Gravitationally compact objects as nucleation sites for first-order vacuum phase transitions

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A characteristic of first-order phase transitions is their ability to be initiated by nucleation sites. In this paper we consider the role that gravitationally compact objects may play as nucleation sites for first-order phase transitions within quantum fields. As the presence of nucleation sites may prevent the onset of supercooling, the existence of nucleation sites for phase transitions within quantum fields may play an important role in some inflationary models of the Universe, in which the Universe is required to exist in a supercooled state for a period of time. In this paper we calculate the Euclidean action for an O(3) bubble nucleating about a gravitationally compact object, taken to be a boson star for simplicity. The gravitational field of the boson star is taken to be a small perturbation on flat space, and the O(3) action is calculated to linear order as a perturbation on the O(4) action. The Euclidean bubble profile is found by solving the (Higgs) scalar field equation numerically; the thin-wall approximation is not used. The gravitationally compact objects are found to have the effect of reducing the Euclidean action of the nucleating bubble, as compared to the Euclidean action for the bubble in flat spacetime. The effect is strongest when the size of the gravitationally compact object is comparable to the size of the nucleating bubble. Further, the size of the decrease in action increases as the nucleating "star" is made more gravitationally compact. Thus, gravitationally compact objects may play the role of nucleation sites. However, their importance to the process of false-vacuum decay is strongly dependent upon their number density within the Universe.

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I. INTRODUCTION

A characteristic of first-order phase transitions in everyday matter is their ability to be initiated by nucleation sites. An example of this is the preferential formation of raindrops around atmospheric dust particles. We are therefore led to ask whether such nucleation sites might occur for first-order vacuum phase transitions in quantum fields. In particular, could gravitationally compact objects act as nucleation sites for false-vacuum decay?

In the absence of nucleation sites, Coleman [1] has shown that first-order phase transitions may occur between a false- and true-vacuum state via the spontaneous nucleation of bubbles of true vacuum within the medium of false vacuum. The "rate" of spontaneous bubble nucleation (i.e., the number of bubbles within a unit four-volume) is given by the expression $\Gamma = Ae^{-B}$. The exponent B in this expression represents the difference between the Euclidean action for the spacetime with and without a nucleating bubble. The coefficient A is a functional determinant which is typically of the order of m^4, where m is the characteristic mass scale of the field undergoing the phase transition.

Coleman's analysis developed the "thin-wall" approximation, where the nucleating bubble has a well-defined core of true vacuum, a thin wall in which the field makes a rapid transition from the true- to the false-vacuum state, and the exterior of the bubble is in the false-vacuum state. This approximation scheme is applicable in the limit where the energy-density difference between the true- and false-vacuum states is small.

Hiscock [2], and Mendell and Hiscock [3], considered the effects of gravitationally compact objects on false-vacuum decay within the "thin-wall" approximation, and further studies have been made by Berezin *et al.* [4], again within the "thin-wall" approximation. In particular, Hiscock [2] considered the effect of black holes, and Mendell and Hiscock [3] considered the effect of gravitationally compact objects such as neutron stars. Their "thin-wall" analyses treated the bubble wall as a surface layer within the Israel [5] formalism, patching together the false- and true-vacuum spacetimes. Thus, for example, when considering the effect of a black hole on the false-vacuum decay from Minkowski to anti–de Sitter space, one would patch together an interior core of Schwarzschild–anti-de Sitter spacetime to an exterior of Schwarzschild spacetime. Their analyses found that gravitationally compact objects have the effect of reducing B, the difference between the Euclidean action for the spacetime with, and without, the nucleating bubble. It would therefore appear that such objects may, in certain situations, increase the false-vacuum decay rate. However, the usual expression for the bubble nucleation rate must be modified when considering O(3) nucleation about a specific site, rather than O(4) nucleation in a homogeneous spacetime. As a result, a reduction in the value of B associated with the nucleation of a bubble around a "star" does not necessarily result in an increase in the bubble nucleation rate within the spacetime. This issue will be addressed at the end of this paper.

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These earlier "thin-wall" analyses had the attractive feature of inherently taking into account the self-gravity of the quantum field undergoing the phase transition. However, as has been shown by Samuel and Hiscock [6,7], the "thin-wall" approximation does not usually describe false-vacuum decay processes very well. Therefore, by adopting a different approach than the "thin-wall" approximation we may be able to gain further insight into the effect of gravitationally compact objects acting as nucleation sites. Such an approach should also allow us to confirm and justify some of the predictions of the "thin-wall" approximate analysis. As a result, in this paper we shall adopt a different approximation than the usual "thin-wall" approach: we shall use a perturbative analysis to study the effect of gravitational nucleation sites. The perturbative analysis will utilize exact, numerically integrated O(4) bubble profiles (i.e., non-thin-wall), associated with bubble nucleation in flat space, as the initial background; a (weakly) self-gravitating boson star will be introduced to act as a nucleation site and the O(3) perturbation to the O(4) bubble action calculated to linear order in perturbation theory. Therefore, our results will be trustworthy only so long as the O(3) perturbation is small; for small perturbations, however, the results are more reliable than the corresponding "thin-wall" analysis, which cannot be generally trusted even in the O(4) case. Our analysis does not take the self-gravity of the bubble into account; only the influence of the changing background gravitational field on the Euclidean action is calculated. The self-gravity effects which we ignore are, however, relevant only when the symmetry-breaking scale is within an order of magnitude of the Planck energy.

In the absence of gravity, Coleman *et al.* [8] have shown that within the O(4)-symmetric Euclideanized Minkowski space, O(4)-symmetric nucleating bubbles have the lowest Euclidean action, and hence are the dominant mode for false-vacuum decay. In this paper we shall consider the O(3)-symmetric background spacetime of a static, spherically symmetric star, and assume that a nucleating bubble centered at the star will have the O(3)-symmetry of the spacetime.

It is illustrative to consider a simple model which may help explain why gravitationally compact objects might act as nucleation sites for false-vacuum decay. The Euclidean action for an O(4)-symmetric nucleating bubble in the "thin-wall" approximation may be written as

$$\mathcal{S}=\sigma\mathcal{A}-\epsilon\mathcal{V}, \quad \text{with } \sigma,\epsilon>0 , \tag{1}$$

where $\mathcal{A}$ corresponds to the surface area of the nucleating bubble and $\mathcal{V}$ to its volume (remembering that these are "three-areas" and "four-volumes"). It is assumed that we have a fixed background spacetime and so the spacetime does not contribute, as such, to B (i.e., the difference between the Euclidean action for the spacetime with, and without, the nucleating bubble); thus we may write $B=\mathcal{S}$. The "decay rate," which is proportional to e^{-B}, may be enhanced by reducing B; so if we could find a way of increasing $\mathcal{V}$ while keeping $\mathcal{A}$ constant (with σ and ϵ also constant) then this reduction in B would be achieved. Obviously, in a flat spacetime once we have fixed the value of $\mathcal{A}$ then $\mathcal{V}$ is automatically defined; however, this need not be true in a curved spacetime. The familiar embedding diagrams of stars [9] illustrate this idea by demonstrating that the volume contained within a fixed proper surface area may be increased by the curved spacetime associated with the "star," as compared to the respective contained volume in flat spacetime (the use of the word "star" will generally refer to the generic group of gravitationally compact objects, e.g., neutron stars, planets, boson stars, monopoles, etc.).

