PCAC consistency. I. Kaon two- and three-body nonleptonic weak decays

R. E. Karlsen and M. D. Scadron
Physics Department, University of Arizona, Tucson, Arizona 85721
(Received 12 November 1991)

We show that the current-algebra–PCAC (partial conservation of axial-vector current) procedure applied consistently to each final-state pion predicts seven $K\to 2\pi$ and $K\to 3\pi$ decay amplitudes that very accurately reproduce the experimental results.

PACS number(s): 13.25.+m, 11.30.Rd, 11.40.Ha

I. INTRODUCTION

In this paper (I), we use chiral-symmetry (CS) and partially conserved axial-vector current (PCAC) techniques [1], extended via "PCAC consistency," to calculate the seven charge modes of $K\to 2\pi,3\pi$ weak decays. In the following paper (II), we will apply similar PCAC consistency methods to 16 two- and three-body charmed-meson weak decay modes. Previously, we briefly sketched the phenomenological application of these current-algebra–PCAC ideas not only to the kaon sector, but also to the charmed-meson and baryon sector [2], for two-body nonleptonic weak decays. Here we go into considerably more detail concerning PCAC consistency and also extend the analysis of kaon two-body decays to three-body decays.

We will use the CS-PCAC reduction in Sec. II to relate the $K\to\pi\pi$ amplitudes to one-body $K\to\pi$ reduced matrix elements, which can be calculated from meson or quark loop graphs. We also include the small effects of final-state interactions in our analysis. In Sec. III the analogous CS-PCAC reduction will be employed to connect the $K\to 3\pi$ amplitudes to the $K\to 2\pi$ transitions that were determined in Sec. II. The result of this study is that the PCAC consistency predictions for the seven $K\to 2\pi,3\pi$ decay amplitudes are consistently in good agreement with the data. We summarize our results in Sec. IV and verify PCAC consistency for $K^0_{2\pi}$ in the Appendix.

Many of the first attempts at explaining the $K\to 2\pi,3\pi$ decays, using PCAC and current-algebra ideas, utilized energy-dependent parametrizations of the amplitudes [3]. By constraining the parameters through PCAC, various ratios of physical observables were predicted. A slightly different approach was used in Ref. [4] to reduce the number of parameters in Ref. [3]. In these studies tadpole graphs were employed to systematically account for the rapid variation of momenta in $K\to 2\pi,3\pi$ decays. There were still unknown parameters in the latter approach, but these were further reduced by overall momentum conservation [1,5]. Still, these rapidly varying pole schemes for two-body decays are difficult to generalize to three-body decays.

In order to avoid such pole complications, in this paper we use PCAC consistency to circumvent the rapidly varying pole terms altogether. We will predict all seven $K\to 2\pi,3\pi$ decay amplitudes, including appropriate $\Delta I=\frac{1}{2},\frac{3}{2}$ parts, with *no* free parameters and without explicit calculation of the pole terms.

To begin, we write the decay amplitude in the form $M=M_P+\overline{M}$, where M_P is the rapidly varying pole contribution and $\overline{M}$ is the background term that varies slowly with momentum. The amplitude $\overline{M}$ is found by letting one of the pions go soft, which results in the amplitude having the on-shell form

$$M=M_{\rm CC}+M_P-M_P(0)\ . \tag{1a}$$

The charge commutator amplitude $M_{\rm CC}$ in (1a) is the amplitude M at the soft point, and the "surface term" $M_P-M_P(0)$ accounts for the rapid variation of pion momenta and modifies $M_{\rm CC}$ in (1a) away from the soft momentum limit.

