa)-xI_{|n\kappa-2|}(x)I'_{|n-2|}(x/\kappa)-2(\kappa-1)I_{|n\kappa-2|}(x)I_{|n-2|}(x/\kappa)}{xK_{|n\kappa-2|}(x)I'_{|n-2|}(x/\kappa)-xK'_{|n\kappa-2|}(x)I_{|n-2|}(x/\kappa)+2(\kappa-1)K_{|n\kappa-2|}(x)I_{|n-2|}(x/\kappa)} . \tag{5.26}$$

Turning now to the ghost equation (5.16), we find that the interior solution which is regular at $r=0$ is

$$\tilde{\Psi}_<^{(3)(4')}(r)=\begin{cases}I_{|n-1|}(sr/\kappa) & \text{for } r<r_0 ,\\ \tilde{A}_1 I_{|n\kappa-1|}(sr)+\tilde{B}_1 K_{|n\kappa-1|}(sr) & \text{for } r>r_0 ,\end{cases} \tag{5.27}$$

where the jump condition at radius r_0 is

$$\lim_{r\to r_0+}\left[\frac{d}{dr}\tilde{\Psi}_<^{(3)(4')}(r)\right]-\kappa\lim_{r\to r_0-}\left[\frac{d}{dr}\tilde{\Psi}_<^{(3)(4')}(r)\right]=-\frac{\kappa-1}{r_0}\tilde{\Psi}_<^{(3)(4')}(r_0) . \tag{5.28}$$

The ghost exterior solution which dies at infinity is

$$\tilde{\Psi}_>^{(3)(4')}(r)=\frac{1}{\tilde{A}_1}K_{|n\kappa-1|}(sr) \quad \text{for } r>r_0 . \tag{5.29}$$

Note that the ghost jump equation (5.28) is identical to the electromagnetic jump equation (4.18) except for a change of sign on the right-hand side. Thus, for $r, r'>r_0$,

$$\tilde{g}_{n,\mathbf{k}}^{(3)(4')}=\tilde{\Psi}_<^{(3)(4')}(r_<)\tilde{\Psi}_>^{(3)(4')}(r_>)$$

$$=K_{|n\kappa-1|}(sr_>)[I_{|n\kappa-1|}(sr_<)+\tilde{C}_1(sr_0,n)K_{|n\kappa-1|}(sr_<)]$$

with $\tilde{C}_1(sr_0,n)=\tilde{B}_1/\tilde{A}_1$ determined by the jump condition (5.28) and continuity of $\tilde{\Psi}_<^{(3)(4')}$ at $r=r_0$ to be

$$\tilde{C}_1(x,n)=\frac{xI'_{|n\kappa-1|}(x)I_{|n-1|}(x/\kappa)-xI_{|n\kappa-1|}(x)I'_{|n-1|}(x/\kappa)+(\kappa-1)I_{|n\kappa-1|}(x)I_{|n-1|}(x/\kappa)}{xK_{|n\kappa-1|}(x)I'_{|n-1|}(x/\kappa)-xK'_{|n\kappa-1|}(x)I_{|n-1|}(x/\kappa)-(\kappa-1)K_{|n\kappa-1|}(x)I_{|n-1|}(x/\kappa)} . \tag{5.30}$$

The ghost and graviton Green's functions have now been completely determined on the flower pot. For the flower-pot model the Ward identities (5.21) take the form

$$\left[\frac{\partial}{\partial r}+\frac{1-n\kappa}{r}\right]g^{(1)(3)(2')(4')}=-\frac{1}{2}\left[\frac{\partial}{\partial r'}+\frac{n\kappa}{r'}\right]\tilde{g}^{(1)(2')} , \tag{5.31a}$$

$$\left[\frac{\partial}{\partial r}+\frac{2-n\kappa}{r}\right]g^{(3)(3)(4')(4')}=-\left[\frac{\partial}{\partial r'}+\frac{n\kappa-1}{r'}\right]\tilde{g}^{(3)(4')} , \tag{5.31b}$$

$$\left[\frac{\partial}{\partial r}+\frac{n\kappa}{r}\right]g^{(1)(2)(3')(4')}(r,r')=-\frac{1}{2}\left[\frac{\partial}{\partial r'}+\frac{(1-n\kappa)}{r'}\right]\tilde{g}^{(3)(4')}(r,r')$$

$$-2(\kappa-1)\tilde{g}^{(3)(4'')}(r,r_0)\lim_{r''\to r_0+}\left[\left[\frac{\partial}{\partial r''}+\frac{n\kappa}{r''}\right]g^{(1'')(2'')(3')(4')}(r'',r')\right] \tag{5.31c}$$

$$g^{(1)(1)(2')(2')}(r,r')=-\tilde{g}^{(1)(2')}(r,r')+2(\kappa-1)\tilde{g}^{(1)(2'')}(r,r_0)g^{(1'')(1'')(2')(2')}(r_0,r') , \tag{5.31d}$$

