Predictive *Ansatz* for fermion mass matrices in supersymmetric grand unified theories

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In this paper we present a new *Ansatz* for fermion mass matrices in the context of supersymmetric grand unified theories. We are able to fit the 13 parameters, associated with quark and lepton masses and mixing angles, and the ratio of Higgs vacuum expectation values (VEV's) which enters any supersymmetric theory, in terms of 8 input parameters; hence, we have 6 predictions. The top quark is predicted to have a mass in the range 176 to 190 GeV, where the upper bound results from the assumption of perturbative unification, and the lower bound is sensitive to the experimental value of V_{cb}. Predictions are also made for m_s, m_s/m_d, $|V_{ub}/V_{cb}|$, the ratio of Higgs VEV's, and the *CP*-violating phase of the Kobayashi-Maskawa matrix.

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I. INTRODUCTION

The standard model describes all known experimental data in terms of 18 phenomenological parameters: 3 gauge couplings, 13 fermion masses and mixing angles, and a Higgs vacuum expectation value (VEV) and mass. Neutrinos are predicted to be massless. Several attempts, using symmetries or specific *Ansätze,* have been made in the past to reduce the number of these independent parameters. Such attempts derive from the firm belief that in more fundamental theory all these phenomenological parameters may be derived from a smaller set of *fundamental* constants. Putting faith aside, there is a more practical reason for these attempts; any reduction in the number of *fundamental* parameters necessarily has predictive power. If experimental data subsequently confirm these predictions, then *perhaps* we have learned some important piece of information about the *fundamental* theory of everything.

In this paper we present a new *Ansatz* for fermion masses with only eight arbitrary parameters. These eight parameters may be associated with the three up-quark masses, three down-quark masses, the *CP*-violating angle, and the ratio of Higgs VEV's, which enters any supersymmetric theory. In order to make progress and go beyond this *Ansatz,* one must necessarily resolve one of the following fundamental problems: understanding the generation hierarchy, the top-bottom mass hierarchy, or the origin of *CP* violation. A solution to any one of these problems would be a phenomenal breakthrough. In the rest of this section, we review the framework for our program [1].

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A. Fritzsch *Ansatz*

In 1978, Fritzsch [2] proposed an *Ansatz* for quark masses which is only now being fully tested. He considered the 3×3 up- and down-quark mass matrices of the form

$$m_U \sim \begin{bmatrix} 0 & C & 0 \\ C & 0 & B \\ 0 & B & A \end{bmatrix} , \tag{1}$$

and m_D is similar to the complex parameters A,B,C replaced by $\tilde{A},\tilde{B},\tilde{C}$. All but two of the phases in these six complex parameters can be rotated away by suitable redefinitions of the quark fields. Thus there are only eight real numbers (six magnitudes and two phases) to describe the six quark masses and four mixing angles in the Kobayashi-Maskawa (KM) matrix and, hence, two predictions. These can be taken to be the *CP*-violating angle and top-quark mass. For example, using the experimental data for the five quark masses and three mixing angles as input, Gilman and Nir [3] found the allowed range for the top-quark mass:

$$77 \le m_t \le 96 \text{ GeV} . \tag{2}$$

This is still consistent with the lower limit on the top-quark mass, $m_t > 89$ GeV, from the Collider Detector at Fermilab (CDF) [4] Collaboration, but outside the preferred range allowed by measurements of electroweak parameters from the CERN e^+e^- collider LEP. For example, L3 recently found $m_t = 193^{+52}_{-69}\pm16$ GeV and ALEPH found $m_t = 170^{+42+21}_{-55-14}$ GeV, where the second error corresponds to a change in the Higgs-boson mass from 50 GeV to 1 TeV [5].

In the Fritzsch *Ansatz*, a *light* top quark is easily seen to follow from the relation between the quark masses and the Cabibbo-Kobayashi-Maskawa (CKM) element V_{cb} given by

$$|V_{cb}| \approx \left| \left[\frac{s}{b}\right]^{1/2} - \left[\frac{c}{t}\right]^{1/2} e^{-i\phi_2} \right| , \quad (3)$$

where ϕ_2 is a free parameter. Since $\sqrt{s/b} \sim 0.18$ and $|V_{cb}| = 0.044 \pm 0.009$, a precise cancellation is required which forces the top-quark mass to be *light*. He and Hou [6] have recently suggested a revised Fritzsch *Ansatz* in which the problematic factor $\sqrt{s/b}$ is eliminated from the above relation. They then find the allowed range for the top-quark mass:

$$200 \le m_t \le 750 \text{ GeV} . \quad (4)$$

This may be overkill [5].

In the following section we present a new *Ansatz* for fermion masses. We are motivated by the desire to predict the largest number of experimentally observable quantities with the fewest number of fundamental parameters. Thus we shall not limit ourselves to quark masses and mixing angles alone, but also include lepton masses. Hence our analysis naturally begins from a grand unified theory (GUT), where quarks and leptons can be considered on the same footing. We are then quickly led to the scheme invented by Georgi and Jarlskog [7], which elegantly reconciles the successful GUT prediction for b/τ with the predictions for the first two families of quarks and leptons.

