Numerical treatment of the spherically symmetric general-relativistic Boltzmann equation for massless and massive particles

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The Arnowitt-Deser-Misner formalism is used to write the Einstein-Boltzmann coupled system of equations. The sources of gravitational field are represented by ordinary matter described by a perfect-fluid approximation together with a particle gas described by a phase-space distribution function obeying the general-relativistic Boltzmann transport equation. Through the use of the Liouville operator in phase space, we obtain a form of the Boltzmann equation that makes it very amenable for numerical treatment. The resulting system of equations can be used for the numerical study of either massless or massive particles interacting with ordinary matter.

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I. INTRODUCTION

Until recently, most studies including general-relativistic effects in cosmology and astrophysics have been carried out within the context of homogeneous and isotropic models. This is not difficult to understand, since the more realistic *inhomogeneous* models are highly complicated, and generally require the solution of partial differential equations, which are far less amenable to analytic solution than are the ordinary differential equations encountered in homogeneous and/or steady-state situations. However, the computational tools available today are powerful enough for us to consider seriously the *numerical* study of the inhomogeneous models. The challenge offered us by these models, together with the desirability of including WIMP's (weakly interacting massive particles, here understood to include massive neutrinos) in them, provided the initial motivation for this work. It is commonly accepted that WIMP's constitute a basic, if not the most important, component of the often-discussed "dark matter" content of the Universe [1–6]. It is also thought that these particles played a most relevant role in the early dynamics of the Universe, with ordinary matter following the WIMP dynamics during, e.g., nucleosynthesis [7,8] and galaxy formation [9–11].

Two distinct types of gravitational field sources are to be considered in this paper: ordinary matter, which will be described by means of a perfect-fluid approximation [12–16], and a gas of particles, either massless (radiation) or massive (e.g., WIMP's), interacting only gravitationally with the fluid, for which a kinetic-theory approach will be used [17–19]. We propose to treat the gas as a collection of identical particles, so they can be described by a phase-space distribution function obeying the general-relativistic Boltzmann transport equation, i.e., we use a kinetic-theory description for their study. This approach is most suitable when the particles undergo only a few or no collisions in the processes in which they play a role. Since WIMP's have very long mean free paths during both the radiation- and the matter-dominated epochs of the Universe, as well as in the (noncosmological) astrophysical situations in which they may take part, the kinetic-theory description is indeed appropriate. It also presents a number of advantages over the conventional fluid description that make its adoption extremely attractive. For example, it naturally incorporates the particle structure of matter, allowing a unified treatment of systems of particles with positive rest mass and those consisting of zero-rest-mass particles (radiation); it offers a straightforward way to complete the system of Einstein field equations with the introduction of the Liouville, or Boltzmann, equation, in such a way that a deterministic model for gravitating material is obtained; and, it forms the basis for relativistic thermodynamics of equilibrium and nonequilibrium systems, e.g., transport processes. Furthermore, the assumptions made in kinetic theory seem reasonably appropriate for a number of real systems, from the very large, such as systems of galaxies, to the very small, such as systems of weakly interacting particles and radiation.

Different topics in relativistic kinetic theory have been studied extensively over the years. Synge [20] gives an exhaustive treatment of kinetic theory in special relativity. Tauber and Weinberg [21], and, independently, Chernikov [22], develop a coordinate-invariant formulation of kinetic theory which could be applied to the curved space times of general relativity. Lindquist [23] develops a general-relativistic form of the Boltzmann equation for particles or radiation interacting with an external medium. Israel [24] presents a review of the elementary properties of the relativistic Boltzmann equation and describes a simple method for the approximate evaluation of relativistic transport coefficients using the moments of the distribution function. From a more mathematical point of view, Ehlers [17] presents an excellent review of the general-relativistic kinetic theory of gases.

Other researchers have studied the subject from a more practical viewpoint. Based on the theoretical framework of Lindquist [23], Wilson [18] studied some aspects of the neutrino transport problem in general relativity by solving the Einstein-Boltzmann equations numerically for axisymmetric space times. Shapiro and Teukolsky [25,26] investigated the dynamical evolution of a collisionless gas of identical particles in general relativity by solving the Vlasov equation for an N-body *particle* simulation which involves the computation of moments of the distribution function, but not the explicit distribution function itself. More recently, Rasio, Shapiro, and Teukolsky [27] have introduced a new numerical method for determining the dynamical evolution of a collisionless system in general relativity. They solved the collisionless Boltzmann (or Vlasov) equation coupled to the Einstein equations for the gravitational field in spherical symmetry. Their approach is different from that of Wilson [18] in that they exploited Liouville's theorem to determine the evolution of the distribution function in phase space, while Wilson actually solved the full Boltzmann equation numerically. Mezzacappa and Matzner [19] studied massless particle (radiation) transport in general relativity by solving the coupled Einstein-Boltzmann system of equations through the introduction of a new *implicit* numerical scheme, the "implicit bordering" method, which appears to be a very promising tool for future research in the subject.

The principal results of our work in this field are twofold: First, based on the work of Lindquist [23] and Wilson [18], we are able to write the Boltzmann equation in a form that lends itself naturally to numerical treatment using the well-known Barton method for numerical transport [14,15]. Our form of the Boltzmann equation will be discussed in full detail in Sec. IV. Second, we generalize the results obtained by Mezzacappa and Matzner [19] to include *massive* particles. The massless particle (radiation) case can then be easily obtained as a special case from the resulting equations.

The Boltzmann equation is coupled to the Einstein equations for the gravitational field so that when the conservation laws for energy-momentum and matter are included, together with an equation of state for the matter variables, we obtain a closed, self-consistent system of equations. The "(3+1)" [or Arnowitt-Deser-Misner (ADM)] formalism [28] is used to implement Einstein's equations as a Cauchy problem. For the sake of completeness, we present a short review of the ADM method in Sec. II, making no assumptions about the explicit form of the metric, except that its signature be $(-,+,+,+)$. We also review, in that section, some general geometrical aspects of space time that will prove useful in the application of the $(3+1)$ formalism.

In Sec. III both source descriptions, as well as the conservation laws the sources must obey, are presented in their most general form. The general equations obtained in Secs. II and III are then specialized to the spherically symmetric case in Sec. IV. The Boltzmann equation is written in a *conservative* form [29] which makes it very amenable to numerical treatment. Section V contains our conclusions.

A computer code that can handle both massless and massive particles was constructed to solve the Einstein-Boltzmann system of equations in spherical symmetry. A description of the general structure of this code, together with a detailed discussion of particular aspects of the coding and the tests that have been carried out with the code so far, are the subject of another paper [30].

II. GEOMETRY

A. The ADM formalism

One of the most useful formalisms available to solve Einstein's equations for the gravitational field, especially from a numerical point of view, is the $(3+1)$ method through which one can recast the Einstein equations by describing space time in terms of a Cauchy, or initial-value, problem. The gravitational field is then considered to be the time history of the geometry of a spacelike three-dimensional hypersurface on which the initial-value problem is solved; a reference system is prescribed, and the dynamical equations are then integrated along the trajectories of the chosen reference system. In addition, one has to take into account any external sources, their evolution equations and equations of state that will complete the description of the situation at hand.

In this section, we will review the ADM formalism as it is used to implement the Cauchy problem for gravitation in its most general form. We will also explore some geometrical aspects of space time that will be helpful in the actual application of the formalism. The analysis presented below follows closely the papers by York [31] and Holcomb [16]. No approximation schemes will be used, as the full Einstein equations will be adopted throughout this work. We will adopt the following conventions: Latin indices from the beginning of the alphabet run from 0 to 3, while indices from the middle of the alphabet run from 1 to 3; the Einstein summation convention is used.

In the ADM formalism space time is decomposed into a family of three-surfaces Σ which are, locally, level surfaces of a scalar function, which we can denote by τ. This family of surfaces is called a foliation, $\{\Sigma\}$. In order for us to specify completely the embedding of the foliation within the larger four-dimensional space time, we need the three-metric on the slices and the extrinsic curvature. The three-metric is defined as

$$\gamma_{ab}=g_{ab}+n_a n_b \ , \tag{2.1}$$

where n_a is the unit normal vector of the slices with $n_a n^a=-1$; this unit vector is timelike and future pointing. The vector $c\mathbf{n}$, where c is the speed of light, can be naturally interpreted as the four-velocity field of the observers that are instantaneously at rest in the slices Σ. These observers are usually referred to as the *Eulerian observers.* The extrinsic curvature is given by

$$K_{ab}=-\tfrac{1}{2}\mathcal{L}_n\gamma_{ab}=-\gamma_a{}^c\gamma_b{}^d\nabla_{(c}n_{d)} \ , \tag{2.2}$$

where $\mathcal{L}_n$ is the Lie derivative along the vector field $\mathbf{n}$. $\mathbf{K}$ may be considered a "velocity" of the spatial metric with respect to the local proper time of the Eulerian observers on a slice. The triple $(\Sigma,\gamma,\mathbf{K})$ represents the initial data in our initial-value problem. From Eq. (2.2), it follows immediately that

$$K \equiv g^{ab}K_{ab} = \gamma^{ab}K_{ab} = -\nabla_a n^a \ , \tag{2.3}$$

i.e., the trace of the extrinsic curvature is the negative of the expansion of the unit normal vector field.