$$g^{(1)(3)(2')(4')}(r,r')=-\frac{1}{2}\tilde{g}^{(3)(4')}(r,r')-2(\kappa-1)\tilde{g}^{(3)(4'')}(r,r_0)g^{(1'')(3'')(2')(4')}(r_0,r') , \tag{5.31e}$$

for $r, r'>r_0$ [where we have used $R(r)=2(\kappa-1)r^{-1}P^{-1}(r)\delta(r-r_0)$]. Replacing the radial Green's functions occurring in (5.31) by their explicit representations in terms of Bessel functions [see, e.g., (5.25)] and making use of the relationships

$$\left[\frac{d}{dz}\pm\frac{\nu}{z}\right]I_\nu(z)=I_{\nu\mp1}(z) , \quad \left[\frac{d}{dz}\pm\frac{\nu}{z}\right]K_\nu(z)=-K_{\nu\mp1}(z) , \tag{5.32}$$

we find that the Ward identities (5.31) reduce to the readily verified identities

$$C_1=-C_0(\xi=0)\ , \tag{5.33a}$$

$$C_2=\begin{cases}-\widetilde{C}_1 & \text{for } n\neq 1\ ,\\ -\widetilde{C}_1+2\dfrac{\sin(\kappa-1)\pi}{\pi} & \text{for } n=1\ ,\end{cases} \tag{5.33b}$$

$$C_0(\xi=0)=-\widetilde{C}_1+2(\kappa-1)[I_{|n\kappa-1|}(sr_0)-C_0(\xi=0)K_{|n\kappa-1|}(sr_0)][I_{|n\kappa-1|}(sr_0)+\widetilde{C}_1K_{|n\kappa-1|}(sr_0)]\ . \tag{5.33c}$$

$$C_0(\xi=1)=C_0(\xi=0)-2(\kappa-1)[I_{\kappa|n|}(sr_0)+C_0(\xi=1)K_{\kappa|n|}(sr_0)][I_{\kappa|n|}(sr_0)+C_0(\xi=0)K_{\kappa|n|}(sr_0)]\ , \tag{5.33d}$$

$$C_1=\widetilde{C}_1-2(\kappa-1)[I_{|n\kappa-1|}(sr_0)+C_1K_{|n\kappa-1|}(sr_0)][I_{|n\kappa-1|}(sr_0)+\widetilde{C}_1K_{|n\kappa-1|}(sr_0)]\ , \tag{5.33e}$$

These equations are not independent, and Eq. (5.33a) was already noted in connection with the electromagnetic Ward identity.

We now evaluate the graviton and ghost Green's functions in the limit as $r_0\to 0$ corresponding to the idealized conical space-time. From (5.26) and (5.30) we deduce that in the limit as $x\to 0$, for $1<\kappa<2$,

$$C_2(x,n)\sim\begin{cases}\dfrac{1}{\Gamma(3-n\kappa)\Gamma(2-n\kappa)}\dfrac{4(\kappa-1)}{\kappa(n-2)}\left[\dfrac{x}{2}\right]^{2(2-n\kappa)} & \text{for } n\le 1\ ,\\ \dfrac{1}{\Gamma(n\kappa)\Gamma(n\kappa-1)}\dfrac{2(\kappa-1)}{\kappa(1-n)}\left[\dfrac{x}{2}\right]^{2(n\kappa-1)} & \text{for } n>1\ ,\end{cases} \tag{5.34}$$

and

$$\widetilde{C}_1(x,n)\sim\begin{cases}\dfrac{1}{\Gamma(3-n\kappa)\Gamma(2-n\kappa)}\dfrac{4(\kappa-1)}{\kappa(2-n)}\left[\dfrac{x}{2}\right]^{2(2-n\kappa)} & \text{for } n<1\ ,\\ 2\dfrac{\sin(\kappa-1)\pi}{\pi} & \text{for } n=1\ ,\\ \dfrac{1}{\Gamma(n\kappa)\Gamma(n\kappa-1)}\dfrac{2(\kappa-1)}{\kappa(n-1)}\left[\dfrac{x}{2}\right]^{2(n\kappa-1)} & \text{for } n>1\ ,\end{cases} \tag{5.35}$$

where the $n=1$ term in (5.35) has arisen from a circumstance identical to that discussed at the end of Sec. IV. [Note that (5.34) and (5.35) are consistent with the Ward identity (5.33b).] It is now clear that, although C_2 vanishes for all n in the ideal conical limit, the same is not true of $\widetilde{C}_1$; as a direct consequence of the crucial change in sign of the Ricci tensor term of the ghost equation (5.6) as compared to the electromagnetic equation (4.5), we see that for $n=1$ (only) $\widetilde{C}_1$ tends to the nonzero quantity

