Gravitational effects of traveling waves along global cosmic strings

Athanasios Economou
Fakultät für Physik der Universität Konstanz, Postfach 5560, D 7750 Konstanz, Germany

Diego Harari
Instituto de Astronomía y Física del Espacio, Casilla de Correo 67, Sucursal 28, 1428 Buenos Aires, Argentina

María Sakellariadou
Université Libre de Bruxelles, RGGR, Faculté des Sciences, CP231, B 1050 Bruxelles, Belgium
(Received 17 June 1991)

Making use of the relationship between the corresponding field configurations, we derive the metric around a straight global cosmic string with traveling waves in terms of the static metric (without traveling waves) in the weak-field limit. We discuss under which conditions the effect of the traveling waves may overcome the repulsive gravitational potential of the static straight global string. We also extend the calculation beyond the weak-field limit combining our result with the recent observation made by Garfinkle and Vachaspati that the exact solution must be of the generalized Kerr-Schild type.

PACS number(s): 98.80.Cq, 04.20.Jb

I. INTRODUCTION

Recent numerical simulations of gauge cosmic-string networks suggest that intercommutation of string segments leads to significant perturbations on scales much smaller than the correlation length of the network [1–3]. On average over distances larger than the typical wavelength of the perturbations, a wiggly string appears smooth, but with mass per unit length, μ, larger and tension T smaller than their unperturbed values, the product μT remaining constant [4]. The exact nature and amount of small-scale structure on a string network is crucial to understanding its evolution and its output in gravitational radiation, which may place a crucial bound on the string mass density through its potential observable effects, such as in millisecond pulsar timing [5,6].

The simplest kind of structure, not necessarily of small scale, are perturbations traveling in one direction at the speed of light along an otherwise straight cosmic string. Although special, since they do not dissipate through gravitational radiation [7] as more general small-scale wiggles do, they constitute a tractable and in some aspects representative case of small-scale structure.

Vachaspati was the first to consider gravitational effects, in the weak-field limit, around a class of solutions to the Nambu-Goto equations of motion representing traveling waves along otherwise straight gauge cosmic strings in the zero-thickness approximation [7]. At distances ρ to the string core larger than the characteristic size of the perturbations, the traveling wave exerts a gravitational force proportional to $1/\rho$. More recently, Garfinkle found an exact solution of vacuum Einstein's equations which reduces to Vachaspati's cosmic-string traveling-wave metric in the weak-field limit, which he interpreted as the metric of a strongly gravitating traveling wave along a gauge cosmic string in the zero-thickness limit [8].

Traveling-wave solutions of field theories are also known for global cosmic strings [9]. In this paper we analyze their gravitational effects. Strings formed after the spontaneous breakdown of a global symmetry have gravitational effects quite different than their local counterparts [10], their energy not being confined to a small tube but rather extending into regions far beyond the central core [11]. A static, straight global string produces, in the weak-field limit, a repulsive logarithmic gravitational potential outside the core, in addition to an angular deficit similar to that of a gauge string but also logarithmically dependent on the distance to the core [10]. It is therefore of interest to analyze whether small-scale structure in the form of traveling waves may eventually overcome the repulsive form of the static global cosmic string.

Rather than directly solving Einstein's equations in the weak-field limit, we will exploit a relationship between the field configurations for static infinite strings and strings with traveling waves, respectively, noticed both for gauge as well as for global strings by Vachaspati and Vachaspati [9]. We will see that there is a simple prescription that allows us to obtain the linearized metric around a gauge or global string with traveling waves directly from the metric of the static string.

The existence of a relatively simple relationship between the static and traveling-wave cosmic string metrics was recently noticed by Garfinkle and Vachaspati, and exploited to obtain the exact metric around gauge cosmic strings with traveling waves [12]. They concluded that the exact metric around a cosmic string with a traveling wave is of the generalized Kerr-Schild type, with the static string metric as the background. Their method actually originates a family of solutions to the full Einstein's equations, coupled to the Abelian-Higgs equation for the string fields, of the form

$$g^{\rm TW}_{\mu\nu}=g^{\rm st}_{\mu\nu}+Fk_\mu k_\nu\ , \tag{1.1}$$

where the superscripts TW and st denote traveling wave and static metrics respectively, k_μ is a null, hypersurface orthogonal, Killing field with respect to both metrics, and F is a scalar function that satisfies a well-defined equation. The prescription of Garfinkle and Vachaspati to find the exact metric around a gauge or global string with traveling waves would be complete given a method to univocally fix the function F in terms of the profiles of the traveling waves. In the case of gauge cosmic strings they managed to fill this gap by imposing on F the asymptotic behavior already known from the weak-field limit. Other vacuum solutions, also of the form (1.1) but with different functions F, were shown to represent traveling waves along cosmic strings interacting with additional gravitational waves, or traveling waves along a set of parallel cosmic strings [13].

In the case of global strings imposing asymptotic conditions on F would be much more difficult, since the exact metric presents a singularity at a finite proper distance away from the core [14]. Our prescription to obtain the traveling-wave metric in the weak-field limit, both for gauge as well as global strings, provides this "missing link," since the weak-field limit can be used to determine uniquely the appropriate function F in (1.1) for given profiles of the traveling waves. The combination of Garfinkle and Vachaspati's general result together with our weak-field calculation will thus allow us to write the exact metric around a global string with traveling waves.