We now bring in the physics through the Einstein equations,

$$G_{ab} = \kappa T_{ab} \ , \tag{2.4}$$

where $\kappa = 8\pi G/c^4$ is the gravitational constant and T_{ab} is the stress-energy tensor for the sources of the gravitational field. In the spirit of the ADM formalism [28], these equations may be recast into four constraint equations which must be satisfied on each slice of the foliation, and six second-order dynamical equations that will refer to the foliation itself, rather than just to the individual slices, so we can use them as evolution equations for the initial data. The constraint equations, which are obtained from the Gauss-Codazzi equations, relate the geometry of the slices to the gravitational source variables and may be written as

$$^{(3)}R + K^2 - \mathbf{K}^2 = 2\kappa\rho_H \tag{2.5}$$

and

$$D_b K^b{}_a - D_a K = \kappa S_a \ , \tag{2.6}$$

where $^{(3)}R$ is the three-scalar curvature, $\mathbf{D}$ is the covariant three-derivative operator, and $\rho_H \equiv n^a n^b T_{ab}$ and $S_a \equiv -\gamma_a{}^b n^c T_{bc}$ are, respectively, the "Hamiltonian" energy density and the momentum flux vector, as seen by the Eulerian observers. Equation (2.5) is the Hamiltonian constraint while the three equations (2.6) are the momentum constraints.

The orthogonal *proper* time interval between surfaces τ and $\tau+\delta\tau$ is $\alpha\delta\tau$, where α is the "lapse" function, therefore $N^a = \alpha n^a$ *is* the natural (orthogonal) vector field connecting the slices with $\alpha = (-g_{ab}N^a N^b)^{1/2}$. One can easily verify that $\mathcal{L}_N \gamma^a{}_b = 0$, and so the time-derivative operator, $\mathcal{L}_N$, when applied to any spatial tensor, will itself be a spatial tensor; here, we have $\mathcal{L}_N = \alpha\mathcal{L}_n$. However, the time vector N is *not* unique. We can choose any vector $\mathbf{t}$ of the form

$$t^a = N^a + \beta^a \ , \tag{2.7}$$

with

$$\beta^a n_a = 0 \ . \tag{2.8}$$

Hence we have an *arbitrary* spatial vector, the *shift vector* β, that represents the remaining kinematical freedom available in describing space time, once a foliation is specified. Since β is spatial, it has at most three nonzero components, which, together with the lapse function, represent the four kinematical degrees of freedom available for us to choose. In view of (2.7), we have $\alpha\mathcal{L}_n = \mathcal{L}_t - \mathcal{L}_\beta$. With all this in mind, the six remaining (dynamical) Einstein equations may now be written as a set of twelve first-order equations. We have six equations of motion for the three-metric, which follow immediately from the definition of K_{ab} [Eq. (2.2)],

$$\mathcal{L}_t \gamma_{ab} = -2\alpha K_{ab} + \mathcal{L}_\beta \gamma_{ab} \ , \tag{2.9}$$

and six equations of motion for the extrinsic curvature, which may be written as

$$\begin{aligned}\mathcal{L}_t K^a{}_b = & -\gamma^{ac} D_c D_b \alpha \\ & + \alpha[{}^{(3)}R^a{}_b + K K^a{}_b - \kappa(S^a{}_b - \tfrac{1}{2}\gamma^a{}_b S) - \tfrac{1}{2}\kappa\rho_H \gamma^a{}_b] \\ & + \mathcal{L}_\beta K^a{}_b \ ,\end{aligned} \tag{2.10}$$

where $^{(3)}R^a{}_b$ is the Ricci three-tensor, $S_{ab} \equiv \gamma_a{}^c \gamma_b{}^d T_{cd}$ is the pressure, or spatial stress tensor, as seen by the Eulerian observers, and $S \equiv \gamma^{ab} S_{ab} = S^a{}_a$. By taking the trace of (2.10) and combining it with the Hamiltonian constraint, (2.5), we obtain

$$\mathcal{L}_t K = -\Delta\alpha + \alpha(\mathbf{K}^2 + \tfrac{1}{2}\kappa\rho_\alpha) + \beta^a D_a K \ , \tag{2.11}$$

where $\Delta \equiv D^a D_a$ is the three-dimensional covariant Laplacian operator, and the quantity $\rho_\alpha \equiv S + \rho_H$ is known as the lapse density. This is a useful equation which can be viewed as an evolution equation for the trace of the extrinsic curvature tensor K. Alternatively, if this trace is given as an initial datum, then (2.11) together with the equations for the shift β^a obtained from (2.9), become a system of coupled equations for α and β^a that must be solved simultaneously. However, if we choose K = constant, these equations become uncoupled and we can then separately compute first α and then β^a. The choice $K = 0$, which is called "maximal slicing," was first studied by Lichnerowicz [32] in 1944, while the more general K = constant slicings have been used extensively more recently [31,33]. In this work we shall adopt a K = constant slicing. Equation (2.11) is then an elliptic equation for the lapse function which is usually referred to as the "lapse equation."

B. Bases and frames

Of all the possible reference frames that one can define, two have proved to be particularly useful: the *coordinate* frame and the *Eulerian-orthonormal* frame; the latter will be referred to simply as the "normal" or "Eulerian" frame. A most natural choice is to let the vector $\mathbf{t}$ be the time leg of the coordinate basis $\{\mathbf{e}_a\}$, which defines the coordinate *frame;* the basis vectors must be such that $\langle \mathbf{e}_a, \mathbf{e}_b \rangle = g_{ab}$. We can then write the components of the coordinate basis vectors simply as $(e_a)^b = \delta_a{}^b$ and $(e_a)_b = g_{ab}$. In the coordinate frame, the Lie derivative along $\mathbf{t}$ becomes simply the partial derivative with respect to t, the coordinate time: $\mathcal{L}_t \to c^{-1}\partial_t$.

The components of the unit normal vector $\mathbf{n}$ in the coordinate frame are

$$n_a = (-\alpha, 0, 0, 0) \ , \tag{2.12}$$

so, for any vector $\mathbf{V}$, we have $n_a V^a = n_0 V^0 = -\alpha V^0$; in particular, if $\mathbf{V}$ is *spatial,* then $n_a V^a = 0$ by definition, and so $V^0 = 0$. The shift vector is spatial, so $\beta^0 = 0$, and thus

$$\beta^a=(0,\beta^i) \; . \tag{2.13}$$

From the fact that $n_a n^a=-1$ and in view of (2.12), we obtain $n^0=\alpha^{-1}$; hence, from the definition $t^a=\alpha n^a+\beta^a$, together with (2.13), we have

$$n^a=(\alpha^{-1},-\alpha^{-1}\beta^i) \; . \tag{2.14}$$

Recall that $c\mathbf{n}$ is the four-velocity of the Eulerian observers. The previous result then offers an interpretation of the shift vector: the quantity $c\alpha^{-1}\beta^i$ is simply the spatial velocity, measured in *proper* time αdt, of the triad $\{\mathbf{e}_i\}$ relative to the normal direction.

Another very useful basis is the Eulerian *orthonormal* basis, $\{\mathbf{e}_{\hat{a}}\}$, which defines the Eulerian frame of reference. In this case $\langle \mathbf{e}_{\hat{a}},\mathbf{e}_{\hat{b}}\rangle=\eta_{ab}$, where η_{ab} is the "Minkowski" tensor. To construct this basis, we require that the unit normal vector be the time leg of the basis, $\mathbf{n}=\mathbf{e}_{\hat{0}}$, or

$$(e_{\hat{0}})^a=n^a=(\alpha^{-1},-\alpha^{-1}\beta^i) \; . \tag{2.15}$$

Since $(e_{\hat{i}})^0=0$, we have

$$g_{ij}(e_{\hat{k}})^i(e_{\hat{l}})^j=\delta_{kl} \; , \tag{2.16}$$

which is a system of linear equations that can be solved for the nine coefficients $(e_{\hat{i}})^j$ in terms of the metric components g_{ij}. These coefficients are the (space) components of the three-vectors $\mathbf{e}_{\hat{i}}$.

The coordinate basis and the normal basis are related by a linear transformation: the components of the normal basis vectors in the coordinate basis $(e_{\hat{a}})^b$ are precisely the transformation coefficients to go from the normal basis to the coordinate basis: $\mathbf{e}_{\hat{a}}=(e_{\hat{a}})^b\mathbf{e}_b$ and, conversely, $\mathbf{e}_b=(e_b)^{\hat{c}}\mathbf{e}_{\hat{c}}$, so it is evident that $(e_c)^{\hat{b}}(e_{\hat{b}})^a=\delta_c{}^a$. It is easy to prove that scalar products of four-vectors are invariant under basis transformations of this kind and, in particular, the transformation preserves the norm of a vector, i.e., if a vector is normalized in one frame, it will also be normalized in the other.

