Structure of the ground state of the electroweak gauge theory in a strong magnetic field

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(Received 16 December 1991)

The structure of the ground state of the Weinberg-Salam model in the presence of an external magnetic field is investigated. As the magnetic induction reaches a critical value B_c, W pairs can be produced with zero energy. The equations for B, Z, W, and the Higgs field Φ are handled, near the transition point, by a perturbative method and solved exactly to first order in the parameter $e(B-B_c)$. Solutions with two types of boundary conditions are considered: (i) B is produced in the interior of a large cylindrical solenoid by electric currents on the surface; (ii) a uniform background magnetic field exists throughout space, above B_c a new phase with W condensates emerges. The latter case admits solutions with lattice symmetry and an integer number of quanta of magnetic flux through each lattice cell. The W wave function is expressed in terms of the Jacobi function $\vartheta_1(z|\tau)$ where τ is the lattice parameter. The average energy density $\overline{\mathcal{H}}$ as a function of τ is shown to be modular invariant. For $M_H > M_Z$, $\overline{\mathcal{H}}$ is minimal for a hexagonal lattice with a simple zero of W at each center. Coherent quantum states are constructed for the W pairs. The relation to type-II superconductivity is discussed.

PACS number(s): 11.15.Kc, 05.70.Fh, 11.15.Ex, 74.60.−w

I. INTRODUCTION

In the presence of a constant uniform magnetic field, the electroweak vacuum of the Weinberg-Salam theory develops an instability [1,2] for a value of the magnetic induction $B_c = M_W^2/e$ ($B_c \approx 10^{24}$G), where M_W is the mass of the charged vector boson W. This is readily understood in terms of the Landau spectrum of a charged particle of mass M, charge e, and spin s in a constant magnetic field B, which we take to be along the direction of the z axis $\mathbf{e}_3$. It is given by

$$E^2 = M^2 + p_3^2 - 2eBs_3 + (2n+1)|e|B \ , \tag{1}$$

where E is the energy, p_3 and s_3 the components of the momentum and of the spin along $\mathbf{e}_3$, and $n \geq 0$ is an integer. Each level with quantum number n has an infinite degree of degeneracy corresponding to eigenstates of the orbital angular momentum along $e\mathbf{B}$ with eigenvalues $m \geq -n$. For $s > \frac{1}{2}$, the magnetic-moment interaction $-2e\mathbf{B}\cdot\mathbf{s}$ may overcome the zero-point energy of the harmonic motion ($n=0$, $p_3=0$) so that, for sufficiently large B, E^2 becomes negative. For $s=1$ and $es_3 = |e|$, E^2 vanishes for $B = B_c = M^2/|e|$. This is the source of the quantum instability. In the one-loop approximation, the quantum correction to the effective energy density develops a logarithmic branch point at this value of the magnetic field (with $M = M_W$) due to W loops [2].

It has been suggested that, beyond this value, the vacuum state may undergo a phase transition from the normal broken phase with $\Phi = \Phi_0$ to the symmetric phase [3] with $\Phi = 0$. Ambjørn and Olesen [4] have proposed a solution of the field equations which would interpolate between the broken and symmetric phases. Their solution contains condensates of W pairs and satisfies periodic boundary conditions on a lattice. Their treatment is, however, valid only for a particular value of the Higgs coupling constant $\lambda = g^2/(8\cos^2\theta_W)$ ($M_H = M_Z$).

In the general case of unconstrained λ, we use a perturbative approach, described in Sec. III, to obtain solutions of the field equations in the neighborhood of the transition value B_c of the magnetic induction, taking $e(B-B_c)$ as the perturbation parameter. This phase transition has a close resemblance to the phase transition at H_{c2} in type-II superconductors; the perturbation method used here is similar to that used to solve the Ginzburg-Landau equations near H_{c2} [5] and was introduced in the present context by Skalozub [6].

Two differences should be pointed out. The first is that, in the model discussed here, the local $|\Psi|^4$ interaction of the Ginzburg-Landau theory is replaced by an effective nonlocal quartic interaction mediated by Z and Φ fields. The second difference is that, in this model, the magnetization is positive, in contrast with a negative magnetization in type-II superconductivity. As a consequence, the ordered phase is above and the normal phase below B_c, which is the reverse of what happens in superconductivity.

We consider two distinct choices of boundary conditions: one corresponds to the production of a uniform field inside a large cylindrical solenoid of radius R by electric currents on the surface, and the other assumes a uniform background magnetic induction $\bar{B}$ throughout space.

The first case is discussed qualitatively in Sec. IV, where we propose a solution with cylindrical symmetry in which negatively charged W's are produced on the solenoid's surface as a result of the decay of electrons into neutrino and W^-. This process will be allowed when the

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strength of the magnetic field reaches a value (below B_c) such that the W's could be created with an energy smaller than the electron mass. It prevents the current from increasing, thus keeping the magnetic induction inside the solenoid below the critical value.

For the second case, dealt with in Sec. V, we investigate general lattice solutions. In lowest order of perturbation theory, the W wave function is an arbitrary superposition W_+ of degenerate modes with zero energy ($s_3=+1$). The other fields and the energy density are obtained in terms W_+. For $\bar{B}>B_c$ and $M_H>M_Z$, creation of W pairs lowers the energy density. Thus, a phase transition occurs with the production of W condensates. The density of W pairs is determined by the quantity $|W_+|^2$ whose space average is the order parameter and is proportional to $e(\bar{B}-B_c)$. For $M_H \leq M_Z$, lowest-order perturbation theory is inadequate, as the energy density in this order becomes unbounded from below.

In order to find the ground state within our perturbative framework, we investigate which lattice solution minimizes the average energy density. The motivation for considering only lattice solutions is that on a macroscopic scale $|W_+|^2$ should be translational invariant. Lattice solutions were first constructed by Abrikosov [5], subject to the flux quantization condition $\bar{B}A=2\pi k/e$, where A is the area of a unit cell and k is an integer equal to the number of zeros of W_+ in a cell. We show that, under certain conditions, the solutions are unique. They are expressed in terms of the Jacobi function ϑ_1. Considered as a function of a complex parameter τ, which specifies the lattice geometry, it is shown in Sec. VI that the energy density, averaged over a lattice cell, is modular invariant. This result is known to hold in the theory of type-II superconductivity [7].

The classical treatment described so far is justified in Sec. VII where the W field is quantized. The order parameter is then the average over a lattice cell of the expectation value $\langle W_+^\dagger W_+ \rangle$, where W_+ is a quantum operator. Coherent quantum states of W pairs are constructed corresponding to the classical solutions.

A numerical calculation in Sec. VIII, including all lattice symmetries of regular polygons and parallelograms, reveals that the lowest-energy solution has the symmetry of a hexagonal lattice with a simple zero of the W_+ wave function at the center of each cell or, equivalently, one magnetic flux quantum per cell. Recalling that the hexagonal lattice is dual to a triangular lattice, this solution has simple zeros at the vertices of triangles.[1] Our results show that the nonlocal effects of the quartic interaction are not negligible.

The average energy density for the regular triangular lattice with a simple zero at the center turns out to coincide with that for a hexagonal lattice with a double zero. This is due to a remarkable identity, Eq. (83), for certain convolutions of ϑ functions.

II. THE FIELD EQUATIONS

We consider the Weinberg-Salam model in the unitary gauge. Let $V^i{}_\mu$ and $V'{}_\mu$ be the vector potentials for the SU(2) and U(1) gauge groups, respectively. The electromagnetic potential A_μ and the neutral vector boson Z_μ are related to $V^3{}_\mu$ and $V'{}_\mu$ by

$$V^3{}_\mu = Z_\mu \cos\theta + A_\mu \sin\theta \,, \tag{2}$$

$$V'{}_\mu = -Z_\mu \sin\theta + A_\mu \cos\theta \,. \tag{3}$$

The vector potential for the positively charged vector boson W is defined by

$$W_\mu = \frac{1}{\sqrt{2}}(V^1{}_\mu - iV^2{}_\mu) \,. \tag{4}$$

The Lagrangian density for the vector potentials and Higgs field Φ is

$$L = -\tfrac{1}{4}F^3_{\mu\nu}F^{3\mu\nu} - \tfrac{1}{2}F^\dagger_{\mu\nu}F^{\mu\nu} - \tfrac{1}{4}F'_{\mu\nu}F'^{\mu\nu} + \partial_\mu\Phi\partial^\mu\Phi - \lambda(\Phi^2-\Phi_0^2)^2 + \left[\frac{1}{2}g^2W^\dagger_\mu W^\mu + \frac{g^2}{4\cos^2\theta}Z_\mu Z^\mu\right]\Phi^2 \,, \tag{5}$$

where θ is the Weinberg angle and

$$F^3_{\mu\nu} = \partial_\mu V^3{}_\nu - \partial_\nu V^3{}_\mu - ig(W^\dagger_\mu W_\nu - W^\dagger_\nu W_\mu) \,, \tag{6}$$

$$F_{\mu\nu} = \mathcal{D}_\mu W_\nu - \mathcal{D}_\nu W_\mu \tag{7}$$

with

$$\mathcal{D}_\mu = \partial_\mu + igV^3{}_\mu \,. \tag{8}$$

From here on we take g and $e=g\sin\theta$ to be positive. The field equations are as follows.