The notion of PCAC consistency dictates that one should obtain the *same* result in (1a) no matter which final-state pion is taken soft (with the kaon always on shell) when computing $M_{\rm CC}$ and $M_P(0)$. That is, in order for (1a) to make physical sense, the $K\to 2\pi$ amplitudes must obey the condition

$$M=M_{\rm CC1}+M_P-M_{P1}(0)=M_{\rm CC2}+M_P-M_{P2}(0)\ , \tag{1b}$$

where the subscripts 1 and 2 denote which pion is taken soft. The requirement (1b) can be satisfied by the two relationships

$$M_P-M_{P1}(0)=M_{\rm CC2},\quad M_P-M_{P2}(0)=M_{\rm CC1}\ , \tag{1c}$$

both found to be empirically valid within the 10% PCAC error. The meaning of (1c) is that the surface term generated by one pion must correspond to the charge commutator due to the second pion. Inserting (1c) into (1b) then leads to the on-shell $K\to 2\pi$ decay amplitude

$$M=M_{\rm CC1}+M_{\rm CC2}+O(m_\pi^2/m_K^2)\ . \tag{2}$$

This double PCAC consistency relation (2) has an advantage over Eq. (1a) because all that needs to be determined is the charge commutator amplitude for each soft pion, which is partially required by Eq. (1a) in any case. In other words, the complicated pole contributions drop out of the analysis when employing the prescription Eq. (2) rather than (1a). Nevertheless, in order to appreciate the true significance of the double PCAC consistency relation (2), it is important to verify in detail that the complicated rapidly varying pole versions of (1a)–(1c) really do lead to the simple form of (2). We carry out this mathematical exercise in the Appendix specifically for $K^0 \to \pi\pi$ decays.

The usual soft-pion CS reduction gives the charge commutator amplitude M_{CC} with the kaon always on mass shell:

$$\langle \pi_j A_f | H_w | A_i \rangle \to M_{CC} = (-i/f_\pi) \langle A_f | [Q_5^j, H_w] | A_i \rangle = (i/f_\pi)[i f_{fjk} \langle A_k | H_w | A_i \rangle - i f_{jik} \langle A_f | H_w | A_k \rangle] , \quad (3)$$

where A denotes a general hadron state (here π or K) and $f_\pi \approx 93$ MeV. The last equality in (3) follows from the usual weak Hamiltonian density (H_w), generated from left-handed currents, being orthogonal to right-handed charges. This latter chiral-symmetry statement corresponds to the current-algebra equal-time commutation relations

$$[Q + Q_5, H_w] = 0 \quad \text{or} \quad [Q_5, H_w] = -[Q, H_w] . \quad (4)$$

Also employed in (3) is the strong-interaction transformation law

$$Q_j | A_i \rangle = i f_{jik} | A_k \rangle . \quad (5)$$

II. $K \to 2\pi$ DECAYS

The first processes to which we apply our double PCAC consistency relation (2) are the $\Delta I = \frac{1}{2}$ dominated $K \to 2\pi$ decays. This prescription, embodied in Eqs. (2) and (3), leads to the real amplitudes [verified from (1) in the Appendix]

$$a_{+-} = i \langle \pi^+ \pi^- | H_w | K^0 \rangle = (1/\sqrt{2} f_\pi) \langle \pi^+ | H_w | K^+ \rangle (1 - m_\pi^2 / m_K^2) , \quad (6a)$$

$$a_{00} = i \langle \pi^0 \pi^0 | H_w | K^0 \rangle = (-1/f_\pi) \langle \pi^0 | H_w | K^0 \rangle (1 - m_\pi^2 / m_K^2) , \quad (6b)$$

$$a_{+0} = i \langle \pi^+ \pi^0 | H_w | K^+ \rangle = (1/2 f_\pi) [\langle \pi^+ | H_w | K^+ \rangle + \sqrt{2} \langle \pi^0 | H_w | K^0 \rangle] (1 - m_\pi^2 / m_K^2) . \quad (6c)$$

The $(1 - m_\pi^2 / m_K^2)$ factors in (6) guarantee that these $K \to 2\pi$ amplitudes vanish in the strict SU(3) limit, as they must [6] because of C-parity considerations. This factor, corresponding to the $O(m_\pi^2 / m_K^2)$ PCAC error in Eq. (2), appears naturally when computing the $K_{2\pi}^0$ amplitudes through Eqs. (1).