We shall consider false-vacuum decay, and bubble nucleation, associated with a quartic polynomial potential (in a similar manner to Samuel and Hiscock [6,7]). This potential is given by

$$U(\phi)=m^2\phi^2-\eta\phi^3+\lambda\phi^4 , \tag{2}$$

which we shall refer to as a $\phi^{2\text{-}3\text{-}4}$ potential. This potential has the attractive feature of being the simplest polynomial potential that allows us to independently vary the relative energy-density separation of the true-vacuum state from the false-vacuum state, and the field separation between the true- and false-vacuum states. For an appropriate choice of parameters (including m^2, η, and $\lambda>0$), this potential will have a false-vacuum state located at $\phi=0$, with zero energy density. We shall label the value of the scalar field at the true-vacuum state by ϕ_+, and the dimensionless vacuum energy density of the true-vacuum state shall be given by $-\tilde{\epsilon}$.

We use this potential only to obtain the O(4)-symmetric bubble profiles associated with the false-vacuum decay in the absence of a nucleation site (i.e., flat space). These bubble profiles are then utilized as the background in the perturbative O(3)-symmetric analysis of bubble nucleation around a "star." For this reason, the analysis and results of this paper are not really sensitive to the actual form of the potential used, and are quite general.

The nucleation site introduced in the background flat space is a simple model of a boson star (for simplicity); this is assumed to be only moderately gravitationally compact so that it may be treated as a perturbation on the background flat-space metric. The Euclidean action of a bubble forming about the star is then calculated as a perturbation on the O(4) solution, obtained by insisting that the bubble have zero total energy and that its action be minimized. These resulting O(3) Euclidean actions are then compared to the O(4) action to see if the presence of the boson star has enhanced the vacuum decay rate. While our analysis is restricted to weakly gravitationally compact objects acting as nucleations sites, it does not in any way depend on the "thin-wall" approximation scheme.

The results of this work show that gravitationally compact objects can definitely enhance the bubble nucleation rate in a first-order vacuum phase transition. Whether the enhancement of the nucleation rate is physically significant is found to depend on the relative sizes of the O(3) and O(4) Euclidean actions and the number density of nucleation sites available of appropriate size for a particular transition. For at least one example we have examined, namely, the possible decay of the present vacuum due to a heavy top quark (or other as yet undiscovered heavy fermionic species), it appears that O(3) nucleation would dominate over the O(4) process only if the mass density of the universe were strongly dominated by ideal nucleation sites, an unlikely scenario.

II. EUCLIDEANIZED SCALAR FIELD EQUATION IN THE PRESENCE OF A GRAVITATIONALLY COMPACT OBJECT

The scalar field equation in Euclidean signature space may be expressed in the general form

$$g^{-1/2}\partial_\mu(g^{1/2}g^{\mu\nu}\partial_\nu)\phi=\frac{dU}{d\phi}\ , \tag{3}$$

where g is the determinant of the metric. The solution to Eq. (3) provides us with the scalar field profile for the nucleating bubble; this solution may then be used to calculate the Euclidean action.

The Euclideanized metric for a static, spherically symmetric spacetime containing a gravitationally compact object may be written (in a $[\tau,r,\theta,\psi]$ coordinate system) as

$$g_{\mu\nu}=\mathrm{diag}[f(r),h(r),r^2,r^2\sin^2\theta]\ , \tag{4}$$

with the determinant

$$g^{1/2}=(fh)^{1/2}r^2\sin\theta\ . \tag{5}$$

This metric does not in general possess the O(4) symmetry of Euclideanized Minkowski space, though it is O(3) symmetric about the origin of the spatial coordinates. Therefore, the proof of Coleman *et al.* [8] which shows that O(4)-symmetric bubbles have the minimum action in Euclideanized Minkowski space no longer applies. However, it seems reasonable to believe, and it shall be our assumption, that the bubbles with the smallest Euclidean action will have the O(3)-symmetry of the metric. If we insert the expression for the metric into Eq. (3) for the scalar field equation, and assume that the scalar field configuration is spherically symmetric (ϕ depending only upon τ and r), we have

$$\frac{1}{f}\frac{\partial^2\phi}{\partial\tau^2}+\frac{1}{h}\frac{\partial^2\phi}{\partial r^2}+\frac{1}{h}\left[\frac{2}{r}+\frac{1}{2f}\frac{df}{dr}-\frac{1}{2h}\frac{dh}{dr}\right]\frac{\partial\phi}{\partial r}=\frac{dU}{d\phi}\ . \tag{6}$$

This equation is substantially more difficult to solve numerically than any of the equations involved with the O(4) analysis [6,7], being a nonlinear partial differential equation rather than an ordinary differential equation. There is additionally uncertainty about the appropriate boundary conditions necessary to solve the equation: we would expect that as $r\to\infty$ the field ϕ should return to the value corresponding to the false vacuum (i.e., $\phi\to 0$ for the $\phi^{2\text{-}3\text{-}4}$ model). However, the boundary condition for finite r, as $\tau\to\infty$ [and similarly, $\tau\to(-)\infty$], is uncertain. A "cylinder" solution with the O(3) symmetry of the metric may have $(\partial\phi/\partial\tau)=0$ for all values of τ, which would complete our required boundary information. We may also consider oscillating cylindrical solutions, with the O(3)-symmetry of the metric; and even solutions where $\phi\to$(false-vacuum value) as $\tau\to(+/-)\infty$ (for all values of r), where the solution would again have the O(3) symmetry of the metric. In this last situation the nucleating bubble might look like a prolate or oblate spheroid with rotational symmetry about the τ axis. Each of these possibilities would correspond to different sets of boundary conditions.