The (3+1) quantities can now be used to write the four-velocities and four-momenta of particles. Suppose that $\mathbf{U}$, the four-velocity of a particle, has components U^a in the coordinate frame and is normalized, such that $U_a U^a=-c^2$. Then we can write [34]

$$U^0=cU\alpha^{-1} \; , \tag{2.17a}$$

$$U^i=cU\alpha^{-1}V^i=cU\alpha^{-1}(\alpha v^i/c-\beta^i) \; , \tag{2.17b}$$

where U is the boost factor between the normal and the coordinate frames, $V^i\equiv cU^i/U^0$ is the "transport velocity," and v^i is the particle three-velocity in the *normal* frame, or the boost velocity of U^a relative to n^a. We can define $V^2\equiv v^a v_a=v^i v_i$, so we find that

$$\frac{V}{c}=\frac{(U^2-1)^{1/2}}{U} \; . \tag{2.18}$$

Let us now consider a particle with four-momentum $\mathbf{p}$, and components p^a in the coordinate basis. The normalization condition for the four-momentum is $p_a p^a=-m^2c^2$, where m is the mass of the particle. The energy of the particle, as measured by an observer with four-velocity cn^a, is given by [35]

$$\hat{E}=-cn_a p^a=c\alpha p^0 \; , \tag{2.19}$$

i.e., $\hat{E}$ is the energy of the particle in the normal frame. The magnitude of the particle three-momentum, as measured by the Eulerian observers, is

$$p=(p_a\gamma^a{}_b p^b)^{1/2} \; , \tag{2.20}$$

so we can write [34]

$$p^0=(v/c)^{-1}p\alpha^{-1} \; , \tag{2.21a}$$

$$p^i=(v)^{-1}p\alpha^{-1}V_p{}^i=(v/c)^{-1}p\alpha^{-1}(\alpha v^i/c-\beta^i) \; , \tag{2.21b}$$

where $V_p{}^i\equiv c(p^i/p^0)$ is the "transport velocity," $v^i=\alpha^{-1}V_p{}^i-cn^i$ is the particle three-velocity, and

$$\frac{v}{c}=\frac{pc}{\hat{E}}=\frac{p}{(m^2c^2+p^2)^{1/2}} \; , \tag{2.22}$$

with $v^2\equiv v_a v^a=v_i v^i$.

We end this section with a short digression about the treatment of the four-momentum when describing *massless,* as opposed to *massive,* particles. Note that if $m=0$, then Eqs. (2.19) and (2.22) yield, respectively, $\hat{E}=pc$ and $v/c=1$. In this case it seems natural to parametrize the four-momentum components (2.21) with the energy of the particles in the Eulerian frame, $\hat{E}$, where $0<\hat{E}<\infty$. For $m\neq 0$, however, we can introduce a dimensionless "momentum parameter," W, such that $p=mcW$, and so

$$\hat{E}=mc^2(1+W^2)^{1/2} \; , \tag{2.23}$$

with $0\leq W<\infty$. Equation (2.22) then becomes

$$\frac{v}{c}=\frac{W}{(1+W^2)^{1/2}} \; . \tag{2.24}$$

The factor $(v/c)^{-1}p$ appearing in (2.21) can then be written as

$$(v/c)^{-1}p=mc(1+W^2)^{1/2} \; . \tag{2.25}$$

These parametrizations turn out to be very useful in the numerical treatment of our equations.

III. SOURCES OF GRAVITATIONAL FIELD

When several sources are considered, each may be characterized by its own distinct stress-energy tensor ${}_ST^{ab}$, where S labels the different sources and, if we assume the validity of a superposition principle for the sources, the *total* stress-energy tensor entering the Einstein equations, T^{ab}, is simply the algebraic sum of the different source tensors, thus

$$T^{ab}=\sum_S ({}_ST^{ab}) \; . \tag{3.1}$$

In the present work, we shall consider two types of distinct sources; on one hand we have ordinary matter, described by a perfect-fluid approximation and, on the other hand, a particle gas described by a phase-space distribution function which satisfies the general-relativistic Boltzmann transport equation. We shall now present the relevant details of these two descriptions within the context of the ADM formalism.

A. The perfect-fluid approximation

In this approach, which has been dealt with extensively in the literature [12–16], matter is characterized by a rest-mass density ρ, sometimes also referred to as the baryon mass density, a specific internal energy per unit mass ε, an average isotropic pressure P, and a bulk, or average four-velocity U^a. These quantities are defined in the fluid rest frame; we define the relativistic enthalpy σ in terms of the rest-frame quantities, as

$$\sigma \equiv \rho + \rho\frac{\varepsilon}{c^2} + \frac{P}{c^2} . \tag{3.2}$$

This quantity plays the role of the "effective inertial mass" of the fluid. The energy-momentum tensor for a perfect fluid is given by

$$_F T^{ab} = \sigma U^a U^b + P g^{ab} , \tag{3.3}$$

however, in the (3+1) formalism the Einstein equations are written in the *Eulerian* frame; hence it is necessary to boost $_F T^{ab}$ from the fluid frame into the Eulerian frame. For the Eulerian observer, the fluid stress-energy tensor is given by

$$_F T^{ab} = {_F\rho_H} n^a n^b + n^a {_F S^b} + n^b {_F S^a} + {_F S^{ab}} . \tag{3.4}$$

The Hamiltonian density ($_F\rho_H$), the momentum flux-density vector ($_F S^b$), and the spatial stress tensor ($_F S^{ab}$) for the fluid, as measured by the *Eulerian* observers, are obtained by projecting $_F T^{ab}$ as given by (3.3). We obtain

$$_F\rho_H = c^2 U(U\sigma) - P , \tag{3.5a}$$

$$_F S^a = c(U\sigma)(Uv^a) , \tag{3.5b}$$

and

$$_F S^a{}_b = U(U\sigma) v^a v_b + \gamma^a{}_b P \tag{3.5c}$$

where (see Sec. II B) U is the boost factor between the normal and the coordinate frames and v^a is the particle three-velocity in the *normal* frame, or the boost velocity of U^a relative to n^a. The fluid lapse density $_F\rho_\alpha$ is

$$_F\rho_\alpha = c^2(U\sigma)(2U - U^{-1}) + 2P . \tag{3.6}$$

From the normalization condition $U_a U^a = -c^2$ and the definition of $_F S_a$ we can obtain an "implicit" equation for U that is useful as an auxiliary equation:

$$U = \left[1 + \frac{_F S_{iF} S^i}{c^2 (cU\sigma)^2} \right]^{1/2} . \tag{3.7}$$

We note here that the quantities $_F\rho_H$ and $_F\rho_\alpha$ are *always* given by (3.5a) and (3.6), regardless of the explicit form of U^a, since they only depend on U^a through the factor U. The quantities $_F S^a$ and $_F S^a{}_b$ do depend, however, on the explicit form of U^a through the three-velocity v^a. Of these two quantities, only $_F S^a$ is used explicitly in the (3+1) Einstein equations. Following Wilson [12], it will be useful in the development of the hydrodynamics equations to define a "mass density" D, and an "internal energy density" E:

$$D \equiv U\rho , \tag{3.8a}$$

$$E \equiv U\rho\varepsilon . \tag{3.8b}$$

A complete description of the fluid must also include the temperature T and the entropy $\mathcal{S}$, or equivalently, the *specific* entropy (entropy per unit mass), s. The relationship between these quantities and the "basic" fluid quantities (ρ, ε, P) is fixed by the second law of thermodynamics, which can be written either as

$$T ds = d\varepsilon + P d(1/\rho) , \tag{3.9a}$$

or as

$$T ds = c^2 dh - (1/\rho) dP , \tag{3.9b}$$

where $h = \sigma/\rho$ is often referred to as the "specific enthalpy." In addition, we must supplement our equations with a constitutive relationship between the fluid variables, i.e., an equation of state such as $P = P(\rho, s)$ or $P = P(\rho, T)$; we also require an auxiliary equation determining either the specific entropy or the temperature. For example, we could introduce a specific heat at constant volume, usually defined as

$$C_V = \frac{d\varepsilon}{dT} , \tag{3.10a}$$

so that

$$\varepsilon = C_V T . \tag{3.10b}$$

In some applications, a barotropic equation of state is quite sufficient, and in such a case, the pressure is uniquely determined by the density, i.e., $P = P(\rho)$. In this work we shall adopt an *adiabatic* equation of state, which may be written as

$$UP = (\Gamma - 1)E , \tag{3.11}$$

where Γ is the adiabatic index. We could still use this equation of state even if the processes being studied were not adiabatic, but in that case, Eq. (3.11) must be considered as the *defining* relationship for Γ, and we would also need an extra equation for P. The choice of Eq. (3.11) is by no means exclusive: other choices may be perfectly feasible depending upon the particular type of problem being studied.

With Eq. (3.11), the "redshifted" enthalpy $(U\sigma)$ appearing in Eqs. (3.5)–(3.7) can now be written as

$$U\sigma = D + \Gamma E / c^2 . \tag{3.12}$$

Notice that with (3.11) and (3.12) we can write $_f\rho_H$, $_F S^i$, $_F S$, and $_F\rho_\alpha$ in terms of D, E, Γ, and U, while U, in turn, is determined by $_F S_{iF} S^i$, D, E, and Γ. So if there is some way for us to choose the *initial* values for D, E, Γ, and $_F S^i$, then the initial values for $_F\rho_H$, $_F S$, $_F\rho_\alpha$, and U would be determined. Alternatively, we could initially choose D, E, Γ, U, and two of the three components of $_F S^i$, and *then* determine the third component plus the other fluid quantities. Once the initial values for these quantities are set, the values they take on future slices will be determined by the evolution equations; these will be discussed in Sec. III C.