To appreciate that the CS-PCAC consistency relations in (6) are of physical significance, we first ignore the small final-state-interaction effects (justified by empirical $\Delta I = \frac{1}{2}$ dominance) and estimate the real transition a_{00} in (6b). Then computing the reduced matrix element $\langle \pi^0 | H_w | K^0 \rangle$ in both a quark loop model [1,7] and a meson loop model [8], we predict

$$\langle \pi^0 | H_w | K^0 \rangle \approx -2.5 \times 10^{-8} \text{ GeV}^2 . \quad (7)$$

Not only does (7) predict from (6b) the $K^0 \to 2\pi^0$ amplitude $a_{00} \approx 25 \times 10^{-8}$ GeV, which is only 5% lower than the experimental result [9], but the scale (7) is likewise compatible with the π^0 pole amplitude for $K_L \to 2\gamma$ decay [1,10] and also needed in the chiral Lagrangian approach [11]. The second important feature of Eqs. (6) is the manifest $\Delta I = \frac{3}{2}$ structure of the right-hand side (RHS) of (6c), as is necessary for the $\Delta I = \frac{3}{2}$ transition $K^+ \to \pi^+ \pi^0$. This satisfying PCAC consistency result does not occur if only one pion is reduced in for $K^+ \to \pi^+ \pi^0$ and the rapidly varying pole term is ignored.

The reduced matrix element $\langle \pi^+ | H_w | K^+ \rangle$ in Eq. (6a) is approximately dominated by a $\Delta I = \frac{1}{2}$ loop graph, which is $\sqrt{2}$ times the scale in (7). However, the $K^+ \to \pi^+$ transition also has a W-pole graph, depicted in Fig. 1, which must be added to the dominant $\Delta I = \frac{1}{2}$ loop contribution from Eq. (7):

$$\langle \pi^+ | H_w | K^+ \rangle = -\sqrt{2} \langle \pi^0 | H_w | K^0 \rangle + (G_F / \sqrt{2}) s_1 c_1 f_\pi f_K m_K^2 \approx 4.0 \times 10^{-8} \text{ GeV}^2 , \quad (8)$$

for $f_K \approx 1.25 f_\pi$, $f_\pi \approx 93$ MeV, and $s_1 c_1 \approx 0.22$.

To relate the CS-PCAC consistency relations (6), using (7) and (8), to those found from experiment, we must lastly include the effects of final-state interactions. The complex experimental amplitudes corresponding to the real CS-PCAC Eqs. (6) can be written as

$$M_{+-} = a_{1/2} e^{i\delta_0} + \tfrac{2}{3} a_{3/2} e^{i\delta_2} , \quad (9a)$$

$$M_{00} = a_{1/2} e^{i\delta_0} - \tfrac{4}{3} a_{3/2} e^{i\delta_2} , \quad (9b)$$

$$M_{+0} = \sqrt{2} a_{3/2} e^{i\delta_2} , \quad (9c)$$

where the subscripts on the real amplitudes $a_{1/2}, a_{3/2}$ refer to the isospin of H_w, while the subscripts on the phase shifts δ refer to the isospin of the final states. To couple the real Eqs. (6) with the complex Eqs. (9), we first express (6) as

$$a_{+-} = a_{1/2} + \tfrac{2}{3} a_{3/2} \approx 28.1 \times 10^{-8} \text{ GeV} , \quad (10a)$$

$$a_{00} = a_{1/2} - \tfrac{4}{3} a_{3/2} \approx 24.8 \times 10^{-8} \text{ GeV} , \quad (10b)$$

$$a_{+0} = \sqrt{2} a_{3/2} \approx 2.3 \times 10^{-8} \text{ GeV} , \quad (10c)$$

where the numerical values on the RHS of (10) come from using the reduced matrix elements (7) and (8) in (6). Finally, inserting the values $a_{1/2} \approx 27.0 \times 10^{-8}$ GeV and $a_{3/2} \approx 1.65 \times 10^{-8}$ GeV, derived from (10), into the magnitudes of the complex Eqs. (9), leads to the magnitudes

$$|M_{+-}| = [a_{1/2}^2 + \tfrac{4}{9} a_{3/2}^2 + \tfrac{4}{3} a_{1/2} a_{3/2} \cos(\delta_0 - \delta_2)]^{1/2} \approx 27.7 \times 10^{-8} \text{ GeV} , \quad (11a)$$

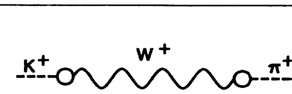

FIG. 1. W-pole graph for the transition $\langle \pi^+ | H_w | K^+ \rangle$.