II. COSMIC-STRING TRAVELING-WAVE CONFIGURATIONS

In Minkowski spacetime with metric $\eta_{\mu\nu} = \mathrm{diag}(1,1,1,-1)$ and Cartesian coordinates $\{x^\mu\} = \{x,y,z,t\}$ we consider *gauge* cosmic strings made of a complex scalar field ϕ and a gauge vector field A_μ via the usual Abelian Higgs Lagrangian

$$\mathcal{L} = -\tfrac{1}{2}(D_\mu\phi)(D^\mu\phi)^* - \tfrac{1}{4}F^{\mu\nu}F_{\mu\nu} - \tfrac{1}{8}\lambda(|\phi|^2-\eta^2)^2 \ . \tag{2.1}$$

Here $D_\mu \equiv \partial_\mu - ieA_\mu$ and $F_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu$. Hereafter, spacetime indices run from one to four unless otherwise stated, and we have adopted the summation convention for repeated indices. The Lagrangian (2.1) will be also used for the description of *global* cosmic strings by setting the gauge field A_μ equal to zero.

To discuss the cosmic-string traveling-wave solutions it is useful to define the coordinate system $\{X^M\} = \{X,Y,Z,T\}$ related to $\{x^\mu\}$ via

$$\{X^M\} \equiv \{X,Y,Z,T\} = \{x - f(t\pm z), y - g(t\pm z), z, t\} \ , \tag{2.2}$$

where f,g are arbitrary functions of the null coordinates u or v defined as

$$u = z - t = Z - T, \quad v = z + t = Z + T \ . \tag{2.3}$$

For definiteness, and without loss of generality, we will assume from now on that the functions f,g depend only on u. It would of course be the same to take them both to be functions of v only. However they should not be a function of both u and v; neither can one depend on u and the other on v.

Let $\{\phi^{\rm st}(x,y), A_\mu^{\rm st}(x,y)\}$ and $\{\phi^{\rm st}(x,y)\}$ be the field configurations that describe a straight static gauge and global cosmic string, respectively, located along the z axis. Note also that because of the assumed symmetries we can choose a gauge such that $A_t^{\rm st} = A_z^{\rm st} = 0$. Vachaspati and Vachaspati have shown [9] that if the field configuration above is an appropriate solution of the field equations representing a static straight string, then the configuration

$$\begin{aligned} \phi^{\rm TW}(x,y,u) &\equiv \phi^{\rm st}(X,Y) \ , \\ A_\mu^{\rm TW}(x,y,u) &\equiv \frac{\partial X^M}{\partial x^\mu} A_M^{\rm st}(X,Y) \end{aligned} \tag{2.4}$$

also solves the field equations. The fields $\{\phi^{\rm TW}(x,y,u), A_\mu^{\rm TW}(x,y,u)\}$ represent traveling waves moving with the speed of light along the cosmic string (i.e., along the z axis), with profiles in the x and y directions characterized by the functions $f(u)$ and $g(u)$, respectively. *A note on our notation:* In this paper, we denote by $q(X,Y)$ the function which results from a function $q(x,y)$ by changing its arguments x and y to $X = x - f(u)$ and $Y = y - g(u)$, respectively.

The stress-energy-momentum tensor of the cosmic-string fields is

$$T_{\mu\nu} = (D_{(\mu}\phi)(D_{\nu)}\phi)^* + F_{\mu\lambda}F_\nu{}^\lambda + \mathcal{L}\eta_{\mu\nu} \ . \tag{2.5}$$

The relationship (2.4) between the field configurations representing a static string and a string with traveling waves, respectively, also implies a relation between their energy-momentum tensors $T_{\mu\nu}^{\rm TW}(x,y,u)$ and $T_{\mu\nu}^{\rm st}(X,Y)$. Introducing the tensor $S_{\mu\nu}$ defined in terms of the stress-energy-momentum tensor as

$$S_{\mu\nu} \equiv T_{\mu\nu} - \tfrac{1}{2}\eta_{\mu\nu}T_\lambda{}^\lambda \tag{2.6}$$

and using Eqs. (2.4) and (2.5), we find this relation to be

$$S_{\mu\nu}^{\rm TW}(x,y,u) = -Q(X,Y)\eta_{\mu\nu} + \frac{\partial X^M}{\partial x^\mu}\frac{\partial X^N}{\partial x^\nu}[S_{MN}^{\rm st}(X,Y) + Q(X,Y)\eta_{MN}] \ , \tag{2.7}$$

where

$$Q(x,y) = S_{tt}^{\rm st}(x,y) = -S_{zz}^{\rm st}(x,y) = \tfrac{1}{4}[F_{\mu\nu}F^{\mu\nu} - \lambda(|\phi|^2-\eta^2)^2] \ . \tag{2.8}$$

It is interesting to note that away from the string core, when $|\phi| \to \eta$ and $F_{\mu\nu} \to 0$, then $Q \to 0$.