$$2\frac{\sin(\kappa-1)\pi}{\pi} \tag{5.36}$$

in the limit as $x\to 0$. Hence, for $n=1$,

$$\begin{aligned}\widetilde{\Psi}^{(3)(4')}_{<}(r)&=\widetilde{A}_1\left[I_{\kappa-1}(sr)+2\frac{\sin(\kappa-1)\pi}{\pi}K_{\kappa-1}(sr)\right]\\ &=\widetilde{A}_1I_{1-\kappa}(sr)\end{aligned} \tag{5.37}$$

in the ideal cone limit, which diverges at the origin as $r^{-(\kappa-1)}$. Bringing together these results one may now determine the graviton and ghost Green's functions in the ideal cone limit. In the graviton case one has

$$g^{(3)(3)(4')(4')}_{n,k}(r,r')=I_{|n\kappa-2|}(sr_<)K_{|n\kappa-2|}(sr_>) \tag{5.38}$$

and so we obtain

$$\begin{aligned}G_C^{(3)(3)(4')(4')}&=\frac{\kappa}{8\pi^2}\frac{1}{rr'\sinh\eta}\sum_{n=-\infty}^{\infty}e^{in\kappa\Delta\phi-|n\kappa-2|\eta}\\ &=\frac{\kappa}{8\pi^2}\frac{1}{rr'\sinh\eta}\left[\sum_{n=-\infty}^{1}e^{in\kappa\Delta\phi+(n\kappa-2)\eta}+\sum_{n=2}^{\infty}e^{in\kappa\Delta\phi-(n\kappa-2)\eta}\right]\end{aligned} \tag{5.39}$$

assuming $1<\kappa<2$. Performing the sum as before we finally find

$$G_C^{(3)(3)(4')(4')}=\frac{\kappa}{8\pi^2}\frac{1}{rr'\sinh\eta}\left[\frac{e^{i\kappa\Delta\phi}\sinh 2(\kappa-1)\eta+e^{2i\kappa\Delta\phi}\sinh(2-\kappa)\eta}{\cosh\kappa\eta-\cos\kappa\Delta\phi}\right]. \tag{5.40}$$

For the ghost field we have

$$\tilde{g}_{n,\mathbf{k}}^{(3)(4')}(r,r')=\begin{cases}I_{1-n\kappa}(sr_<)K_{1-n\kappa}(sr_>) & \text{for } n\le 1\ ,\\ I_{n\kappa-1}(sr_<)K_{n\kappa-1}(sr_>) & \text{for } n>1\end{cases} \tag{5.41}$$

and so

$$\tilde{G}_C^{(3)(4')}=\frac{\kappa}{8\pi^2}\frac{1}{rr'\sinh\eta}\left[\sum_{n=-\infty}^{1}e^{in\kappa\Delta\phi+(n\kappa-1)\eta}+\sum_{n=2}^{\infty}e^{in\kappa\Delta\phi-(n\kappa-1)\eta}\right]. \tag{5.42}$$

Comparing with (4.24), we see that the graviton ghost and electromagnetic Green's functions differ as follows:

$$\begin{aligned}\tilde{G}_C^{(3)(4')}&=G_C^{(3)(4')}+\frac{\kappa}{4\pi^2}\left[\frac{e^{i\kappa\Delta\phi}\sinh(\kappa-1)\eta}{rr'\sinh\eta}\right]\\&=\frac{\kappa}{8\pi^2}\frac{1}{rr'\sinh\eta}\left[\frac{e^{i\kappa\Delta\phi}\sinh(2\kappa-1)\eta-e^{2i\kappa\Delta\phi}\sinh(\kappa-1)\eta}{\cosh\kappa\eta-\cos\kappa\Delta\phi}\right].\end{aligned} \tag{5.43}$$

We stress that had we attempted the foregoing calculation without ever smoothing the conical singularity we would have obtained an incorrect graviton ghost propagator identical to the electromagnetic propagator and become unstuck when the Ward identities for the theory failed to be satisfied. One can perform this calculation *only* on a rounded conical metric.

For completeness we recall from Eqs. (5.10)–(5.12) and (5.15) that the other independent, nontrivial tetrad components of the graviton and ghost Green's functions in the ideal conical limit are

$$\begin{aligned}&G_C^{(1)(1)(2')(2')}=2G_C^{(1)(2)(3')(4')}=-\tilde{G}_C^{(1)(2')}=G_C\ ,\\&G_C^{(1)(3)(2')(4')}=-\tfrac{1}{2}G_C^{(3)(4')}\ ,\end{aligned} \tag{5.44}$$

where G_C and $G_C^{(3)(4')}$ are given explicitly by (3.17) and (4.25), respectively. Although these equations were derived under the assumption that $1<\kappa<2$, they satisfy the appropriate boundary conditions and are valid solutions to the graviton and ghost equations of motion for the entire range of κ.