III. A PERTURBATIVE ANALYSIS OF THE EFFECT OF A GRAVITATIONALLY COMPACT OBJECT ON THE NUCLEATING BUBBLE

The line element corresponding to the metric of Eq. (4) may be written

$$ds^2=f(r)d\tau^2+h(r)dr^2+r^2d\Omega^2\ , \tag{7}$$

where $d\Omega$ is the line element on the unit two-sphere. If $f(r)=h(r)=1$ (i.e., in the absence of a star) the spacetime returns to its O(4)-symmetric state, and the analysis of Samuel and Hiscock [6,7] is appropriate for determining the exact field profile, and Euclidean action of the nucleating bubble.

Consider the situation where we have a star, which is not too compact, in our spacetime (this would rule out black holes for example). We may put this notion on a more precise footing by writing the metric $f(r)$ and $h(r)$ functions as

$$f(r)=1+F(r) \tag{8}$$

and

$$h(r)=1+H(r)\ , \tag{9}$$

and demanding that $|F(r)|\ll 1$, and $|H(r)|\ll 1$, for all r. The star then acts as a weak O(3) perturbation on the background O(4)-symmetric Minkowski space.

We are now forced to make an important assumption about the nucleating bubble in the spacetime of the star. As $F(r)$ and $H(r)$ move away from zero (i.e., from a flat spacetime to a spacetime corresponding to a very "dilute" star), we assume that the nucleating bubble will move away from its O(4)-symmetric state to an O(3)-symmetric state, corresponding to the O(3) symmetry of the metric, in a *continuous* manner. What we mean by this is that when $|F(r)|$ and $|H(r)|$ are very small, the nucleating bubble will have the form of either an oblate or prolate spheroid (i.e., we shall consider the lowest, nonvanishing, multipole perturbation of the O(4) bubble, which will correspond to the "quadrupole" spherical harmonics). The perturbations are required to have the O(3) symmetry of the metric (i.e., the τ axis shall be the axis of rotational symmetry for the spheroid). This assumption appears to be reasonable as it would seem unlikely that the O(4)-symmetric bubble corresponding to $F=H=0$ would, for example, suddenly transform into an infinite cylinder around the τ axis, for an infinitesimally small $F(r)$ and $H(r)$. Such a scenario would perhaps imply an instability of the O(4)-symmetric, flat-spacetime bubble to fluctuations in the metric.

The O(4)-symmetric bubble in the background Minkowski space (i.e., $F=H=0$) has a field profile given by

$$\phi_4=\phi(\rho)\ , \tag{10}$$

with ρ, the O(4) radial variable, being defined by

$$\rho^2=\tau^2+r^2\ . \tag{11}$$

We shall now consider the prolate/oblate spheroidal solution

$$\phi_3=\phi[\bar{\rho}]\ , \tag{12}$$

with

$$\bar{\rho}^2=\rho^2+A\rho^2\cos^2\theta+B\rho^2\sin^2\theta\ , \tag{13}$$

where θ small be the angle measured from the r axis in the two-dimensional (τ,r) space; i.e., $\theta=\arctan(\tau/r)$. Thus the solution to the perturbative analysis will require the determination of the two coefficients A and B. It will be convenient for us to rewrite Eq. (13) as $\bar{\rho}^2=w_\tau\tau^2+w_r r^2$, requiring the determination of the coefficients w_τ and w_r.

The two constraints which allow us to determine the coefficients, w_τ and w_r, are the requirements that the nucleating bubble have zero energy, and that the Euclidean action for the nucleating bubble be an extremum. The energy of the nucleating bubble evaluated on a spacelike hypersurface Σ is given by

$$E=\int_\Sigma T_{\mu\nu}t^\nu d\sigma^\mu\ , \tag{14}$$

where $T_{\mu\nu}$ is the stress-energy tensor for the scalar field, $d\sigma^\mu$ provides the measure for the "surface" integral (remembering that this is a three-surface) and is a normal vector to the hypersurface Σ, and t^ν is the time-translation Killing vector field. The energy defined in this way is conserved, even for time-varying stress-energy tensors, so long as the background spacetime contains a timelike Killing vector field [10].

The stress-energy tensor for a minimally coupled scalar field takes the form

$$T_{\mu\nu}=\partial_\mu\phi\partial_\nu\phi-\tfrac{1}{2}g_{\mu\nu}g^{\alpha\beta}\partial_\alpha\phi\partial_\beta\phi-g_{\mu\nu}U\ . \tag{15}$$

Taking Σ to be the $\tau=0$ hypersurface, we will have $d\sigma^\mu=(f^{-1/2},0,0,0)d\sigma$, where $d\sigma$ is the three-volume element, i.e., $d\sigma=(h^{1/2}r^2\sin\theta)dr\,d\theta\,d\psi$, and $t^\nu=(1,0,0,0)$. The expression for the energy then reduces to (after integrating over the angular coordinates)

$$E=4\pi\int_0^\infty r^2(fh)^{1/2}\left[\frac{1}{2h}\left[\frac{\partial\phi}{\partial r}\right]^2+U\right]dr\ . \tag{16}$$

As we have taken Σ to be the $\tau=0$ hypersurface then the $(\partial\phi/\partial\tau)$ term vanishes due to a reflection symmetry of the bubble profile about this hypersurface. The energy will thus be a function only of the parameter w_r, and the requirement of zero energy (i.e., $E=0$) will therefore allow us to fix this parameter. With w_r known, the Euclidean action will now be a function only of the parameter w_τ. Therefore, by evaluating the Euclidean action for a series of values of w_τ, we may determine the value of w_τ which extremizes the Euclidean action, and hence corresponds to the nucleating bubble.

The Euclidean action for the nucleating bubble is given by

$$S_E=\int d^4x\, g^{1/2}[\tfrac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi+U(\phi)] \tag{17}$$

$$=8\pi\int_0^\infty\int_0^\infty(fh)^{1/2}r^2\left[\frac{1}{2}\frac{1}{f}\left[\frac{\partial\phi}{\partial\tau}\right]^2+\frac{1}{2}\frac{1}{h}\left[\frac{\partial\phi}{\partial r}\right]^2+U(\phi)\right]d\tau\,dr\ . \tag{18}$$

This expression is considerably simplified for the O(4)-symmetric bubble, where we may express the Euclidean action in terms of a one-dimensional integral, associated with the O(4) radial variable, ρ. This gives

$$S_E[\mathrm{O}(4)]=-2\pi^2\int\rho^3U(\phi)d\rho\ . \tag{19}$$

The O(3) Euclidean action for a bubble may be calculated using Eq. (18), and the value of w_τ fixed by minimizing the Euclidean action. The resulting zero-energy, minimized action O(3) bubble may then be compared with the "background" O(4) bubble, with action calculated using Eq. (19). If the O(3) action is smaller than the O(4) action, then the "star" is successfully acting as a nucleation site for the vacuum phase transition.