B. General-relativistic kinetic theory

We begin by defining the distribution function for particles, which forms the basis for their kinetic-theory description. We then proceed with the construction of the relevant quantities that characterize a system of particles. Next, we introduce the Liouville vector, or operator, in phase space; this operator will enable us to write, following Lindquist [23], the relativistic Boltzmann equation in a very compact and elegant form. In a broad sense, this equation describes how the number density of particles belonging to a given species and having a given four-momentum changes as one follows the particles. The number density, as we shall see, is directly related to the distribution function.

1. The particle distribution function

The invariant distribution function for particles $f(\mathbf{x},\mathbf{p})$ may be defined as follows [17,22–24]: consider a thin tube, or beam, of particle world lines, in space time. The particles have four-momenta $\mathbf{p}$ such that $p_a p^a = -m^2c^2$, where these momenta lie within a three-surface element dP in momentum space. The momentum vectors intercept a three-volume element dV at the event $\mathbf{x}$ on some hypersurface Σ with normal unit vector n^a. If the number of particle world lines in the beam is dN, then

$$dN \equiv \frac{g_s}{h^3} f(\mathbf{x},\mathbf{p})(-n_a p^a)dV\,dP \ , \tag{3.13}$$

where g_s is the so-called "Landé" spin factor, or degeneracy index [35,36] and h is Planck's constant, which is introduced so that the distribution function is rendered a dimensionless quantity. The factor $(-n_a p^a)$ is necessary in order that $f(\mathbf{x},\mathbf{p})$ be independent of the orientation of dV, i.e., independent of $\mathbf{n}$. Finally, the invariant volume element dV and the invariant momentum three-surface element dP are defined as

$$dV = \sqrt{-g}\, n^0 d^3x \tag{3.14a}$$

and

$$dP = \sqrt{-g}\, \frac{1}{-p_0} dp^1 dp^2 dp^3 \ , \tag{3.14b}$$

where g is the determinant of the metric g_{ab}. The product $(-\mathbf{n}\cdot\mathbf{p})dV\,dP$ is an invariant, and therefore, since dN is an invariant, the distribution function itself is also an invariant.

2. The stress-energy tensor in kinetic theory

The distribution function defined by (3.14) can now be used to define the stress-energy tensor describing the particles. In any given basis, we can write

$${}_KT^{ab} \equiv c\frac{g_s}{h^3}\int f p^a p^b dP \ . \tag{3.15}$$

The stress-energy tensor for particles as seen by the Eulerian observers takes the form

$${}_KT^{ab} = n^a n^b {}_K\rho_H + n^a {}_KS^b + n^b {}_KS^a + {}_KS^{ab} \ , \tag{3.16}$$

with

$${}_K\rho_H = c\frac{g_s}{h^3}\int f p^2 (v/c)^{-2} dP \ , \tag{3.17a}$$

$${}_KS^a = c\frac{g_s}{h^3}\int f p^2 (v/c)^{-2}(v^a/c) dP \ , \tag{3.17b}$$

and

$${}_KS^{ab} = c\frac{g_s}{h^3}\int f p^2 (v/c)^{-2}(v^a/c)(v^b/c) dP \ , \tag{3.17c}$$

where p and v are given, respectively, by (2.20) and (2.22), while v^a is the particle three-velocity. The lapse density for particles is then

$${}_K\rho_\alpha = c\frac{g_s}{h^3}\int f(2p^2 + m^2c^2) dP \ . \tag{3.18}$$

These results are well defined for both massive and massless particles; in the latter case the factor v/c is equal to unity. The knowledge of $f(\mathbf{x},\mathbf{p})$ *alone* determines all the quantities needed for the $(3+1)$ field equations; therefore, we will need only one equation (per particle species) to determine the evolution of the distribution function $f(\mathbf{x},\mathbf{p})$, and this determines the values of ${}_K\rho_H$, ${}_KS^a$, and ${}_K\rho_\alpha$, as well as ${}_KS^i$, at any time. This equation is the general-relativistic Boltzmann equation, which must be satisfied by $f(\mathbf{x},\mathbf{p})$ (see below).

3. The Liouville operator in phase space

The path of a particle with mass $m \geq 0$ which moves in a gravitational field g_{ab} is described by the geodesic equations of motion

$$\frac{dx^a}{ds} = p^a \tag{3.19a}$$

and

$$\frac{dp^a}{ds} = -\Gamma^a{}_{bc} p^b p^c \ , \tag{3.19b}$$

except, of course, at the collision points, where the slope is discontinuous. We define the affine parameter s by the requirement that p^a is the four-momentum, both for $m=0$ and $m>0$; in the latter case $ms=\tau$ is the proper time. We can see then from (3.19) that the instantaneous state of a particle is determined uniquely by its four-momentum p^a at an event x^a. The set

$$M := \{(\mathbf{x},\mathbf{p}):\ \mathbf{x}\in X,\ \mathbf{p}\in T_{\mathbf{x}},\ \mathbf{p}^2 \leq 0,\ \mathbf{p}\ \text{future directed}\} \ , \tag{3.20}$$

is the one-particle phase space for particles of arbitrary rest masses. In (3.20), X is, in the language of differential geometry, the four-dimensional, oriented, connected, differentiable manifold we usually call "space time," while T_x is the *tangent space* to X at $\mathbf{x}$.

The equations of motion (3.19) define on M a vector field

$$L \equiv p^a \left[\frac{\partial}{\partial x^a} - \Gamma^b{}_{ac} p^c \frac{\partial}{\partial p^b} \right] , \qquad (3.21)$$

which is known as the *Liouville operator,* or vector. The directed (or "coordinatized") and parametrized (through s) integral curves of L, i.e., $(x^a(s),p^a(s))$, form a congruence in the phase space M. We call this congruence the *phase flow* generated by L, and it represents the set of all test-particle motions which are possible in the gravitational field "occurring" in M.

The rest mass of the particles is given by

$$m^2c^2 = -g_{ab}(\mathbf{x})p^a p^b , \qquad (3.22)$$

and it is a scalar function on M, constant on each phase orbit, i.e., $L(m)=0$. The set of phase orbits belonging to a certain mass value $m=$constant generates a hypersurface on M, the *mass shell* M_m; this mass shell is then the phase space for particles of mass m, it has dimension seven, and it is clear that L is tangent to M_m. We take (x^a,p^i) to be the coordinates of M_m. The restriction of L to M_m,

$$L_m \equiv p^a \frac{D}{\partial x^a} = p^a \left[\left. \frac{\partial}{\partial x^a} \right|_{\text{p.s.}} - \Gamma^i{}_{ab} p^b \frac{\partial}{\partial p^i} \right] , \qquad (3.23)$$

is the Liouville operator associated with M_m; in (3.23) we use the subscript "p.s." to stress the fact that these operators "live" in phase space and may be different from the usual $\partial/\partial x^a$ in space time. Also, it is important to stress the fact that (3.23) is coordinate invariant despite the appearance of the Christoffel symbols, since for any function $f=f(\mathbf{x},\mathbf{p})$, $L_m f$ is the directional derivative of f along the phase flow, which was defined above. The fact that x^a and p^i are the coordinates in phase space means they are *independent* in phase space and therefore

$$\left. \frac{\partial p^i}{\partial x^a} \right|_{\text{p.s.}} = 0 \qquad (3.24a)$$

and

$$\frac{\partial x^a}{\partial p^i} = 0 . \qquad (3.24b)$$

This has the effect that, for a function which depends only on the space-time coordinates x^a or on the momenta p^a, the operator L_m reduces to the usual space-time operator d/ds. In particular,

$$L_m x^a = p^b \frac{\partial x^a}{\partial x^b} = \frac{dx^a}{ds} = p^a \qquad (3.25a)$$

and

$$L_m p^i = p^a \frac{Dp^i}{\partial x^a} = -\Gamma^i{}_{ab} p^a p^b = \frac{dp^i}{ds} . \qquad (3.25b)$$

In (3.25) we have used the equations of motion (3.19).

The Liouville operator is particularly useful in writing the relativistic version of the Boltzmann equation [17,22–24]; the equation becomes simply

$$L_m f = \left[\frac{Df}{ds} \right]_{\text{coll}} , \qquad (3.26)$$

where $f=f(x^a,p^i)$ is the particle distribution function and the collision term $(Df/ds)_{\text{coll}}$ denotes the change in f due to particle interactions such as scattering, absorption, or emission.

4. *The collision term*

Following Wilson [18] and Mezzacappa and Matzner [19], we will treat the collision term phenomenologically by writing it as a combination of an emission term and an absorption/scattering term, i.e.,

$$\left[\frac{Df}{ds} \right]_{\text{coll}} = e - of , \qquad (3.27)$$

where e is the invariant emissivity, o is the invariant opacity, and f is the particle distribution function. In any particular frame, we define the invariant emissivity in terms of a matter emissivity η and the particle energy E, *both measured in the same frame,* as

$$e \equiv \frac{h^2c}{g_s} \frac{1}{E^2} \eta . \qquad (3.28a)$$

Similarly, the invariant opacity is defined in terms of a matter opacity χ and the particle energy E as

$$o \equiv \frac{E}{c} \chi , \qquad (3.28b)$$

with $\chi = 1/\lambda$, where λ is the mean free path of the particles. The collision term (3.27) then becomes

$$\left[\frac{Df}{ds} \right]_{\text{coll}} = \frac{E}{c} \left[\frac{h^2c^2}{g_s} \frac{1}{E^3} \eta - \chi f \right] . \qquad (3.29)$$

In some applications, one can define a *local thermodynamic equilibrium* [37] (LTE), and in that case the emissivity η can be written as

$$\eta = \chi B_E(T) , \qquad (3.30)$$

where T is the temperature of the matter, and

$$B_E(T) = \frac{g_s}{h^2c^2} E^3 f_P(E;T) , \qquad (3.31a)$$

with

$$f_P(E;T) = \frac{1}{\exp(E/T) - \epsilon} . \qquad (3.31b)$$

In the above, $B_E(T)$ is the blackbody distribution function and $f_P(E;T)$ is the *invariant* Planck equilibrium distribution function, such that

$$\epsilon = \begin{cases} +1, & \text{bosons (e.g., photons)}, \\ 0, & \text{Maxwell-Boltzmann distribution}, \\ -1, & \text{fermions (e.g., neutrinos)}. \end{cases} \qquad (3.32)$$

Notice that in (3.31b) we have set the Boltzmann constant k_B equal to unity so that the temperature is measured in energy units. If the LTE conditions are met, the collision term then takes the simple form

$$\left[\frac{Df}{ds}\right]_{\text{coll}}=\frac{E}{c}\chi(f_P-f)\ ; \tag{3.33}$$

this form for the collision term was adopted by Wilson [18], among others.

Consider now two frames $\mathbf{F}$ and $\mathbf{F}'$ which can be related by a Lorentz transformation $\mathbf{\Lambda}$. From the invariance of the emissivity e and the opacity o, we can easily obtain relationships between quantities measured in $\mathbf{F}$ and those same quantities measured in $\mathbf{F}'$. In what follows, primed (unprimed) quantities will denote those measured in $\mathbf{F}'$ ($\mathbf{F}$). Equations (3.28) then imply that

$$\frac{\eta}{E^2}=\frac{\eta'}{(E')^2}\Longrightarrow\eta=\left[\frac{E}{E'}\right]^2\eta' \tag{3.34a}$$

and

$$E\chi=E'\chi'\Longrightarrow\chi=\left[\frac{E'}{E}\right]\chi'\ . \tag{3.34b}$$

Furthermore, notice that in the LTE case we have

$$\frac{e}{o}=f_P\ , \tag{3.35}$$

so it is clear that, as was previously claimed, f_P is an invariant; from (3.31b) we then see that the ratio E/T must also be an invariant, so

$$\frac{E}{T}=\frac{E'}{T'}\Longrightarrow T=\left[\frac{E}{E'}\right]T'\ . \tag{3.36}$$

Since the opacity χ is nothing more than the inverse of the mean free path λ of the particles, it is often expressed in the literature as

$$\chi=\kappa\rho\ , \tag{3.37}$$

where κ is the absorption coefficient per unit rest mass, with units of area per unit mass (not to be confused with the gravitational constant), and ρ is the proper rest mass density, a relativistic scalar by definition, with units of mass per unit volume. Since ρ is a scalar (and therefore, an invariant), combining (3.26) and (3.34b) yields

$$\kappa=\left[\frac{E'}{E}\right]\kappa'\ . \tag{3.38}$$

If the explicit form of the Lorentz transformation $\mathbf{\Lambda}$ between the frames $\mathbf{F}$ and $\mathbf{F}'$ is known, we can use the fact that $E=cp^0$ in any frame in order to obtain an explicit expression for the ratio (E/E') which appears in (3.34)–(3.38), thereby allowing us to translate quantities measured in one frame into their counterparts in the other frame.

The usefulness of these relationships becomes evident when we realize that the emissivity η, opacity χ, temperature T, and absorption coefficient κ are usually measured *physically* in the rest frame of the matter (or fluid), while the (3+1) field and evolution equations are written from the *Eulerian observers*' point of view. Thus we need the Lorentz transformation between the Eulerian and fluid frames in order to be able to write the collision term in the Eulerian frame, but in terms of quantities measured in the fluid frame. If we use hatted quantities to denote those measured in the Eulerian frame and primed quantities to denote those measured in the fluid frame, the collision term (3.29) may be written as

$$\begin{aligned}\left[\frac{Df}{ds}\right]_{\text{coll}}&=\frac{\hat{E}}{c}\left[\frac{h^2c^2}{g_s}\frac{1}{\hat{E}^3}\hat{\eta}-\hat{\chi}f\right]\\&=\alpha p^0\left[\frac{E'}{\hat{E}}\right]\left[\frac{h^2c^2}{g_s}\frac{1}{\hat{E}^3}\left[\frac{\hat{E}}{E'}\right]^3\eta'-\chi'f\right]\ ,\end{aligned} \tag{3.39}$$

where we have used Eq. (2.19) to write $\hat{E}/c=\alpha p^0$, together with (3.34a), (3.34b) to write the Eulerian frame quantities in terms of those measured in the fluid frame. If LTE can be assumed, the collision term reduces to

$$\left[\frac{Df}{ds}\right]_{\text{coll}}=\alpha p^0\left[\frac{E'}{\hat{E}}\right]\chi'(f_P-f)\ . \tag{3.40}$$

In Sec. IV we will obtain an explicit expression for the ratio $(E'/\hat{E})$ in the spherically symmetric case.

C. Conservation laws for the sources

We shall assume that the law of baryon conservation for the fluid holds, i.e.,

$$\nabla_a(\rho U^a)=0\ , \tag{3.41}$$

where ρ is the rest-mass density of the fluid and U^a is the normalized fluid four-velocity; Eq. (3.41) is also known as the continuity equation. In addition, the conservation laws for energy and momentum are contained in the four equations

$$\nabla_a T^{ab}=0\ , \tag{3.42}$$

where T^{ab} is the total stress-energy tensor for the sources. In our case, it is the sum of the fluid and the particle stress-energy tensors: $T^{ab}={}_FT^{ab}+{}_KT^{ab}$. We can then write (3.42) as

$$\nabla_a({}_FT^{ab})=-\nabla_a({}_KT^{ab})\ ; \tag{3.43}$$

if we define

$$\nabla_a({}_KT^{ab})\equiv J^b\ , \tag{3.44}$$

then (3.43) becomes

$$\nabla_a({}_FT^{ab})=-J^b\ . \tag{3.45}$$

The quantity J^b appearing as a source term in (3.45) can be interpreted as the particle four-force density vector, so that J^0 is related to the net rate of energy transfer, per unit volume per unit time, while J^i is related to the net rate of momentum transfer, per unit volume per unit time, between the fluid and the particles.

Equations (3.41) and (3.45) provide us with the evolution equations for D, E, and ${}_FS_i$ (see, e.g., Wilson [12] and Centrella and Wilson [14,15]), with the source term

J^b. Now, since the particle stress-energy tensor is a function of the space-time coordinates only, we can use the properties of the Liouville operator together with the explicit expression for ${}_K T^{ab}$, Eq. (3.15), to write

$$J^b = \nabla_a \left[c \frac{g_s}{h^3} \int f p^a p^b dP \right]$$
$$= c \frac{g_s}{h^3} \int (L_m f) p^b dP \ . \qquad (3.46)$$

In view of the Boltzmann equation (3.26), we then get

$$J^b = c \frac{g_s}{h^3} \int \left[\frac{Df}{ds} \right]_{\text{coll}} p^b dP \ , \qquad (3.47)$$

where $(Df/ds)_{\text{coll}}$ is given by expression (3.39), or (3.40) if we assume LTE.

IV. SPHERICALLY SYMMETRIC SPACETIMES

Up to this point, all the equations we have obtained are completely general as far as the metric tensor g_{ab} is concerned. In order to advance further in our search for a solution to the equations, we must choose the metric tensor and the coordinates to be used. Once this is done, the necessary vector bases and other geometric objects can be constructed explicitly. The field sources must also be specified, and this is done by constructing the stress-energy tensor that describes them. The (3+1) field equations are then cast in their "final" form, ready to be solved. In this section we carry out the steps to fulfill this program in the special case of spherical symmetry; we shall focus our attention, however, around the Boltzmann equation. The coordinates will be labeled in the usual way: $x^0 := ct$, $x^1 := r$, $x^2 := \theta$, and $x^3 := \varphi$. The metric will be assumed to be a function of the time t and the radial coordinate r, i.e., this will be an inhomogeneous, one-dimensional model.