II. NEW *ANSATZ*

In the preceding section we discussed the Fritzsch *Ansatz* for fermion masses which required a top quark with mass ≤ 96 GeV and is now apparently ruled out by electroweak limits on the top-quark mass [5]. We also considered a revised Fritzsch *Ansatz* [6], which allowed for a heavier top quark, albeit perhaps a bit too heavy. In both models a simple parametrization of the quark mass matrices was used to obtain predictions for quark-mixing angles in terms of quark masses. In any renormalizable field theory, the numerical value for the elements of the quark mass matrix depends on the renormalization scale. Up until now we have not explicitly emphasized this point. However, we have (as is normally done) implicitly defined the mass matrices at a scale of order the top-quark mass. It is at this scale that we define the mixing angles via unitary transformations which diagonalize the mass matrices. Below the top-quark mass m_t, we use an effective-field theory with five quarks (defined by integrating out the top quark) to run the light-quark masses down to m_b. Similarly, from m_b to m_c, the bottom quark is integrated out and a four-quark effective-field theory is used for the running. Finally, a three-quark model is used to run the light-quark masses down to 1 GeV.

One may ask, is there any *a priori* reason to define the quark mass *Ansatz* at the top-quark mass. In general, Yukawa matrices satisfy renormalization-group equations, and a particular *Ansatz* at one scale may be significantly changed at a new scale. The answer to this rhetorical question is that since the original *Ansatz* is arbitrary, no generality is lost by defining it at an arbitrary scale. In this paper we are guided by a desire to find the minimal parameter set fitting *all* fermion masses. In particular, we want to relate quark and lepton masses. This is most naturally accomplished in a grand unified gauge theory with an *Ansatz* for mass matrices defined at the scale M_G, where the grand unified gauge symmetry is restored.

Even with grand unification it is not mandatory to have the form of the *Ansatz* destroyed beneath the unification scale. For example, global symmetries which enforce the form of the *Ansatz* might survive beneath the GUT scale. There are several reasons why we prefer not to do this. Such symmetries require more than the minimal number of light Higgs doublets, which will upset the successful predictions of $\sin^2\theta_W$. The low-energy theory would not be the simplest possible. Furthermore, if the global symmetries were broken at a scale not far above the weak scale, one would expect this would lead to unacceptable flavor-changing neutral currents. In general, having an *Ansatz* apply near the weak scale is very dangerous: Flavor-changing problems are likely to appear on construction of the full theory, which has the *Ansatz* guaranteed by symmetries. It will turn out that it is crucial for our scheme that the *Ansatz* applies only at very high scales: It is the renormalizations away from the original *Ansatz* that produce an acceptable value for the top-quark mass.

In constructing our framework we are guided by the following practical constraints: it has become evident that a supersymmetric (SUSY) GUT [8–10] is the simplest model which fits the accurate LEP data for α_s, α, and $\sin^2\theta_W$ [11]; in a SUSY GUT the good b/τ prediction can be maintained in the presence[1] of a heavy top quark [12]; to obtain reasonable relations for e, μ, d, and s masses with a minimal number of parameters, we use the successful scenario of Georgi and Jarlskog [7] with the following proviso; the GUT gauge group is assumed to be SO(10) or larger.

The last constraint enhances the predictive power of our framework by allowing the mass matrices to be naturally symmetric, thus minimizing the number of parameters.

[1]The second consideration provides additional evidence in favor of SUSY GUT's and therefore deserves some explanation. A large top-quark Yukawa coupling has, in general, a significant effect on the renormalization-group running. In a one-Higgs-boson model, such as in a minimal non-SUSY GUT, it contributes to the running of both quarks and leptons. It has the effect of increasing the b/τ ratio, such that for a top-quark mass ~ 140 GeV the b/τ ratio is too large. In a minimal SUSY GUT, there are at least two Higgs doublets. As a consequence, the top-quark Yukawa coupling contributes, at one loop, only to the running of the quarks with the effect of decreasing the b/τ ratio. This leads to acceptable b/τ ratios with m_t as large as ~ 190 GeV.

We now discuss our *Ansatz* for the Yukawa matrices at the GUT scale. Since we do not attempt to relate the up-quark matrix to the down-quark matrix, the Yukawa matrices must involve at least seven real parameters: three for the hierarchy of eigenvalues of the up-quark mass matrix, three more for the down-type quark masses, and the *CP*-violating phase. Let us temporarily ignore *CP* violation and concentrate on the form of the down-quark and charged-lepton mass matrices. With just three free parameters, it is easy to show that the only way of obtaining the GUT scale mass relations $m_b=m_\tau$, $m_s=m_\mu/3$, and $m_d=3m_e$ is to use the Georgi-Jarlskog *Ansatz* for the down-quark and charged-lepton mass matrices [7]. There are several ways of arranging the three parameters of the up-quark Yukawa matrix to get the desired hierarchy of up-type quark masses. However, present measurements of the KM angles immediately rule out all but one possibility. Furthermore, when we reintroduce *CP* violation by allowing all the parameters of the Yukawa matrices to be complex, this last remaining acceptable possibility has just one irremovable phase, as we will shortly demonstrate. Thus we find that there is a unique GUT *Ansatz* for the Yukawa matrices which incorporates the Georgi-Jarlskog mass relations for down quarks and charged leptons and which has the minimal number of seven free parameters. This *Ansatz* is given by

$$m_U=\mathbf{U}\frac{V\sin\beta}{\sqrt{2}},\quad m_D=\mathbf{D}\frac{V\cos\beta}{\sqrt{2}},\quad m_E=\mathbf{E}\frac{V\cos\beta}{\sqrt{2}}, \tag{5}$$

where $\tan\beta$ is the ratio of Higgs VEV's, $V=246$ GeV, and the 3×3 Yukawa matrices $\mathbf{U},\mathbf{D},\mathbf{E}$ are given by