III. THE WEAK GRAVITATIONAL FIELD OF COSMIC-STRING TRAVELING WAVES

Let us first review briefly the weak-field linearized Einstein equations [15]. In this limit, one can find coordinates $\{x^\mu\}$, such that the space-time metric is written as

$$g_{\mu\nu}=\eta_{\mu\nu}+h_{\mu\nu}(\{x^\lambda\}) , \tag{3.1}$$

where $\eta_{\mu\nu}=\eta^{\mu\nu}=\mathrm{diag}(1,1,1,-1)$ and $|h_{\mu\nu}|<<1$. Hereafter, within first order in $|h_{\mu\nu}|$, space-time indices will be raised and lowered using the Minkowski metric $\eta_{\mu\nu}$.

Preserving the weak-field form (3.1), one can always choose nearly Cartesian coordinates $\{x^\mu\}$ in such a way that the harmonic gauge conditions

$$\partial^\mu(h_{\mu\nu}-\tfrac{1}{2}\eta_{\mu\nu}h^\lambda_\lambda)=0 \tag{3.2}$$

are satisfied. In this gauge the linearized field equations are simply

$$\Box h_{\mu\nu}=-16\pi G S_{\mu\nu} . \tag{3.3}$$

Here G is the gravitational constant.

Let $\{S^{\mathrm{st}}_{\mu\nu}(x,y),h_{\mu\nu}(x,y)\}$ represent, according to the field equations (3.2) and (3.3), the energy content and the gravitational field of a spacetime with a straight static (gauge or global) cosmic string along the z axis. With the assumed symmetries and in the weak-gravity limit it should be possible to write the space-time metric in the form

$$ds^2=[1-w(x,y)](-dt^2+dz^2)+[\delta_{ij}+h_{ij}(x,y)]dx^i dx^j . \tag{3.4}$$

Here δ_{ij} is the Kronecker delta while the indices i,j run from 1 to 2. The coordinate system $\{x^\mu\}$ has been chosen to satisfy the gauge conditions (3.2) and the quantities $h_{\mu\nu}(x,y)$ satisfy the field equations (3.3). In particular, because of (2.8), the function $w=w(x,y)\equiv h_{tt}=-h_{zz}$ satisfies

$$\Box w(x,y)=-16\pi G Q(x,y) . \tag{3.5}$$

Consider now in the $\{x^\mu\}$ coordinate system the metric

$$g^{\mathrm{TW}}_{\mu\nu}(x,y,u)=[1-w(X,Y)]\eta_{\mu\nu} + \frac{\partial X^M}{\partial x^\mu}\frac{\partial X^N}{\partial x^\nu}[h_{MN}(X,Y)+w(X,Y)\eta_{MN}] . \tag{3.6}$$

This metric represents the gravitational field of the cosmic string modulated by a traveling wave moving with the speed of light in the z direction with profiles in the x and y directions characterized by the functions $f(u)$ and $g(u)$, respectively. Indeed, it is not very difficult to check that, since the harmonic conditions (3.2) are satisfied in the metric (3.4), the same is also true in the metric (3.6). In addition, from the field equations (3.3) and Eq. (3.5), we find that the metric (3.6) has as a source precisely the tensor $S^{\mathrm{TW}}_{\mu\nu}(x,y,u)$ of Eq. (2.7). This is most easily proved by expressing the $\Box$ operator of Eq. (3.3) in the $\{X^M\}$ coordinate system of Eq. (2.2) and noting that for any function $q(x,y,u)$, $\Box_{(X^M)}q(X,Y,u) =\{\Box_{(x^\mu)}q(x,y,u)\}|_{x=X,y=Y}$.

A. Gauge cosmic strings

Equation (3.6) expresses, in the weak-field limit, the metrics of cosmic strings with traveling waves, be them gauge or global, in terms of the corresponding static metrics, which are known. Gauge cosmic strings with traveling waves have already been discussed by Vachaspati [7] in the weak-field limit and Garfinkle [8,12] in the general case. Let us anyhow consider gauge cosmic strings in this subsection, just for completeness and for later comparison with the global strings.

The metric around a static, straight gauge cosmic string along the z axis reads [16,17]

$$ds^2_{\mathrm{st}}=-dt^2+dz^2+d\rho^2+\rho^2(1-4G\mu)^2 d\theta^2 , \tag{3.7}$$

where μ is the string mass per unit length. The metric (3.7) is actually flat, with a conical singularity. It describes the spacetime away from the string core, when the fields that make up the string reach their asymptotic values, and is also valid beyond the linear approximation. Of course the exact solution to Einstein's equations coupled to the string fields is much more complicated, and difficult to find analytically [18], except for very special values of the parameters in the field-theory model for the string [19]. But Eq. (3.7) is an excellent approximation to the exterior metric, unless the string parameters are very close to the Planck scale [20]. Using harmonic coordinates, Eq. (3.7) can be written as

$$ds^2_{\mathrm{st}}=-dt^2+dz^2+[1+h(x,y)](dx^2+dy^2) , \tag{3.8}$$

with

$$h(x,y)=\left[\frac{x^2+y^2}{\gamma^2}\right]^{-4G\mu}-1\approx-4G\mu\ln[(x^2+y^2)/\gamma^2] . \tag{3.9}$$