A. Geometry

The most general three-dimensional line element consistent with spherical symmetry [16,25,26] is given by

$$dl^2 = A^2 dr^2 + B^2 r^2 (d\theta^2 + \sin^2\theta d\varphi^2) \ , \qquad (4.1)$$

while the shift vector takes the form

$$\beta^a = (0, \beta, 0, 0) \ ; \qquad (4.2)$$

this is so because it is the only form for the shift vector with which the three-metric will be kept diagonal throughout the evolution. Therefore, our full (3+1) line element is

$$ds^2 = -(\alpha^2 - A^2\beta^2) c^2 dt^2 + 2A^2 \beta c \, dt \, dr$$
$$+ A^2 dr^2 + B^2 r^2 (d\theta^2 + \sin^2\theta d\varphi^2) \ . \qquad (4.3)$$

The unit normal vector in spherical symmetry is given by

$$n^a = (\alpha^{-1}, -\alpha^{-1}\beta, 0, 0) \ . \qquad (4.4)$$

The three-metric can then be constructed using Eq. (2.1); its determinant, γ, is found to be

$$\sqrt{\gamma} = AB^2 r^2 \sin\theta \ . \qquad (4.5)$$

At this point we may use our coordinate freedom to impose the "isotropic" gauge developed by Wilson [38] and Dykema [39] and used by Evans [40,41], Shapiro and Teukolsky [25,26], Holcomb [16], and Mezzacappa and Matzner [19], among others. This gauge is set by simply requiring that $A = B$. For the sake of generality, however, we shall obtain all of our equations for $A \neq B$.

The unit normal vector, given by (4.4), defines the time leg of the Eulerian basis, i.e., $\mathbf{n} = \mathbf{e}_{\hat{0}}$. Now, using the nine equations (2.16) we obtain the nonzero components of the other three vectors, $\mathbf{e}_{\hat{i}}$, of that basis. The transformation matrix to go from the Eulerian basis to the coordinate basis, $\mathbf{L}$, is simply formed by the components $(e_{\hat{a}})^b$ of the Eulerian basis vectors, we get

$$L_{\hat{b}}{}^c = (e_{\hat{a}})^b = \begin{bmatrix} \alpha^{-1} & -\alpha^{-1}\beta & 0 & 0 \\ 0 & A^{-1} & 0 & 0 \\ 0 & 0 & (Br)^{-1} & 0 \\ 0 & 0 & 0 & (Br\sin\theta)^{-1} \end{bmatrix} , \qquad (4.6)$$

while the inverse transformation, $\mathbf{L}^{-1}$, with coefficients $L_b{}^{\hat{c}} = (e_b)^{\hat{c}}$, is obtained from $e_{\hat{a}}{}^b e_b{}^{\hat{c}} = \delta_{\hat{a}}{}^{\hat{c}}$.

We will be dealing with two kinds of sources: ordinary matter with a bulk four-velocity U^a and particles with four-momentum p^a. In spherical symmetry it is usual to require that ordinary matter should have a four-velocity $\mathbf{U}$ in the form

$$U^a = (U^0, U^1, 0, 0) \ , \qquad (4.7)$$

that is, only radial motions are allowed for the fluid. In view of (2.17), we have

$$U^0 = cU\alpha^{-1} \qquad (4.8a)$$

and

$$U^1 = cU \left[\frac{(U^2-1)^{1/2}}{AU} - \frac{\beta}{\alpha} \right] . \qquad (4.8b)$$

The Lorentz transformation between the fluid rest frame and the Eulerian frame can now be obtained quite easily. If we write the four-velocity of the fluid in the Eulerian frame by applying the transformation defined in (4.6)–(4.8), we obtain

$$U^{\hat{a}} = U^b L_b{}^{\hat{a}} = cU \left[1, \frac{(U^2-1)^{1/2}}{U}, 0, 0 \right] , \qquad (4.9)$$

so we immediately recognize that

$$\gamma_v \equiv U \qquad (4.10a)$$

and

$$V \equiv c \frac{(U^2-1)^{1/2}}{U} \qquad (4.10b)$$

are the boost factor and velocity that define the Lorentz transformation, $\mathbf{\Lambda}$, between the Eulerian frame and the fluid rest frame. Thus, the components of a four-vector $\mathbf{v}$ in the Eulerian frame, $v^{\hat{a}}$, are related to its components in the fluid rest frame, $v^{a'}$, by

$$v^{\hat{a}}=\Lambda^{\hat{a}}{}_{a'}v^{a'} , \tag{4.11}$$

where

$$\Lambda^{\hat{a}}{}_{a'}=\begin{bmatrix} \gamma_v & \gamma_v V/c & 0 & 0 \\ \gamma_v V/c & \gamma_v & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} , \tag{4.12}$$

due to the fact that the fluid is moving only in the radial direction. The inverse Lorentz transformation is obtained simply by letting $V\rightarrow -V$. The boost velocity V may also be written in terms of the coordinate-frame quantities as

$$V=A\alpha^{-1}(c\beta+V^1) , \tag{4.13}$$

where V^1 is the only nonzero component of the transport velocity.

We now move on to the particle four-momentum. Since we are considering spherically symmetric space times, it is only natural that we introduce spherical coordinates in momentum space as well. We define the "cosine angle parameters" μ and λ in the Eulerian frame as

$$\mu=\cos\theta_p \tag{4.14a}$$

and

$$\lambda=\cos\varphi_p , \tag{4.14b}$$

where θ_p and φ_p are the usual polar and azimuthal angles of spherical coordinates in momentum space. Using the "$\mathbf{e}_{\hat{1}}$ axis" as the polar axis instead of the more usual "$\mathbf{e}_{\hat{3}}$ axis" the components of the three-momentum ${}^{(3)}\mathbf{p}$, as measured by the Eulerian observers can be written as

$$p^{\hat{1}}=p\mu , \tag{4.15a}$$

$$p^{\hat{2}}=p(1-\mu^2)^{1/2}\lambda , \tag{4.15b}$$

$$p^{\hat{3}}=p(1-\mu^2)^{1/2}(1-\lambda^2)^{1/2} , \tag{4.15c}$$

while the normalization condition $p^{\hat{a}}p_{\hat{a}}=-m^2c^2$, together with (2.19) and (2.22), give

$$p^{\hat{0}}=(m^2c^2+p^2)^{1/2}=\frac{\hat{E}}{c}=p(v/c)^{-1} . \tag{4.15d}$$

With the transformation (4.6) and the expression (2.22) for v, the components of the four-momentum in the coordinate basis become

$$p^0=p(v/c)^{-1}\alpha^{-1} , \tag{4.16a}$$

$$p^1=p(v/c)^{-1}\alpha^{-1}\left[\frac{\mu\alpha}{A}(v/c)-\beta\right] , \tag{4.16b}$$

$$p^2=p\frac{1}{Br}(1-\mu^2)^{1/2}\lambda , \tag{4.16c}$$

$$p^3=p\frac{1}{Br\sin\theta}(1-\mu^2)^{1/2}(1-\lambda^2)^{1/2} . \tag{4.16d}$$

We stress the fact that p, μ, and λ are quantities measured by the Eulerian observers.

B. The sources in spherical symmetry

We now proceed to write the results obtained in Sec. III for the source quantities in the special case of spherical symmetry.

1. *Ordinary matter*

While the Hamiltonian density and the lapse density for the fluid are given directly by (3.5a) and (3.6), respectively, the only independent nonzero component of the momentum flux vector in the coordinate basis is

$${}_FS_r=c^2(U\sigma)A(U^2-1)^{1/2} , \tag{4.17}$$

with ${}_FS_0=\beta({}_FS_r)$ and ${}_FS_\theta={}_FS_\varphi=0$. With (4.17), Eq. (3.7) becomes

$$U=\left[1+\frac{({}_FS_r)^2}{c^2(cU\sigma)^2A^2}\right]^{1/2} . \tag{4.18}$$

The only nonvanishing component of the transport velocity is, in this case,

$$V^1=cU^1/U^0=c(\alpha v^1/c-\beta) , \tag{4.19}$$

with $v^1=c(U^2-1)^{1/2}/AU$. The equation of state is specified, we recall, by (3.11) and the "red-shifted" enthalpy, $U\sigma$, is given by (3.12).

2. *Weakly interacting particles*

In spherical coordinates, the momentum three-surface element, (3.14b), can be written as

$$dP=p(v/c)dp\,d\mu\,d\varphi_p , \tag{4.20}$$

where φ_p is the azimuthal angle in momentum space. With this expression for dP, the particle stress-energy tensor, (3.15), becomes

$${}_KT^{ab}=c\frac{g_s}{h^3}\int fp^ap^bp(v/c)dp\,d\mu\,d\varphi_p . \tag{4.21}$$

The (3+1) particle quantities ${}_K\rho_H$ and ${}_K\rho_\alpha$ [cf. Eqs. (3.17), (3.18)] then become

$${}_K\rho_H=2\pi c\frac{g_s}{h^3}\int fp^3(v/c)^{-1}dp\,d\mu \tag{4.22a}$$

and

$${}_K\rho_\alpha=2\pi c\frac{g_s}{h^3}\int fp(v/c)(2p^2+m^2c^2)dp\,d\mu . \tag{4.22b}$$

The only independent, nonvanishing component for the particle momentum flux vector is

$${}_KS_r=2\pi cA\frac{g_s}{h^3}\int fp^3dp\,\mu\,d\mu , \tag{4.23}$$

with ${}_KS_0=\beta({}_KS_r)$ and ${}_KS_\theta={}_kS_\varphi=0$. Since in spherical symmetry f cannot depend on the azimuthal angle φ_p, we were able to carry out the φ_p integrations in Eqs. (4.22), (4.23) immediately.