$$\mathbf{U}=\begin{bmatrix}0 & C & 0\\ C & 0 & B\\ 0 & B & A\end{bmatrix},\quad \mathbf{D}=\begin{bmatrix}0 & F & 0\\ F & E & 0\\ 0 & 0 & D\end{bmatrix},\quad \mathbf{E}=\begin{bmatrix}0 & F & 0\\ F & -3E & 0\\ 0 & 0 & D\end{bmatrix}, \tag{6}$$

defined at a scale $\mu=M_G$. The parameters A–F are, in general, complex. We now use the freedom of field redefinition to eliminate some of the phases as follows: The U and D Yukawa matrices have nine nonvanishing entries. We have nine fields at our disposal—three doublets and six singlets; thus eight relative phases that can be used to get rid of all but one of the complex phases. For convenience we use this phase freedom to make A, B, C, D, and E real and keep F complex and the mass matrices Hermitian. Thus we have seven real parameters A,B,C,D,E, the magnitude of F (call it F from now on), and its phase ϕ. A, B, and C describe the hierarchy of up-quark masses, D, E, and F that of down-quark masses or electrons. The lepton mass matrix E can easily be made real by using the phase freedom of the six fields—three doublets and three charged singlets. In this paper we will not consider neutrino masses. After these field redefinitions, we obtain the Yukawa matrices

$$\mathbf{U}=\begin{bmatrix}0 & C & 0\\ C & 0 & B\\ 0 & B & A\end{bmatrix},\quad \mathbf{D}=\begin{bmatrix}0 & Fe^{i\phi} & 0\\ Fe^{-i\phi} & E & 0\\ 0 & 0 & D\end{bmatrix},\quad \mathbf{E}=\begin{bmatrix}0 & F & 0\\ F & -3E & 0\\ 0 & 0 & D\end{bmatrix}. \tag{7}$$

The zeros in these matrices are assumed to be constrained by a discrete symmetry defined in the GUT. This discrete symmetry is, however, necessarily broken in the low-energy theory, which has at most two Higgs doublets [8–11]. In order to compare with low-energy data, we must then use the renormalization-group equations (RGE's) (see next subsection) to obtain the fermion mass matrices at the scale m_t. As a result of the RG running, the zeros in the *Ansatz* are now only approximate, and in addition, two significant terms are generated in the above mass matrices.

We find (at the scale $\sim m_t$)

$$\mathbf{U}=\begin{bmatrix}0 & C & 0\\ C & \delta_u & B\\ 0 & B & A\end{bmatrix},\quad \mathbf{D}=\begin{bmatrix}0 & Fe^{i\phi} & 0\\ Fe^{-i\phi} & E & \delta_d\\ 0 & 0 & D\end{bmatrix},\quad \mathbf{E}=\begin{bmatrix}0 & F & 0\\ F & -3E & 0\\ 0 & 0 & D\end{bmatrix}. \tag{8}$$

It is these matrices which are diagonalized by the unitary matrices V_u, V_d^L, V_d^R, V_e defined by $\mathbf{U}^{\rm diag}=V_u\mathbf{U}V_u^\dagger$, $\mathbf{D}^{\rm diag}=V_d^L\mathbf{D}V_d^{R\dagger}$, $\mathbf{E}^{\rm dia}=V_e\mathbf{E}V_e^\dagger$, and $\mathbf{U}^{\rm diag}$, etc., are real diagonal matrices. The CKM mixing matrix is then given by $V_{\rm CKM}=V_uV_d^{L\dagger}$.

The matrix elements satisfy the inequalities $A\gg B\sim\delta_u\gg C$ and $D\gg E\sim\delta_d\gg F$. Therefore we can obtain approximate eigenvalues

$$t\sim A,\quad c\sim-\delta_u+\frac{B^2}{A},\quad u\sim\frac{C^2}{c},\qquad b\sim D,\quad s\sim E,\quad d\sim\frac{F^2}{s} \tag{9}$$

and approximate mixing matrices

$$V_u=\begin{bmatrix}c_2 & s_2 & 0\\ -s_2 & c_2 & 0\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}1 & 0 & 0\\ 0 & c_3 & s_3\\ 0 & -s_3 & c_3\end{bmatrix},$$
$$V_d^L=\begin{bmatrix}c_1 & -s_1 & 0\\ s_1 & c_1 & 0\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}1 & 0 & 0\\ 0 & c_4 & s_4\\ 0 & -s_4 & c_4\end{bmatrix}\begin{bmatrix}1 & 0 & 0\\ 0 & e^{i\phi} & 0\\ 0 & 0 & e^{i\phi}\end{bmatrix}, \tag{10}$$

where

$$s_3 \sim -\frac{B}{A} \sim \left[\frac{c-|\delta_u|}{t}\right]^{1/2}, \quad s_2 \sim \left[\frac{u}{c}\right]^{1/2}, \quad s_1 \sim \left[\frac{d}{s}\right]^{1/2}, \quad s_4 \sim -\frac{\delta_d}{D} .$$

The CKM matrix elements are then determined in terms of the above angles. We find

$$V_{\rm CKM} = \begin{bmatrix} c_1c_2 - s_1s_2e^{-i\phi} & s_1 + c_1s_2e^{-i\phi} & s_2(s_3-s_4) \\ -c_1s_2 - s_1e^{-i\phi} & -s_1s_2 + (c_1c_2c_3c_4 + s_3s_4)e^{-i\phi} & (s_3-s_4) \\ s_1(s_3-s_4) & -c_1(s_3-s_4) & (c_3c_4+s_3s_4)e^{i\phi} \end{bmatrix} , \tag{11}$$

where $s_1 \sim \lambda$, $s_2 \sim s_3 \sim s_4 \sim \lambda^2$, and the expression is valid to order λ^4. The parameter λ can now be identified with Wolfenstein's parameter $\lambda = |V_{cd}|$. Recall the Wolfenstein [13] form for the CKM matrix:

$$V_{\rm CKM} = \begin{bmatrix} 1-\frac{1}{2}\lambda^2 & \lambda & \lambda^3 A(\rho - i\eta) \\ -\lambda & 1-\frac{1}{2}\lambda^2 & \lambda^2 A \\ \lambda^3 A(1-\rho-i\eta) & -\lambda^2 A & 1 \end{bmatrix} . \tag{12}$$