Here γ is an arbitrary length scale, that can be changed rescaling x and y. It should get fixed by matching the exterior solution to a realistic core, and is expected to be of the order of the core width. The relation of the harmonic coordinates x and y to the radial proper distance ρ is given by

$$x^2+y^2=\gamma^2[(1-4G\mu)\rho/\gamma]^{2/(1-4G\mu)} . \tag{3.10}$$

Using Eq. (3.6) (notice that now $w=0$ and $h_{ij}=h\delta_{ij}$) we get, for the traveling-wave metric,

$$ds^2_{\mathrm{TW}}=-dt^2+dz^2+dx^2+dy^2+h(X,Y)[(\dot f^2+\dot g^2)(dz-dt)^2-2\dot f dx(dz-dt)-2\dot g dy(dz-dt)+dx^2+dy^2] . \tag{3.11}$$

Here an overdot denotes a derivative with respect to the argument $u=z-t$. Our derivation of Eq. (3.6) is only valid in the weak-field limit, so we should take $h=-4G\mu\ln\{[(x-f)^2+(y-g)^2]/\gamma^2\}$ in Eq. (3.11), and the result agrees with Vachaspati's direct calculation [7]. It is nevertheless interesting to remark that the metric (3.11), with the exact function $h(X,Y)$ given by (3.9), is for $(X,Y)\neq(0,0)$ an exact solution of vacuum Einstein's equations, as shown by Garfinkle [8], who also noticed that it is of the type known as *pp* waves. It can be interpreted as the exact metric of a gauge cosmic string with traveling waves in the limit of zero thickness of the string core.

The geodesic equations for a nonrelativistic test particle in the traveling-wave metric (3.11) were already discussed in Ref. [7], while motion of light rays was considered in Ref. [21]. Consider a localized traveling wave (a traveling pulse), acting upon a test particle that is initially at rest with respect to the string, while the pulse is making its approach from far away. When the pulse has already passed by and is far away in the other direction, the particle will be moving towards the string at a speed proportional to the integral of $\dot f^2+\dot g^2$ over the pulse profile [22]. In order to visualize this effect in a simple case, consider the acceleration felt by a slowly moving test particle located at $y=0$ at a distance x to the string much larger than the amplitude of the traveling waves, $x\gg f,g$. Notice however, that now radial trajectories are not necessarily geodesics, since the traveling waves break the rotational symmetry around the string axis, and may also force test particles to move in the direction parallel to the string. To lowest order there is no difference between t and proper time, and then

$$\frac{d^2x}{dt^2}\approx-4G\mu\left[\frac{\dot f^2+\dot g^2}{x}-2\frac{\partial}{\partial t}\left[\dot f\ln\frac{x-f}{\gamma}\right]\right]. \tag{3.12}$$

The first term on the right-hand side is a $1/x$ attractive force, analogous to what is expected from an additional mass per unit length of string proportional to $\mu(\dot f^2+\dot g^2)$. The second term can have different signs at different times. However, being a time derivative it has no net effect upon integration between times before and after a traveling pulse has passed by the particle. The force on the test particle increases as we approach the string. Although Eq. (3.12) holds for sufficiently large distances from the string, the $1/x$ dependence of the acceleration clearly indicates that near the string test particles experience quite large forces. In fact, as shown by Garfinkle [23], for the gauge cosmic-string traveling-wave metric (3.11), the tidal forces diverge at the string's core $(X,Y)=(0,0)$.

B. Global cosmic strings

Consider now global cosmic strings. In the weak-field limit the metric of a straight static string reads [10]

$$ds^2_{\rm st}=\left[1-4G\mu\left[\ln\frac{\rho}{\delta}+c+\frac{5}{4}\right]\right](-dt^2+dz^2)+d\rho^2+\rho^2\left[1-8G\mu\left[\ln\frac{\rho}{\delta}+c\right]\right]d\theta^2. \tag{3.13}$$

Here $\mu\equiv\pi\eta^2$ for the model Lagrangian of Eq. (2.1) (with no gauge fields), $\delta\approx(\eta^2\lambda)^{-1/2}$ is the core width, and c is a constant of order unity that takes into account the effect of the string core. Remember that for the global string, the energy per unit length in the Nambu-Goldstone-boson mode up to a distance ρ outside the core of the string is $\mu\ln(\rho/\delta)$, and that for most relevant cases this logarithmic factor is large (as large as 130 if ρ is the present Hubble distance and $\eta\approx10^{15}$ GeV). The constant factor proportional to $(c+5/4)$ in front of $(-dt^2+dz^2)$ corresponds to a choice of scaling on t and z appropriate for later comparison with the exact metric considered in the next section.