$$|M_{00}|=[a_{1/2}^2+\tfrac{16}{9}a_{3/2}^2-\tfrac{8}{3}a_{1/2}a_{3/2}\cos(\delta_0-\delta_2)]^{1/2}$$
$$\approx 25.8\times10^{-8}\ \text{GeV}\ , \qquad (11b)$$
$$|M_{+0}|=\sqrt{2}a_{3/2}\approx 2.3\times10^{-8}\ \text{GeV}\ , \qquad (11c)$$

where we have employed the experimental $\pi\pi$ phase shifts at the K mass [12], $\delta_0-\delta_2\approx 55°$.

The $K\to\pi\pi$ predictions (11) compare quite well with the experimental decay amplitudes [9]

$$|M_{+-}|_{\text{expt}}=(27.65\pm0.07)\times10^{-8}\ \text{GeV}\ , \qquad (12a)$$
$$|M_{00}|_{\text{expt}}=(26.26\pm0.12)\times10^{-8}\ \text{GeV}\ , \qquad (12b)$$
$$|M_{+0}|_{\text{expt}}=(1.834\pm0.007)\times10^{-8}\ \text{GeV}\ . \qquad (12c)$$

Of course, the $\Delta I=\frac{3}{2}$ prediction (11c) is further removed from the experimental results than (11a) and (11b). This is due to the extremely delicate cancellation in (6c), and so (11c) is really only accurate to one significant figure.

III. $K\to 3\pi$ DECAYS

We now extend our double PCAC consistency prescription (2) for two-body $K\to2\pi$ decays to include the three-body $K\to3\pi$ decays. When one considers the transitions $i\langle\pi\pi\pi|H_w|K\rangle=A$ in the context of Feynman amplitudes, the final-state pions are treated as independent, and statistical factors of $N=2!$ or $3!$ for identical pions are included to reduce the experimental phase space. However, when employing our PCAC consistency procedure, independent final-state pions would suggest the symmetrized form $M_{\text{CC1}}+M_{\text{CC2}}+M_{\text{CC3}}$. But then the reduced matrix element $\langle\pi\pi|H_w|K\rangle$ would "know" about the above symmetrization, whereas Feynman amplitudes continue to treat the pions as independent. This mismatch means that we must multiply $M_{\text{CC1}}+M_{\text{CC2}}+M_{\text{CC3}}$ by an additional factor of $\frac{1}{2}$ to obtain the Feynman three-body transition (where the pions are again treated as independent and the kaon is always on mass shell):

$$\langle\pi_1\pi_2\pi_3|H_w|K\rangle$$
$$=\tfrac{1}{2}(M_{\text{CC1}}+M_{\text{CC2}}+M_{\text{CC3}})+O(m_\pi^2/m_K^2)\ . \qquad (13)$$

In contrast with two-body transitions, when the soft-pion reduction is performed on three-body decays, the charge operator Q acts back on a two-pion final state. To cope with this situation, we let Q operate on each final-state pion and add the results together in this extension of the strong-interaction transformation law (5):

$$\langle A_iA_j|Q_k=if_{ikl}\langle A_lA_j|+if_{jkl}\langle A_iA_l|\ . \qquad (14)$$

It turns out that the symmetry of Eqs. (13) and (14) and the antisymmetry of the structure constants f_{ijk} in (14) cause the terms, arising from the charge operator acting to the left on the final states, to cancel. This leaves only those terms where Q operates to the right on the kaon initial state, which greatly simplifies the analysis.

Applying the triple PCAC consistency prescription (13) to the four $K\to3\pi$ decays yields the Feynman amplitudes