We then obtain the identities

$$\begin{aligned} &\lambda = (s_1^2 + s_2^2 + 2s_1s_2\cos\phi)^{1/2} = |V_{cd}| = |V_{us}| , \\ &\lambda^2 A = s_3 - s_4 = |V_{cb}| , \\ &\lambda(\rho^2+\eta^2)^{1/2} = s_2 = |V_{ub}/V_{cb}| = \left[\frac{u}{c}\right]^{1/2} , \\ &\eta = s_1s_2\sin\phi/\lambda^2 . \end{aligned} \tag{13}$$

A. Input data

We have seven fundamental parameters in our fermion Yukawa coupling *Ansatz* (A, B, C, D, E, F, and ϕ) and one ratio of Higgs VEV's given by $\tan\beta$. We shall use the eight best measured low-energy parameters (e,μ,τ,c, $b|V_{cb}|,\mu/d,|V_{cd}|$) as input to fix the seven Yukawa couplings and $\tan\beta$. We then quote predictions for d, s, t, $|V_{ub}/V_{cb}|$, $\sin\beta$, and the *CP*-violating angle.

Let us now discuss the acceptable ranges for the experimentally measured parameters. The charged-lepton masses e, μ, and τ are accurately known to be 0.511, 105.658, and $1784.1^{+2.7}_{-3.6}$ MeV, respectively. Gasser and Leutwyler [14] give values for the running masses of the three light quarks u, d, and s, determined via current algebra and QCD sum rules: (at 1 GeV) 5.1 ± 1.5, 8.9 ± 2.6, and 175 ± 55 MeV, respectively. Finally, for the heavy quarks, they find $m_c(m_c)=1.27\pm0.05$ GeV and $m_b(m_b)=4.25\pm0.1$ GeV. Note that the ratios s/d and u/d are constrained by chiral-Lagrangian analyses [15]. Kaplan and Manohar [15] find $0.2 \le u/d \le 0.7$ and $15 \le s/d \le 25$, where larger values of s/d correspond to smaller values of u/d. Leutwyler [15], using additional constraints, finds tighter bounds.

Limits on the weak mixing angles are found by the Particle Data Group [16]. The 90% confidence limits are $|V_{us}| = |V_{cd}| = 0.221\pm0.003$, $|V_{cb}| = 0.044\pm0.014$, and $|V_{ub}| = 0.004\pm0.003$. At 1σ we have $|V_{cb}| = 0.044 \pm 0.009$ and $|V_{ub}/V_{cb}| = 0.09\pm0.04$.

Quark and lepton masses must be renormalized to a common scale $\mu = m_t$ in order to be compared directly with the results of the next section. For this paper we include the two-loop effects of QCD and the one-loop effects of QED in the running.

B. Scaling Yukawa matrices from grand to weak scales

At the grand unification scale M_G, we assume that there is some symmetry structure which ensures the Yukawa coupling matrices have the form given in Eq. (6). Beneath M_G we take this symmetry to be broken, so that the form is not preserved as the matrices are renormalization-group scaled to the weak scale. For many parameters this scaling is unimportant. This can be because the parameter was in any case arbitrary at the grand unified scale (e.g., $U_{21}=U_{12}=C$) or because, even though it was determined at the unification scale, it gets generated at a level too small to affect comparison with data (e.g., U_{13} and U_{31}). For the purposes of our mass and mixing-angle predictions, there are six important RG equations.

(a) The top Yukawa coupling $\lambda_t \equiv U_{33} = A$ evolves toward its infrared fixed point.

(b) The charm Yukawa coupling $\delta_u = U_{22}$ becomes nonzero and gives an important contribution to m_c.

(c) The coupling $\delta_d = D_{23}$ becomes nonzero and gives a contribution to V_{cb}.

(d) The running of $B = U_{23} = U_{32}$ is important since it appears as a source term in generating δ_d and δ_u.

(e) The running of $\lambda_b = D_{33}$ is important since it also appears as a source term in generating δ_d.

(f) The running of $R = \lambda_b/\lambda_\tau = D_{33}/E_{33}$ leads to the prediction for m_b/m_τ. Also, the running of $R' = \lambda_s/\lambda_\mu = D_{33}/E_{22}$ leads to the prediction for m_s. However, the solution for R' is obtained trivially from that for R.

To perform these six RG scalings, we need a model which successfully unifies: That is, the three gauge couplings g_i ($i=1,2,3$) all merge at some scale M_G which is larger than 10^{15} GeV. The best-motivated such model is the minimal supersymmetric standard model (MSSM). In this paper we take all superpartners and the second Higgs doublet to be degenerate at $M_S = 160$ GeV. We assume no threshold effects at intermediate or superheavy scales, and we take the lightest Higgs boson degenerate with the Z. A less idealistic superpartner spectrum will not substantially alter our predictions. We do not expect large changes in our predictions to occur unless many of the superpartners are made much heavier than a TeV.