Harmonic coordinates can be chosen such that the above metric takes the form of Eq. (3.4) with

$$\begin{aligned} w\equiv h_{tt}&=-h_{zz}=4G\mu\ln\frac{r}{\gamma},\\ h_{xx}&=-2G\mu\left[\ln\frac{r}{\gamma}+2\ln^2\left[\frac{r}{\gamma}\right]-2\sin^2\theta+\alpha\right],\\ h_{yy}&=-2G\mu\left[\ln\frac{r}{\gamma}+2\ln^2\left[\frac{r}{\gamma}\right]-2\cos^2\theta+\alpha\right],\\ h_{xy}&=-4G\mu\sin\theta\cos\theta, \end{aligned} \tag{3.14}$$

where now $r^2\equiv x^2+y^2$, $\theta\equiv\arctan(y/x)$, and $\gamma=\delta\exp(-c-\frac{5}{4})$. In h_{xx} and h_{yy}, α is an arbitrary constant which can be transformed away with an appropriate rescaling of r. In what follows, we will set $\alpha=0$. It is now straightforward to obtain the metric of a global string with traveling waves using Eq. (3.6). The result can be written as

$$ds^2_{\rm TW}=[\eta_{\mu\nu}+h_{\mu\nu}(X,Y)]dx^\mu dx^\nu+du[\dot f(w+h_{xx})(\dot f du-2dx)+\dot g(w+h_{yy})(\dot g du-2dy)+2h_{xy}(\dot f\dot g du-\dot g dx-\dot f dy)], \tag{3.15}$$

where, as already explained, the $h_{\mu\nu}$ here are the same functions as in (3.14) but with $X=x-f$ and $Y=y-g$ as arguments, rather than x and y. As usual, $u=z-t$.

Let us now evaluate, as we did for gauge strings, the acceleration in the x direction felt by a slowly moving test particle located at $y=0$, assuming the distance to the string core much larger than the amplitude of the traveling waves, $x\gg f,g$. We will also neglect terms of order unity against $\ln x/\gamma$, which we assume to be large. Then

$$\frac{d^2x}{dt^2}\approx2G\mu\left[\frac{1}{x}-2\frac{\dot f^2+\dot g^2}{x}\ln\frac{x}{\gamma}+2\frac{\partial}{\partial t}\left[\dot f\ln^2\frac{x-f}{\gamma}\right]\right]. \tag{3.16}$$

The first term, independent on the traveling waves, is the repulsive $1/x$ force that a static global string exerts [10]. The second term, proportional to $(\dot{f}^2+\dot{g}^2)$, is an attractive $1/x$ force analogous to the gauge-string case. But now this term has an additional factor $\ln x/\gamma$. This extra logarithmic dependence can be understood taking into account that while for a gauge string the traveling waves affect the stress-energy tensor just along the string core, in the global-string case it is affected everywhere, and global strings have mass per unit length and tensions that depend logarithmically on the distance to the string core. The last term has different signs at different times, but as in the gauge-string case is a time derivative that can be dropped off when evaluating the integrated effect of a traveling pulse.

IV. STRONGLY GRAVITATING TRAVELING WAVES

The metric around a static and cylindrically symmetric cosmic string that lies along the z axis, clearly admits two null hypersurface orthogonal Killing vectors pointing along the $z\pm t$ directions. As the weak field Eq. (3.6) implies, traveling waves propagating along the $(z-t)$ direction break the symmetry corresponding to the Killing vector ∂_{z+t} but not the one corresponding to ∂_{z-t}, since the latter remains a Killing vector in the traveling-wave metric. In fact this property holds also beyond the weak-field limit. Garfinkle and Vachaspati have shown [12] that the exact metric of cosmic strings with traveling waves must be of a generalized Kerr-Schild type with background the static cosmic-string metric; i.e., that the metric must have the form of Eq. (1.1). As already mentioned in Sec. I, k_μ of Eq. (1.1) is a null and hypersurface-orthogonal Killing vector with respect to both static and traveling-wave metrics. F is a scalar function which satisfies, with respect to the static metric, the condition

$$k^\mu \nabla_\mu F=0 \; , \tag{4.1}$$

and the wave equation

$$\Box(e^A F)=0 \; . \tag{4.2}$$

Here A is a scalar determined from $\nabla_\mu k_\nu = k_{[\nu}\nabla_{\mu]}A$. The existence of A follows from the fact that k_μ is a Killing and hypersurface-orthogonal vector field. Equivalently, if we write the static metric in coordinates adapted to the two Killing fields, say $\{u,v,y^1,y^2\}$, then e^A is the g_{uv} component of the metric written in the form

$$ds^2_{\rm st}=2e^{A(y^1,y^2)}du\,dv+g_{ij}(y^1,y^2)dy^i dy^j \; , \tag{4.3}$$

with i,j running from one to two. Here one of the obvious null Killing vectors is $k^\mu=\delta^\mu_v, k_\mu=e^A\delta^u_\mu$ (the other is obtained from k_μ by interchanging u and v). Thus in this coordinate system, Eq. (4.1) simply states that F is independent of v.

The function F is not uniquely determined from Eq. (4.2) alone. One needs further information, such as asymptotic conditions, to appropriately choose the solution that describes the spacetime around a string with traveling waves characterized by the functions f and g. In the global-string case the situation might appear, at first sight, even more problematic [12], because the static metric becomes singular at a finite proper distance from the core. Hence, it may not be possible to give asymptotic conditions on the function F. In this section we show how to select the function F among all possible solutions by imposing the appropriate weak-field limit. Our strategy will be to explicitly write the weak-field traveling-wave metric of Eq. (3.6) as a generalized Kerr-Schild metric with background the corresponding static one. In so doing, we will find the weak-field value of the function F. Nothing new will be learned for the gauge-string case, already discussed in Ref. [12]. For the global string, though, this will allow us to choose the function F adequate to the exact metric with traveling waves.