$$A^{++-}=i\langle\pi^+\pi^+\pi^-|H_w|K^+\rangle$$
$$=(1/\sqrt{2}f_\pi)\langle\pi^+\pi^-|H_w|K^0\rangle\ , \qquad (15a)$$
$$A^{00+}=i\langle\pi^0\pi^0\pi^+|H_w|K^+\rangle$$
$$=(1/2\sqrt{2}f_\pi)[\langle\pi^0\pi^0|H_w|K^0\rangle+\sqrt{2}\langle\pi^+\pi^0|H_w|K^+\rangle]$$
$$=(1/2\sqrt{2}f_\pi)\langle\pi^+\pi^-|H_w|K^0\rangle\ , \qquad (15b)$$
$$A_L^{+-0}=i\langle\pi^+\pi^-\pi^0|H_w|K_L^0\rangle$$
$$=(-1/2\sqrt{2}f_\pi)[\langle\pi^+\pi^-|H_w|K^0\rangle-\sqrt{2}\langle\pi^+\pi^0|H_w|K^+\rangle]$$
$$=(-1/2\sqrt{2}f_\pi)\langle\pi^0\pi^0|H_w|K^0\rangle\ , \qquad (15c)$$
$$A_L^{000}=i\langle\pi^0\pi^0\pi^0|H_w|K_L^0\rangle$$
$$=(-3/2\sqrt{2}f_\pi)\langle\pi^0\pi^0|H_w|K^0\rangle\ , \qquad (15d)$$

where we have used the $K\to\pi\pi$ sum rule $M_{+-}-M_{00}=\sqrt{2}M_{+0}$ to deduce the RHS equalities in Eqs. (15b) and (15c). If we then input our theoretical predictions for the $K^0\to2\pi$ amplitudes [Eqs. (11a) and (11b)] into Eqs. (15), we obtain the theoretical magnitudes of the $K\to3\pi$ transitions:

$$|A^{++-}|=1.94\times10^{-6}\ , \qquad (16a)$$
$$|A^{00+}|=0.97\times10^{-6}\ , \qquad (16b)$$
$$|A_L^{+-0}|=0.91\times10^{-6}\ , \qquad (16c)$$
$$|A_L^{000}|=2.72\times10^{-6}\ . \qquad (16d)$$

Here we have multiplied (15) by the factor $(1-m_\pi^2/m_K^2)$ in computing (16) [though the latter is not required by *CP* invariance and *C*-parity symmetry, it makes Eqs. (16) compatible with (13) as well as with Eqs. (6)]. Two-pion final-state interactions enter Eqs. (15) and (16) through Eqs. (11), but for simplicity we have ignored the explicit three-pion final-state interactions in these equations as is traditionally done [13].

To compute the experimental $K\to3\pi$ amplitudes A, we employ the usual three-body phase-space integral [14] (N is the statistical factor mentioned previously)

$$\Gamma=[2/N(8\pi M)^3]|A|^2\int_{4\mu^2}^{(M-m)^2}ds\left\{\frac{[s-4\mu^2][s-(M+m)^2][s-(M-m)^2]}{s}\right\}^{1/2}$$
$$=I|A|^2\ , \qquad (17)$$

where M is the kaon mass, m is the odd-pion mass, and μ is the non-odd-pion mass. We have assumed that the amplitude A in (17) is constant, which is empirically valid to 5%. The integrals I in (17) for the four cases of interest are $I(++-)=0.798$, $I(00+)=0.996$, $I(+-0)=1.95$, and $I(000)=0.397$ in units of 10^{-6} GeV. With these values for I and the observed decay rates [9], we find the experimental $K\to 3\pi$ Feynman amplitudes to be

$$|A^{++-}|_{\rm expt}=(1.93\pm0.01)\times10^{-6}\ , \tag{18a}$$

$$|A^{00+}|_{\rm expt}=(0.96\pm0.01)\times10^{-6}\ , \tag{18b}$$

$$|A_L^{+-0}|_{\rm expt}=(0.90\pm0.01)\times10^{-6}\ , \tag{18c}$$

$$|A_L^{000}|_{\rm expt}=(2.63\pm0.05)\times10^{-6}\ , \tag{18d}$$

which agree very well with the theoretical predictions (16).

IV. CONCLUSION

In summary, we have employed the concepts of chiral-symmetry and PCAC consistency to explain all seven $K\to 2\pi, 3\pi$ weak decays. We began in Sec. II by introducing our procedure for $K\to 2\pi$ decays where it is shown that the decay amplitudes have the double PCAC consistency form $M_{\rm CC1}+M_{\rm CC2}$, while also accounting for the momentum variation in these decays. This prescription gives the $\Delta I=\frac{1}{2}$ scale of the $K^0\to\pi\pi$ decays (justified in detail in the Appendix), as well as the $\Delta I=\frac{3}{2}$ structure of the $K^+\to\pi^+\pi^0$ decay.