(1) For W_ν,

$$\mathcal{D}^\mu F_{\mu\nu} - igF^3_{\mu\nu}W^\mu + \tfrac{1}{2}g^2\Phi^2W_\nu = 0 \,. \tag{9}$$

(2) For $V^3{}_\nu$,

$$\partial^\mu F^3_{\mu\nu} - ig(W^{\mu\dagger}F_{\mu\nu} - W^\mu F^\dagger_{\mu\nu}) + \frac{g^2}{2\cos\theta}\Phi^2Z_\nu = 0 \,. \tag{10}$$

(3) For $V'{}_\nu$,

$$\partial^\mu F'_{\mu\nu} - \frac{g^2}{2\cos^2\theta}\sin\theta\Phi^2Z_\nu = 0 \,. \tag{11}$$

(4) For Φ,

$$\left[-\partial^\mu\partial_\mu - 2\lambda(\Phi^2-\Phi_0^2) + \tfrac{1}{2}g^2W^\dagger_\mu W^\mu + \frac{g^2}{4\cos^2\theta}Z_\mu Z^\mu\right]\Phi = 0 \,. \tag{12}$$

The integrability conditions are

$$\mathcal{D}^\nu(W_\nu\Phi^2) - \frac{ig}{\cos\theta}Z^\nu W_\nu\Phi^2 = 0 \,, \tag{13}$$

$$\partial^\nu(Z_\nu\Phi^2) = 0 \,. \tag{14}$$

[1] Throughout this paper we call hexagonal what is usually called triangular in condensed-matter theory (and vice versa) due to our choice of lattice cells with their centers, rather than vertices, at the positions of zeros of the order parameter.

Because of gauge invariance these conditions are satisfied by solutions of the field equations (9)–(12) for which they hold at the boundaries. We look for static solutions of these equations which are translational invariant in the z direction. We shall also assume that the only nonvanishing components of the vector potentials are spacelike on the plane normal to the z axis. This reduces the problem to two dimensions in the xy plane. Additional symmetries are dictated by boundary conditions.

In the case discussed in Ref. [4], the coupling constants were constrained by $\lambda=g^2/(8\cos^2\theta)$ or $M_H=M_Z$. For $B>B_c$, the field equations then have a particular solution in which the W field has only one polarization with $s_3=1$ and the field strength $F_{\mu\nu}$ vanishes. In the case of unconstrained λ, the two polarizations $s_3=\pm1$ do not decouple. To solve the equations in this general case we use a perturbative method in a neighborhood of the transition point, taking $e(B-B_c)$ as the perturbation parameter.

III. PERTURBATIVE METHOD

Let $W_\pm=(W_1\mp iW_2)/\sqrt{2}$ be the wave function for polarized states with spin projection $s_3=\pm1$. In the absence of a Z field and with $\Phi=\Phi_0$, the subsidiary condition [Eq. (13)] reduces to $\mathcal{D}^\mu W_\mu=0$. Using this condition in the equation for W_μ in the presence of a uniform magnetic field $\mathbf{B}=B\mathbf{e}_3$ and vector potential $\mathbf{A}=\frac{1}{2}\mathbf{B}\times\boldsymbol{\rho}$, the linearized equations for polarization states $W_\pm$ with energy eigenvalues $E_\pm$ decouple and become

$$(-\Delta-eBL_3+\tfrac{1}{4}e^2B^2\rho^2\mp2eB+M_W^2)W_\pm=E_\pm^2W_\pm\ , \tag{15}$$

where L_3 is the z component of the orbital angular momentum. The symmetric gauge used here for the vector potential $\mathbf{A}$ differs from that used in Refs. [5,6]; it allows for the separation of Eq. (15) in terms of eigenfunctions of L_3.

Above the critical value of the magnetic field the lowest eigenvalue will be zero. W pairs can then be produced with zero energy. Since the Landau spectrum [Eq. (1)] is continuous, in order to isolate the zero modes, one has to make the spectrum discrete by imposing periodic boundary conditions in the z direction on two boundary surfaces perpendicular to the z axis and a distance L apart. Then the momentum p_3 becomes quantized with values $(2\pi k/L)$ for integer k.

In cylindrical coordinates (z,ρ,φ) the eigenfunctions, normalized to L and labeled by the quantum numbers k, n and m, are

$$f_{knm}(\boldsymbol{\rho})=\left[\frac{eB}{2\pi}\right]^{1/2}\exp[i(m\varphi+2\pi kz/L)]\times\left[\frac{n_\rho!}{(n_\rho+|m|)!}\right]^{1/2}\xi^{|m|/2}L_{n_\rho}^{|m|}(\xi)\times\exp(-\xi/2)\ , \tag{16}$$

where $\xi=\frac{1}{2}eB\rho^2$, $n_\rho=n+(m-|m|)/2$, and $L_{n_\rho}^{|m|}$ is a Laguerre polynomial (we are here taking the limit of a cylinder of infinite radius). The lowest-energy states have $k=0$, $s_3=1$, $m\geq0$, and E^2 will have a gap of size $(2\pi/L)^2$. The corresponding eigenfunctions are

$$f_m(\boldsymbol{\rho})=\left[\frac{eB}{2\pi m!}\right]^{1/2}\left[\left[\frac{eB}{2}\right]^{1/2}\rho e^{i\varphi}\right]^m\times\exp(-\tfrac{1}{4}eB\rho^2) \tag{17}$$

with degenerate eigenvalue $E_+^2=M_W^2-eB$. They satisfy the summation formula

$$\sum_{m=0}^{\infty}f_m^*(\boldsymbol{\rho})f_m(\boldsymbol{\rho})=\frac{eB}{2\pi}\ . \tag{18}$$

The general lowest-energy solution is of the form

$$W_+(\boldsymbol{\rho})=\sum_{m=0}^{\infty}\lambda_m f_m(\boldsymbol{\rho})\ . \tag{19}$$

This solution satisfies $F_{12}=0$, that is $(\mathcal{D}_1+i\mathcal{D}_2)W_+=0$.

We shall use a perturbative method in the neighborhood of the critical value of the magnetic field, taking Eq. (19) as the zeroth-order W wave function. Notice that, if $eB>M_W^2$, then $E_+^2<0$. Therefore, the perturbation method will be valid only if the first-order correction to E_+^2 is positive. The density of W pairs, as measured by $|W_+|^2$, can be determined so that, up to this order, $E_+^2=0$. This yields a density $|W_+|^2$ of order $(eB-M_W^2)\equiv e(B-B_c)$. The W pairs then become a source for the other fields which can be expanded in powers of $|W_+|^2$.

The degeneracy of E_+ in the linearized field equation for W_+ will be lifted by the interaction terms in the exact energy density $\mathcal{H}$. For the static fields considered here this is given by

$$\mathcal{H}=\tfrac{1}{2}(F_{12}^3)^2+\tfrac{1}{2}(F'_{12})^2+|F_{12}|^2+\nabla\Phi\cdot\nabla\Phi+\lambda(\Phi^2-\Phi_0^2)^2+\left[\frac{1}{2}g^2(|W_+|^2+|W_-|^2)+\frac{g^2}{4\cos^2\theta}\mathbf{Z}^2\right]\Phi^2\ . \tag{20}$$

The first two terms can be written in terms of A_μ and Z_μ as

$$\tfrac{1}{2}[A_{12}+e(|W_+|^2-|W_-|^2)]^2+\tfrac{1}{2}[Z_{12}+g\cos\theta(|W_+|^2-|W_-|^2)]^2\ . \tag{21}$$

We want to compute $\mathcal{H}$ up to second order in $|W_+|^2$. To this end we have to solve Eqs. (10)–(12) for $\mathbf{A}$, $\mathbf{Z}$, and $\Phi_1=\Phi-\Phi_0$, taking $|W_+|^2$ as a source in addition to the external magnetic field. The suppressed component $W_-(\boldsymbol{\rho})$ can be obtained from the subsidiary condition (13). Since it is of third order in W_+, its contribution to the energy density $\mathcal{H}$ is only in terms third order in $|W_+|^2$ which are here being neglected.