We conclude this section by tabulating the Lorentzian space-time components of the graviton and ghost propagator in the ideal conical limit $r_0\to 0$. The 64 nonzero components of the graviton Feynman propagator

$$G^{\mu\nu\rho'\tau'}(x,x')=\frac{i}{32\pi G}\langle 0|T[h^{\mu\nu}(x)h^{\rho'\tau'}(x')]|0\rangle$$

are given by

$$\begin{aligned}G_C^{ttt't'}&=G_C^{ttz'z'}=G_C^{zzt't'}=G_C^{zzz'z'}=G_C^{ttr'r'}=G_C^{rrt't'}=r^2G_C^{\phi\phi t't'}=r'^2G_C^{tt\phi'\phi'}\\&=-G_C^{tzt'z'}=-G_C^{tzz't'}=-G_C^{ztt'z'}=-G_C^{ztz't'}=-G_C^{rrz'z'}=-G_C^{zzr'r'}\\&=-r^2G_C^{\phi\phi z'z'}=-r'^2G_C^{zz\phi'\phi'}=-\tfrac{1}{2}G_C^{tt'}=\frac{i\kappa}{16\pi^2}\frac{1}{rr'\sinh\eta}\left[\frac{\sinh\kappa\eta}{\cosh\kappa\eta-\cos\kappa\Delta\phi}\right],\end{aligned}$$

$$\begin{aligned}G_C^{rzr'z'}&=G_C^{rzz'r'}=G_C^{zrr'z'}=G_C^{zrz'r'}=rr'G_C^{\phi z\phi'z'}=rr'G_C^{\phi zz'\phi'}=rr'G_C^{z\phi\phi'z'}\\&=rr'G_C^{z\phi z'\phi'}=-G_C^{trt'r'}=-G_C^{trr't'}=-G_C^{rtt'r'}=-G_C^{rtr't'}=-rr'G_C^{t\phi t'\phi'}\\&=-rr'G_C^{t\phi\phi't'}=-rr'G_C^{\phi tt'\phi'}=-rr'G_C^{\phi t\phi't'}=\tfrac{1}{2}G_C^{rr'}\\&=\frac{i\kappa}{16\pi^2}\frac{1}{rr'\sinh\eta}\left[\frac{\sinh\eta\cos\kappa\Delta\phi+\sinh(\kappa-1)\eta}{\cosh\kappa\eta-\cos\kappa\Delta\phi}\right],\end{aligned}$$

$$\begin{aligned}r'G_C^{rz\phi'z'}&=r'G_C^{rzz'\phi'}=r'G_C^{zr\phi'z'}=r'G_C^{zrz'\phi'}=rG_C^{t\phi t'r'}=rG_C^{t\phi r't'}=rG_C^{\phi tt'r'}=rG_C^{\phi tr't'}\\&=-r'G_C^{trt'\phi'}=-r'G_C^{tr\phi't'}=-r'G_C^{rtt'\phi'}=-r'G_C^{rt\phi't'}=-rG_C^{\phi zr'z'}=-rG_C^{\phi zz'r'}\\&=-rG_C^{z\phi r'z'}=rG_C^{z\phi z'r'}=\frac{r'}{2}G_C^{r\phi'}=\frac{i\kappa}{16\pi^2}\frac{1}{rr'\sinh\eta}\left[\frac{\sinh\eta\sin\kappa\Delta\phi}{\cosh\kappa\eta-\cos\kappa\Delta\phi}\right],\end{aligned}$$

$$G_C^{rrr'r'}=r^2r'^2G_C^{\phi\phi\phi'\phi'}=rr'G_C^{r\phi r'\phi'}=rr'G_C^{r\phi\phi'r'}=rr'G_C^{\phi rr'\phi'}=rr'G_C^{\phi r\phi'r'}=-r'^2G_C^{rr\phi'\phi'}=-r^2G_C^{\phi\phi r'r'}$$
$$=\frac{i\kappa}{16\pi^2}\frac{1}{rr'\sinh\eta}\left[\frac{\sinh 2(\kappa-1)\eta\cos\kappa\Delta\phi+\sinh(2-\kappa)\eta\cos 2\kappa\Delta\phi}{\cosh\kappa\eta-\cos\kappa\Delta\phi}\right],$$

$$r'G_C^{rrr'\phi'}=r'G_C^{rr\phi'r'}=rr'^2G_C^{r\phi\phi'\phi'}=rr'^2G_C^{\phi r\phi'\phi'}=-rG_C^{r\phi r'r'}=-rG_C^{\phi rr'r'}=-r^2r'G_C^{\phi\phi r'\phi'}=-r^2r'G_C^{\phi\phi\phi'r'}$$
$$=\frac{i\kappa}{16\pi^2}\frac{1}{rr'\sinh\eta}\left[\frac{\sinh 2(\kappa-1)\eta\sin\kappa\Delta\phi+\sinh(2-\kappa)\eta\sin 2\kappa\Delta\phi}{\cosh\kappa\eta-\cos\kappa\Delta\phi}\right],$$

where $G_C^{\mu\nu'}$ is the electromagnetic Green's function (4.26). The nonzero components of the graviton ghost Feynman function