Let us first express the metric (3.6) in the coordinates $\{X,Y,u,v\}$ defined in Eqs. (2.2) and (2.3). We easily find that

$$\begin{aligned} ds^2_{\rm TW}=&(1-w)du\,dv+(\delta_{ij}+h_{ij})dX^i dX^j \\ &+(1-w)[(\dot{f}^2+\dot{g}^2)du+2\dot{f}dX+2\dot{g}dY]du \; , \end{aligned} \tag{4.4}$$

with i,j running from one to two. By changing in a last step the v coordinate to $\hat{v}$,

$$2\hat{v}=v+\int(\dot{f}^2+\dot{g}^2)du+2\dot{f}X+2\dot{g}Y \; , \tag{4.5}$$

we find

$$\begin{aligned} ds^2_{\rm TW}=&2(1-w)du\,d\hat{v}+(\delta_{ij}+h_{ij})dX^i dX^j \\ &-2(1-w)(\ddot{f}X+\ddot{g}Y)du^2 \; . \end{aligned} \tag{4.6}$$

In this form, the cosmic-string traveling-wave metric can be compared to the static one (3.4). We observe that it contains the static part (3.4), with the coordinates $X, Y, \hat{v}+u/2$ and $\hat{v}-u/2$ playing, respectively, the role of x, y, z, and t. The modification with respect to the static metric is of the (generalized) Kerr-Schild type since the metric (4.6) can be written as

$$g^{\rm TW}_{\mu\nu}=g^{\rm st}_{\mu\nu}+Fk_\mu k_\nu \; , \tag{4.7}$$

with $k^\mu=\delta^\mu_{\hat{v}}$ a null, hypersurface orthogonal Killing field with respect to both static and traveling-wave metrics. Since $k_\mu=(1-w)\delta^u_\mu$, we conclude by comparison of (4.6) and (4.7) that the function F in (4.7) is in the weak-gravity limit equal to

$$F=-2\frac{\ddot{f}X+\ddot{g}Y}{1-w} \; . \tag{4.8}$$

A. Gauge cosmic strings

This case has already been discussed in the literature [8]. Here we will just briefly reproduce the result for the exact metric around a gauge cosmic string with traveling waves in the zero-thickness approximation, for completeness and to exemplify how our method works before moving into global strings.

The exact metric around the static string is given by Eq. (3.8) with the exact function h of Eq. (3.9). Comparing it with Eq. (4.3) we find that $A=0$. It should then be obvious that the function F which satisfies Eq. (4.2) and has as weak-field limit the Eq. (4.8) with $w=0$, is simply $F=-2(\ddot{f}X+\ddot{g}Y)$. This result agrees with that of Ref. [8], obtained there in a different way.

B. Global cosmic strings

The exact metric around a static global string reads [14]

$$ds^2_{\rm st}=\frac{\xi}{\xi_0}(-dt^2+dz^2)+\gamma^2\left[\frac{\xi_0}{\xi}\right]^{1/2}\exp\left[\frac{\xi_0^2-\xi^2}{\xi_0}\right](d\xi^2+d\Theta^2)\ . \qquad (4.9)$$

Here, with regard to the expression that appears in Ref. [14], we have made the notational changes $\{u,u_0\}\to\{\xi,\xi_0\}$. The parameter γ is the same as in Eq. (3.14). Finally, the parameter ξ_0 is

$$\frac{1}{\xi_0}=4G\mu\ll 1 \qquad (4.10)$$

with $G\mu$ the same as in Eqs. (3.13) and (3.14). We point out here that the metric in Eq. (4.9), as well as its weak-field limit in Eq. (3.13), are not solutions of Einstein equations coupled to the equations for the scalar field that makes up the string. Rather, they are solutions to Einstein equations with the scalar field configuration fixed to its asymptotic value everywhere outside the core, which is in most cases a sensible approximation.

The metric (4.9) has two singularities, one at $\xi=\infty$ and another at $\xi=0$. The former is located at the center of the string's core, where the metric (4.9) is not expected to describe correctly the spacetime. Its validity starts from the string's core at $\xi\approx\xi_0$ and ranges down to the outer singularity at $\xi=0$.

The weak-field limit is obtained for

$$\xi\to\xi_0\gg 1\ . \qquad (4.11)$$

To see this, let us first perform a coordinate transformation of the radial coordinate ξ to R:

$$\xi-\xi_0=-\left[1-\frac{1}{2\xi_0}\right]\ln\frac{R}{\gamma}\ . \qquad (4.12)$$

It is then not very difficult to check that with the usual coordinate transformation $X=R\cos\Theta, Y=R\sin\Theta$ we obtain in the limit (4.11), to first order in ξ_0^{-1}, the weak-field metric for the static global string as in Eq. (3.14). Consequently, the coordinates t, z, (R,Θ), X, and Y of this subsection, agree in the weak-field limit (4.11) with the corresponding $t,z,(r,\theta),x,y$ harmonic coordinates of Sec. III B.

