Nontrivial vacua from equal time to the light cone

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Kinematic arguments suggest that the perturbative vacuum may be an eigenstate of the full Hamiltonian for light-cone-quantized field theories. Nevertheless, properties such as spontaneous symmetry breaking can be accommodated in this approach, by applying a quantization which interpolates between equal-time and light-cone quantization and in which the quantization surface may approach the light cone as a limit. In several simple two-dimensional models presented here, including the Gross-Neveu and Schwinger models, the difference between the full and perturbative vacuum vanishes in this limit. Nonzero vacuum expectation values, however, are preserved by singularities in the fields near $k_-=0$. Furthermore, this procedure provides a simple treatment for massless fields and nontrivial tests of Lorentz invariance, and may be applied to models, such as that of Gross and Neveu, for which conventional light-cone quantization is difficult to implement. Finally, the connection between long distances and short times suggests that vacuum effects may be incorporated in an effective Hamiltonian.

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I. INTRODUCTION

Light-cone quantization has proved to be a successful formalism [1] for computing a large array of quantities, mainly perturbative, in QCD. It possesses some troubling features, however, when applied in nonperturbative calculations, especially those sensitive to vacuum properties. In conventional light-cone quantization, one treats $x^+=(x^0+x^3)/\sqrt{2}$ as time and $x^-=(x^0-x^3)/\sqrt{2}$ and $\mathbf{x}_\perp=(x^1,x^2)$ as spatial variables. For free particles of mass m, their conjugate momenta are

$$p_+=\frac{1}{\sqrt{2}}(p_0+p_3)=\frac{m^2+p_\perp^2}{2p_-}\ ,$$
$$p_-=\frac{1}{\sqrt{2}}(p_0-p_3),\quad \mathbf{p}_\perp=(p_1,p_2)\ . \tag{1}$$

Both the energy p_+ and longitudinal momentum p_- are evidently positive. The light-cone Hamiltonian P_+ conserves three-momentum, including p_-. Therefore interactions which connect the perturbative vacuum, which is the ground state of the free theory devoid of particles and carrying zero p_-, to states with particles each carrying positive p_-, are absent. This is in large part responsible for the simplicity of light-cone perturbation theory observed in [2]. The perturbative vacuum is then an eigenstate of the full, and not just the free, Hamiltonian.

Also, the quanta which one might use to construct a nontrivial vacuum must carry three-momenta which sum to zero to maintain the vacuum's Lorentz invariance. While transverse momenta pose no difficulty, the longitudinal momenta p_- of individual quanta are non-negative. The only possibility is that each p_- is zero, leaving one to try to build a vacuum with quanta whose energies are divergent. Particles of zero mass and $p_\perp^2$ are possibly an exception, but this requires dealing with states at a single point in momentum space. Some sort of limiting procedure seems in order.

Nevertheless, the QCD vacuum is believed to be quite complex, generating chiral-symmetry breaking and its consequent pseudo Goldstone boson, the pion. It is not clear how this can emerge in the usual light-cone treatment (though, for a hint, see [3,4]). Furthermore, the Higgs phenomenon, in which gauge bosons become massive through spontaneous symmetry breaking and the acquisition of a vacuum expectation value for the scalar Higgs field, seems precluded.

There are other troubling features which are magnified in $1+1$ dimensions where this discussion will focus. In the massless Schwinger model, for example, the fermion field has one left-moving and one right-moving component. The constraint equation for massive fermions in the $A^+=0$ gauge,

$$i\partial_-\psi_L=\frac{m}{\sqrt{2}}\psi_R\ , \tag{2}$$

implies that ψ_L is only a function of ψ_R rather than a dynamical degree of freedom. However, ψ_L appears to decouple completely from the dynamics when m vanishes. Among other problems, if ψ_L is discarded, the minus component of the gauge current vanishes. In that case the gauge and axial currents are identical, with

$$J_5^+=J^+=\sqrt{2}\psi_R^\dagger\psi_R\ ,$$
$$-J_5^-=J^-=\sqrt{2}\psi_L^\dagger\psi_L=0\ . \tag{3}$$

While J^μ must be conserved, J_5^μ is anomalous, and it is not clear what is going on.

The origin of these problems is straightforward. The surface $x^+=0$ is a characteristic surface of the wave equation and is inadequate to fully specify initial conditions or commutation relations [5–8]. While right-moving quanta intercept this surface, those moving left simply run parallel to it and cannot be initialized.

II. INTERPOLATING FROM EQUAL TIME TO THE LIGHT CONE

In order to study questions such as the structure of the vacuum on the light cone, this paper will present some simple $(1+1)$-dimensional systems in which light-cone quantization will be defined as a limit of a more conventional procedure. This is in the spirit of the old infinite-momentum-frame approach, but without the choice of a particular frame. At each step the quantization will be performed and the Hamiltonian constructed on a spacelike surface; the quantization will be canonical and unambiguous. Unlike the light-cone approach, there will be no new constraint equations. As a result, some of the formal simplicity of light-cone quantization will be sacrificed for the sake of better control over quantities such as the vacuum and singularities peculiar to the light cone.

In equal-time quantization, fields and commutation relations are initialized at $x^0=0$ to coincide with free fields, while the Hamiltonian constructed from these evolves the system to subsequent x^0 slices. In this paper I will define the initial surface to interpolate from $x^0=0$ to $x^0+x^1=0$. The angle which the initial or quantization surface makes relative to $x^0=0$ will be left as a parameter. Lorentz-invariant quantities such as masses must in the end be independent of this angle, while in intermediate stages this angle may be chosen for convenience. Such a procedure was applied to the study of two-dimensional QCD in a generalized axial gauge in [9], and similar interpolations have been employed recently to study the Dirac equation [10] and perturbation theory [11]. Earlier extensions of light-cone quantization to surfaces close to the light cone appeared in [12,13], and such an extension was used recently in [14] and [15] to study two-dimensional QED and QCD.

Specifically, the time coordinate x^+ and space coordinate x^- are defined to be

$$\begin{bmatrix} x^+ \\ x^- \end{bmatrix} \equiv \begin{bmatrix} \sin\theta/2 & \cos\theta/2 \\ \cos\theta/2 & -\sin\theta/2 \end{bmatrix} \begin{bmatrix} x^0 \\ x^1 \end{bmatrix} , \tag{4}$$

with $(\pi-\theta)/2$ the angle between the quantization surface and $x^0=0$. The equal-time limit is at $\theta=\pi$, while $\theta=\pi/2$ is the light-cone limit. Because I will focus on the light cone, I have chosen to retain the conventional light-cone notation: x^+ for time and x^- for space, with p_+ and p_- their conjugate momenta. It should be kept in mind that throughout this paper what these mean depends on this angle. In the equal-time limit, $x^+\to x^0$ and $x^-\to -x^1$; in the light-cone limit, $x^+\to(x^0+x^1)/\sqrt{2}$ and $x^-\to(x^0-x^1)/\sqrt{2}$. The change in sign for x^1 is necessary for the light-cone limit to coincide with the usual convention.

The metric [9]

$$g^{\mu\nu}=g_{\mu\nu}=\begin{bmatrix} -\cos\theta & \sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \tag{5}$$

also depends on θ and is seen to interpolate between the usual equal-time $(g_{00}=-g_{11}=1)$ and light-cone $(g_{+-}=g_{-+}=1)$ metrics. It has both on- and off-diagonal elements, and so lowering or raising indices is somewhat involved. To avoid confusion I will use upper indices for coordinates and lower indices for their conjugate momenta. