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Density fluctuations in extended inflation

Alan H. Guth
Center for Theoretical Physics, Laboratory for Nuclear Science, and Department of Physics, Massachusetts Institute of Technology, Cambridge, Massachusetts 02139 and Harvard-Smithsonian Center for Astrophysics, 60 Garden Street, Cambridge, Massachusetts 02138

Bhuvnesh Jain
Center for Theoretical Physics, Laboratory for Nuclear Science, and Department of Physics, Massachusetts Institute of Technology, Cambridge, Massachusetts 02139
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We estimate the density perturbation spectrum $\delta\rho/\rho$ in the extended inflationary model, in which the scalar curvature is coupled to a Brans-Dicke field. Through a conformal transformation and a redefinition of the Brans-Dicke field, the action of the theory is cast into a form with no coupling to the scalar curvature and a canonical kinetic term for the redefined field. Following Kolb, Salopek, and Turner, we calculate $\delta\rho/\rho$ using the transformed action and the standard recipe developed for conventional inflation. This recipe is expected to give a valid order-of-magnitude estimate, but a precise calculation would require a more careful treatment of several aspects of the problem. The spectrum behaves as a positive power of the wavelength, a feature that might be useful in building models to account for the observed large-scale structure of the universe. Our result for the overall amplitude of density perturbations differs slightly from that of the previous authors, and the reasons for these differences are discussed. We also point out that the conformal transformation method can be applied to a wider class of generalized gravity theories.

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I. INTRODUCTION

Extended inflation is a new model of inflation, proposed by La and Steinhardt [1]. Its key feature is that the effective gravitational constant G varies with time due to the nonminimal coupling of a scalar field to the scalar curvature. As first proposed [1], it was based on the Brans-Dicke [2] theory of gravity, for which the action is given by [3]

$$S = \int d^4x\sqrt{-g}\left(-\frac{R}{16\pi}\Phi + \frac{\omega}{16\pi}g^{\mu\nu}\frac{\partial_\mu\Phi\partial_\nu\Phi}{\Phi} + \mathcal{L}_{\rm matter}\right) . \quad (1.1)$$

With $\Phi \equiv 2\pi\phi^2/\omega$, the kinetic term for the scalar field can be written in the standard way:

$$S = \int d^4x\sqrt{-g}\left(-\frac{R}{8\omega}\phi^2 + \frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi + \mathcal{L}_{\rm matter}\right) . \quad (1.2)$$

We shall work with (1.1), the form originally introduced by Brans and Dicke.

The Brans-Dicke field Φ couples to gravity and is responsible for the time variation of G. The inflaton field σ contributes to $\mathcal{L}_{\rm matter}$ and provides the nearly constant vacuum energy density that drives inflation. ω is a dimensionless parameter of the theory: Brans-Dicke gravity becomes identical to Einstein gravity as ω approaches infinity.

In contrast to the exponential expansion of standard inflation, the time variation of G in extended inflation leads to a power-law expansion of the scale factor $a(t)$. The Hubble parameter $H \equiv \dot{a}/a$ is therefore decreasing with time. Once H becomes sufficiently small, the transition to a radiation-dominated universe can be completed by bubble nucleation, providing the possibility of a graceful exit to the false-vacuum phase. If H changes too slowly, however, then the problems of the original inflationary scenario remain—a nearly scale-invariant distribution of bubbles is formed, resulting in large inhomogeneities and distortions of the cosmic background radiation. These distortions are unacceptably large unless $\omega \lesssim 25$ [4, 5], whereas time-delay experiments constrain ω to be $\gtrsim 500$ [6]. This problem can be avoided by introducing a potential for the Φ field, with a minimum at $\Phi = G_N^{-1}$, where G_N is the present value of the gravitational constant. Thus, a scalar field that couples to gravity can be used to construct an interesting cosmological model. The physics of this coupling is interesting in any case, because a number of particle theories—superstring, supergravity, and Kaluza-Klein theories, for example—involve such a coupling. In general, terms with higher-order couplings of Φ to the scalar curvature are also possible. Steinhardt and Accetta [7] have studied a generalization of extended inflation, called hyperextended inflation, in which the consequences of such higher-order coupling terms are explored.

In this paper we compute the density perturbation spectrum $\delta\rho/\rho$ in the context of the original model of extended inflation of La and Steinhardt. Specifically, we compute the curvature fluctuations that arise from quantum fluctuations in the Φ field. We work with the simple Brans-Dicke action because it provides tractable equations of motion.

We begin in Sec. II by obtaining the equations of motion in the Jordan frame, i.e., the frame defined by the action (1.1). In Sec. III, following Holman *et al.* [8], we make a conformal transformation that takes the action to the standard Einstein-Hilbert form. In this conformally rescaled frame, known as the Einstein frame, a rescaled time variable is introduced and the equations of motion are derived. A new field Ψ, obeying the equations of motion of a minimally coupled scalar field, is defined in terms of Φ. As pointed out by Kolb, Salopek, and Turner [9] (hereafter called KST), this form of the action allows us to directly apply the results for $\delta\rho/\rho$ obtained in standard inflation [10–13]. The calculation of $\delta\rho/\rho$ is carried out in Sec. IV. We point out some subtleties in the application of the standard density perturbation results, but we leave the investigation of these subtleties to a future paper. We nonetheless argue that the present result should be acceptable as an order-of-magnitude estimate. In Sec. V our result is compared with that obtained by naively applying the standard formalism in the Jordan frame. A calculation similar to ours is carried out in KST, but our result differs from theirs by a factor that depends on ω. This discrepancy vanishes in the limit of large ω, a limit in which both results agree with the answer that would be obtained naively in the Jordan frame. We point out what we believe are the reasons for the discrepancy. We also demonstrate that the action for a more general class of gravity theories can in principle be transformed to the form for a minimally coupled scalar field with a canonical kinetic term. We summarize in Sec. VI.

II. JORDAN FRAME RESULTS

In this section we summarize the homogeneous background solutions for $\Phi(t)$ and the scale factor $a(t)$ for the Jordan frame action (1.1), assuming a flat (i.e., $k = 0$) Robertson-Walker metric. We follow the notation of KST to facilitate comparison of results.

From the action (1.1), the equations of motion for $\Phi(t)$ and $a(t)$ are given by

$$\ddot{\Phi} + 3H\dot{\Phi} = \frac{8\pi}{2\omega + 3}(\rho - 3p), \tag{2.1}$$

$$H^2 \equiv \left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi\rho}{3\Phi} + \frac{\omega}{6}\left(\frac{\dot{\Phi}}{\Phi}\right)^2 - H\frac{\dot{\Phi}}{\Phi}. \tag{2.2}$$

The energy density ρ and the pressure p are determined by $\mathcal{L}_{\rm matter}$, which describes the inflaton field σ and all other matter fields:

$$\mathcal{L}_{\rm matter} = \tfrac{1}{2}g^{\mu\nu}\partial_\mu\sigma\partial_\nu\sigma - V(\sigma) + \cdots . \tag{2.3}$$

In extended inflation $V(\sigma)$ provides the nearly constant false-vacuum energy density that dominates the energy density of the universe during inflation. Since the σ field stays anchored very near its false-vacuum value, its kinetic energy is negligible. Thus, during inflation we have $\rho \approx \rho_{\rm vac}$ and $p \approx -\rho_{\rm vac}$, where $\rho_{\rm vac} \equiv M^4$ is the value of $V(\sigma)$ in the false vacuum. The desired solution can then be written

$$\Phi(t) = \Phi_0(Bt)^2, \tag{2.4}$$

$$a(t) = a_0(Bt)^{\omega+1/2}, \tag{2.5}$$

$$H(t) = \frac{\omega + \frac{1}{2}}{t}, \tag{2.6}$$

where

$$B\Phi_0^{1/2} = \frac{M^2}{q\omega} \quad \text{and} \quad q = \sqrt{\frac{(6\omega+5)(2\omega+3)}{32\pi\omega^2}} . \tag{2.7}$$

(Readers comparing with KST will note that we have chosen a different origin for the time variable t.) Unlike exponential inflation, the Hubble parameter H in this case is time dependent.

III. EINSTEIN FRAME RESULTS

In this section we make the conformal transformation [8] that defines the Einstein frame in terms of the Jordan frame described above. The Einstein frame quantities will be indicated by an overbar.

Define a new metric $\bar{g}_{\mu\nu}$ as

$$g_{\mu\nu}(\mathbf{x},t) = \Omega^2(t)\bar{g}_{\mu\nu}(\mathbf{x},t), \tag{3.1}$$

where

$$\Omega^2(t) = \frac{m_{\rm Pl}^2}{\Phi(t)} , \tag{3.2}$$

and $m_{\rm Pl} \equiv G_N^{-1/2}$ is the present value of the Planck mass. Define a field Ψ in terms of Φ by

$$\Psi = \Psi_0 \ln\left(\frac{\Phi}{m_{\rm Pl}^2}\right), \tag{3.3}$$

where

$$\Psi_0 = \sqrt{\frac{2\omega+3}{16\pi}}\, m_{\rm Pl} . \tag{3.4}$$

The field Ψ is introduced so that the kinetic term also takes the canonical form. Carrying out the conformal transformation (3.1) (see, e.g., Birrell and Davies [14] for the transformation of $R[g_{\mu\nu}]$) yields

$$\bar{S} = \int d^4x\sqrt{-\bar{g}}\Big(-\frac{\bar{R}}{16\pi G_N} + \frac{1}{2}\bar{g}^{\mu\nu}\partial_\mu\Psi\partial_\nu\Psi + \frac{1}{2}e^{-\Psi/\Psi_0}\bar{g}^{\mu\nu}\partial_\mu\sigma\partial_\nu\sigma - e^{-2\Psi/\Psi_0}M^4\Big), \tag{3.5}$$

where we have used $V(\sigma) = M^4$. Notice that the gravitational part of $\bar{S}$ has the usual Einstein form, and that the kinetic term for Ψ also takes the canonical form. Since the kinetic energy of the σ field is negligible, $\bar{S}$ takes the form of the action for a minimally coupled scalar field Ψ with an exponential potential:

$$V(\Psi) = M^4 e^{-2\Psi/\Psi_0}. \tag{3.6}$$

In $\bar{S}$, Ψ plays the role of the inflaton field—this identification simplifies the calculation of $\delta\rho/\rho$ [15].

We write the equations of motion in terms of a rescaled time variable $\bar{t}$ so that the metric takes the Robertson-Walker form

$$d\bar{s}^2 = d\bar{t}^2 - \bar{a}(\bar{t})^2 \, d\bar{x}^2 = \Omega^{-2}(t) ds^2 \;, \tag{3.7}$$

where

$$d\bar{t} = \Omega^{-1} \, dt, \tag{3.8}$$

$$\bar{a}(\bar{t}) = \Omega^{-1} a(t), \tag{3.9}$$

$$d\bar{x} = dx \; . \tag{3.10}$$

In these coordinates the equations of motion are

$$\ddot{\Psi}(\bar{t}) + 3\bar{H}\dot{\Psi}(\bar{t}) + \frac{dV}{d\Psi} = 0, \tag{3.11}$$

$$\bar{H}^2 \equiv \left(\frac{\dot{\bar{a}}(\bar{t})}{\bar{a}(\bar{t})}\right)^2 = \frac{8\pi}{3m_{\rm Pl}^2}\left(\frac{1}{2}\dot{\Psi}^2(\bar{t}) + V(\Psi)\right) \; . \tag{3.12}$$

In (3.11) and (3.12) and in all subsequent equations, an overdot indicates a derivative with respect to $\bar{t}$.

Using Eqs. (3.2) and (2.4), one sees that the desired conformal transformation is given by

$$\Omega(t) = \frac{m_{\rm Pl}}{B\Phi_0^{1/2} t} \; . \tag{3.13}$$

The relation between $\bar{t}$ and t can then be found by integrating Eq. (3.8), yielding

$$C\bar{t} = (Bt)^2, \tag{3.14}$$

with

$$C = \frac{2Bm_{\rm Pl}}{\Phi_0^{1/2}} \; . \tag{3.15}$$

By combining Eqs. (2.4), (3.3), and (3.14), the Jordan frame solution for $\Phi(t)$ can be transformed to give

$$\Psi(\bar{t}) = \Psi_0 \ln\left(\frac{C\Phi_0 \bar{t}}{m_{\rm Pl}^2}\right) \; . \tag{3.16}$$

Equations (3.9), (3.13), and (3.14) lead to

$$\bar{a}(\bar{t}) = \bar{a}_0 (C\bar{t})^{(2\omega+3)/4} \;, \tag{3.17}$$

where

$$\bar{a}_0 = a_0 \frac{\Phi_0^{1/2}}{m_{\rm Pl}} \; . \tag{3.18}$$

The time dependence of the Hubble parameter can be obtained by differentiating Eq. (3.17), yielding

$$\bar{H} = \frac{2\omega + 3}{4\bar{t}} \; . \tag{3.19}$$

It is straightforward to verify that the equations of motion (3.11) and (3.12) are satisfied by these transformed solutions.

IV. CALCULATION OF $\delta\rho/\rho$

The equation of motion (3.11) for Ψ is the same as that for a minimally coupled scalar field in standard inflation. This identification [9, 16] allows us to use the results [10–13] for density perturbations arising from quantum fluctuations of a minimally coupled scalar field. The density perturbation amplitude for a scale coming inside the Hubble length in the late universe is then given by

$$\left(\frac{\delta\rho}{\rho}\right)_{\rm Hubble} \approx \left.\frac{\bar{H}(\bar{t})^2}{\dot{\Psi}(\bar{t})}\right|_{\bar{t}=\bar{t}_h} \;, \tag{4.1}$$

where the right-hand side is to be evaluated at the time $\bar{t}_h$ when the scale crossed outside the Hubble length during inflation.

While the conformal transformation has eliminated the coupling between the scalar field and gravity, we must still ask whether Eq. (4.1) is adequate for our problem. There are several issues that must be considered.

(i) Even in the original context of standard inflation, the formula is only an approximation. It can be obtained, as in Ref. [11], by matching together an approximate solution valid at early times and an approximate solution valid at late times. The matching is done at the time of Hubble length crossing, a time when neither solution is highly reliable. Alternatively, as in Ref. [10], it can be obtained by fixing the amplitude of the late-time solution by using a rough estimate of quantum fluctuations at early times. The approximation is good enough for most purposes, but here we face the problem that the effects we will be studying are quite small—see, for example, Fig. 1 below. To properly justify the consideration of such small effects, one wants to know that the other uncertainties are even smaller. A rough estimate of the uncertainty in formula (4.1) can be obtained by recognizing that the precise time at which the right-hand side is to be evaluated has not been carefully thought out. While the standard convention holds that it should be evaluated at $\lambda_{\rm physical} = H^{-1}$, one might just as well have decided to evaluate it when $\lambda_{\rm physical} = 2H^{-1}$. This modification of the rules, however, would produce an ω-dependent correction that is comparable to the size of the effects that will be considered below.

(ii) The standard derivations of Eq. (4.1) assumed that the $\ddot{\Psi}$ term of the equation of motion for Ψ is negligible, while we will find that this is not the case when ω is small. Again there is no problem if Eq. (4.1) is considered an approximation, but the accuracy that we desire will merit a more careful look at this approximation.

(iii) Equation (4.1) was derived originally for exponential inflation, while here we are applying it to power-law inflation, with $a(t) \propto t^p$. The application to power-law inflation has been investigated by Lucchin and Matarrese [17], who conclude that the standard formula is correct. This conclusion, however, is valid only as an approximation. Abbott and Wise [18] have shown, for example, that the two-point function that is used to calculate the scalar field quantum fluctuations depends on the exponent p in a complicated way. Moreover, if H depends on time, any answer that depends on H must specify precisely the time at which it should be evaluated.

These issues, however, are separate from the question of evaluating the right-hand side of Eq. (4.1). In this paper we will carry out this evaluation, postponing the investigation of the subtle issues to a future paper. We believe that the answer obtained below is a valid order-of-magnitude estimate (similar in its accuracy to the standard results [10–13] in conventional inflation), but it is not a precise calculation.

Since the equation of motion is obtained from the Einstein action with $\bar{t}$ as the time variable, we must be careful to evaluate the right-hand side of Eq. (4.1) by using $\bar{t}$ and the Einstein frame Hubble parameter $\bar{H}(\bar{t})$. In order to express $(\delta\rho/\rho)_{\rm Hubble}$ as a function of a present-day length scale, we use the ratio of the scale factors at the time of Hubble length crossing and the present time. In doing so we assume that the transition from inflation to radiation domination occurs instantaneously at a temperature $T \approx M$, and that the field $\Phi(t)$ does not vary significantly after the end of inflation. Therefore we approximate

$$\Phi(t_e) \approx \frac{1}{G_N} \equiv m_{\rm Pl}^2 \ , \tag{4.2}$$

where t_e denotes the time at the end of inflation. We assume that Eq. (4.2) holds also for all $t \gtrsim t_e$. Then $\Omega^2(t) = m_{\rm Pl}^2/\Phi(t) \approx 1$ for all $t \gtrsim t_e$, so after the end of inflation the Einstein and Jordan frames coincide. The evolution of the perturbation amplitude after inflation is therefore the same in both frames. We will need an expression for $\bar{t}_e$ (the rescaled time variable at the end of inflation), which can be found by combining Eqs. (2.4) and (3.14) to obtain $\Phi(\bar{t}_e) = C\Phi_0\bar{t}_e = m_{\rm Pl}^2$. Then using Eqs. (3.15) and (2.7), one has

$$\bar{t}_e = \frac{q\omega m_{\rm Pl}}{2M^2} \ . \tag{4.3}$$

To evaluate (4.1) we use (3.19) for $\bar{H}(\bar{t})$ and (3.16) for $\Psi(\bar{t})$ to obtain

$$\left(\frac{\delta\rho}{\rho}\right)_{\rm Hubble} \approx \left.\frac{\bar{H}^2(\bar{t})}{\dot{\Psi}(\bar{t})}\right|_{\bar{t}=\bar{t}_h} = \frac{(2\omega+3)^2}{16\Psi_0\bar{t}_h} \ . \tag{4.4}$$

To solve for $\bar{t}_h$, the rescaled time variable at the moment of Hubble length crossing, we use

$$\lambda_{\rm physical}(\bar{t}_h) = \bar{a}(\bar{t}_h)\lambda_c = \bar{H}(\bar{t}_h)^{-1} \ , \tag{4.5}$$

where λ_c denotes the comoving wavelength. This can be rewritten as

$$\frac{\bar{a}(\bar{t}_h)}{\bar{a}(\bar{t}_e)}\frac{\bar{a}(\bar{t}_e)}{\bar{a}(\bar{t}_0)}\lambda_c = \bar{H}(\bar{t}_h)^{-1} \ , \tag{4.6}$$

where we have set the present value of the scale factor $\bar{a}(\bar{t}_0) = 1$. Since $\bar{a}(\bar{t}) \propto T^{-1}$, $\bar{a}(\bar{t}_e)/\bar{a}(\bar{t}_0) = T_0/M$, where $T_0 \approx 2.7\,{\rm K}$ is the present photon temperature. Also, $\bar{a}(\bar{t}_h)/\bar{a}(\bar{t}_e) = (\bar{t}_h/\bar{t}_e)^{(2\omega+3)/4}$ from (3.17). We substitute these relations into (4.6) to get

$$\left(\frac{\bar{t}_h}{\bar{t}_e}\right)^{(2\omega+3)/4}\left(\frac{T_0}{M}\right)\lambda_c = \frac{4\bar{t}_h}{2\omega+3} \ . \tag{4.7}$$

Solving for $\bar{t}_h$ and substituting in (4.4) we obtain

$$\left(\frac{\delta\rho}{\rho}\right)_{\rm Hubble} \approx \frac{(2\omega+3)^2}{16\Psi_0}\left[\left(\frac{2\omega+3}{4}\right)\left(\frac{T_0}{M}\right)\lambda_c\right]^{4/(2\omega-1)} \times \left(\frac{1}{\bar{t}_e}\right)^{(2\omega+3)/(2\omega-1)} \ . \tag{4.8}$$

We now eliminate Ψ_0 and $\bar{t}_e$ by using Eqs. (3.4) and (4.3), obtaining $\delta\rho/\rho$ in terms of known quantities:

$$\left(\frac{\delta\rho}{\rho}\right)_{\rm Hubble} \approx \sqrt{2\pi}\left(\frac{2\omega+3}{2}\right)^{(6\omega+5)/2(2\omega-1)}\left(\frac{1}{q\omega}\right)^{(2\omega+3)/(2\omega-1)}\left(\frac{M}{m_{\rm Pl}}\right)^{2(2\omega+1)/(2\omega-1)}(\lambda_c T_0)^{4/(2\omega-1)} . \tag{4.9}$$

Since we set $\bar{a}(\bar{t}_0) = 1$, λ_c is the physical wavelength at the present time. Remembering that we are using units for which $\hbar = c = k = 1$, one has the conversion $\lambda_c T_0 = \lambda_{\rm Mpc} \times 1\ {\rm Mpc} \times 2.7\,{\rm K} = 3.64 \times 10^{25}\lambda_{\rm Mpc}$. Thus,

$$\left(\frac{\delta\rho}{\rho}\right)_{\rm Hubble} \approx \sqrt{2\pi}\left(\frac{2\omega+3}{2}\right)^{(6\omega+5)/2(2\omega-1)}\left(\frac{1}{q\omega}\right)^{(2\omega+3)/(2\omega-1)}\left(\frac{M}{m_{\rm Pl}}\right)^{2(2\omega+1)/(2\omega-1)}\left(3.64 \times 10^{25}\lambda_{\rm Mpc}\right)^{4/(2\omega-1)} , \tag{4.10}$$

where q is defined in Eq. (2.7). This is our main result. Notice that beyond using $\delta\rho/\rho \approx \bar{H}^2(\bar{t})/\dot{\Psi}(\bar{t})$, the only approximation that we have made is to neglect the evolution of $\Phi(t)$ after the end of inflation.

In agreement with KST, we find that the perturbations are proportional to $\lambda^{4/(2\omega-1)}$. This means that extended inflation might be an attractive way to account for the astronomical observations that show evidence for increased power on large scales.

V. COMMENTS

A. Comparison with the naive Jordan frame result

To see the effect on this calculation of the transformation to the Einstein frame, it is interesting to compare our result (4.10) with the answer that would be obtained by naively applying the standard formalism in the Jordan frame. Using an asterisk to denote the naive calculation, we have

$$\left(\frac{\delta\rho}{\rho}\right)^*_{\rm Hubble} \equiv \left.\frac{H^2(t)}{d\phi(t)/dt}\right|_{t=t_h^*} , \qquad (5.1)$$

where ϕ is the field defined canonically by the action (1.2), and t_h^* is the time of Hubble length crossing as seen in the Jordan frame. In the following we denote the time of Hubble length crossing as seen in the Einstein frame by $t = t_h$ (and $\bar{t} = \bar{t}_h$ for the rescaled time variable), while the time of Hubble length crossing in the Jordan frame is denoted by $t = t_h^*$ (and $\bar{t} = \bar{t}_h^*$). Our result for $(\delta\rho/\rho)_{\rm Hubble}$ differs from the naive result for two reasons.

(i) At a given time the quantities $H^2/(d\phi/dt)$ and $\bar{H}^2/(d\Psi/d\bar{t})$ are not equal. Using the formulas from Sec. II, one easily finds that

$$\frac{H^2(t)}{d\phi(t)/dt} = \frac{\sqrt{2\pi\omega}(2\omega+1)^2 q}{4M^2t^2} . \qquad (5.2)$$

For comparison, the right-hand side of Eq. (4.4) can be expressed in terms of t by using Eq. (3.14). One then finds

$$\frac{\bar{H}^2(\bar{t})}{d\Psi(\bar{t})/d\bar{t}} = \sqrt{\frac{2\omega}{2\omega+3}}\left(\frac{2\omega+3}{2\omega+1}\right)^2 \left.\frac{H^2(t)}{d\phi(t)/dt}\right|_{t=t(\bar{t})} . \qquad (5.3)$$

(ii) The time of Hubble length crossing itself is different in the two frames. In the Jordan frame this time is evaluated using

$$a(t_h^*)\lambda_c = H(t_h^*)^{-1} , \qquad (5.4)$$

which is not equivalent to the Einstein frame relation (4.5). Using Eq. (3.14) to evaluate t_h in terms of $\bar{t}_h$, one finds the relation

$$\left(\frac{1}{t_h^2}\right) = \left(\frac{2\omega+3}{2\omega+1}\right)^{4/(2\omega-1)} \left(\frac{1}{t_h^{*2}}\right) . \qquad (5.5)$$

To put together the two sources of discrepancy, note that the correct expression for $(\delta\rho/\rho)_{\rm Hubble}$ is obtained by evaluating the left-hand side of Eq. (5.3) at $\bar{t}_h$, which implies that the right-hand side is evaluated at t_h. According to Eq. (5.2) this expression is proportional to $1/t_h^2$, which can be replaced by the right-hand side of Eq. (5.5). The factors occurring in Eqs. (5.3) and (5.5) are then multiplied to give

$$\left(\frac{\delta\rho}{\rho}\right)_{\rm Hubble} = F(\omega)\left(\frac{\delta\rho}{\rho}\right)^*_{\rm Hubble} , \qquad (5.6)$$

where the correction factor is given by

$$F(\omega) = \sqrt{\frac{2\omega}{2\omega+3}}\left(\frac{2\omega+3}{2\omega+1}\right)^{2(2\omega+1)/(2\omega-1)} \qquad (5.7)$$

The correction factor $F(\omega)$ is plotted in Fig. 1. It decreases monotonically with ω, approaching one as ω approaches infinity.

We emphasize again that in the Einstein frame the field Ψ behaves as a minimally coupled scalar field, and the rescaled time variable $\bar{t}$ and scale factor $\bar{a}(\bar{t})$ correspond to a Robertson-Walker metric; therefore *these* functions, not the original ones, must be used in applying the standard methods to calculate $(\delta\rho/\rho)_{\rm Hubble}$.

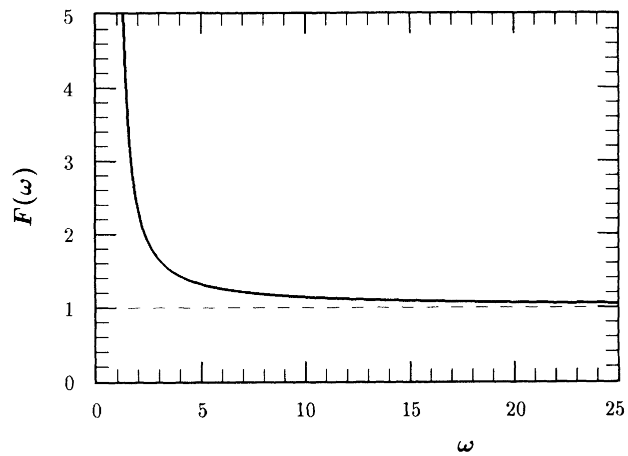

FIG. 1. The effect of transforming to the Einstein frame. The correct answer for $\delta\rho/\rho$ is larger by the factor $F(\omega)$ than the answer that would be obtained by naively applying the standard formalism in the Jordan frame.