Comparing Eqs. (4.9) and (4.3) we see that $e^{A}\propto\xi$. Therefore, using Eq. (4.2) and the metric (4.9), we find that the function F for the global string satisfies

$$\partial_\xi[\xi\partial_\xi(\xi F)]+\partial_\Theta[\xi\partial_\Theta(\xi F)]=0\ . \qquad (4.13)$$

According to Eq. (4.8), in the weak-field limit (4.11) F must behave, to first order in ξ_0^{-1}, as

$$F\to-2\frac{R(\ddot{f}\cos\Theta+\ddot{g}\sin\Theta)}{1-(1/\xi_0)\ln(R/\gamma)}\ . \qquad (4.14)$$

Guided by this large-ξ behavior, Eq. (4.13) can be easily solved by separation of variables and an intermediate ansatz $F=\hat{F}/\xi$, leading to a zero-order modified Bessel differential equation for the ξ-dependent part of $\hat{F}$. The result is

$$F=-2Q\frac{K_0(\xi)}{\xi}(\ddot{f}\cos\Theta+\ddot{g}\sin\Theta),\quad \text{with } Q\equiv\left[\frac{2\xi_0^3}{\pi}\right]^{1/2}\gamma e^{\xi_0}\left[1+\frac{1}{8\xi_0}\right]\ . \qquad (4.15)$$

Here $K_0(\xi)$ denotes the zero-order modified Bessel function. That this expression reduces appropriately to (4.14), can be checked using Eq. (4.12) and the large-argument asymptotic behavior of $K_0(\xi)$:

$$K_0(\xi)\to\left[\frac{\pi}{2\xi}\right]^{1/2}e^{-\xi}\left[1-\frac{1}{8\xi}+O(\xi^{-2})\right]\ .$$

Thus, according to Eq. (4.7), the exact metric for a global string with traveling waves can be written as

$$ds^2_{\rm TW}=2\frac{\xi}{\xi_0}du\,d\hat{v}+\gamma^2\left[\frac{\xi_0}{\xi}\right]^{1/2}\exp\left[\frac{\xi_0^2-\xi^2}{\xi_0}\right](d\xi^2+d\Theta^2)-\frac{2Q}{\xi_0^2}\xi K_0(\xi)(\ddot{f}\cos\Theta+\ddot{g}\sin\Theta)du^2\ . \qquad (4.16)$$

To go back to the $x,y,(r,\theta),z,t$ coordinate system, where f and g attain their natural interpretation as profiles of the traveling waves in the x and y direction, respectively, one has to perform in reversed order the transformations of Eqs. (4.5) and (2.2), using (4.12) to go from ξ,Θ to X,Y.

An interesting point to note here is that $\xi K_0(\xi)$ vanishes at $\xi=0$ as $\xi\ln\xi$. As mentioned above, at $\xi=0$ there is a curvature singularity of the static global-string metric [14]. This singularity appears to be unavoidable in standard field-theoretical models of a global string [24]. Nonsingular solutions around a static global string (which have an event horizon rather than a singularity) can be found only if invariance under boosts in the string direction is given up [25,26]. It is thus worthwhile to investigate what is the effect of traveling waves on the singular structure of the static global-string metric. In the Appendix we evaluate the nonvanishing components of the Riemman tensor for the spacetime around a global string with traveling waves. We conclude that more components of the Riemann tensor diverge at the singularity ($\xi=0$). Thus, we may say that the effect of the traveling waves is to increase the strength of this singularity, at least in the sense of increasing the tidal forces in its neighborhood. However one should note that curvature scalars, like the one given in the Appendix, appear to be entirely insensitive to the presence of the traveling waves.

V. SUMMARY AND CONCLUSIONS

A traveling wave along an otherwise straight cosmic string exerts, in the linear approximation to general relativity, a net attractive effect upon a nonrelativistic test particle. We have seen that in the case of global cosmic strings these effects are larger than for gauge strings by a factor with logarithmic dependence on the distance to the string core. This is not so surprising, since the mass per unit length and tensions of the static global string also grow logarithmically. The static global string exerts a repulsive force, inversely proportional to the distance off the core. The traveling waves may have even larger gravitational effects if $(\dot f^{\,2}+\dot g^{\,2})\ln(\rho/\gamma)$ is larger than unity.

Combining Garfinkle and Vachaspati's proof that the exact metric around a string with traveling waves must be of the generalized Kerr-Schild type, i.e., as in Eq. (4.7), with our result for the linearized metric, we found the exact metric around a global string with traveling waves to be specifically that of Eq. (4.16). The metric around a static global string is singular at a finite proper distance off the core. Traveling waves can not remedy this situation. On the contrary, more components of the Riemann tensor diverge at the singularity. Curvature invariants, though, blow up at the singularity exactly as in the static case.

ACKNOWLEDGMENTS

We are grateful to T. Vachaspati and A. Vilenkin for pointing out a mistake in our manuscript, and for very useful comments about the gravitational effects of a traveling pulse. This work was supported by the Directorate General for Science, Research and Development of the Commission of the European Communities under Contract No. C 11-0540-M(TT). The work of D.H. was also supported by Consejo Nacional de Investigaciones Científicas y Técnicas (Argentina) (CONICET), Universidad de Buenos Aires and Fundación Antorchas.

APPENDIX

Here we give for the metric (4.16) the nonvanishing components of the Riemann tensor:

$$
\begin{aligned}
R_{u\hat v u\hat v} &= (2\xi_0\gamma)^{-2}\left[\frac{\xi}{\xi_0}\right]^{1/2}\exp\left[\frac{\xi^2-\xi_0^2}{\xi_0}\right] ,\\
R_{u\xi u\xi} &= \frac{(4\xi^2+\xi_0)\xi K_0'(\xi)+[4\xi^2(\xi_0+1)-\xi_0]K_0(\xi)}{4\xi_0^3\xi}Q(\ddot g\sin\Theta+\ddot f\cos\Theta) ,\\
R_{u\Theta u\Theta} &= -\frac{(4\xi^2+\xi_0)\xi K_0'(\xi)+[4\xi^2(\xi_0+1)+\xi_0]K_0(\xi)}{4\xi_0^3\xi}Q(\ddot g\sin\Theta+\ddot f\cos\Theta) ,\\
R_{u\xi u\Theta} &= -\frac{4\xi_0\xi K_0'(\xi)+(4\xi^2+3\xi_0)K_0(\xi)}{4\xi_0^3}Q(\ddot f\sin\Theta-\ddot g\cos\Theta) ,\\
R_{u\Theta\hat v\Theta} &= \frac{4\xi^2+\xi_0}{8\xi_0^2\xi} ,\\
R_{\xi\Theta\xi\Theta} &= \gamma^2\frac{4\xi^2-\xi_0}{4\xi_0\xi^2}\left[\frac{\xi_0}{\xi}\right]^{1/2}\exp\left[\frac{\xi_0^2-\xi^2}{\xi_0}\right] .
\end{aligned}
\tag{A1}
$$

Here prime denotes derivative with respect to ξ.

The only nonvanishing component of the Ricci tensor is

$$R_{\Theta\Theta}=\frac{2}{\xi_0} . \tag{A2}$$

From Eq. (A1) we can obtain after a lengthy but straightforward calculation the curvature scalar

$$R_{\mu\nu\rho\eta}R^{\mu\nu\rho\eta}=\frac{32\xi^4-8\xi_0\xi^2+3\xi_0^2}{4\gamma^2\xi_0^3\xi^3}\exp\left[2\frac{\xi^2-\xi_0^2}{\xi_0}\right] . \tag{A3}$$

Note that it has no dependence on the traveling waves. It blows up at $\xi=0$ and $\xi=\infty$ demonstrating thus the real nature of these singularities.

[1] A. Albrecht and N. Turok, Phys. Rev. D **40**, 973 (1989).
[2] D. P. Bennett and F. R. Bouchet, Phys. Rev. Lett. **63**, 2776 (1989).
[3] B. Allen and E. P. S. Shellard, Phys. Rev. Lett. **64**, 119 (1990).
[4] A. Vilenkin, Phys. Rev. D **41**, 3038 (1990).
[5] C. Hogan and M. Rees, Nature (London) **311**, 109 (1984).
[6] D. R. Stinebring, M. F. Ryba, J. H. Taylor, and R. W. Romani, Phys. Rev. Lett. **65**, 285 (1990).
[7] T. Vachaspati, Nucl. Phys. **B277**, 593 (1986).
[8] D. Garfinkle, Phys. Rev. D **41**, 1112 (1990).
[9] Vachaspati and T. Vachaspati, Phys. Lett. B **238**, 41 (1990).
[10] D. Harari and P. Sikivie, Phys. Rev. D **37**, 3438 (1988).
[11] A. Vilenkin, Phys. Rep. **121**, 263 (1985).
[12] D. Garfinkle and T. Vachaspati, Phys. Rev. D **42**, 1960 (1990).
[13] V. P. Frolov and D. Garfinkle, Phys. Rev. D **42**, 3980 (1990).
[14] A. Cohen and D. Kaplan, Phys. Lett. B **215**, 67 (1988).
[15] S. Weinberg, *Gravitation and Cosmology* (Wiley, New York, 1972).
[16] A. Vilenkin, Phys. Rev. D **23**, 852 (1981).
[17] J. R. Gott, Astrophys. J. **288**, 422 (1985).
[18] D. Garfinkle, Phys. Rev. D **32**, 1323 (1985).
[19] B. Linet, Phys. Lett. A **124**, 240 (1987).
[20] P. Laguna and D. Garfinkle, Phys. Rev. D **40**, 1011 (1989).
[21] D. N. Vollick and W. G. Unruh, Phys. Rev. D **42**, 2621 (1990).
[22] A. Vilenkin, in *300 Years of Gravitation,* edited by S. W. Hawking and W. Israel (Cambridge University Press, Cambridge, England, 1987).
[23] D. Garfinkle, Phys. Rev. D **40**, 1801 (1989).
[24] R. Gregory, Phys. Lett. B **215**, 663 (1988).
[25] D. Harari and A. Polychronakos, Phys. Lett. B **240**, 55 (1990).
[26] G. W. Gibbons, M. E. Ortiz, and F. Ruiz Ruiz, Phys. Lett. B **240**, 50 (1990).