We solve the one-loop RG equations $dg_i/dt = b_i g_i^3/16\pi^2$, where $b_i=(\frac{33}{5},1,-3)$ above M_S and $t=\ln\mu$. Taking inputs of $\alpha^{-1}=127.8$ and $\sin^2\hat\theta_W=0.233$, we find that α_1 and α_2 merge at $M_G=1.0\times10^{16}$ GeV, where they are equal to $\alpha_G=1/25.1$. These numbers correspond to $m_t=160$ GeV, which will turn out to be at the lower end of our predicted range. Evolving α_3 down to weak scales, we predict $\alpha_3(M_Z)=0.109$.

In the MSSM the six one-loop RG equations for Yukawa couplings that we need are

$$\frac{d\lambda_t}{dt}=\frac{\lambda_t}{16\pi^2}(-c_i g_i^2+6\lambda_t^2)\ , \tag{14a}$$

$$\frac{d\delta_u}{dt}=\frac{\delta_u}{16\pi^2}\left[-c_i g_i^2+3\lambda_t^2+3B^2\frac{\lambda_t}{\delta_u}\right]\ , \tag{14b}$$

$$\frac{d\delta_d}{dt}=\frac{\delta_d}{16\pi^2}\left[-c_i' g_i^2+B\lambda_t\frac{\lambda_b}{\delta_d}\right]\ , \tag{14c}$$

$$\frac{dB}{dt}=\frac{B}{16\pi^2}(-c_i g_i^2+6\lambda_t^2)\ , \tag{14d}$$

$$\frac{d\lambda_b}{dt}=\frac{\lambda_b}{16\pi^2}(-c_i' g_i^2+\lambda_t^2)\ , \tag{14e}$$

$$\frac{dR}{dt}=\frac{R}{16\pi^2}(-d_i g_i^2+\lambda_t^2)\ , \tag{14f}$$

where $c_i=(\frac{13}{15},3,\frac{16}{3})$, $c_i'=(\frac{7}{15},3,\frac{16}{3})$, and $d_i=(-\frac{4}{3},0,\frac{16}{3})$. The exact solutions to these equations are

$$\frac{\lambda_t^2}{\lambda_{t_G}^2}=\frac{\eta}{1+(3/4\pi^2)\lambda_{t_G}^2 I}=\eta\left[1-\frac{3}{4\pi^2}\lambda_t^2\frac{I}{\eta}\right]\ , \tag{15a}$$

$$\frac{\delta_u}{\lambda_t}=-\frac{B^2}{\lambda_t^2}\left[\left[\frac{\lambda_{t_G}}{\lambda_t}\right]^{1/2}\eta^{1/4}-1\right]\ , \tag{15b}$$

$$\frac{\delta_d}{\lambda_b}=-\frac{B}{\lambda_t}\left[\left[\frac{\lambda_{t_G}}{\lambda_t}\right]^{1/6}\eta^{1/12}-1\right]\ , \tag{15c}$$

$$\frac{B}{B_G}=\frac{\lambda_t}{\lambda_{t_G}}\ , \tag{15d}$$

$$\frac{\lambda_b}{\lambda_{b_G}}=\left[\frac{\lambda_t}{\lambda_{t_G}}\right]^{1/6}\eta^{5/12}\left[\frac{\alpha_1}{\alpha_G}\right]^{1/33}\ , \tag{15e}$$

$$R=\left[\frac{\lambda_t}{\lambda_{t_G}}\right]^{1/6}\eta^{5/12}\left[\frac{\alpha_2}{\alpha_G}\right]^{3/2}\left[\frac{\alpha_1}{\alpha_G}\right]^{1/6}$$

$$\equiv\lambda_b/\lambda_\tau\ , \tag{15f}$$

where all couplings without subscripts are evaluated at μ, which we later take to be m_t, and those with a G subscript are evaluated at $\mu=M_G$. Also,

$$\eta(\mu)=\prod_i\left[\frac{\alpha_G}{\alpha_i}\right]^{c_i/b_i} \tag{16a}$$

and

$$I(\mu)=\int_\mu^{M_G}\eta(\mu')d\ln\mu'\ . \tag{16b}$$

Using values of $\alpha_i(M_Z)$ quoted earlier, we find $\eta(\mu=160\ \text{GeV})=10.1$ and $I(\mu=160\ \text{GeV})=112$. In the following analysis, the parameters η, I and the ratio of gauge couplings should be self-consistently evaluated at $\mu=m_t$. We shall, however, neglect the small changes in these parameters resulting from varying m_t away from 160 GeV.

III. MASS AND MIXING-ANGLE PREDICTIONS

We now discuss the predictions which follow from these solutions. We begin by considering the heavy two generations, since these predictions are sensitive to the RG scaling and can be studied independent of the first generation. There are five free parameters in the mass matrices of the second and third generations: the four independent parameters in the Yukawa matrices at M_G and the ratio of Higgs-doublet VEV's $v_2/v_1=\tan\beta$. These five parameters can be fixed by specifying m_μ, m_τ, m_c, m_b, and V_{cb}, leaving two predictions for m_s and m_t. It turns out, however, that the additional constraint of perturbativity will also imply that the b mass is predicted to be in a narrow range.