$$\tilde{G}^{\nu\rho'}(x,x')=\frac{i}{32\pi G}\langle 0|T[\bar{c}^{\nu}(x)c^{\rho'}(x')]|0\rangle$$

are

$$\tilde{G}_C^{tt'}=-\tilde{G}_C^{zz'}=-G_C=-\frac{i\kappa}{8\pi^2}\frac{1}{rr'\sinh\eta}\left[\frac{\sinh\kappa\eta}{\cosh\kappa\eta-\cos\kappa\Delta\phi}\right],$$

$$\tilde{G}_C^{rr'}=rr'\tilde{G}_C^{\phi\phi'}=\frac{i\kappa}{8\pi^2}\frac{1}{rr'\sinh\eta}\left[\frac{\sinh(2\kappa-1)\eta\cos\kappa\Delta\phi-\sinh(\kappa-1)\eta\cos 2\kappa\Delta\phi}{\cosh\kappa\eta-\cos\kappa\Delta\phi}\right],$$

$$r'\tilde{G}_C^{r\phi'}=-r\tilde{G}_C^{\phi r'}=\frac{i\kappa}{8\pi^2}\frac{1}{rr'\sinh\eta}\left[\frac{\sinh(2\kappa-1)\eta\sin\kappa\Delta\phi-\sinh(\kappa-1)\eta\sin 2\kappa\Delta\phi}{\cosh\kappa\eta-\cos\kappa\Delta\phi}\right],$$

where G_C is the scalar Green's function (3.18). Note that, in these Lorentzian formulas, η is given by Eq. (3.18b).

Having obtained the above explicit formulas for the Lorentzian Green's functions, it is a straightforward, if laborious, procedure to determine the renormalized vacuum expectation value of the stress-energy tensor for the system. As before, it proves most economical to return to the tetrad formalism in order to compute the regular parts of the Green's functions

$$W^{(a)(b)(c)(d)}=-8\pi^2 ig^{(c)}{}_{(c')}g^{(d)}{}_{(d')}[G_C^{(a)(b)(c')(d')}-(G_C^{(a)(b)(c')(d')})_{\kappa=1}] \tag{5.45a}$$

and

$$\tilde{W}^{(a)(b)}=-8\pi^2 ig^{(b)}{}_{(b')}[\tilde{G}_C^{(a)(b')}-(\tilde{G}_C^{(a)(b')})_{\kappa=1}]\ . \tag{5.45b}$$

The next step is to determine the first three coefficients occurring in the covariant Taylor expansion of each of $W^{(a)(b)(c)(d)}$ and $\tilde{W}^{(a)(b)}$. This task is facilitated by first writing (5.45) as a power series in the coordinate differences Δx^{μ}; we obtain

$$W^{(1)(1)(2)(2)}=W^{(2)(2)(1)(1)}=2W^{(1)(2)(3)(4)}=2W^{(3)(4)(1)(2)}=W\ ,$$
$$W^{(1)(3)(2)(4)}=W^{(2)(3)(1)(4)}=-\tfrac{1}{2}W^{(3)(4)}\ , \tag{5.46a}$$
$$W^{(3)(3)(4)(4)}=\frac{\kappa-1}{360r^2}\Bigg[60(13\kappa-23)+60(13\kappa-23)\frac{\Delta r}{r}+240i(\kappa-2)(3\kappa-4)\Delta\phi$$
$$+(119\kappa^3-601\kappa^2+829\kappa-251)\frac{-(\Delta t)^2+(\Delta z)^2}{r^2}+(119\kappa^3-601\kappa^2+1609\kappa-1631)\frac{(\Delta r)^2}{r^2}$$
$$+240i(\kappa-2)(3\kappa-4)\frac{\Delta r\Delta\phi}{r}-3(119\kappa^3-601\kappa^2+959\kappa-481)(\Delta\phi)^2+O((\Delta x^{\mu})^3)\Bigg],$$

and

$$\tilde{W}^{(1)(2)}=\tilde{W}^{(2)(1)}=-W\ ,$$
$$\tilde{W}^{(3)(4)}=W^{(3)(4)}+\frac{2\kappa(\kappa-1)}{r^2}\Bigg[1+\frac{\Delta r}{r}+i(\kappa-1)\Delta\phi+\kappa(\kappa-2)\frac{-(\Delta t)^2+(\Delta z)^2}{6r^2} \tag{5.46b}$$
$$+[\kappa(\kappa-2)+6]\frac{(\Delta r)^2}{6r^2}+i(\kappa-1)\frac{\Delta r\Delta\phi}{r}-(\kappa-1)^2\frac{(\Delta\phi)^2}{2}+O((\Delta x^{\mu})^3)\Bigg]$$

with the only other nonvanishing components being those obtainable from symmetries $W^{(a)(b)(c)(d)}=W^{((a)(b))((c)(d))}$ and the interchange of (3) and (4) under complex conjugation. In (5.46), W and $W^{(3)(4)}$ are to be replaced by the expansions (3.21) and (4.33b), respectively. By substituting these power series into the formulas