C. The field equations

In spherical symmetry the system of equations simplifies considerably. The only equations that remain are the Hamiltonian constraint, one momentum constraint, the lapse equation, and two "auxiliary" equations obtained from the definition of the extrinsic curvature. Now, the three-metric is not only diagonal, but at most only two of its components are independent, with only one independent component if the isotropic gauge ($A=B$) is imposed. In view of equations (2.10) (which may be considered as the definition of the extrinsic curvature in terms of the time derivative of the three-metric), the extrinsic curvature itself will also have at most only two independent components. Let us begin with the extrinsic curvature: its two independent components will be represented by the trace of the extrinsic curvature, K, and an extrinsic curvature variable defined as $K^*\equiv K^1{}_1-\frac{1}{3}K$. Explicitly, they are given by

$$K\equiv K^i{}_i=-(A^{-2}\Delta_A+2B^{-2}\Delta_B) \tag{4.24}$$

and

$$K^*\equiv-\tfrac{2}{3}(A^{-2}\Delta_A-B^{-2}\Delta_B)\ , \tag{4.25}$$

where

$$\Delta_A\equiv\frac{A^2}{\alpha}\left[\frac{\dot A}{A}-\beta\frac{A'}{A}-\beta'\right] \tag{4.26a}$$

and

$$\Delta_B\equiv\frac{B^2}{\alpha}\left[\frac{\dot B}{B}-\beta\frac{B'}{B}-\frac{\beta}{r}\right]\ ; \tag{4.26b}$$

here a dot means $\partial/c\partial t$ and a prime means $\partial/\partial r$.

We will assume that K is a given datum initially and also that it is a function of time only, so that $K=K(t)$; i.e., it is a constant on each slice of the foliation $\{\Sigma\}$. This condition is known as constant-mean-curvature slicing. Physically, it simply means we are effectively demanding that all Eulerian observers, who are at rest in the slices, measure the same Hubble constant at a given time; thus, in the absence of anisotropy and inhomogeneities, the metric reduces to the Friedmann-Robertson-Walker metric. With this choice the lapse equation, (2.11), uncouples from the equations for the three-metric, (2.9), and thus these equations can be solved independently for the lapse function α and for the shift β.

Using the explicit forms of Δ_A and Δ_B, equations (4.24) and (4.25) may be written, after some algebra, as

$$\frac{\partial}{\partial t}(AB^2)+\frac{1}{r^2}\frac{\partial}{\partial r}(-c\beta r^2AB^2)=-c\alpha\tilde K \tag{4.27}$$

and

$$\frac{\partial}{\partial r}\left[\frac{1}{r}c\beta AB^{-1}\right]-\frac{1}{r}\frac{\partial}{\partial t}(AB^{-1})=\tfrac{3}{2}c\alpha\frac{1}{r}(AB^{-1})K^*\ . \tag{4.28}$$

In (4.27) we have defined $\tilde K\equiv(AB^2)K$. These are the two auxiliary equations that were mentioned above. Notice that (4.27) is clearly an evolution equation (in the form of a transport equation, with transport velocity "$-c\beta$") for the conformal factor (AB^2), with source term $-c\alpha\tilde K$. The fluid evolution equations are actually written for *conformalized* quantities, e.g., $\tilde D=(AB^2)D$, etc., so that the solution of (4.27) would allow us to recover the "bare" quantities after the evolution has been carried out. In order to make any further progress with Eq. (4.28), we must fix the gauge, i.e., we need a relationship between the metric functions A and B. For the simple choice (of which the isotropic gauge with $A=B$ is, of course, a special case)

$$B=f_r(r)A\ , \tag{4.29}$$

where $f_r(r)$ is a well-behaved, known function of the radial coordinate only, we would have $AB^{-1}=f_r{}^{-1}$, while the conformal factor is $AB^2=A^3f_r{}^2$. Equation (4.28) then becomes an ordinary differential equation for β on each slice which may be readily integrated to yield

$$\beta=-\tfrac{3}{2}(rf_r)\int_r^\infty\alpha(r'f_r)^{-1}K^*dr'\ . \tag{4.30}$$

To summarize, the auxiliary equation (4.27) gives us the evolution of the conformal factor AB^2, provided that the trace of the extrinsic curvature $K=K(t)$ is a given initial datum, while the auxiliary equation (4.28), together with a gauge choice of the type (4.29), gives us an expression for the shift β.

The field equations remaining to be recast into spherical symmetry are the Hamiltonian constraint, (2.5), the momentum constraint equations, (2.6), and the lapse equation, (2.11). We stress the fact that, in these equations, the quantities ρ_H, ρ_α, and S_j are the total source quantities formed with a fluid part and a particle part. The Hamiltonian constraint equation becomes

$$\frac{\partial}{\partial r}\left[\frac{B}{A}r\frac{\partial}{\partial r}\sqrt{Br}\right]=\tfrac{1}{4}A(Br)^{3/2}[(Br)^{-2}+\tfrac{1}{3}K^2-\tfrac{3}{4}(K^*)^2-\kappa\rho_H]\ , \tag{4.31}$$

which is a differential equation for the quantity Br; of course, some kind of knowledge about A and B, e.g., a gauge condition, would be necessary in order to make progress towards its solution [16,30]. Of the three-momentum constraint equations, only the one for S_r is relevant in spherical symmetry, the other two being trivial identities. From (2.6) we obtain an equation for K^* which can be integrated to yield

$$K^*=\frac{\kappa}{(Br)^3}\int_0^r(Br')^3S_rdr'+\frac{2}{3}\frac{1}{(Br)^3}\int_0^r(Br')^3\left[\frac{\partial}{\partial r'}K\right]dr'\ . \tag{4.32}$$

Notice that we did not restrict ourselves to a constant-mean-curvature slicing to obtain (4.32); when we do, the last term on the right-hand side drops out, and we obtain an expression for K^* in terms of the momentum flux S_r. Finally, the lapse equation (2.11) becomes

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left[\frac{B^2}{A}r^2\left[\frac{\partial}{\partial r}\alpha\right]\right]=\tfrac{1}{2}\alpha AB^2[\tfrac{2}{3}K^2+3(K^*)^2+\kappa\rho_\alpha]-AB^2\left[\frac{1}{c}\frac{\partial}{\partial t}K-\beta\frac{\partial}{\partial r}K\right]; \tag{4.33}$$

again, if we assume a constant-mean-curvature slicing, i.e., $K=K(t)$, the last term on the right-hand side drops out.

D. Evolution equations for the sources

The evolution equations for the matter "density" D, the matter "energy" E, and the matter momentum flux ${}_FS_i$ in spherical coordinates, which can be obtained from Eqs. (3.41) and (3.45), have been already worked out elsewhere (see, e.g., Ref. [19]) so we shall not repeat those calculations here. On the other hand, the Boltzmann equation, which is really an evolution equation for the particle distribution function and is given in its most general form by Eq. (3.26), together with the definition of the Liouville operator specified by (3.23), may be written in a form that turns out to be highly amenable to numerical treatment. Using the properties of the Liouville operator we can write the Boltzmann equation in "conservative" form, in the sense defined by, for instance, Mihalas and Mihalas [29], meaning that each term in the resulting equation vanishes when integrated over its full range. To that effect, it will prove useful to define

$$\mathcal{H}_p\equiv p^{-1}(p^0)^{-1}(L_m p)=-\frac{\mu\alpha'}{A}(v/c)^{-1}+\alpha[\tfrac{1}{3}K+\tfrac{1}{2}K^*(3\mu^2-1)] \tag{4.34a}$$

and

$$\mathcal{H}_\mu\equiv(p^0)^{-1}(L_m\mu)=(1-\mu^2)\left[-\frac{\alpha'}{A}(v/c)^{-1}+\tfrac{3}{2}\alpha\mu K^*+\alpha(v/c)\frac{1}{A}\frac{1}{Br}(Br)'\right], \tag{4.34b}$$

where $L_m p$ and $L_m\mu$ are computed by expressing p and μ in terms of the four-momentum components p^i as given by Eqs. (4.16) and using (3.25). The quantities $\mathcal{H}_p$ and $\mathcal{H}_\mu$ turn out to be the "advection velocities" in the momentum and angle "transport" terms, respectively, in the conservative form of the Boltzmann equation.