Thus we have a framework by which we may perform a perturbative analysis of the effects of a gravitationally compact object on false-vacuum decay. In order to perform an actual calculation, however, it will be necessary to have the metric functions $f(r)$ and $h(r)$ for the star.

IV. THE METRIC FUNCTIONS FOR A "TOY" BOSON STAR

It would not be sensible to construct a realistic stellar model as a first step in an attempt to model the effects of a gravitationally compact object on false-vacuum decay. This is because the determination of the metric functions $f(r)$ and $h(r)$ for a realistic stellar model is quite a complex task, requiring a knowledge of the equations of state for the matter that comprises the star, for example. While we have some knowledge of the nuclear equation of state at densities relevant to neutron-star models, we have little detailed knowledge of the equation of state for possible microphysical compact objects, such as boson stars or monopoles. What we require is a simple model star, where we are not overburdened by a large number of free, stellar parameters; thus a simple model which would only have two parameters such as the mass of the star and the "compactness" of the star would be appropriate.

Further, we may expect (and this will later be verified) that if a star is going to have a significant effect on the nucleating bubble, then the size of the star would have to be comparable to the size of the nucleating bubble. In a cosmological setting this would imply that the star would have to have a size corresponding to a "grand unified length scale"; thus we are dealing with "stellar" candidates such as boson stars and monopoles. The model star should also be a reasonable model for these candidates.

One exact analytic solution to the Einstein equations which is an obvious candidate for our model is the interior Schwarzschild solution, which represents a constant density star. These solutions are, however, undesirable for our model "star" because they do not act as good models for boson stars and monopoles. Also, the radial derivatives of the metric functions $f(r)$ and $h(r)$ are not continuous at the stellar boundary; this may lead to complications in the numerical analysis, which would not be an attractive feature in an initial model.

The following functions will be used for the metric of the "model star":

$$h(r)=\frac{1}{1-\frac{2\kappa}{r}\left\{1-\exp\left[-\left[\frac{r}{2\kappa\gamma}\right]^3\right]\right\}}, \tag{20}$$

$$f(r)=\begin{cases}1-\frac{2\kappa}{r}, & r\geq 2\kappa\gamma, \quad (21)\\ 1-\frac{e^{1/2}}{\gamma}\exp\left[\frac{-1}{2}\left[\frac{r}{2\kappa\gamma}\right]^2\right], & r<2\kappa\gamma . \quad (22)\end{cases}$$

These metric functions are desirable because they possess a simple algebraic form; they are continuous, together with their derivative functions (i.e., sufficiently smooth to realistically model a boson star); they have a small parameter space (i.e., only two parameters κ and γ); and generally provide a qualitatively reasonable and tractable model for a stellar candidate such as a boson star (see, for example, Seidel and Suen [11] for a discussion of boson star metrics).

The mass of the star is given by the parameter κ, and the parameter γ represents the compactness of the star. As is typical of boson stars, this model does not possess a well-defined boundary; however, a characteristic size of the star [as we may observe below in Eq. (23)] is given by $2\kappa\gamma$; this illustrates the role that γ plays in representing the compactness of the "star."

We may write the mass function for the star as

$$M(r)=\kappa\left\{1-\exp\left[-\left[\frac{r}{2\kappa\gamma}\right]^3\right]\right\}, \tag{23}$$

where $M(r)$ is the usual mass function defined in terms of the metric function $h(r)$ by $M(r)=r(h-1)/(2h)$. This expression gives the mass of the star contained within the radius r, so

$$M(r\rightarrow\infty)=\kappa , \tag{24}$$

and the fraction of the mass contained within the "boundary" of the star is given by

$$M(r=2\kappa\gamma)=(1-e^{-1})\approx 63\% . \tag{25}$$

The (t,t) component of the stress-energy tensor for the metric provides the information about the local energy density for the matter which comprises the star. This may be expressed in terms of the metric $h(r)$ function using the Einstein equations

$$-8\pi T_t^t=\frac{1}{rh^2}\frac{dh}{dr}+\frac{1}{r^2}\left[1-\frac{1}{h}\right]. \tag{26}$$

For the metric under consideration this reduces to

$$-T_t^t=\rho=\frac{3}{32\pi\kappa^2\gamma^3}\exp\left[-\left[\frac{r}{2\kappa\gamma}\right]^3\right], \tag{27}$$

where ρ represents the energy density of the stellar model. This tells us that we have an exponentially decaying positive energy density as we move away from the center of the star. Figures 1–3 display the metric function $f(r)$, the mass function $M(r)$, and the corresponding energy-density function $\rho(r)$ for the model star as functions of r, with parameters $\kappa=1$, and $\gamma=5$.

The metric function $f(r)$ is piecewise constructed such that "outside" the star (i.e., $r>2\kappa\gamma$) the metric function corresponds to the exterior Schwarzschild solution. [N.B. The metric function $h(r)$ does not correspond to the Schwarzschild solution "outside" the star; thus the exterior is *not* described by the vacuum Schwarzschild solution.] At $r=2\kappa\gamma$ an interior function which is

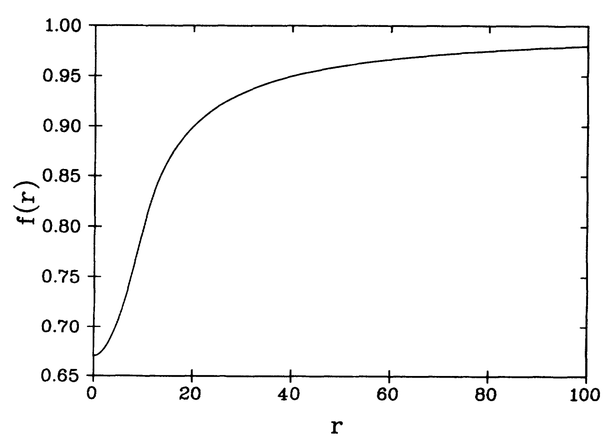

FIG. 1. Metric function $f(r)$ for a model boson star with metric parameters $\kappa=1.0$ and $\gamma=5.0$; this would result in a dimensionless stellar radius $r_{\text{star}}=10$.