We also obtain the correct $\Delta I=\frac{1}{2},\frac{3}{2}$ structure of the $K\to 3\pi$ decays in Sec. III when the CS-PCAC consistency technique is extended to three-body decays. In this case the Feynman amplitude has the form $\frac{1}{2}(M_{\rm CC1}+M_{\rm CC2}+M_{\rm CC3})$, the logical extension of the $K\to 2\pi$ amplitude. The factor of $\frac{1}{2}$ is due to symmetrization effects of the three final-state pions. This procedure gives the correct magnitudes for the four $K\to 3\pi$ amplitudes.

In short, CS-PCAC consistency is essentially a model-independent scheme which explains the seven nonleptonic weak kaon decay rates with no new parameters and no dynamical assumptions except for the scale of the single reduced matrix element $\langle\pi^0|H_w|K^0\rangle/f_\pi$. (The latter can be computed from quark or meson loops [1,7,8].) Alternatively, we give our $K\to 2\pi$ predictions (11) and $K\to 3\pi$ predictions (16) by listing the seven branching ratios in Table I. We will extend this PCAC consistency procedure to charmed-meson decays in paper II.

TABLE I. Nonleptonic kaon decay branching ratios.

	$\mathrm{BR_{th}}$ (%)[a]	$\mathrm{BR_{expt}}$ (%)[b]
$K_S\to\pi^+\pi^-$	68.8	68.61±0.28
$K_S\to\pi^0\pi^0$	30.3	31.39±0.28
$K^+\to\pi^+\pi^0$	33	21.17±0.16
$K^+\to\pi^+\pi^+\pi^-$	5.64	5.59±0.05
$K^+\to\pi^0\pi^0\pi^+$	1.76	1.73±0.04
$K_L\to\pi^+\pi^-\pi^0$	12.7	12.38±0.21
$K_L\to\pi^0\pi^0\pi^0$	23.1	21.6±0.8

[a]We have used the observed K^+,K_S,K_L lifetimes in computing the branching ratios, although the K_S lifetime is also predicted in our analysis.
[b]See Ref. [9].

ACKNOWLEDGMENT

The authors appreciate the partial support of the U.S. Department of Energy.

APPENDIX

Here we verify that the (tedious) rapidly varying pole prescription in Eqs. (1) leads to the total on-shell $K^0\to\pi^i\pi^j$ amplitudes as obtained using the much simpler double PCAC prescription (2). First, we compute the off-shell $K^0\to\pi^i\pi^j$ decay amplitude for the tadpole graph of Fig. 2 [1,5,15]:

$$M_P^{ij}=\frac{\langle 0|H_w|K^0\rangle}{4f_\pi^2m_K^2}\times\{-i\epsilon^{ij3}[2k\cdot(p_i-p_j)-p_i^2+p_j^2]+\delta^{ij}[(p_i+p_j)^2+p_i^2+p_j^2-2m_\pi^2]\}\ . \tag{A1}$$

When one conserves the four-momentum $k=p_i+p_j$, even off the pion mass shell, so that $p_j^2\to m_K^2$ as $p_i\to 0$, then (A1) requires the surface terms for the various charge modes $K^0\to\pi^0\pi^0$ and $K^0\to\pi^+\pi^-$ (M^{+-}:$p_{\pi^+}\to 0$) or $K^0\to\pi^-\pi^+$ (M^{-+}:$p_{\pi^-}\to 0$):

$$M_P^{00}-M_P^{00}(0)=-\frac{\langle 0|H_w|K^0\rangle}{4f_\pi^2}\left[1-\frac{2m_\pi^2}{m_K^2}\right]\ , \tag{A2a}$$

$$M_P^{+-}-M_P^{+-}(0)=\frac{\langle 0|H_w|K^0\rangle}{2f_\pi^2}\frac{m_\pi^2}{m_K^2}\ , \tag{A2b}$$

$$M_P^{-+}-M_P^{-+}(0)=-\frac{\langle 0|H_w|K^0\rangle}{2f_\pi^2}\left[1-\frac{m_\pi^2}{m_K^2}\right]\ . \tag{A2c}$$