The prediction for the strange-quark mass is very straightforward, but is the least interesting as there is considerable uncertainty in the value extracted from data. The prediction follows from the solution for R', which is the same as that for R given in (15f) except the $(\lambda_t/\lambda_{t_G})^{1/6}$ factor is absent; there is a suppression of a factor of 3 coming from $R'(M_G)$ and the power of η has changed:

$$m_s-m_d=\frac{1}{3}\eta_s m_\mu \eta^{1/2}\left[\frac{\alpha_2}{\alpha_G}\right]^{3/2}\left[\frac{\alpha_1}{\alpha_G}\right]^{1/6}$$

$$=149\left[\frac{\eta_s}{2.02}\right]\text{MeV}\ , \tag{17}$$

where η_s is the factor which results from renormalizing (m_s/m_μ) from m_t to 1 GeV. This is the successful Georgi-Jarlskog prediction, with the precise number appropriate to unification of the MSSM. Later, we shall combine this result with the Georgi-Jarlskog prediction for the ratio m_s/m_d [see Eq. (31)], thus obtaining the values for m_s and m_d, separately.

Diagonalization of the up-quark mass matrix in the heavy two-generation sector gives the ratio of current quark masses,

$$\frac{m_c}{m_t}=\eta_c\left[\frac{B^2}{\lambda_t^2}-\frac{\delta_u}{\lambda_t}\right]=\eta_c\frac{B^2}{\lambda_t^2}\left[\frac{\lambda_{t_G}}{\lambda_t}\right]^{1/2}\eta^{1/4}\ , \tag{18}$$

where η_c is the RG scaling factor of $m_c(\mu)$ from $\mu=m_c$ to m_t and all couplings on the right-hand side are, in principle, to be evaluated at $\mu=m_t$, though with negligible error we fix $\mu=160$ GeV throughout. Using $\alpha_3(M_Z)=0.109$, we find $\eta_c=1.94$ ($m_c=1.27$ GeV, $m_t=160$ GeV). The diagonalization also gives

$$V_{cb}=s_3-s_4=\frac{B}{\lambda_t}-\frac{\delta_d}{\lambda_b}=\frac{B}{\lambda_t}\left[\frac{\lambda_{t_G}}{\lambda_t}\right]^{1/6}\eta^{1/12} . \quad (19)$$

Combining Eqs. (18) and (19) gives

$$m_t=\frac{m_c}{\eta_c V_{cb}^2}\left[\left[\frac{\lambda_t}{\lambda_{t_G}}\right]\frac{1}{\sqrt{\eta}}\right]^{1/6} . \quad (20)$$

The factor in square brackets is the net effect on m_t of the RG scaling of the Yukawa matrices from M_G to m_t. It reduces the predicted value of m_t, but not by so much as one might have hoped. This is because, although the generation of δ_u considerably reduces m_t, this is largely offset by the generation of δ_d, resulting in the small value of $\frac{1}{6}$ for the index. Nevertheless, as a result of the RG running from M_G to m_t, our prediction for m_t is significantly smaller than that of He and Hou [see Eq. (4)] [6].

This equation is not our final prediction for m_t because it still depends on the free parameter λ_{t_G}. This can be removed by studying the solution for m_b. From Eq. (15f),

$$m_b=\eta_b m_\tau\left[\frac{\lambda_t}{\lambda_{t_G}}\right]^{1/6}\eta^{5/12}\left[\frac{\alpha_2}{\alpha_G}\right]^{3/2}\left[\frac{\alpha_1}{\alpha_G}\right]^{1/6} . \quad (21)$$

The dependence on $(\lambda_t/\lambda_{t_G})$ can now be eliminated between Eqs. (20) and (21) to give

$$m_t=\frac{m_b m_c}{m_\tau V_{cb}^2}\frac{1}{\eta_b\eta_c\sqrt{\eta}}\left[\frac{\alpha_G}{\alpha_2}\right]^{3/2}\left[\frac{\alpha_G}{\alpha_1}\right]^{1/6} . \quad (22)$$

This is the prediction for the top-quark mass in terms of the inputs m_b, m_c, m_τ, and V_{cb}. Numerically, we find

$$m_t=(176\ \text{GeV})\left[\frac{m_b}{4.15\ \text{GeV}}\right]\left[\frac{m_c}{1.22\ \text{GeV}}\right]\times\left[\frac{0.053}{V_{cb}}\right]^2\left[\frac{1.41}{\eta_b}\right]\left[\frac{1.94}{\eta_c}\right] . \quad (23)$$

As expected, the top-quark mass tends to be very heavy. We have inserted low values for m_c and m_b and a high value for V_{cb} in (23) so that this equation can be viewed as a lower bound on m_t. Note that the lower bound is particularly sensitive to V_{cb}: Changing V_{cb} to 0.058 decreases m_t from 176 to 147 GeV. A decrease in the experimental error bar for V_{cb} will make our lower bound for m_t much more restrictive. Using central values for the bottom and charm masses, we have $m_t=188$ GeV, which we shall soon see is at its perturbative "upper bound."

The two predictions of the second and third generations have been given in Eqs. (17) and (22), where m_s-m_d and m_t are given in terms of the inputs m_μ, m_τ, m_b, m_c, and V_{cb}. However, we can obtain considerably more information by requiring that the top-quark Yukawa coupling is everywhere perturbative. This need not be the case in nature, but is a necessary condition for our ability to make predictions. The top-quark mass is given by

$$m_t=\frac{v}{\sqrt{2}}\lambda_t\sin\beta . \quad (24)$$

From Eq. (23) we learn that the top quark is heavy, and thus for (24) to be true neither λ_t nor $\sin\beta$ can be much less than 1. Hence the solution (15a) for λ_t is usefully rewritten as

$$\lambda_t=\left[\frac{4\pi^2}{3I}\eta\left[1+\frac{4\pi^2}{3I}\frac{1}{\lambda_{t_G}^2}\right]\right]^{1/2}=1.09\left[1+\frac{0.12}{\lambda_{t_G}^2}\right]^{1/2} , \quad (25)$$

so that

$$m_t=\frac{190\sin\beta\ \text{GeV}}{(1+0.12/\lambda_{t_G}^2)^{1/2}} . \quad (26)$$

If we take perturbativity to imply that $\lambda_{t_G}<2$, we derive an "upper bound"

$$m_t<187\ \text{GeV} . \quad (27)$$

Using solution (15a) in (21) gives

$$m_b=(4.4\ \text{GeV})\left[\frac{\eta_b}{1.41}\right]\frac{1}{\lambda_{t_G}^{1/6}}\left[\frac{1}{1+0.12/\lambda_{t_G}^2}\right]^{1/12} . \quad (28)$$

The perturbativity constraint $\lambda_{t_G}<2$ is equivalent to $m_b>3.9$ GeV. Indeed, we could have replaced the perturbativity constraint by the requirement that the b quark be heavier than 3.9 GeV.