$$w^{(a)(b)(c)(d)}(x)=\lim_{x\to x'}[W^{(a)(b)(c)(d)}(x,x')]\ ,$$
$$w^{(a)(b)(c)(d)}{}_{\mu}(x)=\lim_{x\to x'}[-g_{\mu}{}^{\rho'}(x,x')W^{(a)(b)(c)(d)}(x,x')_{;\rho'}]\ ,$$
$$w^{(a)(b)(c)(d)}{}_{\mu\nu}(x)=\lim_{x\to x'}[g_{\mu}{}^{\rho'}(x,x')g_{\nu}{}^{\tau'}(x,x')\times W^{(a)(b)(c)(d)}(x,x')_{;\rho'\tau'}]\ , \tag{5.47a}$$

and

$$\tilde{w}^{(a)(b)}(x)=\lim_{x\to x'}[\tilde{W}^{(a)(b)}(x,x')]\ ,$$
$$\tilde{w}^{(a)(b)}{}_{\mu}(x)=\lim_{x\to x'}[-g_{\mu}{}^{\rho'}(x,x')\tilde{W}^{(a)(b)}(x,x')_{;\rho'}]\ ,$$
$$\tilde{w}^{(a)(b)}{}_{\mu\nu}(x)=\lim_{x\to x'}[g_{\mu}{}^{\rho'}(x,x')g_{\nu}{}^{\tau'}(x,x')\times \tilde{W}^{(a)(b)}(x,x')_{;\rho'\tau'}]\ , \tag{5.47b}$$

we quickly find that the required coefficients are

$$\begin{aligned}
w^{(1)(1)(2)(2)}{}_{\alpha\cdots\beta}&=w^{(2)(2)(1)(1)}{}_{\alpha\cdots\beta}\\
&=2w^{(1)(2)(3)(4)}{}_{\alpha\cdots\beta}\\
&=2w^{(3)(4)(1)(2)}{}_{\alpha\cdots\beta}=w_{\alpha\cdots\beta}\ ,\\
w^{(1)(3)(2)(4)}{}_{\alpha\cdots\beta}&=w^{(2)(3)(1)(4)}{}_{\alpha\cdots\beta}\\
&=-\tfrac{1}{2}w^{(3)(4)}{}_{\alpha\cdots\beta}\ ,\\
w^{(3)(3)(4)(4)}&=\frac{1}{6r^2}(\kappa-1)(13\kappa-23)\ ,\\
w^{(3)(3)(4)(4)}{}_{r}&=\frac{1}{6r^3}(\kappa-1)(13\kappa-23)\ ,\\
w^{(3)(3)(4)(4)}{}_{\phi}&=\frac{2i}{3r^2}(\kappa-1)(\kappa-2)(3\kappa-4)\ ,\\
w^{(3)(3)(4)(4)}{}_{tt}&=-w^{(3)(3)(4)(4)}{}_{zz}\\
&=-\frac{1}{180r^4}(\kappa-1)(119\kappa^3-601\kappa^2+829\kappa-251)\ ,\\
w^{(3)(3)(4)(4)}{}_{rr}&=\frac{1}{180r^4}(\kappa-1)(119\kappa^3-601\kappa^2+1609\kappa-1631)\ ,\\
w^{(3)(3)(4)(4)}{}_{r\phi}&=w^{(3)(3)(4)(4)}{}_{\phi r}\\
&=\frac{4i}{3r^3}(\kappa-1)(\kappa-2)(3\kappa-4)\ ,\\
w^{(3)(3)(4)(4)}{}_{\phi\phi}&=-\frac{1}{60r^2}(\kappa-1)(119\kappa^3-601\kappa^2+1089\kappa-711)\ ,
\end{aligned} \tag{5.48a}$$

and

$$\begin{aligned}
\tilde{w}^{(1)(2)}{}_{\alpha\cdots\beta}&=\tilde{w}^{(2)(1)}{}_{\alpha\cdots\beta}=-\tilde{w}_{\alpha\cdots\beta}\ ,\\
\tilde{w}^{(3)(4)}&=w^{(3)(4)}+\frac{2\kappa(\kappa-1)}{r^2}\ ,\\
\tilde{w}^{(3)(4)}{}_{r}&=w^{(3)(4)}{}_{r}+\frac{2\kappa(\kappa-1)}{r^3}\ ,\\
\tilde{w}^{(3)(4)}{}_{\phi}&=w^{(3)(4)}{}_{\phi}+\frac{2i\kappa(\kappa-1)^2}{r^2}\ ,\\
\tilde{w}^{(3)(4)}{}_{tt}&=-\tilde{w}^{(3)(4)}{}_{zz}\\
&=w^{(3)(4)}{}_{tt}-\frac{2\kappa^2(\kappa-1)(\kappa-2)}{3r^4}\ ,\\
\tilde{w}^{(3)(4)}{}_{rr}&=w^{(3)(4)}{}_{rr}+\frac{2\kappa(\kappa-1)(\kappa^2-2\kappa+6)}{3r^4}\ ,\\
\tilde{w}^{(3)(4)}{}_{r\phi}&=\tilde{w}^{(3)(4)}{}_{\phi r}=w^{(3)(4)}{}_{r\phi}+\frac{4i\kappa(\kappa-1)^2}{r^3}\ ,\\
\tilde{w}^{(3)(4)}{}_{\phi\phi}&=w^{(3)(4)}{}_{\phi\phi}-\frac{2\kappa(\kappa-1)(\kappa^2-2\kappa+2)}{r^2}
\end{aligned} \tag{5.48b}$$

[with the only other nonvanishing components being those obtainable from the above via

$$w^{(a)(b)(c)(d)}{}_{\alpha\cdots\beta}=w^{((a)(b))((c)(d))}{}_{\alpha\cdots\beta}$$

and complex conjugation which interchanges (3) and (4)]. It remains to determine

$$\begin{aligned}
s^{\mu\nu\rho\tau}{}_{\alpha\cdots\beta}&=\tfrac{1}{2}(e^{\mu}_{(a)}e^{\nu}_{(b)}e^{\rho}_{(c)}e^{\tau}_{(d)}+e^{\rho}_{(a)}e^{\tau}_{(b)}e^{\mu}_{(c)}e^{\nu}_{(d)})\times w^{(a)(b)(c)(d)}{}_{\alpha\cdots\beta}\ ,\\
a^{\mu\nu\rho\tau}{}_{\alpha\cdots\beta}&=\tfrac{1}{2}(e^{\mu}_{(a)}e^{\nu}_{(b)}e^{\rho}_{(c)}e^{\tau}_{(d)}-e^{\rho}_{(a)}e^{\tau}_{(b)}e^{\mu}_{(c)}e^{\nu}_{(d)})\times w^{(a)(b)(c)(d)}{}_{\alpha\cdots\beta}\ ,
\end{aligned} \tag{5.49a}$$

$$\begin{aligned}
\tilde{s}^{\mu\nu}{}_{\alpha\cdots\beta}&=\tfrac{1}{2}(e^{\mu}_{(a)}e^{\nu}_{(b)}+e^{\nu}_{(a)}e^{\mu}_{(b)})\tilde{w}^{(a)(b)}{}_{\alpha\cdots\beta}\ ,\\
\tilde{a}^{\mu\nu}{}_{\alpha\cdots\beta}&=\tfrac{1}{2}(e^{\mu}_{(a)}e^{\nu}_{(b)}-e^{\nu}_{(a)}e^{\mu}_{(b)})\tilde{w}^{(a)(b)}{}_{\alpha\cdots\beta}\ ,
\end{aligned} \tag{5.49b}$$

check that all is well by verifying the identities [18]

$$\begin{aligned}
s^{\mu\nu\rho\tau\alpha}{}_{\alpha}&=a^{\mu\nu\rho\tau\alpha}{}_{;\alpha}=0\ ,\\
s^{\mu\nu\rho\tau\alpha\beta}{}_{;\beta}&=\tfrac{1}{4}\Box(s^{\mu\nu\rho\tau;\alpha})\ ,
\end{aligned} \tag{5.50a}$$

$$\begin{aligned}
\tilde{s}^{\mu\nu\alpha}{}_{\alpha}&=\tilde{a}^{\mu\nu\alpha}{}_{;\alpha}=0\ ,\\
\tilde{s}^{\mu\nu\alpha\beta}{}_{;\beta}&=\tfrac{1}{4}\Box(\tilde{s}^{\mu\nu;\alpha})
\end{aligned} \tag{5.50b}$$

[which arise from the wave equations (5.5) and (5.6)], and finally insert our explicit expressions for the coefficients (5.49) into the relevant formulas of Ref. [12] for the renormalized stress-energy tensor, being careful to choose those formulas which have not yet been simplified by means of Ward identities (since as we have seen these identities are invalid on the idealized space-time). We find that the renormalized vacuum expectation value of the stress-energy tensor for the linearized gravitational field in the vicinity of an idealized cosmic string is given by

$$\langle 0|T_{\mu}{}^{\nu}|0\rangle_R=\frac{\kappa-1}{720\pi^2r^4}\ \mathrm{diag}\ [f(\kappa),g(\kappa),-3g(\kappa),f(\kappa)]_{\mu}{}^{\nu}\ , \tag{5.51a}$$

where $f(\kappa)$ and $g(\kappa)$ are the polynomials

$$\begin{aligned}
f(\kappa)&=121\kappa^3+121\kappa^2+1421\kappa-1459\ ,\\
g(\kappa)&=121\kappa^3+121\kappa^2-829\kappa+251\ .
\end{aligned} \tag{5.51b}$$

VI. THE SCALAR FIELD REEXAMINED

We have seen in the preceding two sections that we can be forced to introduce solutions of the wave equation which are singular at $r=0$. In light of this, it is worth reexamining the case of scalar fields on the idealized string space-time allowing such behavior. In particular, let us allow the $n=0$ mode of the Green's function on the ideal cone to have the form

$$g_{0,\mathbf{k}}(r,r')=K_0(sr_>)[I_0(sr_<)+AK_0(sr_<)] , \tag{6.1}$$

where A is a constant that does not depend on $\mathbf{k}$. For $\kappa=1$, $A\neq 0$ is ruled out by regularity, but for $\kappa\neq 1$ we have seen above that there is no reason not to allow such (mild) singular behavior.