In the linear approximation, i.e., to lowest order in W_+, Eqs. (10)–(12) reduce to

$$\Delta\mathbf{A}-g\sin\theta\mathbf{e}_3\times\nabla|W_+|^2=0\ , \tag{22}$$

$$\Delta \mathbf{Z} - g\cos\theta \mathbf{e}_3 \times \nabla |W_+|^2 - M_Z^2 \mathbf{Z} = 0 , \tag{23}$$

$$\Delta \Phi_1 - \tfrac{1}{2} g^2 \Phi_0 |W_+|^2 - M_H^2 \Phi_1 = 0 , \tag{24}$$

where M_H is the Higgs-boson mass given by $M_H^2 = 4\lambda\Phi_0^2$. These equations can be immediately solved in terms of the Green's functions for Laplace's and Poisson's equations in two dimensions. We obtain

$$\mathbf{A} = \tfrac{1}{2}\mathbf{H} \times \boldsymbol{\rho} + g\sin\theta \mathbf{e}_3 \times \frac{1}{2\pi} \int \nabla \ln(|\boldsymbol{\rho} - \boldsymbol{\rho}'|) |W_+(\boldsymbol{\rho}')|^2 d^2\rho' , \tag{25}$$

$$\mathbf{Z} = -g\cos\theta \mathbf{e}_3 \times \nabla U_{M_Z}(\boldsymbol{\rho}) , \tag{26}$$

$$\Phi_1 = -\tfrac{1}{2} g^2 \Phi_0 U_{M_H}(\boldsymbol{\rho}) , \tag{27}$$

where

$$U_M(\boldsymbol{\rho}) = \frac{1}{2\pi} \int K_0(M|\boldsymbol{\rho} - \boldsymbol{\rho}'|) |W_+(\boldsymbol{\rho}')|^2 d^2\rho' , \tag{28}$$

and $K_0(z)$ is a modified Bessel function of the third kind. For z real and positive, K_0 is a positive, monotonically decreasing function.

The integrations in Eqs. (25) and (28) extend over the whole plane. In Eq. (25), $\mathbf{H}$ is a constant vector in the direction of $\mathbf{e}_3$. It should be identified as the magnetic field strength. The field $\mathbf{B} = \nabla \times \mathbf{A}$ is then given by

$$\mathbf{B} = \mathbf{H} + e|W_+|^2 \mathbf{e}_3 . \tag{29}$$

The average, or macroscopic, induction is

$$\bar{B} = H + e\overline{|W_+|^2} , \tag{30}$$

where $e\overline{|W_+|^2}$ is the macroscopic magnetization.

Using the solutions of the field equations for the real fields, one can reexpress $\mathcal{H}$ as a sum of derivative terms, which would contribute only at the boundary, and terms which give, up to second order in $|W_+|^2$,

$$\mathcal{H}(\boldsymbol{\rho}) = \tfrac{1}{2}(B - e|W_+|^2)^2 + M_W^2 |W_+|^2 + \tfrac{1}{2} g^2 M_W^2 U(\boldsymbol{\rho}) |W_+|^2 , \tag{31}$$

where

$$U(\boldsymbol{\rho}) = U_{M_Z}(\boldsymbol{\rho}) - U_{M_H}(\boldsymbol{\rho}) = \frac{1}{2\pi} \int_{M_Z}^{M_H} d\mathcal{M} \int d^2\rho' |\boldsymbol{\rho}'| K_1(\mathcal{M}|\boldsymbol{\rho}'|) |W_+(\boldsymbol{\rho}' + \boldsymbol{\rho})|^2 . \tag{32}$$

In the next two sections, we consider two distinct formulations of the boundary conditions.

(i) If one assumes that the uniform magnetic field is produced in the interior of a large solenoid, then the constant field H is determined by the current per unit length on the solenoid's surface, which is to be considered as the external source.

(ii) There is a uniform background magnetic field throughout space. This is a mathematical idealization. The total energy is infinite and one has to consider the average energy density. Above the critical value of the background magnetic field, W's can be produced, forming pair condensates. To find the ground state in the perturbative method, the next step calls for the minimization of the energy using the lowest-order expression (19) for W_+. It is clear that the ensuing configuration should have translational invariance on a macroscopic scale. Thus, we are led to look for solutions with lattice symmetry and investigate which lattice configuration gives the minimal average energy density for a given value of the average magnetic field $\bar{B}$.

IV. MAGNETIC FIELD OF A SOLENOID

In this section we assume that the magnetic field is produced in the interior of a large solenoid of radius R and look for solutions with cylindrical symmetry. The external source is the electron current on the solenoid or more precisely the voltage producing that current.

As the magnetic field reaches the critical value, the ground-state energy of W's in this field becomes zero. The angular momentum degeneracy of the previous case is lifted in the case of a finite solenoid but the shifts in energy are extremely small up to values of $m \approx (M_W R)^2/2$. There will be a maximum value of m above which the W's are no longer bound since they would be outside the solenoid where the field is zero.

Taking into account the electron current, the equation for the magnetic field becomes

$$\nabla \times \mathbf{H} = \mathbf{j}_{\rm el} + \mathbf{j}_W , \tag{33}$$

where

$$\mathbf{H} = \nabla \times \mathbf{A} - e|W_+|^2 \mathbf{e}_3 \tag{34}$$

and

$$\mathbf{j}_W = ie(\mathbf{W}^\dagger \times \mathbf{F} - \mathbf{W} \times \mathbf{F}^\dagger) . \tag{35}$$

To lowest order in $|W_+|^2$, $\mathbf{j}_W$ vanishes since $\mathbf{F} \equiv -F_{12}\mathbf{e}_3 = \mathbf{0}$. Therefore, the magnetic field strength H inside the solenoid is constant and its magnitude is equal to the current per unit length in the solenoid.

The energy density is, to lowest order in $|W_+|^2$,

$$\mathcal{H} = \tfrac{1}{2}H^2 + M_W^2 |W_+|^2 \tag{36}$$

and for a fixed value of H it increases with W production.

But, for $eH \geq M_W^2$, W's would be created with $E^2 \leq 0$, which is the signal of an instability.

In order to resolve this paradox, we suggest that one should take into account the interaction $(g/\sqrt{2})\bar{\nu}\gamma^\mu e W_\mu$. This allows electrons in the solenoid to decay into a neutrino and a negatively charged W, provided that the magnetic field H is large enough (but below B_c) that the W^-'s can be produced near the surface with an energy smaller than $M_e c^2$. They will be produced in a state of angular momentum $m \approx eHR^2/2$ and their electric current will vanish to lowest order in $|W^-|^2$. Equation (17), with $B = H$, describes approximately the wave function for this state, which for large $m \approx eHR^2/2$ has the following behavior:

$$|f_m|^2 \sim \frac{eH}{2\pi} \frac{1}{\sqrt{2\pi m}} \exp\left[-\frac{m}{2} \left[\frac{\rho^2}{R^2} - 1 \right]^2 \right] . \tag{37}$$

This is sharply peaked at $\rho=R$.

The process qualitatively described above provides a mechanism for preventing the electric current in the solenoid from increasing beyond a critical value as the voltage is increased, thus preventing a phase transition from occurring.

V. LATTICE SOLUTIONS

In this section we discuss the general properties of perturbative solutions with lattice symmetry and proceed with their explicit construction. By introducing the complex variable $z=\rho e^{i\varphi}$, we can rewrite the general solution, Eqs. (19) and (17), as

$$W_+(\boldsymbol{\rho})=\left[\frac{eB}{2\pi}\right]^{1/2}\exp(-\tfrac{1}{4}eB\rho^2)G(z)\ , \tag{38}$$

where

$$G(z)=\sum_{m=0}^{\infty}\frac{\lambda_m}{\sqrt{m!}}\left[\frac{eB}{2}\right]^{m/2}z^m\ . \tag{39}$$

We shall assume that the function $G(z)$ is analytic in some domain $\mathcal{D}$ containing a lattice cell centered at $z=0$. Then translational invariance of $|W_+|$ by a lattice vector implies the following.

Theorem 1: Flux quantization. The magnetic flux through a lattice cell of area A is given by $BA=2\pi k/e$, where k is the number of zeros of W_+ inside the cell. This is the equivalent of the result (19) of Ref. [4].