The (equal-time) charge commutators are found from (3) with $f_\pi\approx 93$ MeV:

$$M_{\rm CC}^{00}=(-i/f_\pi)\langle\pi^0|[Q_5^3,H_w]|K^0\rangle=(i/2f_\pi)\langle\pi^0|H_w|K^0\rangle\ , \tag{A3a}$$

$$M_{\rm CC}^{+-}=(-i/f_\pi)\langle\pi^-|[Q_5^{(1-i2)/\sqrt{2}},H_w]|K^0\rangle=(i/f_\pi)\langle\pi^0|H_w|K^0\rangle\ , \tag{A3b}$$

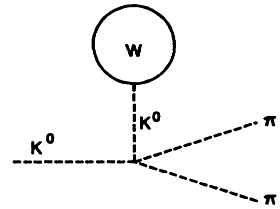

FIG. 2. Rapidly varying tadpole for $K^0\to\pi\pi$ decays.

$$M_{CC}^{-+}=(-i/f_\pi)\langle\pi^+|[Q_5^{(1+i2)/\sqrt{2}},H_w]|K^0\rangle$$
$$=(-i/f_\pi)[\langle\pi^0|H_w|K^0\rangle+2^{-1/2}\langle\pi^+|H_w|K^+\rangle]\ . \tag{A3c}$$

The charge commutator reduced matrix elements $\langle\pi|H_w|K\rangle$ in (A3) can be further reduced to $\langle 0|H_w|K^0\rangle$ by again applying Eq. (3), but without any rapidly varying tadpole component, resulting in

$$\langle\pi^0|H_w|K^0\rangle\to(-i/f_\pi)\langle 0|[Q_5^3,H_w]|K^0\rangle$$
$$=(i/2f_\pi)\langle 0|H_w|K^0\rangle\ , \tag{A4a}$$
$$\langle\pi^+|H_w|K^+\rangle\to(-i/f_\pi)\langle 0|[Q_5^{(1-i2)/\sqrt{2}},H_w]|K^+\rangle$$
$$=(-i/\sqrt{2}f_\pi)\langle 0|H_w|K^0\rangle\ . \tag{A4b}$$

Finally, inserting (A4a) into (A2) and adding the result to (A3), according to the rapidly varying pole prescription $M=M_{CC}+M_P-M_P(0)$ of Eq. (1a), leads to the on-shell $\Delta I=\frac{1}{2}K_{2\pi}^0$ amplitudes

$$M^{00}=\langle\pi^0\pi^0|H_w|K^0\rangle$$
$$=(i/f_\pi)\langle\pi^0|H_w|K^0\rangle(1-m_\pi^2/m_K^2)\ , \tag{A5a}$$

$$M^{+-}=\langle\pi^+\pi^-|H_w|K^0\rangle$$
$$=(i/f_\pi)\langle\pi^0|H_w|K^0\rangle(1-m_\pi^2/m_K^2)\ , \tag{A5b}$$

$$M^{-+}=\langle\pi^-\pi^+|H_w|K^0\rangle$$
$$=(-i/\sqrt{2}f_\pi)\langle\pi^+|H_w|K^+\rangle(1-m_\pi^2/m_K^2)\ . \tag{A5c}$$

Note that (A5b) is identical with (A5c), since Eqs. (A4) imply that $\langle\pi^0|H_w|K^0\rangle$ is equivalent to $(-1/\sqrt{2})\langle\pi^+|H_w|K^+\rangle$. Thus the three final forms for the rapidly varying pole $K_{2\pi}^0$ amplitudes indicated in (1) and explicitly displayed in (A5) are indeed the same as obtained in Eqs. (6) using the much simpler double PCAC consistency prescription Eq. (2). However, Eqs. (A5) are pure $\Delta I=\frac{1}{2}$, while Eqs. (6) also contain $\Delta I=\frac{3}{2}$ pieces. Also note that $M^{+-}=M^{-+}$ even though the component M_{CC} and $M_P-M_P(0)$ terms are each different in the two cases. This is an explicit example of PCAC consistency: The overall on-shell $K^0\to\pi^+\pi^-$ amplitude remains unchanged regardless of which pion is reduced. Also see Ref. [16].