Let us denote the corresponding Euclidean Green's function by G^A and recall from Sec. III that

$$G^0(x,x')=\frac{\kappa}{8\pi^2}\frac{1}{rr'\sinh\eta}\frac{\sinh\kappa\eta}{\cosh\kappa\eta-\cos\kappa\Delta\phi} , \tag{6.2}$$

where η is the non-negative solution of

$$\cosh\eta=\frac{(\Delta t)^2+(\Delta z)^2+r^2+r'^2}{2rr'} . \tag{6.3}$$

A straightforward calculation (using GR 6.578.10 and 8.715.1) then reveals that

$$G^A(x,x')=G^0(x,x')+A\frac{\kappa}{8\pi^2}\frac{\eta}{rr'\sinh\eta} . \tag{6.4}$$

In contrast with the case when $A=0$, for $A\neq 0$ this Green's function diverges logarithmically when one of the points approaches the string.

It is again straightforward to determine that the renormalized expectation value of φ^2 is

$$\langle\varphi^2\rangle^A=\langle\varphi^2\rangle^0+A\frac{\kappa}{8\pi^2r^2}=\frac{\kappa^2+6A\kappa-1}{48\pi^2r^2} \tag{6.5}$$

and that the renormalized stress tensor is

$$\begin{aligned}\langle T_\mu{}^\nu\rangle^A&=\langle T_\mu{}^\nu\rangle^0+A\frac{\kappa}{24\pi^2r^4}(6\xi-1)\,\mathrm{diag}\,(2,-1,3,2)_\mu{}^\nu\\&=\frac{1}{1440\pi^2r^4}[(\kappa^4-1)\,\mathrm{diag}\,(1,1,-3,1)_\mu{}^\nu+10(\kappa^2+6A\kappa-1)(6\xi-1)\,\mathrm{diag}\,(2,-1,3,2)_\mu{}^\nu] .\end{aligned} \tag{6.6}$$

For $A=0$ this is in agreement with Eq. (3.26).

VII. CONCLUSIONS

The major lesson of this work is that the δ-function curvature singularities present in the space-time geometry of an ideal cosmic string cannot be ignored in certain cases of physical interest: they can couple to the quantum fields around the string.

For the case of a scalar field propagating with equation of motion $(-\Box+\xi R)\varphi=0$ in the vicinity of a cosmic string of finite thickness, the effects of the coupling term do disappear in the limit as the core radius r_0 tends to zero, albeit much more slowly than might naively be expected [9,20]. More precisely, for all values of the coupling constant ξ, the Feynman two-point function G_ξ becomes independent of ξ in the limit as $r_0\to 0$.

The same is not true of the vector wave equation $(-\Box A_\mu+\xi R_\mu{}^\nu A_\nu)=0$ for all values of ξ however. Whereas the $\xi=1$ case (corresponding to the electromagnetic field) does yield the same Feynman propagator as the case $\xi=0$ in the ideal conical limit, we have shown that, when $\xi=-1$ (corresponding to the graviton ghost), the coupling of the ghost field to the background curvature changes the Green's function from what it would be for $\xi=0$ *even in the limit as* $r_0\to 0$. This change is shown explicitly in Eq. (5.43) and would have been missed had we not carried out our calculations as the limiting case of a rounded cosmic string.

Finally, it is instructive to compare the expressions we have obtained for the renormalized stress-energy tensors of all three quantum fields considered in this paper. Realistic cosmic strings have mass per unit length $\mu\sim 10^{-6}$, so we can write $\kappa=(1-4\mu)^{-1}\approx 1+4\mu$ in Eqs. (3.26), (4.38), and (5.51) and obtain the following approximate results for the energy densities of the quantum fields:

$$-\langle 0|T_t{}^t|0\rangle_R\approx-\frac{\mu}{90\pi^2r^4}\begin{cases}1 & \text{spin } 0\ (\xi=\frac{1}{6}) ,\\ 12 & \text{spin } 1 ,\\ 102 & \text{spin } 2 .\end{cases}$$

This result lends further weight to the conjecture that the influence of nonconformally invariant quantum fields strongly dominates that of conformally invariant fields as one approaches a space-time singularity (see [19] for already existing evidence of this conjecture) and so encourages an investigation of the behavior of the linearized gravitational field in the neighborhood of other physically interesting space-time singularities, such as the spherical singularity formed by a collapsing star or the initial singularity occurring in cosmological space-times.

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