Proof. Since $\ln[G(z)]$ is analytic inside $\mathcal{D}$ except for branch points at the zeros z_i of $G(z)$, one has

$$\Delta\ln|W_+|=-eB+2\pi\sum_i\delta^{(2)}(\boldsymbol{\rho}-\boldsymbol{\rho}_i)\ . \tag{40}$$

The right-hand side represents a uniform charge distribution with density $-eB$ and point charges at the positions $\boldsymbol{\rho}_i$ as sources for the potential function $\ln|W_+|$. Since $|W_+|$ is invariant under translation by a lattice vector, it follows that the total charge inside a cell must be zero. Integration over a cell proves the theorem. We remark that a zero on the edge of a cell counts as $\frac{1}{2}$, and at a vertex it counts as

$$(\text{angle between the two sides at the vertex})/2\pi\ .$$

Theorem 2: Uniqueness theorem [8]. For a lattice of regular polygons or a rectangular lattice there exists a unique solution for $W_+(\boldsymbol{\rho})$ (up to a multiplicative constant) with the following properties: (i) W_+ has a zero of order $k>0$ at the center of the cell; (ii) $|W_+|$ is invariant under reflections in the sides of the polygon.

Proof. Let us take the real axis perpendicular to a side of the polygon with origin at the center of the cell. Set $G(z)=c_kz^k\exp[f(z)]$, where $f(z)=u+iv$ is analytic in the domain $\mathcal{D}$, $f(0)=0$, and c_k is an arbitrary constant. Because of the symmetry property (ii), the normal derivative of $\ln|W_+|$ at the boundary of a cell vanishes. Therefore,

$$-\tfrac{1}{2}eBh+kh/(h^2+s^2)+\partial u/\partial n=0\ , \tag{41}$$

where h is the distance from the center of the polygon to a side, s is a coordinate along the side with origin at the middle point and running counterclockwise, and $\partial u/\partial n$ is the derivative of u in the direction of the outward normal to the side, at the position s. Since $f(z)$ is analytic in $\mathcal{D}$, $\partial u/\partial n=\partial v/\partial s$. Then, on each side of the polygon, we have

$$v(h,s)=\tfrac{1}{2}eBhs-k\arctan(s/h)\ . \tag{42}$$

The condition

$$\oint\frac{\partial v}{\partial s}ds=0 \tag{43}$$

along the perimeter of the cell is equivalent to the flux quantization condition.

Now one can use Poisson's formula to determine $f(z)$ inside the cell:

$$f(z)=\frac{i}{2\pi}\int_{-\pi}^{\pi}v(h,s(e^{i\theta}))\frac{e^{i\theta}+\xi(z)}{e^{i\theta}-\xi(z)}d\theta\ , \tag{44}$$

where $\xi(z)$ is the map of the interior of the polygon onto the interior of the unit circle, with the origins and directions of the real axes coinciding and $s(e^{i\theta})$ determined by the inverse map. Then $W_+(\boldsymbol{\rho})$ inside $\mathcal{D}$ will be given by

$$W_+(\boldsymbol{\rho})=c_k\left[\frac{eB}{2\pi}\right]^{1/2}\exp(-\tfrac{1}{4}eB\rho^2)z^k\exp[f(z)]\ , \tag{45}$$

which is unique up to the multiplicative constant c_k.

The theorem can be generalized to the cases in which W_+ has other zeros inside the cell, provided that the positions of the zeros preserve the reflection symmetry (ii).

A proof of the existence of periodic solutions for the nonlinear equations of Ref. [4] was given by Spruck and Yang [9].

For lattices of regular polygons, triangular, square, and hexagonal ($n=3,4,6$) in addition to the translation and reflection symmetries one has invariance of $|W_+|$ under rotations C_n by multiples of $\alpha_n=2\pi/n$. Such lattices have maximal symmetry C_{4V} for the square and C_{6V} for the lattices of equilateral triangles and regular hexagons. Any other symmetry group of a two-dimensional lattice is a subgroup of either C_{4V} or C_{6V}.

Let us denote by W_n and G_n ($n=3,4,6$) the functions W_+ and G, with $k=1$, for these three regular lattices. A representation for the function $G_n(z)$ which explicitly exhibits the rotational symmetry was given in Ref. [10]:

$$G_n(z)=c_nz\prod_j\left[1-\left[\frac{z}{z_j}\right]^n\right]\ , \tag{46}$$

where the product extends over the centers z_j of all the cells within the domain $-\alpha_n/2<\mathrm{Arg}(z)\le\alpha_n/2$, excluding the origin. For $n=3,4,6$, this infinite product is uniformly convergent throughout the complex plane, defining an entire function. Under a rotation by α_n, $G_n(z)$ clearly transforms to $e^{i\alpha_n}G_n(z)$. The triangular lattice is the dual of the hexagonal lattice. It may be considered as a hexagonal lattice with the zeros of W_+ at the vertices of the hexagons. Then, if one takes a hexagon as the lattice cell, it has two zeros per cell. As shown in Ref. [10], $W_3(\boldsymbol{\rho})$ can be expressed in terms of a product

of two functions W_6 with their arguments shifted by constant vectors.

Here we shall generalize the results of Refs. [4,6,10] and consider lattices of arbitrary identical parallelograms with one zero of W_+ at the center of each cell. Let us take one parallelogram in the complex z plane, with four vertices at $\mp(a/2)(1\mp\tau)$, where a is the length of one side along the real axis and $\tau=\tau_R+i\tau_I$ is a complex number with positive imaginary part τ_I. The center of the parallelogram is at the origin $z=0$. Its area is given by $A=a^2\tau_I$, which by the flux quantization condition determines the parameter a. This is a unit cell of the lattice of parallelograms. There is then a one-to-one correspondence between lattices of parallelograms and the upper half τ plane. For a solution with the symmetry of a lattice of parallelograms, $|W_+|$ must be doubly periodic with periods a and τa in the z plane. Therefore, the function $G(z)$ should have a representation in terms of Jacobi ϑ functions [11]. Indeed, let

$$G(z/a,\tau)=(2\tau_I)^{1/4}\exp\left[\frac{\pi}{2\tau_I}(z/a)^2\right]\vartheta_1(\pi z/a\,|\tau) \tag{47}$$

and

$$\mathcal{W}_+(\rho,\tau)=\exp\left[-\frac{\pi}{2A}\rho^2\right]G(z/a,\tau)\ , \tag{48a}$$

where $\tau=\tau_R\mathbf{e}_1+\tau_I\mathbf{e}_2$. Then

$$W_+(\rho,\tau)=c\left[\frac{eB}{2\pi}\right]^{1/2}\mathcal{W}_+(\rho,\tau) \tag{48b}$$

has the following properties under translations:

$$\text{(i)}\quad W_+(\rho+a\mathbf{e}_1,\tau)=-\exp\left[i\frac{\pi a}{A}y\right]W_+(\rho,\tau)\ ,$$

$$\text{(ii)}\quad W_+(\rho+a\tau,\tau)=-\exp\left[i\frac{\pi a}{A}(-x\tau_I+y\tau_R)\right]W_+(\rho,\tau)\ . \tag{49}$$

As shown in Ref. [8] these are precisely the gauge transformations of W_+ corresponding to the transformation of the gauge potential $\mathbf{A}_1=\frac{1}{2}e\mathbf{B}\times\rho_1$ to $\mathbf{A}_2=\frac{1}{2}e\mathbf{B}\times\rho_2$, where $\mathbf{a}_{12}\equiv\rho_2-\rho_1$ is a constant vector corresponding, in the complex plane, to the displacements a and τa, respectively. It is clear that $|W_+|$ is left invariant by these transformations.

The function $G(z/a,\tau)$ defined by Eq. (47) coincides (up to a constant factor) with $G_n(z)$ given by Eq. (46) for the values of τ corresponding to the square ($n=4$, $\tau=i$) and hexagonal [$n=6$, $\tau=(1+i\sqrt{3})/2$] lattices. The triangular lattice does not correspond to a simple lattice of parallelograms with a single zero at the center but to one with two zeros in each cell. Therefore, as already explained, it can be represented in terms of a product of two ϑ functions. The functions

$$W_+^{(k)}(\rho,\tau)\equiv c_k\left[\frac{eB}{2\pi}\right]^{1/2}[\mathcal{W}_+(\rho,\tau)]^k$$

are also solutions with the same lattice symmetry but with a zero of order k at the center of each parallelogram.

VI. AVERAGE ENERGY DENSITY

We now proceed to express the average energy density in terms of the lattice solutions. In expression (31), the first term is a constant $\frac{1}{2}H^2$. As given by Eq. (30), $H=\bar{B}-e\overline{|W_+|^2}$, where $\bar{B}$ is taken as the external background field. The bar over a function indicates the space average of that function, which for lattice solutions equals the average over a unit cell. The average energy density becomes

$$\overline{\mathcal{H}}=\tfrac{1}{2}\bar{B}^2-(e\bar{B}-M_W^2)\overline{|W_+|^2}+\tfrac{1}{2}e^2(\overline{|W_+|^2})^2+\tfrac{1}{2}g^2M_W^2\overline{U(\rho)|W_+|^2}\ . \tag{50}$$

The last two terms are of order $(\overline{|W_+|^2})^2$ so that, if $e\bar{B}>M_W^2$, the average energy density decreases with the production of W pairs. Therefore, a phase transition occurs with $|W_+|^2$ playing the role of an order parameter.