One can use the "upper bound" on m_t of (27) in (23) to derive an "upper bound" on the b-quark mass:

$$m_b<(4.4\ \text{GeV})\left[\frac{1.22\ \text{GeV}}{m_c}\right]\left[\frac{V_{cb}}{0.053}\right]^2\frac{\eta_b}{1.41}\frac{\eta_c}{1.94} . \quad (29)$$

Hence we conclude this discussion of the heavy two-generation predictions by summarizing our t and b results: The b-quark mass is larger than 3.9 GeV and has an "upper bound" given by (29); the t-quark mass has an "upper bound" of 187 GeV coming from perturbativity and can range down to a lower value given in (23), which is sensitive to m_c/V_{cb}^2. The range of λ_{t_G} is 1–2 with the lower bound being extremely sensitive to m_c/V_{cb}^2.

Finally, we obtain our prediction for $\sin\beta$, which appears in much phenomenology of the MSSM. We find

$$\sin\beta = 0.89 \left[\frac{m_b}{4\text{ GeV}}\right]\left[\frac{m_c}{1.22\text{ GeV}}\right]\left[\frac{0.053}{V_{cb}}\right]^2\left[1+\frac{0.12}{\lambda_{t_G}^2}\right]^{1/2}$$
$$= 0.95\left[\frac{4.25\text{ GeV}}{m_b}\right]^5\left[\frac{m_c}{1.22\text{ GeV}}\right]\left[\frac{0.053}{V_{cb}}\right]^2\left[\left[\frac{4.25\text{ GeV}}{m_b}\right]^{12}-0.08\right]^{-1/2}, \tag{30}$$

where the first equation results from combining Eqs. (23)–(25) and the second makes use of Eq. (28) to fix the value of λ_{t_G} in terms of m_b. Using values for $m_b=4.25$ GeV and $m_c=1.22$ GeV, we find $\sin\beta \sim 1$. Note that demanding $\sin\beta < 1.0$ [Eq. (30)] gives us an important lower bound on $V_{cb} > 0.052$, with $m_c=1.22$, $m_b=4.15$.

Next, we consider the predictions which involve the lightest generation. The theory has two additional real parameters to be determined: C and F. We will fix these by inputting m_u/m_d and m_e/m_μ. Three further predictions will result: m_s/m_d, θ_1, and θ_2.

Diagonalizing the e/μ and d/s subspaces of the mass matrices gives the Georgi-Jarlskog relation

$$\frac{d/s}{(1-d/s)^2} = 9\frac{e/\mu}{(1-e/\mu)^2}, \tag{31}$$

which yields a very accurate prediction for $m_s/m_d = 25.15$. At $\mu=1$ GeV, Eq. (17) predicts $m_s - m_d = 149$ MeV. Combining these two results, we find

$$m_d = 6.2\text{ MeV},$$
$$m_s = 156\text{ MeV}. \tag{32}$$

This m_s/m_d ratio is larger than is obtained using leading-order chiral perturbation theory [14]. However, Kaplan and Manohar [15] have shown that it is acceptable when second-order chiral effects are included for a wide range of u/d. A recent calculation by Leutwyler [15], which makes additional assumptions to obtain the coefficients of these second-order terms, prefers m_s/m_d to be less than 22.

The rotation angle in the d/s sector is obtained very accurately by $\tan\theta_1 = \sqrt{m_d/m_s}$ or

$$s_1 = 0.196. \tag{33}$$

Diagonalization in the u/c sector gives

$$s_2 \simeq \left[\frac{m_u(\mu)}{m_c(\mu)}\right]^{1/2}, \tag{34}$$

allowing a prediction for s_2 which involves the uncertainty in the value of the input parameter m_u/m_d:

$$s_2 \simeq 0.05\left[\frac{m_u/m_d}{0.6}\,\frac{1.25\text{ GeV}}{m_c}\right]^{1/2}. \tag{35}$$

From the Kobayashi-Maskawa matrix of Eq. (11), we see that $s_2 \simeq |V_{ub}/V_{cb}|$, so that (35) should be compared with the data $|V_{ub}/V_{cb}| = 0.09 \pm 0.04$ [16]. We have completed the three additional predictions which involve the first generation: m_s/m_d [and therefore m_d as given in Eq. (32)], θ_1 of Eq. (33), and θ_2 of Eq. (35).

There is still one remaining parameter of the original seven unknown Yukawa couplings which still has to be determined: the phase ϕ. It is fixed by inputting the Cabibbo angle:

$$\sin\theta_C = V_{us} = |s_1 + c_1 s_2 e^{-i\phi}| = 0.221 \pm 0.003. \tag{36}$$

Keeping track of the experimental uncertainty on $\sin\theta_C$ and the dependence of s_2 on m_u/m_d gives

$$\cos\phi = (0.53^{+0.07}_{-0.06})\left[\left[\frac{0.6}{m_u/m_d}\right]\left[\frac{m_c}{1.25\text{ GeV}}\right]\right]^{1/2}$$
$$-0.13\left[\left[\frac{m_u/m_d}{0.6}\right]\left[\frac{1.25\text{ GeV}}{m_c}\right]\right]^{1/2}, \tag{37}$$

where the upper (lower) error is correlated with the upper (lower) error in $\sin\theta_C$.