In terms of the conformalized distribution function, the left-hand side of the Boltzmann equation, (3.26), is given by

$$L_m f=(AB^2)^{-1}\left[L_m\tilde{f}-\tilde{f}\left[\frac{1}{A}\frac{dA}{ds}+\frac{2}{B}\frac{dB}{ds}\right]\right], \tag{4.35}$$

where

$$\tilde{f}=\tilde{f}(t,r,p,\mu)=AB^2 f, \tag{4.36}$$

since in spherical symmetry the distribution function cannot depend on the azimuth λ. Now, applying the Liouville operator to (4.36), we can write

$$L_m\tilde{f}=p^0\frac{1}{c}\frac{\partial}{\partial t}(\tilde{f})+p^1\frac{\partial}{\partial r}(\tilde{f})+(L_m p)\frac{\partial}{\partial p}(\tilde{f})+(L_m\mu)\frac{\partial}{\partial\mu}(\tilde{f}), \tag{4.37}$$

where we simply used the chain rule, along with (3.25a). Using the definitions (4.69) and combining Eq. (4.37) with Eq. (4.35), the left-hand side of the Boltzmann equation now becomes

$$\begin{aligned}L_m f=&\frac{p^0}{AB^2}\left[\frac{1}{c}\frac{\partial}{\partial t}(\tilde{f})+\frac{1}{r^2}\frac{\partial}{\partial r}\left[r^2\frac{p^1}{p^0}\tilde{f}\right]+\frac{1}{p^2}\frac{\partial}{\partial p}(p^2\mathcal{H}_p\tilde{f})+\frac{\partial}{\partial\mu}(\mathcal{H}_\mu\tilde{f})\right]\\&-\frac{p^0}{AB^2}\tilde{f}\left[\frac{1}{r^2}\frac{\partial}{\partial r}\left[r^2\frac{p^1}{p^0}\right]+\frac{1}{p^2}\frac{\partial}{\partial p}(p^3\mathcal{H}_p)+\frac{\partial}{\partial\mu}(\mathcal{H}_\mu)+\frac{1}{p^0}\frac{1}{A}\frac{dA}{ds}+\frac{1}{p^0}\frac{2}{B}\frac{dB}{ds}\right].\end{aligned} \tag{4.38}$$

It can be shown that the second term in square brackets on the right-hand side of (4.38) is identically zero; this calculation, although tedious, is rather straightforward so it will not be repeated here. The Boltzmann equation, (3.26), finally takes the simple form

$$\frac{\partial}{\partial t}(\tilde{f})+\frac{1}{r^2}\frac{\partial}{\partial r}(r^2V_p{}^1\tilde{f})+\frac{1}{p^2}\frac{\partial}{\partial p}(p^3c\mathcal{H}_p\tilde{f})+\frac{\partial}{\partial\mu}(c\mathcal{H}_\mu\tilde{f})=c\frac{1}{p^0}(AB^2)\left[\frac{Df}{ds}\right]_{\text{coll}}, \tag{4.39}$$

where

$$V_p{}^1 = cp^1/p^0 = c\left[\frac{\mu\alpha}{A}(v/c) - \beta\right] . \tag{4.40}$$

From Eq. (4.39), it now becomes clear that $\mathcal{H}_p$ and $\mathcal{H}_\mu$, which are given explicitly by Eqs. (4.34), in effect play the role of "advection velocities" in, respectively, momentum and angle, as was previously stated. Equation (4.39) is the *conservative* form of the Boltzmann equation in spherical symmetry. In the isotropic gauge, $(A \to B)$, it is a straightforward exercise to prove that for massless particles and maximal slicing, i.e., $m=0$ and $K=0$, Eq. (4.39) can be recast into the form used by Mezzacappa and Matzner [19].

The collision term appearing on the right-hand side of Eq. (4.39) is given, in general, by (3.39) or, if LTE is assumed, by (3.40). In order to specialize this term to spherical symmetry, we need to obtain an explicit expression for the factor $(E'/\hat{E})$ appearing in (3.39) and (3.40). This can be easily achieved through the use of the Lorentz transformation between the fluid and Eulerian frames, which is specified by (4.12) with $V \to -V$, together with the components of the particle four-momentum in the Eulerian frame, as given by Eqs. (4.15); we also use the facts that $E' = cp^{0'}$ and $\hat{E} = cp^{\hat{0}}$. We then have

$$\frac{E'}{c} = p^{0'} = \Lambda^{0'}{}_{\hat{a}} p^{\hat{a}} = \frac{\hat{E}}{c} U\left[1 - \mu\frac{V}{c}\frac{v}{c}\right] , \tag{4.41}$$

whence we find

$$\frac{E'}{\hat{E}} = U\left[1 - \mu\frac{V}{c}\frac{v}{c}\right] . \tag{4.42}$$

Recall that the boost velocity of the fluid, V/c, is given by (2.18), while the three-velocity of the particles, v/c, with respect to the Eulerian observers, is given by (2.22). The right-hand side of Eq. (4.39) then becomes

$$c\frac{1}{p^0}(AB^2)\left[\frac{Df}{ds}\right]_{\rm coll} = c\alpha\left[\frac{E'}{\hat{E}}\right]\left[\frac{h^2c^2}{g_s}\frac{1}{\hat{E}^3}\left[\frac{\hat{E}}{E'}\right]^3 \tilde{\eta}' - \chi'\tilde{f}\right] , \tag{4.43}$$

where, as usual, we have defined $\tilde{\eta}' \equiv AB^2\eta'$, $\tilde{f}$ is the conformalized distribution function, and $(E'/\hat{E})$ is given by (4.42). In the LTE case, (4.43) reduces to:

$$c\frac{1}{p^0}(AB^2)\left[\frac{Df}{ds}\right]_{\rm coll} = c\alpha\left[\frac{E'}{\hat{E}}\right]\chi'[\tilde{f}_p - \tilde{f}] \tag{4.44}$$

[see Eq. (3.33)]. With (3.39) and (3.47), together with (4.16a), the conformalized four-force density vector $\tilde{J}_a$ is given by

$$\tilde{J}_a = 2\pi c\frac{g_s}{h^3}\int\left[\frac{E'}{\hat{E}}\right]\left[\frac{h^2c^2}{g_s}\frac{1}{\hat{E}^3}\left[\frac{\hat{E}}{E'}\right]^3\tilde{\eta}' - \chi'\tilde{f}\right] \times p_a p^2 dp\, d\mu . \tag{4.45}$$

Let us now take a closer look at the matter opacity χ'. This opacity may be written in terms of an absorption coefficient, κ', and the *invariant* proper rest mass density of the matter, ρ [cf. Eqs. (3.37) and (3.38)], as

$$\chi' = \kappa'\rho . \tag{4.46}$$

Now, the absorption coefficient κ' in the fluid proper frame can be written as [23] $\kappa' = \sigma_a(E') + \sigma_s(E')$, where $\sigma_a(E')$ and $\sigma_s(E')$ are, respectively, the absorption and scattering cross sections per unit mass. Notice that in this frame, κ' is a function of *only* the particle energy E', not of direction. The angle dependence of the absorption coefficient κ in an arbitrary frame can be obtained through the invariant combination $p^a U_a$, where U_a is the four-velocity of the external medium (e.g., fluid) and p^a is the four-momentum of the particles; we would then get the relationship (3.38), i.e., $\kappa = (E'/E)\kappa'$, which we had obtained by using invariance considerations. In some applications, such as neutrino opacities [42], the cross sections are proportional to the square of the energy, so we can write

$$\kappa' = \kappa_o(E')^2 , \tag{4.47}$$

where κ_o is a constant. In this case, (4.46) becomes

$$\chi' = \kappa_o\rho(E')^2 = \kappa_o\rho\hat{E}^2\left[\frac{E'}{\hat{E}}\right]^2 . \tag{4.48}$$

In light of (4.42) and the definition for the matter "density," $D = U\rho$, we then get

$$\chi' = \kappa_o D\hat{E}^2 U\left[1 - \mu\frac{V}{c}\frac{v}{c}\right]^2 . \tag{4.49}$$

This expression can then be used to compute (and update) the matter opacity χ'. Finally, in the Eulerian frame the mean free path will be given by

$$\hat{\lambda} = \frac{1}{\hat{\chi}} , \tag{4.50}$$

where [cf. Eq. (3.34b)]

$$\hat{\chi} = \left[\frac{E'}{\hat{E}}\right]\chi' \tag{4.51}$$

is the opacity as measured by the Eulerian observers.

V. SUMMARY AND CONCLUSIONS

We have written the general-relativistic Boltzmann equation in its most general form within the context of the (3+1) formalism. The sources of gravitational field were taken to be a perfect fluid, representing ordinary matter, interacting gravitationally with a particle "gas," described by a distribution function obeying the general-relativistic Boltzmann transport equation. We then specialized this equation to the spherically symmetric case.

Ultimately we wrote the Boltzmann equation in a conservative [29] form which can be used for both massless and massive particles. This was achieved by first expressing the spacelike components of the particle momentum in spherical coordinates, i.e., $p^i \to (p, \mu, \lambda)$, where p is the magnitude of the three-momentum in the Eulerian frame, and μ, λ are the angle cosines of the polar and azimuthal angles in momentum space, respectively; then, using the Liouville operator in phase space, as described in Secs. III B and IV D, we obtain the final form for the equation. It was found that, in this form, the Boltzmann equation lends itself naturally to numerical treatment through the use of well-established radiation-hydrodynamics techniques, together with a new "implicit bordering" method due to Mezzacappa and Matzner [19].

The fact that Wilson [18] obtained a "conservative" form of the Boltzmann equation in axisymmetric space-times, even though he used a parametrization for the particle "momentum" that does not have a direct physical interpretation (as does our choice of the variable p) leads us to believe that the Boltzmann equation can *always* be written in conservative form by following our method for the application of the Liouville operator, regardless of the (momentum) coordinates used. Further investigation is required about this point.

Among the problems that can be studied with this model, the cosmological collapse of a combination of weakly interacting massive particles and a fluid presents a most interesting challenge. Also, from the purely numerical point of view, other gauge choices for which $A \neq B$ should be investigated, since the equations, as we present them, are "gauge ready," so to speak, in this respect.

A description of a computer code for the Einstein-Boltzmann system of equations in spherical symmetry using the isotropic gauge, i.e., with $A = B$, that can handle both massive and massless particles, together with the results obtained so far with this code, is the subject of a separate paper [30].

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