The stability of the system depends on the quartic terms. We distinguish two cases.

(i) $M_H<M_Z$. In this case, $U(\rho)$ is negative. One can show that there are configurations of the W_+ field with lattice symmetry for which the quartic terms become negative. As the density of W pairs in such configurations increases, the average energy density becomes increasingly negative, that is unbounded from below, and the system would be unstable. However, this instability is an artifact of the perturbation method. One can, in fact, verify that the system is always stable but the perturbative method cannot be used in this case.

(ii) $M_H\geq M_Z$. In this case, $U(\rho)$ is positive and the quartic interaction is positive definite. The system is stable and the average energy density will have a minimum with $|W_+|^2\neq 0$.

The minimal average energy density, in lowest order of $|W_+|^2$, is obtained by varying Eq. (50) at a fixed value of $\bar{B}$ with respect to the parameters λ_m. One obtains the set of equations

$$\int[g^2M_W^2U(\rho)+e^2\overline{|W_+|^2}-(e\bar{B}-M_W^2)]W_+^*(\rho)f_m(\rho)d^2\rho=0\ . \tag{51}$$

If one assumes that the minimal average energy density corresponds to a particular lattice solution, then a sufficient condition for (51) to hold is that, for this solution and all $m\geq 0$, one has

$$\tfrac{1}{2}\sum_{\varepsilon=\pm1}\int|W_+(\rho'+\varepsilon\rho)|^2W_+^*(\rho')f_m(\rho')d^2\rho'=\mathcal{A}(\rho)\int W_+^*(\rho')f_m(\rho')d^2\rho'\ , \tag{52a}$$

where

$$\mathcal{A}(\rho)=\int_{\text{cell}}|W_+(\rho'+\rho)|^2|W_+(\rho')|^2d^2\rho'\Big/\int_{\text{cell}}|W_+(\rho')|^2d^2\rho' \ . \tag{52b}$$

Using the properties of the ϑ functions, one can show [12] that such identities do indeed hold for the solutions with $k=1$ on the hexagonal and square lattices.

On the other hand, for a particular lattice solution, $W_+(\rho)$ is known up to a normalizing constant c which sets the magnitude of the density of W pairs. This overall constant can then be determined by the condition

$$(M_W^2-e\bar{B})\overline{|W_+|^2}+g^2[\sin^2\theta(\overline{|W_+|^2})^2+M_W^2\overline{U(\rho)|W_+|^2}]=0 \tag{53}$$

from which one obtains the following expression for the average value of the energy density in a cell:

$$\overline{\mathcal{H}}=\tfrac{1}{2}\bar{B}^2-\frac{1}{2g^2}(e\bar{B}-M_W^2)^2\eta(M_Z/M_H) \ , \tag{54}$$

where

$$\eta(M_Z/M_H)=1/[M_W^2\overline{U(\rho)|W_+|^2}/(\overline{|W_+|^2})^2+\sin^2\theta] \ . \tag{55}$$

This parameter η is also a function of $(e\bar{B}/M_W^2)$ but to the order of magnitude we are calculating one can set this equal to unity.

This result would qualitatively agree with Skalozub's [6] were it not for a sign discrepancy in the magnetization. Our sign agrees with that of Ref. [4]. Skalozub also makes an approximation which effectively replaces the nonlocal quartic interaction by a local one. Our exact calculation in Sec. VIII shows that the effects of nonlocality can change the values of η by as much as 10% as compared to differences of about 0.25% between the square and hexagonal lattices. Our conclusion is that, for any $M_H>M_Z$, the hexagonal lattice has the lowest energy density.

Lattices of parallelograms are specified by the complex parameter τ with $\text{Im}\tau>0$. We shall now show the following important result.

Theorem 3: Modular invariance of the average energy density. With $\mathcal{W}_+$ given by Eq. (48a), or some power k thereof, the average energy density $\overline{\mathcal{H}}$ considered as a function of the lattice parameter $\tau=te^{i\beta}$ is modular invariant.

The proof is based on the following properties of the function ϑ_1:

$$\text{(i)}\quad \vartheta_1(\pi z/a|\tau)=e^{-in\pi/4}\vartheta_1(\pi z/a|\tau+n) \ , \qquad n=\text{integer} \ , \tag{56}$$

$$\text{(ii)}\quad \vartheta_1(\pi z/a|\tau)=e^{-i\pi/4}\tau^{-1/2}\exp\left[i\pi\frac{z^2}{a^2}\tau'\right]\times\vartheta_1(\pi(z/a)\tau'|\tau'), \quad \tau'=-1/\tau \ . \tag{57}$$

From (i) it immediately follows that $G(z/a,\tau)=e^{-in\pi/4}G(z/a,\tau+n)$.

Let us introduce in (ii) the variable $z'/a'=(z/a)\tau'$. Because of the flux quantization condition, we set $A'=(a')^2\tau_I'=A=a^2\tau_I$, and therefore $(a')^2=a^2t^2$ and $z=-z'e^{i\beta}$. The factor $\exp[(\pi/2\tau_I)(z/a)^2]$ in the definition of $G(z/a,\tau)$ combines with $\exp[i\pi(z^2/a^2)\tau']$ in (ii) to give $\exp[(\pi/2\tau_I')(z'/a')^2]$. Moreover, $\tau_I^{1/4}\tau^{-1/2}=e^{-i\beta/2}(\tau_I')^{1/4}$. Therefore, from (ii), one obtains

$$G(z/a,\tau)=e^{-i(\beta/2+\pi/4)}G(z'/a',\tau') \ . \tag{58}$$

Since $\tau\to\tau+n$ and $\tau\to-1/\tau$ are the generators of the modular group, it follows that the modulus of $G(z/a,\tau)$ is invariant under the modular group with the variable z/a transforming under each inversion as $z'/a'=(z/a)\tau'$. Therefore, by Eqs. (48), it follows that the average energy density is invariant under modular transformations. A simple geometric interpretation of this invariance was given in [7].

Theorem 4: Reflection symmetry. The average energy density is invariant under the transformation $\tau\to-\tau^*$. In fact, $G(z/a,\tau)^*=G(z^*/a,-\tau^*)$. Since ρ^2 and A are unchanged by the transformations $z\to z^*$, $\tau\to-\tau^*$, the theorem follows from Eqs. (48), (54), and (55).

A numerical analysis searching for the lattice configuration for which Eq. (54) is an absolute minimum is deferred to Sec. VIII. In the next section the classical treatment we have been following so far will be justified by an approach where the W field is quantized.

VII. QUANTIZATION OF THE W FIELD

So far we have given a completely classical treatment of the instability of the electroweak theory against the production of W-pair condensates in a strong magnetic field above the critical value $eB_c=M_W^2$.

Since the W field is complex, representing a charged particle, it is not an observable and therefore does not have a classical limit. The expectation value of W in any physical state is always zero. Therefore, this classical treatment has to be justified by a quantum field approach. We shall here follow a simplified approach in which the real fields **A**, **Z**, and Φ are treated classically while the W field is quantized. One should then look for the quantum state of an ensemble of W's that corresponds to the minimal eigenstate of the energy density operator. For the lattice solutions, this quantum state will be one with W-pair condensates.

The field equations for the real fields should be interpreted as classical equations, in which the W-dependent source terms are to be taken as expectation values in the state $|\Omega\rangle$ that minimizes the energy. On the other hand, the equation for the W field is operator valued.

We shall quantize the field W_+ in a constant magnetic

field $\bar{B}$, taking the set of eigenfunctions $f_{knm}(z,\rho,\varphi)$ given in Eq. (16) with degenerate eigenvalues

$$E_{kn}^2 = M_W^2 + (2\pi k/L)^2 + (2n-1)e\bar{B} \tag{59}$$

as a basis for the one-particle states. It will be given by

$$W_+(z,\rho,\varphi) = \sum_{k=-\infty}^{\infty} \sum_{n=0}^{\infty} \sum_{m=-n}^{\infty} \frac{1}{\sqrt{2LE_{kn}}} (a_{knm} e^{-iE_{kn}t} + b_{knm}^\dagger e^{iE_{kn}t}) \times f_{knm}(z,\rho,\varphi) . \tag{60}$$

The components W_- and W_3 should be quantized in the same way, but since they do not develop zero modes we do not have to consider them here.