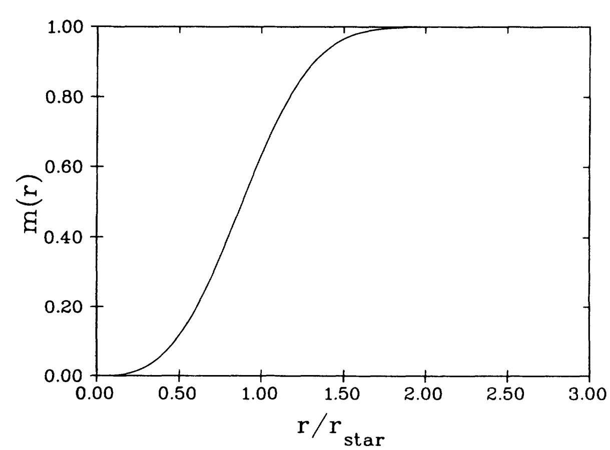

FIG. 2. The stellar mass function $M(r)$ with metric parameters $\kappa=1.0$ and $\gamma=5.0$.

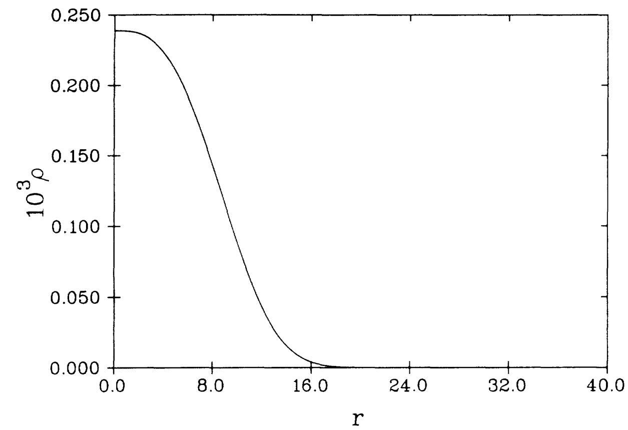

FIG. 3. The energy density for the matter comprising the star as a function of radius, with metric parameters $\kappa=1.0$ and $\gamma=5.0$.

analytically simple and roughly appropriate to the description of a boson star is joined to the exterior Schwarzschild function so that the overall metric function $f(r)$ together with its derivative df/dr are continuous at this point. The form of the interior function is such that it admits a minimum value for the γ parameter, beyond which the perturbative analysis intrinsically breaks down. This value occurs at $\gamma=e^{1/2}\cong 1.65$, for which $f(r=0)=0$ [i.e., $F(r)=-1$, which strongly violates the condition, $|F(r)|\ll 1$, necessary for the perturbative analysis]. Of course, the validity of the perturbative analysis will be brought into question before this value of γ is reached.

V. RESULTS

Figure 4 shows the ratio of B_3 to B_4 (evaluated numerically) as a function of the stellar mass κ for a series of values of γ; the scalar field potential parameters are held fixed at the values of $\tilde{\varepsilon}=1.0$ and $\psi_+=1.0$. B_3 is the Euclidean action for the bubble in the spacetime of the gravitationally compact object and B_4 is the Euclidean action for the bubble in the background flat spacetime. Note that, as the analysis adopts a fixed background spacetime, there is no contribution from the spacetime, as such, to the value of B (the difference between the Euclidean action for the spacetimes with and without the nucleating bubble); we may therefore write $B_3=S_3$ and $B_4=S_4$.

We observe a phenomenon which may be described as a "resonance" for the B_3/B_4 curves. As the mass of the star goes to zero the Euclidean action of the nucleating bubble approaches the O(4)-symmetric value, as would be expected. Similarly, for large mass stars, with a constant value of γ (e.g., this might correspond to microscopic bubble forming at the center of the Earth), the Euclidean action for the nucleating bubble approaches the O(4)-symmetric value. However, there is a range of masses for which the O(3) Euclidean action of the nucleating bubble drops significantly below the O(4)-symmetric value. The prominence of this drop away from the O(4)-symmetric value increases with a decrease in the value of γ; i.e., the effect increases with an increase in the compactness of the star. The value of κ at which the minimum in the Euclidean action occurs is dependent upon the compactness of the star, i.e., upon γ; we shall later find the exact relationship between κ and γ for the minimum in the Euclidean action.

Figures 5 and 6 show the w_τ and w_r values corresponding to the metric and potential parameters of Fig. 4. These curves imply that the minimum action O(3) nucleating bubble is a prolate spheroid [prolate in the τ-r Euclidean space; the bubble is of course still O(3) spherically symmetric]. The O(3) radius (i.e., the size of the bubble on the $\tau=0$ slice) is slightly less than the corresponding radius for the O(4)-symmetric bubble (i.e., $w_r>1$). We may think of this as the effect of the gravitational field of the boson star "pulling in" the bubble wall.

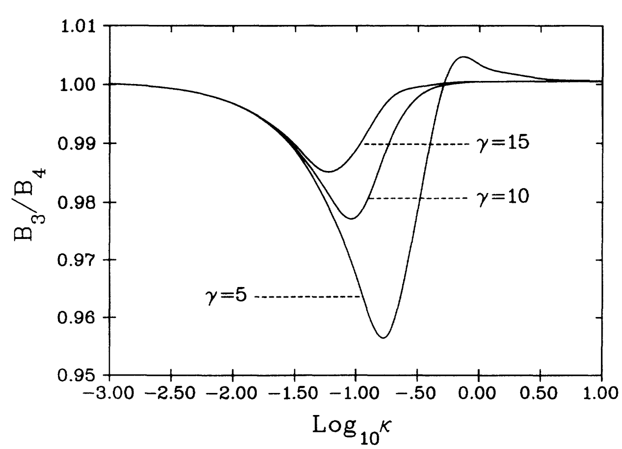

FIG. 4. The ratio of the O(3) to O(4) Euclidean actions (B_3/B_4) for a range of metric parameters κ and γ; the scalar field potential parameters are fixed at the values of $\tilde{\varepsilon}=1.0$ and $\psi_+=1.0$.

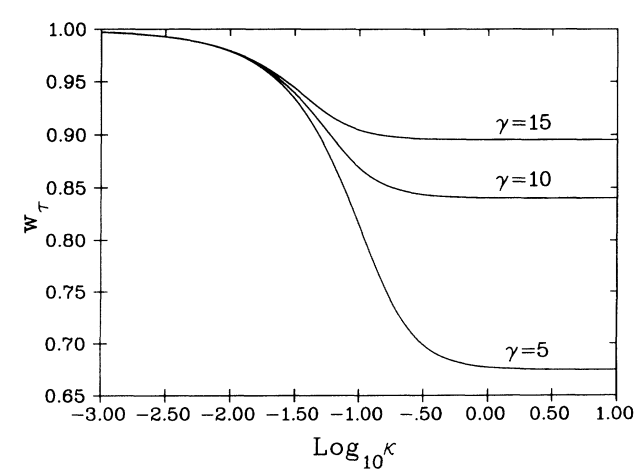

FIG. 5. Deformation of the bubble in the τ direction (w_τ) for a range of metric parameters κ and γ; the scalar field potential parameters are fixed at the values of $\tilde{\varepsilon}=1.0$ and $\psi_+=1.0$.