With hindsight, the resulting weak scales of $K^0\to\pi^0\pi^0$ and $K^0\to\pi^+\pi^-$ in (A5a) and (A5b) are double the original values [17] obtained by assuming constant matrix elements with no momentum variation. Not only do the latter weak scales fail to explain model calculations, but such a constant matrix element approach is also incompatible with our PCAC consistency scheme in Eqs. (2) and (6) [which obviously doubles the weak scale in (6b) by Bose statistics].

[1] See, e.g., the current-algebra–PCAC review by M. D. Scadron, Rep. Prog. Phys. **44**, 213 (1981). Throughout that review and the present paper, we use the phase conventions $|\pi^\pm\rangle=|\pi_1\pm i\pi_2\rangle/\sqrt{2}$ and $\langle\pi^+|H_w^{1/2}|K^+\rangle=-\sqrt{2}\langle\pi^0|H_w^{1/2}|K^0\rangle$ along with $\langle\pi^0|H_w^{3/2}|K^0\rangle=\sqrt{2}\langle\pi^+|H_w^{3/2}|K^+\rangle$.

[2] R. E. Karlsen and M. D. Scadron, Phys. Rev. D **43**, 1739 (1991).

[3] C. Bouchiat and Ph. Meyer, Phys. Lett. **25B**, 282 (1967); B. R. Holstein, Phys. Rev. **171**, 1668 (1968); **177**, 2417 (1969); **183**, 1228 (1969).

[4] P. C. McNamee and M. D. Scadron, Phys. Rev. D **10**, 2280 (1974).

[5] M. D. Scadron, Nuovo Cimento A **91**, 87 (1986).

[6] N. Cabibbo, Phys. Rev. Lett. **12**, 62 (1964); M. Gell-Mann, *ibid.* **12**, 155 (1964).

[7] R. Delbourgo and M. D. Scadron, Nuovo Cimento Lett. **44**, 193 (1985); N. H. Fuchs and M. D. Scadron, Nuovo Cimento A **93**, 205 (1986); V. Elias, D. McKeon, and M. D. Scadron, Can. J. Phys. **68**, 1330 (1990); V. Elias, R. Mendel, and M. D. Scadron, Phys. Rev. D **42**, 1591 (1990).

[8] R. E. Karlsen and M. D. Scadron, Mod. Phys. Lett. A **6**, 543 (1991).

[9] Particle Data Group, J. J. Hernández *et al.*, Phys. Lett. B **239**, 1 (1990).

[10] G. Eilam and M. D. Scadron, Phys. Rev. D **31**, 2263 (1985).

[11] J. Cronin, Phys. Rev. **161**, 1483 (1967).

[12] G. Grayer *et al.*, Nucl. Phys. **B75**, 189 (1974); N. Durusoy *et al.*, Phys. Lett. **45B**, 517 (1973).

[13] T. J. Devlin and J. O. Dickey, Rev. Mod. Phys. **51**, 237 (1979).

[14] See, e.g., R. L. Thews, Phys. Rev. D **10**, 2993 (1974).

[15] In Refs. [1,5] the matrix elements were defined as $M=-\langle\pi\pi|H_w|K\rangle$. Here we delete this additional sign and write $M=\langle\pi\pi|H_w|K\rangle$.

[16] As an aside, we observe that, although $M_P-M_P(0)$ in (A2b) vanishes as $m_\pi^2\to 0$ (also noted in the Appendix of the fourth reference in [3]), M_{CC} in (A3b) as well as the on-shell $K^0\to\pi^+\pi^-$ amplitude in (A5b) do *not* vanish. This is reconfirmed because (A2c) and (A5c) are also not zero even though now (A3c) vanishes by virtue of (A4). Thus, in the limit $m_\pi^2\to 0$, (A2b) equals (A3c) and (A2c) equals (A3b) as postulated in (1c).

[17] See, e.g., R. E. Marshak, Riazuddin, and C. P. Ryan, *Theory of Weak Interactions in Particle Physics* (Wiley, New York, 1969).