The annihilation operators a,b and their Hermitian conjugates $a^\dagger, b^\dagger$ have the following nonvanishing canonical commutators:

$$[a_{knm}, a_{k'n'm'}^\dagger] = \delta_{kk'}\delta_{nn'}\delta_{mm'} ,$$
$$[b_{knm}, b_{k'n'm'}^\dagger] = \delta_{kk'}\delta_{nn'}\delta_{mm'} . \tag{61}$$

The operators a,b annihilate a state $|\ \rangle$ which is the vacuum of the interaction representation. The charge operator is given by

$$Q = e \sum_{k,n,m} (a_{knm}^\dagger a_{knm} - b_{knm}^\dagger b_{knm}) . \tag{62}$$

Above the critical value of the magnetic field, the lowest eigenvalue $(E_{00})^2$ becomes negative and the expansion (60) does not make sense since E_{00} becomes imaginary. In order to circumvent this difficulty we shall add a mass counterterm $M_C^2 W_+^\dagger W_+$ to the unperturbed Hamiltonian density and subtract the same term from the interaction part. Then $(E_{00})^2 = M_W^2 - e\bar{B} + M_C^2$, which is positive for a properly chosen M_C^2. One finds that the minimum of the energy density occurs for $(E_{00})^2 = 0$ and M_C^2 equal to the shift in E^2 due to the perturbation.

In what follows we shall be interested in the sector $\mathcal{S}_0$ of the Hilbert space spanned by the zero-mode operators. Therefore, we shall restrict ourselves to the set of quantum numbers $\{k=0, n=0, m\geq 0\}$ and shall drop from our notation the indices k and n. In $\mathcal{S}_0$, W_+ reduces to

$$W_+ = \frac{1}{\sqrt{2LE}} \sum_{m=0}^{\infty} [a_m(t) + b_m^\dagger(t)] f_m , \tag{63}$$

where $a_m(t) = a_m e^{-iEt}$, $b_m(t) = b_m e^{-iEt}$. These operators satisfy the same commutation relations (61) as a_m, b_m. In the limit $E \to 0$, $a_m(t) \to a_m$ and $b_m(t) \to b_m$. From now on we shall write a_m and b_m for $a_m(t)$ and $b_m(t)$ with the understanding that only in the limit $E \to 0$ do they become time independent. Since we are now dealing with time-dependent solutions of the linearized equations for the W field, we have to add to the energy density the kinetic term $\dot{W}_+^\dagger \dot{W}_+$.

In the static limit, the energy density will become a function of $W_+^\dagger W_+$. Therefore, one should look for eigenstates of this operator of the form [13]

$$W_+^\dagger W_+ |\Omega\rangle = \mathcal{J}^* \mathcal{J} |\Omega\rangle , \tag{64}$$

where $\mathcal{J} = (1/\sqrt{2LE}) \sum_m \lambda_m f_m$ is a classical solution.

In order to construct these eigenstates it is convenient to perform a Bogoliubov transformation

$$\alpha = a\Lambda^\dagger, \quad \alpha^\dagger = \Lambda a^\dagger ,$$
$$\beta = \Lambda b, \quad \beta^\dagger = b^\dagger \Lambda^\dagger , \tag{65}$$

where Λ is a unitary matrix, a and $b^\dagger$ are row vectors, $a^\dagger$ and b are column vectors, and one makes the identification $\Lambda_{0m} = \lambda_m/\lambda$, with

$$|\lambda|^2 \equiv \sum_m \lambda_m^* \lambda_m = 2LE \int \mathcal{J}^* \mathcal{J} d^2\rho . \tag{66}$$

For the lattice solutions this integral diverges. It is then necessary to introduce a cutoff m_0 in m: $m \leq m_0$. This cutoff roughly corresponds to quantizing inside a cylinder of radius R such that $m_0 \approx \frac{1}{2} e\bar{B} R^2$. In all that follows we shall neglect terms of order m_0^{-1}.

In terms of the operators defined by Eq. (65), $W_+^\dagger W_+$ is

$$W_+^\dagger W_+ = \frac{1}{2LE} \sum_{m,n} (\alpha_m^\dagger + \beta_m)(\alpha_n + \beta_n^\dagger) F_m^* F_n , \tag{67}$$

where $F = \Lambda f$. In order to obtain an eigenstate of $W_+^\dagger W_+$ of the form (64), the following conditions must be satisfied:

$$\text{(i)} \quad m \neq 0: \quad (\alpha_m^\dagger + \beta_m)|\Omega(\lambda)\rangle = 0 ,$$
$$(\alpha_m + \beta_m^\dagger)|\Omega(\lambda)\rangle = 0 ,$$
$$\text{(ii)} \quad (\alpha_0^\dagger + \beta_0)(\alpha_0 + \beta_0^\dagger)|\Omega(\lambda)\rangle = \lambda^*\lambda|\Omega(\lambda)\rangle . \tag{68}$$

The general solution of these equations such that $|\Omega(\lambda)\rangle$ can be expressed as a power series in the set $(a^\dagger, b^\dagger)$ acting on the state $|\ \rangle$ is [14]

$$|\Omega(\lambda)\rangle = Z^{-1/2} \exp\left[-\sum_{m=0}^{m_0} \alpha_m^\dagger \beta_m^\dagger \right] \frac{1}{2\pi} \times \int_0^{2\pi} du\, g(u) \exp(\lambda\alpha_0^\dagger e^{-iu} + \lambda^* \beta_0^\dagger e^{iu}) |\ \rangle , \tag{69}$$

where Z is a constant. The Fourier decomposition of $g(u)$ gives a decomposition of $|\Omega(\lambda)\rangle$ into eigenstates of the charge operator with eigenvalues equal to the index of the Fourier component of $g(u)$. For a solution corresponding to a condensate of W pairs the total charge is zero. This state is obtained with $g(u) = 1$. Then the integral becomes

$$\frac{1}{2\pi} \int_0^{2\pi} du \exp(\lambda\alpha_0^\dagger e^{-iu} + \lambda^* \beta_0^\dagger e^{iu}) = I_0(2\sqrt{\lambda^*\lambda\alpha_0^\dagger\beta_0^\dagger}) , \tag{70}$$

where I_0 is a modified Bessel function of the third kind.

The state $|\Omega(\lambda)\rangle$ as defined is not normalizable. Nevertheless, one can use a simple device which allows for the construction of normalized states dependent on a parameter ε; in the limit $\varepsilon \to 0$ they go over to the eigenstates $|\Omega(\lambda)\rangle$. One can then use these states to compute expectation values and only after this is done take the limit $\varepsilon \to 0$. In fact, for ε such that $|1-\varepsilon| < 1$, the states [14]

$$|\Omega(\lambda,\varepsilon)\rangle = Z^{-1/2}\exp\left[-(1-\varepsilon)\sum_{m=0}^{m_0}\alpha_m^\dagger\beta_m^\dagger\right]\frac{1}{2\pi}\times\int_0^{2\pi}du\,\exp(\lambda\alpha_0^\dagger e^{-iu}+\lambda^*\beta_0^\dagger e^{iu})|\rangle \quad (71)$$

are normalized to unity by taking

$$Z=\left[\frac{1}{1-|1-\varepsilon|^2}\right]^{m_0+1}\exp\left[-\lambda^*\lambda\frac{2-\varepsilon-\varepsilon^*}{1-|1-\varepsilon|^2}\right]\times I_0(2\lambda^*\lambda/(1-|1-\varepsilon|^2)) . \quad (72)$$

As $\varepsilon\to 0$, $Z\sim(1-|1-\varepsilon|^2)^{-m_0-1/2}\to\infty$.

The expectation value of the energy density in the state $|\Omega\rangle$ is

$$\langle\mathcal{H}\rangle=\tfrac{1}{2}(B-e\langle W_+^\dagger W_+\rangle)^2+M_W^2\langle W_+^\dagger W_+\rangle+\tfrac{1}{2}g^2M_W^2U(\rho)\langle W_+^\dagger W_+\rangle+\tfrac{1}{2}g^2[\langle(W_+^\dagger W_+)^2\rangle-\langle W_+^\dagger W_+\rangle^2]+\langle\dot W_+^\dagger\dot W_+\rangle . \quad (73)$$

In this expression $U(\rho)$ is given by (31) with $|W_+|^2$ replaced by $\langle W_+^\dagger W_+\rangle$.

We need to compute the expectation values of the operators $W_+^\dagger W_+$, $(W_+^\dagger W_+)^2$, and $\dot W_+^\dagger\dot W_+$. This rather laborious computation is omitted here. We shall give only the results.

Take ε real, let $\Delta=1-|1-\varepsilon|^2=2\varepsilon-\varepsilon^2$ and $\chi=\varepsilon^2/(2LE\Delta)$. We consider two regimes: $\lambda^*\lambda\ll\Delta$ and $\lambda^*\lambda\gg\Delta$. First, at the onset of the phase transition, one has the following.