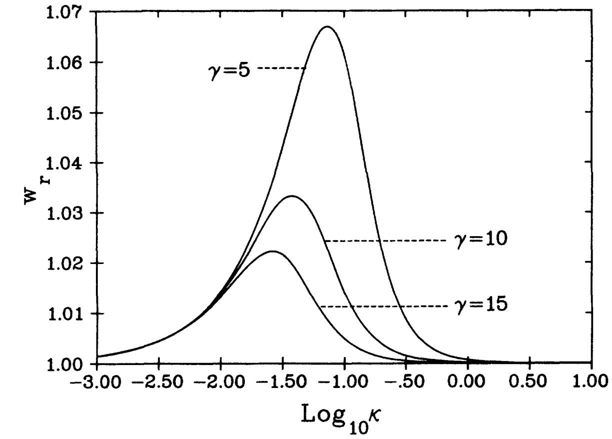

FIG. 6. Deformation of the bubble in the radial direction (w_r) for a range of metric parameters κ and γ; the scalar field potential parameters are fixed at the values of $\bar{\epsilon}=1.0$ and $\psi_+=1.0$.

The "size" of the bubble along the τ axis is larger than that corresponding O(4)-symmetric bubble, thus giving a prolate appearance. Figure 7 illustrates the evolution of the shape of the nucleating bubble from the O(4)-symmetric configuration to the O(3)-symmetric, resonant configuration.

When the star is much larger than the nucleating bubble (i.e., $2\kappa\gamma$ is much larger than the dimensionless radius of the bubble), we may write the metric line element, as seen by the bubble nucleating at the center of the star, in the approximate form

$$ds^2=f_0 d\tau^2+dr^2+r^2d\Omega^2 , \tag{28}$$

where

$$f_0=1-\frac{e^{1/2}}{\gamma} ; \tag{29}$$

i.e., $f_0=f(r=0)$, for the given set of metric parameters.

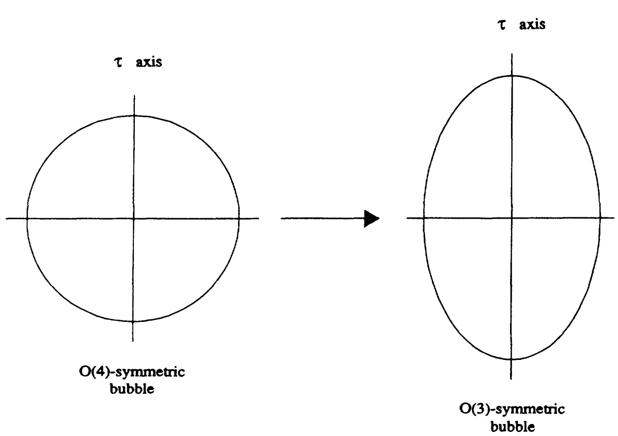

FIG. 7. Characteristic evolution of the shape of the nucleating bubble at, and near, resonance (maximum decrease in action), from the O(4)-symmetric form in a flat spacetime to an O(3) prolate spheroid (with τ as the rotational symmetry axis), in the spacetime of a "dilute" gravitationally compact object.

We may make this approximation because the variation in $f(r)$ over the spatial extent of the bubble will be negligible. Similarly, the coefficient of dr^2 is taken to be $h(r=0)=1$; the spatial variation of this metric function is also negligible over the spatial extent of the bubble.

Defining a new variable, $\bar{\tau}=f^{1/2}\tau$, we may rewrite the approximate line element as

$$ds^2=d\bar{\tau}^2+dr^2+r^2d\Omega^2 . \tag{30}$$

This line element is O(4) symmetric in the $(\bar{\tau},r,\theta,\psi)$ space, and so the solution to the Euclideanized scalar field equation is just the usual $\phi(R)$, where $R^2=\bar{\tau}^2+r^2$. In terms of the original (τ,r,θ,ϕ) coordinate system, the nucleating bubble solution is again $\phi(R)$, where we now express R^2 as $R^2=f_0\tau^2+r^2$. Thus the nucleating bubble will be a prolate spheroid, with the τ axis being the axis of rotational symmetry; the perturbation parameters w_τ and w_r take the values f_0 and 1.0, respectively. Figures 5 and 6 verify this analysis, where, for large $\kappa, w_r\rightarrow 1$ and $w_\tau\rightarrow f_0$.

If the nucleating bubble is much larger than the characteristic size of the star (i.e., $\kappa\rightarrow 0$) then the volume "surplus" due to the star will be negligible compared to the volume of the bubble. In this case we would not expect the star to have a significant effect upon the nucleating bubble. This is verified with the results shown in Figs. 4, 5, and 6, where $B_3\rightarrow B_4$, $w_\tau\rightarrow 1$, and $w_r\rightarrow 1$ as $\kappa\rightarrow 0$.

The quantity $(2\kappa\gamma)$ represents the characteristic size of the star. It may, therefore, be useful to plot the ratio

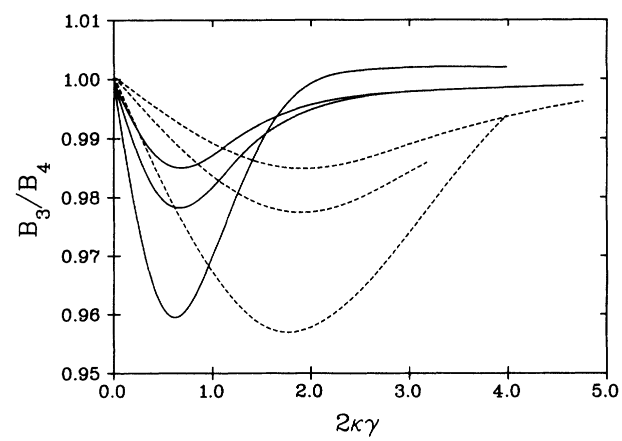

FIG. 8. Superposition of two sets of B_3/B_4 curves, the solid curves corresponding to potential parameters $\bar{\epsilon}=1.0$ and $\psi_+=1.0$, and the dashed curves corresponding to potential parameters $\bar{\epsilon}=0.2$ and $\psi_+=1.0$. The ratio of the Euclidean actions, (B_3/B_4), is plotted against the characteristic size for the star, $2\kappa\gamma$. The curves, for a given set of potential parameters, correspond to the γ parameters: $\gamma=5$ (deepest trough), $\gamma=10$ (middle trough), and $\gamma=15$ (shallowest trough). The potential parameters $\bar{\epsilon}=1.0$, $\psi_+=1.0$ result in a nucleating bubble of characteristic (dimensionless) radius $r_{\text{bubble}}=1.3$, and the potential parameters $\bar{\epsilon}=0.2$, $\psi_+=1.0$ result in a nucleating bubble of characteristic (dimensionless) radius $r_{\text{bubble}}=4.3$. The maximum decrease in the Euclidean action is seen to occur when the star is typically about half the size of the nucleating bubble, independent of the scalar potential.