B. Comparison with KST's results

KST (Ref. [9]) have also worked with the Einstein frame action, but nonetheless their answer [Eq. (2.21) of their paper] differs from ours: it is equal to our answer times the factor

$$\sqrt{\frac{6\omega+5}{3(2\omega+3)}}\left(\frac{2\omega}{2\omega+3}\right)^{4/(2\omega-1)} . \qquad (5.8)$$

This discrepancy is due to the following reasons.

(i) They evaluate $\bar{H}^2(\bar{t})/\dot{\Psi}(\bar{t})$ at the time of Hubble length crossing in the Jordan frame, while we maintain that the time of Hubble length crossing must be evaluated in the Einstein frame. According to Eq. (5.5), this causes their result to contain an additional factor $[(2\omega+1)/(2\omega+3)]^{4/(2\omega-1)}$.

(ii) They use a "slow-rollover" approximation $\dot{\Psi}(\bar{t}) \approx -(dV/d\Psi)/3\bar{H}(\bar{t})$, while we evaluate $\dot{\Psi}(\bar{t})$ by differentiating the exact solution for $\Psi(\bar{t})$. This causes their result to contain the additional factor $\dot{\Psi}_{\rm exact}/\dot{\Psi}_{\rm approx} = 3(2\omega+3)/(6\omega+5)$.

(iii) They evaluate $\bar{H}$ by using $\bar{H}^2 \approx 8\pi V/3m_{\rm Pl}^2$ (neglecting the kinetic energy), while we used the exact expression. This causes their result to contain an additional factor $(\bar{H}_{\rm approx}/\bar{H}_{\rm exact})^3 = [(6\omega+5)\,/\,3(2\omega+3)]^{3/2}$.

(iv) They omit a factor $2\omega/(2\omega+1)$ that should appear on the top line of their Eq. (2.9). This causes their result to contain an additional factor $[2\omega/(2\omega+1)]^{4/(2\omega-1)}$.

Each of these discrepancy factors approaches one as ω approaches infinity, but in this limit the effect of transforming to the Einstein frame disappears altogether.

The discrepancy factor (5.8) carries over into the formula for the temperature fluctuations of the cosmic background radiation, $(\delta T/T)_{\theta>1^\circ} \simeq \bar{H}^2(\bar{t})/15\dot{\Psi}(\bar{t})$, given as Eq. (2.25) in KST. For the same reasons, we would differ with KST's results for graviton perturbations, Eqs. (3.4) and (3.7) in their paper. For the dimensionless amplitude of a gravitational-wave perturbation as it comes inside the Hubble length in the late universe, we obtain

$$h_\lambda \approx \frac{\bar{H}}{m_{\rm Pl}} = \left(\frac{M}{m_{\rm Pl}}\right)^{2(2\omega+1)/(2\omega-1)} \times \left(\frac{2\omega+3}{2q\omega}\right)^{(2\omega+3)/(2\omega-1)} \times \left(3.64\times10^{25}\lambda_{\rm Mpc}\right)^{4/(2\omega-1)} . \quad (5.9)$$

C. Application to generalized gravity theories

We have obtained the density perturbation spectrum for a simple model of extended inflation. The method we have used, however, is applicable to a wide class of generalized gravity theories that involve a scalar field coupled to gravity. Suppose that the action can be written as

$$S = \int d^4x\sqrt{-g}[-f(\phi)R + \tfrac{1}{2}g^{\mu\nu}T(\phi)\partial_\mu\phi\partial_\nu\phi + \mathcal{L}_{\rm matter}] , \quad (5.10)$$

where $f(\phi)$ and $T(\phi)$ are arbitrary functions. From Eq. (1.2) it follows that for Brans-Dicke gravity $f(\phi) = \phi^2/8\omega$ and $T(\phi) = 1$. If $T(\phi) = 1$, then for all $f(\phi) > 0$ [the condition for a general $T(\phi)$ is given below] the conformal transformation to the Einstein frame can be performed and, through a redefinition of fields, the action can be cast in the form of the action for a minimally coupled scalar field. We first demonstrate this for a general $f(\phi)$, and then consider the analytically tractable case of $f(\phi) = \phi^4$.