Hence taking $m_u/m_d = 0.6 \pm 0.2$ gives

$$\cos\phi = 0.41^{-0.15}_{+0.22}$$
or
$$\sin\phi = 0.91^{+0.05}_{-0.13}. \tag{38}$$

The *CP*-violating phase in this model is therefore determined to be large. Note that there is no quadrant ambiguity for the angles of the KM matrix. Without losing any generality we have chosen the signs of $A, \ldots, F$ so that θ_1, θ_2, and θ_3 are all in the first quadrant. We then find that ϕ is determined to also lie in the first quadrant.

We finish this section by presenting our predictions for the parameters that appear in the Wolfenstein form of the KM matrix. For us, λ and A are both input parameters. Since $m_t < 187$ GeV implies large values for V_{cb}, this implies that A will also be large:

$$A = 1.09\left[\frac{V_{cb}}{0.053}\right]\left[\frac{0.221}{V_{cd}}\right]^2. \tag{39}$$

The quantity $\rho^2+\eta^2$ is related to s_2, and so

$$(\rho^2+\eta^2)^{1/2} = \frac{s_2}{\lambda} = 0.23\left[\frac{0.221}{\lambda}\right]\left[\left[\frac{m_u/m_d}{0.6}\right]\left[\frac{1.25\text{ GeV}}{m_c}\right]\right]^{1/2}, \tag{40}$$

and η is related to $\sin\phi$:

$$\eta=0.2\left[\frac{0.221}{\lambda}\right]^2\left[\left[\frac{m_u/m_d}{0.6}\right]\left[\frac{1.25\text{ GeV}}{m_c}\right]\right]^{1/2}\sin\phi=(0.12\text{--}0.23)\ . \tag{41}$$

IV. DISCUSSION AND CONCLUSIONS

We have presented a framework for describing fermion masses in supersymmetric grand unified theories. We obtain all 13 low-energy fermion masses and mixing angles in terms of 7 real parameters in the Yukawa matrices at M_G and the ratio of Higgs VEV's $\tan\beta$. We fit these 8 arbitrary parameters using, as much as possible, the best-measured low-energy parameters $(e,\mu,\tau,c,b,|V_{cb}|,u/d,|V_{cd}|)$. We then make 6 predictions: d and s [Eq. (32)], t [Eq. (23)], $|V_{ub}/V_{cb}|$ [Eq. (35)], $\sin\beta$ [Eq. (30)], and the *CP*-violating angle [Eq. (38) or (41)]. It is important to realize that the relations for m_d, m_s, and m_b in terms of m_e, m_μ, and m_τ are those of Georgi and Jarlskog. The important new predictions that we make are for m_t, $|V_{ub}/V_{cb}|$, $\sin\beta$, and the *CP*-violating angle. Harvey, Ramond, and Reiss [19], who studied the Georgi-Jarlskog *Ansatz* in an SO(10) model, were the first to realize that it led to a prediction for m_t in terms of V_{cb} and that the *Ansatz* led to a KM matrix which violated CP. However, they do not RG scale the Yukawa couplings to obtain predictions for m_t, $|V_{ub}/V_{cb}|$, or the *CP*-violating angle.

The down- and strange-quark masses are within the errors quoted by Gasser and Leutwyler [14]. The ratio $m_s/m_d=25.15$ is fixed by the Georgi-Jarlskog relation [Eq. (31)]. This result is consistent with the chiral-Lagrangian analysis of Kaplan and Manohar [15], but somewhat larger than is allowed by a more recent analysis of Leutwyler [15] using additional constraints.

The ratio $|V_{ub}/V_{cb}|\sim 0.05$ is at the low end of the acceptable range 0.09 ± 0.04 [16]. More recent analyses, however, seem to favor the upper end [17]. A better determination of this ratio will provide a solid test of our framework.

The top-quark mass is predicted to be large, between 176 and 190 GeV. The lower bound, however, depends sensitively on the experimental value of $|V_{cb}|=0.044\pm0.009$ [16]. We have quoted our results with $V_{cb}=0.053$, at the upper 1σ bound. In fact, we cannot tolerate a value which is much lower than this, as is discussed following Eq. (30).

Finally, in a forthcoming paper [18] we shall present an analysis of the consequences of our model for K and B physics. We just remark here that our results are consistent with all measured quantities. In addition, we make very specific predictions for future measurements of *CP*-violating asymmetries in neutral B-meson decays.

All our results are subject to a number of theoretical uncertainties. We have stated the dependence on experimentally observed inputs explicitly. We have, however, additional uncertainties which we have not evaluated. In particular, there will always be threshold effects coming at both the weak and GUT scales. At the weak scale we have the Higgs boson, in addition to a multitude of supersymmetric partners of ordinary particles. We have assumed a common new particle threshold at 160 GeV. Note that these low-energy thresholds may someday be measured experimentally. At the GUT scale, on the other hand, there are many more particles, with different masses, which affect the boundary conditions of our RGE equations at M_G. The combined effects could easily change our results by several percent. We note, however, that the predictions for the ratios m_s/m_d and $|V_{ub}/V_{cb}|$ are insensitive to these uncertainties.

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