(B_3/B_4) against $(2\kappa\gamma)$ to provide some insight into the relationship between the stellar size and the minimum Euclidean action O(3) bubble. Figure 8 provides such a plot in which we have superimposed the solution curves corresponding to two sets of potential parameters: namely, $\tilde{\epsilon}=1.0$ and $\tilde{\epsilon}=0.2$, both with $\psi_+=1.0$. The first thing that is noticed from Fig. 8 is that the maximum decrease in the Euclidean action always occurs at roughly the same value of $(2\kappa\gamma)$ for a given set of potential parameters. In particular, if we refer to the size of the O(4)-symmetric bubble for the given set of potential parameters, we notice that the maximum decrease occurs when the star is roughly half the size of the nucleating bubble. This is roughly what one would expect from general intuitive arguments. Again, as with Fig. 4, the maximum decrease in the O(3) Euclidean action increases with an increase in the compactness of the star; this also agrees with general intuitive arguments.

For the scalar field potential parameters under consideration, a stellar compactness of $\gamma=5$ results in a characteristic 4% to 5% reduction in the Euclidean action of the nucleating bubble, as compared to the O(4)-symmetric bubble action. This is in qualitative agreement with the predictions of the "thin-wall" analysis. It is not possible to make a direct quantitative comparison between the perturbative results and the "thin-wall" results due to fundamental differences in the approximations used in the formulation of the two analyses. However, the "thin-wall" analysis of Mendell and Hiscock, in which the effect of a neutron star on false-vacuum decay was considered, resulted in values of B_3 which were characteristically 30% to 50% lower than B_4. If we considered smaller values of γ in the perturbative analysis (i.e., more compact stars) then we might expect reductions in B_3 which would be closer to the "thin-wall" analysis. However, the validity of the perturbative approach may then be brought into question.

The "thin-wall" analysis of Hiscock, and Mendell and Hiscock, considered solutions to the coupled Euclideanized scalar field and Einstein equations (via the Israel formalism) which had the form of infinite, oscillating cylinders, where the τ axis was the axis of symmetry. The perturbative analysis has shown that the nucleating bubble, at resonance, takes the form of a prolate spheroid [Fig. 7 illustrates the evolution of the O(4)-symmetric bubble in the flat spacetime to the form of a prolate spheroid in the stellar spacetime]. The extent of this prolation is increased with an increase in the stellar compactness. It is therefore conceivable that a transition takes place where the nucleating bubble transforms from a prolate spheroid to an oscillating cylinder of the "thin-wall" analysis for a sufficiently compact star.

We may now address the question of whether gravitationally compact objects will have a significant effect on false-vacuum decay.

VI. FALSE-VACUUM "DECAY RATE" IN THE PRESENCE OF A GRAVITATIONALLY COMPACT OBJECT

In an O(4)-symmetric spacetime, the number of O(4)-symmetric nucleating bubbles per unit four-volume, associated with false-vacuum decay, is given by

$$\Gamma = Ae^{-B_4} , \tag{31}$$

where B_4 is the difference between the Euclidean action for the spacetime with and without the nucleating bubble. The coefficient A typically has an order of magnitude given by the field mass to the fourth power, i.e., m^4.

For example, if we had a square box with sides of length L, and volume L^3, the characteristic time that one would have to wait for a nucleating bubble to appear is

$$T_4 = L^{-3} m^{-4} e^{B_4} . \tag{32}$$

Of course, the nucleating bubble may appear anywhere within the box; there is no preferred location for the formation of the bubble. However, when we consider the effect of a gravitationally compact object on false-vacuum decay, we make the assumption that the bubble will be forming around the compact object. The nucleating bubble field profile and Euclidean action are obtained under this assumption. We must therefore use a modified form of Eq. (32) to calculate the characteristic time associated with the formation of a nucleating bubble around the compact object. Of course, the nucleating bubble could form anywhere within the stellar spacetime. However, the Euclidean action B_3 and the associated "decay rate" formula relate to the formation of the bubble around the star. We may use, as a good approximation, Eq. (32) to estimate the rate of bubble nucleation in a stellar spacetime for bubbles not forming around the star.

If we place one compact object (star) into our originally flat, O(4)-symmetric spacetime, where the Euclidean action for a nucleating bubble around the star is given by B_3, then the characteristic time associated with the formation of the nucleating bubble around the star is

$$T_3 = Ce^{B_3} , \tag{33}$$

where the coefficient C typically has an order of magnitude given by m^{-1}. If there is more than one star in the spacetime then we may make a "dilute-gas" approximation in order to calculate T_3. The dilute-gas approximation basically assumes that the stars do not interfere with one another, with regards to the bubble formation process. This requires the typical distance between the stars to be much greater than the characteristic size of the stars, and also much greater than the characteristic size of the nucleating bubbles. Thus the spacetime "between the stars" is also approximately flat.

If there are N similar stars (i.e., same values of κ and γ within our stellar model) in an otherwise empty spacetime, and B_3 is the Euclidean action associated with the formation of a nucleating bubble around a star, then the characteristic time for a single bubble to form around one of the stars is

$$T_3 = N^{-1} m^{-1} e^{B_3} . \tag{34}$$

To address the question of whether gravitationally compact objects play an important role in false-vacuum decay, it is necessary to compare to characteristic times for nucleating bubble formation, i.e., T_3 and T_4. Thus, if

$T_3 < T_4$ then bubbles formed about gravitationally compact objects will dominate the false-vacuum decay process. However, if $T_3 > T_4$ then gravitationally compact objects will not play an important role in false-vacuum decay. It is not sufficient merely to compare the values of B_3 and B_4 in order to determine the dominant mechanism for false-vacuum decay.

Consider a spacetime with three-volume V and N similar stars. The ratio of T_3 to T_4 is given by

$$\frac{T_3}{T_4} = \frac{Vm^3}{N}\exp(B_3 - B_4) . \qquad (35)$$

For known values of m, B_3, and B_4, this expression provides us with the number density of "stars" needed for the gravitationally compact objects to play the dominant role in false-vacuum decay [i.e., stars will play the dominant role in the decay mechanism if $(N/V) > m^3\exp(B_3 - B_4)$].