(i) $\lambda^*\lambda\ll\Delta$:

$$\langle W_+^\dagger W_+\rangle=\frac{e\bar B}{2\pi}\chi+\tfrac{1}{2}\mathcal{F}^*\mathcal{F} , \quad (74a)$$

$$\langle(W_+^\dagger W_+)^2\rangle-\langle W_+^\dagger W_+\rangle^2=\left[\frac{e\bar B}{2\pi}\right]^2\chi^2+\frac{e\bar B}{2\pi}\chi\mathcal{F}^*\mathcal{F} , \quad (74b)$$

$$\langle\dot W_+^\dagger\dot W_+\rangle=\frac{e\bar B}{2\pi}\frac{1}{4L^2}\frac{1}{\chi}-\frac{1}{8L^2}\frac{1}{\chi^2}\mathcal{F}^*\mathcal{F} , \quad (74c)$$

where we used Eq. (18).

Assuming a lattice solution, the energy density becomes a function of only two parameters: $\lambda^*\lambda/(2LE)$ and χ. Averaging over a unit cell and varying with respect to these two parameters, one obtains the conditions for the minimum.

Variation with respect to $\lambda^*\lambda$ gives

$$-\tfrac{1}{2}(e\bar B-M_W^2)\overline{\mathcal{F}^*\mathcal{F}}+\tfrac{1}{2}g^2\frac{e\bar B}{2\pi}\chi\overline{\mathcal{F}^*\mathcal{F}}+\tfrac{1}{2}e^2\left[\frac{e\bar B}{2\pi}\chi+\frac{1}{2}\overline{\mathcal{F}^*\mathcal{F}}\right]\overline{\mathcal{F}^*\mathcal{F}}+\tfrac{1}{2}g^2M_W^2\overline{U(\rho)\mathcal{F}^*\mathcal{F}}-\frac{1}{8L^2}\frac{1}{\chi^2}\mathcal{F}^*\mathcal{F}=0 . \quad (75)$$

Dividing by $\frac{1}{2}\overline{\mathcal{F}^*\mathcal{F}}$ and taking the limit $\lambda^*\lambda\to 0$, one obtains

$$-(e\bar B-M_W^2)+g^2\left[2-\frac{M_W^2}{M_H^2}\right]\frac{e\bar B}{2\pi}\chi-\frac{1}{4L^2}\frac{1}{\chi^2}=0 . \quad (76)$$

Variation with respect to χ in the limit $\lambda^*\lambda\to 0$ gives the same equation.

If one assumes that the energy density is continuous through the transition point, one finds that, for finite L, the critical magnetic field is shifted to

$$e\bar B_c=M_W^2\Big/\left[1-\frac{3g^2}{8\pi}\left[2-\frac{M_W^2}{M_H^2}\right]\chi_c\right] \quad (77)$$

with

$$\chi_c\approx\left[g^2L^2\frac{M_W^2}{2\pi}\left[2-\frac{M_W^2}{M_H^2}\right]\right]^{-1/3} . \quad (78)$$

Next, consider the following regime.

(ii) $\lambda^*\lambda\gg\Delta$. In lowest order in χ,

$$\langle W_+^\dagger W_+\rangle=\frac{e\bar B}{2\pi}\chi+\mathcal{F}^*\mathcal{F} , \quad (79a)$$

$$\langle(W_+^\dagger W_+)^2\rangle-\langle W_+^\dagger W_+\rangle^2=\left[\frac{e\bar B}{2\pi}\right]^2\chi^2+2\frac{e\bar B}{2\pi}\chi\mathcal{F}^*\mathcal{F} , \quad (79b)$$

$$\langle\dot W_+^\dagger\dot W_+\rangle=\frac{1}{4L^2}\frac{e\bar B}{2\pi}\frac{1}{\chi} . \quad (79c)$$

Variation of the average energy density with respect to $\lambda^*\lambda$ gives

$$-(e\bar B-M_W^2)\overline{\mathcal{F}^*\mathcal{F}}+g^2\frac{e\bar B}{2\pi}\chi\overline{\mathcal{F}^*\mathcal{F}}+g^2M_W^2\overline{U(\rho)\mathcal{F}^*\mathcal{F}}+e^2\left[\frac{e\bar B}{2\pi}\chi+\overline{\mathcal{F}^*\mathcal{F}}\right]\overline{\mathcal{F}^*\mathcal{F}}=0 . \quad (80)$$

Variation of the average energy density with respect to χ gives

$$-(e\bar B-M_W^2)+g^2M_W^2\overline{U(\rho)}+e^2\left[\frac{e\bar B}{2\pi}\chi+\overline{\mathcal{F}^*\mathcal{F}}\right]-\frac{1}{4L^2}\frac{1}{\chi^2}+g^2\left[\frac{e\bar B}{2\pi}\chi+\overline{\mathcal{F}^*\mathcal{F}}\right]=0 . \quad (81)$$

Dividing Eq. (80) by $\overline{\mathcal{F}^*\mathcal{F}}$ and subtracting from Eq. (81), one obtains

TABLE I. Values of the parameter $\eta(M_Z/M_H)$ resulting from four different distributions of $|W_+|^2$, in units of $(e\bar{B}-M_W^2)^2/(2g^2)$.

M_Z/M_H	Triangle	Square	Hexagon	Uniform
0.000 0	0.936 86	0.977 15	0.981 66	1.000 00
0.062 5	0.940 44	0.980 58	0.985 06	1.003 06
0.125 0	0.951 24	0.990 92	0.995 31	1.012 34
0.187 5	0.969 39	1.008 30	1.012 54	1.028 20
0.250 0	0.995 28	1.033 10	1.037 14	1.051 25
0.312 5	1.029 60	1.066 07	1.069 88	1.082 45
0.375 0	1.073 53	1.108 49	1.112 03	1.123 20
0.437 5	1.128 89	1.162 27	1.165 56	1.175 50
0.500 0	1.198 47	1.230 29	1.233 35	1.242 24
0.562 5	1.286 46	1.316 80	1.319 64	1.327 66
0.625 0	1.399 25	1.428 25	1.430 90	1.438 20
0.687 5	1.547 01	1.574 77	1.577 26	1.583 96
0.750 0	1.746 63	1.773 22	1.775 56	1.781 74
0.812 5	2.028 42	2.053 66	2.055 84	2.061 52
0.875 0	2.452 44	2.475 53	2.477 49	2.482 54
0.937 5	3.156 84	3.174 79	3.176 30	3.180 12
1.000 0	4.545 45	4.545 45	4.545 45	4.545 45

$$g^2\left[1+\frac{1}{\eta_0}-\frac{1}{\eta}\right]\overline{\mathcal{F}^*\mathcal{F}}-\frac{1}{4L^2}\frac{1}{\chi^2}=0\ , \tag{82}$$

where the parameter η is defined by Eq. (55) with $|W_+|^2=\mathcal{F}^*\mathcal{F}$ and given in Table I for the triangular, square, and hexagonal lattices and $\eta_0=[1-(M_W^2/M_H^2)]^{-1}$.

Now one can take the limit $E\to 0$, $\varepsilon\to 0$ at fixed ratio $1/(4L\chi)$. Then $|\Omega\rangle$ becomes an energy eigenstate.

In the limit $L\to\infty$, $\chi\sim 1/(2gL\sqrt{\overline{\mathcal{F}^*\mathcal{F}}})$ goes to zero. In this limit the energy density (73) averaged over a unit cell reduces to the classical expression (50).

VIII. NUMERICAL ANALYSIS AND CONCLUSIONS

Using Eq. (55), the parameter η in the expression (54) for the average energy density was calculated numerically for lattice configurations with maximal symmetry, namely, lattices of regular triangles, squares, and hexagons. The results were given in Ref. [11]. Because of a small error in the computer code, the numbers given there for the triangular and hexagonal lattices were accurate only to three decimal places. We have now been able to do the integrations analytically [15].

The corrected values of the parameter η, for $\sin^2\theta=0.22$, are reproduced in Table I for completeness. The conclusions are independent of the value of the Weinberg angle θ. The last column in this table was calculated taking $|W_+|^2=$const. Such a uniform distribution does not correspond to a classical solution; it could result from a coherent quantum state of W pairs [as given by Eq. (71) with $\lambda=0$], but the quantum fluctuations, given by Eq. (74b) with $\mathcal{F}^*\mathcal{F}=0$, would raise the energy density above the values for lattices of regular polygons.