We make the conformal transformation $g_{\mu\nu} = \Omega^2\bar{g}_{\mu\nu}$, where

$$\Omega^2 = \frac{m_{\rm Pl}^2}{16\pi}\frac{1}{f(\phi)} . \quad (5.11)$$

The action (5.10) then takes the form (with $\mathcal{L}_{\rm matter} = 0$ for convenience)

$$\bar{S} = \int d^4x\sqrt{-\bar{g}}\left(-\frac{m_{\rm Pl}^2}{16\pi}R + \frac{1}{2}g^{\mu\nu}K(\phi)\partial_\mu\phi\partial_\nu\phi\right) , \quad (5.12)$$

where $K(\phi)$ is given by Salopek, Bond, and Bardeen [19] as

$$K(\phi) = \frac{m_{\rm Pl}^2}{16\pi}\frac{1}{f(\phi)^2}[3f'(\phi)^2 + f(\phi)T(\phi)] . \quad (5.13)$$

The first term on the right-hand side of (5.12) comes from the conformal transformation of the scalar curvature term in (5.10), and the second term comes from the original kinetic term. If we define a field $\Psi(\phi)$ such that $\Psi'(\phi) = \sqrt{K(\phi)}$, then

$$\partial_\mu\Psi(\phi)\partial_\nu\Psi(\phi) = \Psi'(\phi)^2\partial_\mu\phi\partial_\nu\phi = K(\phi)\partial_\mu\phi\partial_\nu\phi . \quad (5.14)$$

So in terms of $\Psi(\phi)$ the kinetic term is canonical and the action takes the form for a minimally coupled scalar field. For $K(\phi) > 0$ the integral

$$\Psi(\phi) = \int\sqrt{K(\phi)}d\phi \quad (5.15)$$

is well defined and is a monotonically increasing function of ϕ, so there is a unique value of $\Psi(\phi)$ (up to an additive constant) for every value of ϕ. Therefore one expects that the quantum theory for Ψ gives the standard result, Eq. (4.1), for the density perturbation spectrum. In general the integral for $\Psi(\phi)$ and the solutions of the equations of motions must be obtained numerically.

For Brans-Dicke gravity, the integral for $\Psi(\phi)$ is simple and the result is given by (3.3). $\Psi(\phi)$ can also be obtained in closed form for the case $f(\phi) = \phi^4$, $T(\phi) = 1$. From Eq. (5.13) one has

$$K(\phi) = \frac{m_{\rm Pl}^2}{16\pi}\left(\frac{48\phi^2+1}{\phi^4}\right) , \quad (5.16)$$

which can be integrated according to Eq. (5.15) to give

$$\Psi(\phi) = \sqrt{\frac{3m_{\rm Pl}^2}{\pi}}\left(-\sqrt{\frac{48\phi^2+1}{48\phi^2}} + \ln(\sqrt{48}\phi + \sqrt{48\phi^2+1})\right) . \quad (5.17)$$

In hyperextended inflation a term of this form may dominate the $f(\phi)R$ coupling during a cosmologically important epoch, so it is of some interest to study its density perturbation spectrum [9].

A nonminimally coupled scalar field with $f(\phi) = 1 - \alpha\phi^2$ has been studied by Futamase and Maeda [20], who have obtained $\Psi(\phi)$ for all $\alpha > 0$.

VI. CONCLUSION

We have estimated the density perturbation spectrum in the original model of extended inflation, with Brans-Dicke gravity. Curvature fluctuations arising from quantum fluctuations in the Brans-Dicke field contribute a significant amplitude of density perturbations. They are a slowly increasing function of the scale, a feature that might be useful in building models to account for the observed large-scale structure of the universe. We have performed the calculation by transforming to the Einstein conformal frame, then applying the standard procedures used in conventional inflationary models. We have pointed out some subtleties associated with this procedure, but we nonetheless believe that the result is valid as an order-of-magnitude estimate.

We have compared our density perturbation amplitude to the answer that would be obtained by working naively in the Jordan frame—our answer is larger by a factor that is near unity, but which becomes large for very small values of the Brans-Dicke parameter ω. If the calculation is done correctly in both frames, however, one should of course expect to obtain the same answer. Indeed, part of our motivation was to lay some groundwork toward a consistent calculation in the two frames. The success of such a calculation would give us confidence that the field theory is being treated correctly, and that the conformal transformation method is valid at the quantum (or at least semiclassical) level as well as the classical level. The question of consistency between the two frames has been addressed in two recent papers [21].

As pointed out earlier, the model we have studied must be modified if it is to satisfy experimental constraints. One possibility is to add a small mass term for the Brans-Dicke field—the evolution of Φ during the inflationary period would not be significantly affected, but the mass term could still freeze the value of the field in the present epoch so that the theory would be consistent with observation. As pointed out by KST, in this scenario the Φ particles would have to be unstable in order to prevent the mass density of the universe from becoming dominated by them. If the model is repaired in this fashion, then the calculation of density fluctuations presented in this paper would remain valid. One can also imagine more substantial modifications to the model, in which case our calculation would no longer be valid in detail. It would nonetheless serve as an illustration of a technique to compute $(\delta\rho/\rho)_{\rm Hubble}$ for models with a scalar field coupled to gravity.

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