As an example, we may apply the above analysis to possible first-order false-vacuum decay associated with a phase transition at the electroweak scale. The form of the electroweak potential is strongly dependent upon the mass of the top quark (see, for example, Mahanthappa and Sher [12], and Flores and Sher [13]). For a sufficiently "light" top quark, our current vacuum state is a true-vacuum state, and therefore stable. However, for a more massive top quark, our current vacuum state is rendered a false-vacuum state, and therefore unstable to quantum decay.

As the shape of the electroweak potential is a function of the top-quark mass then the lifetime of a possible false-vacuum state will be a function of the top-quark mass. We know that our current vacuum state has been in existence for about 10^{10} yr, and so this allows us to place an upper bound upon the mass of the top quark (e.g., the mass of the top quark cannot be so large that it would have resulted in a false-vacuum decay after a few seconds).

We may therefore ask whether nucleated false-vacuum decay is likely to play a dominant, or important, role associated with a possible false-vacuum decay at the electroweak energy scale, *at this current epoch* within the Universe, if the mass of the top quark is assumed to be large enough that we are currently living in a supercooled false-vacuum state.

Let us write the Euclidean action associated with a bubble nucleating around a gravitationally compact object [an O(3) bubble] in terms of the Euclidean action for an O(4) bubble, and a "deficit" parameter α:

$$B_3 = (1-\alpha)B_4 . \qquad (36)$$

The perturbative analysis performed within this paper resulted in deficit parameters in the region of $\alpha \approx 0.05$, whereas the "thin-wall" analysis of Hiscock resulted in larger deficit parameters (e.g., $\alpha \approx 0.25$) associated with black holes. The analysis of Mendell and Hiscock was able to generate even larger deficit parameters, within a "thin-wall" analysis, but the model "star" that they used would be a very unrealistic model of a "microscopic" gravitationally compact object.

We shall assume an optimal scenario where the entire mass density within the Universe is in the form of "optimal" nucleation sites, and where the deficit associated with the nucleation sites is α. To obtain an estimate of the mass of a nucleation site we shall assume that it is a black hole, with a Schwarzschild radius equal to the characteristic length scale of the field undergoing the phase transition (i.e., the electroweak scale). Thus the gravitationally compact object will be comparable in size to the nucleating bubble, which is required for it to act efficiently as a nucleation site. As the size of the black-hole nucleation sites are known, then their mass is also known (i.e., a characteristic length scale $\approx 10^{-16}$ cm, giving a characteristic mass for the nucleation sites $M_{\rm nuc} \approx 10^{12}$ g).

The maximum allowed mass density of the Universe, within observational constraints, is $\rho_{\rm max} \approx 10^{-28}$ g cm^{-3}; this gives a bound on the maximum number density of nucleation sites:

$$n_{\rm max} < \frac{\rho_{\rm max}}{M_{\rm nuc}} = 10^{-40}\ {\rm cm}^{-3} , \qquad (37)$$

where $n_{\rm max}$ is the maximum allowed number density of nucleation sites.

For the nucleation process to dominate over the spontaneous creation of bubbles, we require a number density of nucleation sites given by

$$n_{\rm req} > m^3\exp(-\alpha B_4) . \qquad (38)$$

Inserting $m \approx 10^{16}$ cm^{-1} and $\alpha = 0.25$ (from the "thin-wall" analysis of Hiscock) into Eq. (38), and setting $B_4 \approx 400$, the value associated with O(4) spontaneous bubble nucleation and false-vacuum decay at the present epoch [i.e., from placing $T_4 = 10^{10}$ yr into Eq. (32), and solving for B_4], we find

$$n_{\rm req} > 10^3\ {\rm cm}^{-3} , \qquad (39)$$

which is much greater than the observationally allowed value of 10^{-40} cm^{-3}. Thus we conclude that *nucleated electroweak false-vacuum decay* at this current epoch within the Universe is far less likely than spontaneous false-vacuum decay. It should be pointed out that this sort of conclusion is very strongly dependent on the particular value of α; if we had assumed that an O(3) action decrease corresponding to $\alpha = 0.5$ were possible in this case, then the limit would be much weaker: $n_{\rm req} > 10^{-39}$ cm^{-3}. This strong dependence illustrates the need for further exact O(3) Euclidean action calculations (without the use of the "thin-wall" or any other approximation) for physically interesting vacuum phase transitions.

VII. CONCLUSIONS

We have found that gravitationally compact objects such as "stars" can have the effect of reducing the Euclidean action for the nucleating bubble of a first-order phase transition, as compared to the Euclidean action in the absence of the star. Thus, such objects will act as nucleation sites. Within the perturbative analysis, the reduction in the Euclidean action is seen to be at its greatest when the size of the compact object is comparable in size to the nucleating bubble, and the reduction in the Euclidean action increases with an increase in the compactness of the star. As characteristic sizes of nucleating bubbles are given by the characteristic mass scale of the field undergoing the phase transition, for any realistic cosmological phase transitions, possible candidates for nucleation sites would have to be microscopic in dimensions. For example, if the electroweak phase transition was a first-order transition, then the most effective nucleation sites would have a size corresponding to an energy scale of 10^2 GeV, i.e., of the order 10^{-16} cm. Boson stars, topological defects such as grand-unified-theory (GUT) monopoles, and microscopic black holes seem to be the only plausible candidates for such nucleation sites. The importance of nucleation sites to false-vacuum decay is strongly dependent upon their number density within the Universe. Bubble nucleation associated with false-vacuum decay may occur anywhere within the Universe; however, nucleated decay may, by definition, only occur at the nucleation sites. It may often be the case that, though bubble nucleation is more difficult without a nucleation site, there is far more empty "space" for an O(4) bubble to nucleate in than the corresponding few number of nucleation sites for O(3) bubbles (i.e., there is a competition between "ease of nucleation" and "available space" for nucleation). Certainly, for false-vacuum decay at the electroweak scale in the present epoch (caused by heavy fermions), the number density of nucleation sites required so that nucleated decay would be the dominant process is such that we would have to have more than the observed mass density of the Universe within nucleation sites.

The results of this work, obtained using perturbation theory about the O(4)-symmetric Euclidean bubble solutions, but without using the generally suspect "thin-wall" approximation, show that gravitationally compact objects can definitely enhance the bubble nucleation rate in a first-order vacuum phase transition. Further work, in which the full O(3) scalar field equations are solved numerically (i.e., without using perturbation theory or the "thin-wall" approximation), is clearly called for.

ACKNOWLEDGMENTS

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