For M_H in the range $0<M_Z/M_H<1$ the results can be summarized as follows: (i) For each lattice symmetry the energy density is lowest for $k=1$ and increases monotonically with the order k of zeros of W_+; (ii) the lowest energy density is obtained for the hexagonal lattice with one zero of W_+ at the center of each cell; and (iii) the average energy density for the triangular lattice with $k=1$ is exactly equal to the energy density for a hexagonal lattice with $k=2$. This rather surprising result comes about by virtue of the following remarkable identity, involving ϑ functions.

The wave functions $\omega_3(\boldsymbol{\rho})$ and $\omega_6(\boldsymbol{\rho})$ corresponding to these two lattices are (up to a constant factor)

$$\omega_6(\boldsymbol{\rho})=[\mathcal{W}_+(\boldsymbol{\rho},\tau)]^2\ , \tag{83a}$$

$$\omega_3(\boldsymbol{\rho})=\mathcal{W}_+(\boldsymbol{\rho}+\mathbf{e}_2a/\sqrt{3},\tau)\mathcal{W}_+(\boldsymbol{\rho}-\mathbf{e}_2a/\sqrt{3},\tau) \tag{83b}$$

with $\mathcal{W}_+(\boldsymbol{\rho},\tau)$ defined by Eq. (48a) and $\boldsymbol{\tau}=\frac{1}{2}\mathbf{e}_1+(\sqrt{3}/2)\mathbf{e}_2$. Then we find numerically that

$$\frac{\int_{\text{cell}}|\omega_6(\boldsymbol{\rho}')|^2|\omega_6(\boldsymbol{\rho}'+\boldsymbol{\rho})|^2d^2\boldsymbol{\rho}'}{\left[\int_{\text{cell}}|\omega_6(\boldsymbol{\rho}')|^2d^2\boldsymbol{\rho}'\right]^2}=\frac{\int_{\text{cell}}|\omega_3(\boldsymbol{\rho}')|^2|\omega_3(\boldsymbol{\rho}'+\boldsymbol{\rho})|^2d^2\boldsymbol{\rho}'}{\left[\int_{\text{cell}}|\omega_3(\boldsymbol{\rho}')|^2d^2\boldsymbol{\rho}'\right]^2}\ . \tag{83c}$$

For $k=1$ we have also investigated the behavior of the average energy density for lattices of parallelograms, specified by the complex parameter $\tau=\tau_R+i\tau_I$ with $\tau_I>0$. Because of the modular invariance one needs to investigate the average energy density only in a fundamental domain of τ space. Such a domain can be specified by $\{\tau:-0.5<\tau_R\le 0.5, |\tau|\ge 1\}$. However, because of the additional reflection symmetry under $\tau\to-\tau^*$, one needs to consider only the region $\mathcal{D}=\{\tau:\ 0\le\tau_R\le 0.5, |\tau|\ge 1\}$ (see Fig. 1).

The minimal average energy density corresponds to a configuration for which $\int_{M_Z}^{M_H}d\mathcal{M}\,V(\mathcal{M},\tau)$ is a minimum [cf. Eqs. (32), (54), and (55)], where

TABLE II. Sample of values for $v(\mathcal{M}/\sqrt{e\bar{B}},\tau)$ for lattices of parallelograms.

$\mathcal{M}/\sqrt{e\bar{B}}$	$\lvert\tau\rvert$	$\tau_R=0$	0.25	0.5
0.5	1	1.000 26	1.000 22	1.000 18
	2	1.002 26	1.002 20	1.002 02
	3	1.008 01	1.007 93	1.007 69
1	1	1.003 30	1.002 92	1.002 34
	2	1.024 28	1.023 70	1.021 96
	3	1.073 64	1.073 02	1.071 17
1.5	1	1.012 19	1.010 89	1.008 94
	2	1.072 60	1.071 05	1.066 42
	3	1.189 62	1.188 27	1.184 23
2	1	1.026 59	1.024 01	1.020 15
	2	1.130 93	1.128 43	1.120 89
	3	1.305 52	1.303 64	1.297 94
3	1	1.061 25	1.056 28	1.048 90
	2	1.229 95	1.226 21	1.214 89
	3	1.470 05	1.467 61	1.460 24
∞	1	1.180 34	1.171 95	1.159 60
	2	1.424 80	1.419 71	1.404 24
	3	1.732 61	1.729 61	1.720 53

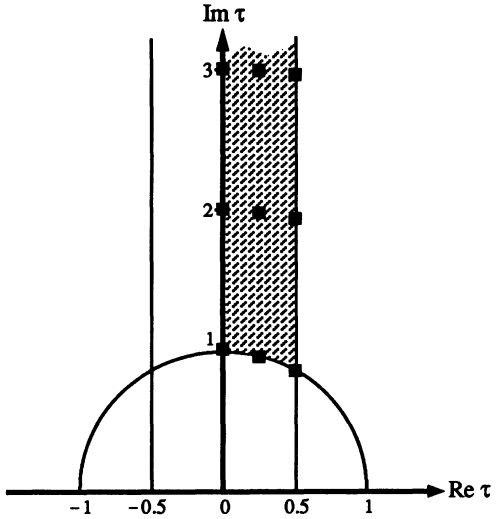

FIG. 1. The hatched area is the semifundamental domain of τ space corresponding to the set of parameters of inequivalent lattice solutions. The solid squares indicate points sampled in Table II.

$$V(\mathcal{M},\tau)=\frac{\int d^2\rho'|\rho'|K_1(\mathcal{M}|\rho'|)\int_{\text{cell}}d^2\rho|W_+(\rho)|^2|W_+(\rho'+\rho)|^2}{\left[\int_{\text{cell}}|W_+(\rho)|^2d^2\rho\right]^2}\ . \tag{84}$$

The integrals were carried out analytically [15]. Using Eqs. (47) and (48) and the condition $e\bar{B}A=2\pi$, one obtains

$$\int_{\text{cell}}|W_+(\rho)|^2d^2\rho=|c|^2 \tag{85}$$

and

$$V(\mathcal{M},\tau)=-\frac{d}{d\mathcal{M}}\sum_{k,l=-\infty}^{\infty}\frac{1}{\mathcal{M}^2/e\bar{B}+(2\pi/\tau_I)|k\tau-l|^2}\times\exp\left[-\frac{\pi}{\tau_I}|k\tau-l|^2\right]\ . \tag{86}$$

The sum converges very rapidly.

We have scanned this function numerically for τ in the region $\mathcal{D}$. It has the following properties for all positive values of $\mathcal{M}$: (i) For fixed $|\tau|$ it decreases monotonically with τ_R; (ii) for fixed τ_R it increases monotonically with $|\tau|$. The point $|\tau|=1$, $\tau_R=0.5$ that corresponds to the hexagonal lattice gives the minimum of this function. The point $|\tau|=1$, $\tau_R=0$, corresponding to the square lattice, is a saddle point.

The function $V(\mathcal{M},\tau)$ behaves as

$$(2e\bar{B}/\mathcal{M}^3)v(\mathcal{M}/\sqrt{e\bar{B}},\tau)$$

where v is a slowly varying function of the arguments and $v(0,\tau)\equiv 1$. A sample of numerical values of $v(\mathcal{M}/\sqrt{e\bar{B}},\tau)$ is given in Table II. The limit $\mathcal{M}\to\infty$ corresponds to a local $|W|^4$ interaction and v becomes a function only of the geometrical parameter τ. In this limit, v coincides with the parameter β_A in the Ginzburg-Landau theory of type-II superconductivity near the transition point H_{c2} [7]. The values of v in Table II for $M=\infty$ and $|\tau|=1$, $v=1.180\,34$ for the square and $v=1.159\,60$ for the hexagonal lattice, agree with the values of β_A calculated by Abrikosov [5] for the square and by Kleiner, Roth, and Autler [16] for the hexagonal lattice.

This treatment of the electroweak phase transition is mathematically a generalization of the Ginzburg-Landau theory in that the quartic interaction mediated by the Z and Φ fields is nonlocal. The propagator, in coordinate space, for the respective interactions is given by the function $(1/2\pi)K_0(M|\rho-\rho'|)$, where the inverse of the mass parameter M ($=M_Z,M_H$) determines the range of the interaction.

Our analysis supports the conclusion that the hexagonal lattice with $k=1$ has the lowest average energy density. This result is valid for more general quartic interactions of the form

$$\int d^2\rho'\int d^2\rho|W_+(\rho')|^2\mathcal{K}(|\rho'-\rho|)|W_+(\rho)|^2 \tag{87}$$

where $\mathcal{K}(\rho)=\int_0^\infty g(\mathcal{M})K_0(\mathcal{M}\rho)d\mathcal{M}$, with $g(\mathcal{M})\geq 0$ for all $\mathcal{M}$.

ACKNOWLEDGMENTS

We would like to thank Feza Gürsey, Hubert Saleur, Subir Sachdev, and Charles Sommerfield for useful discussions.

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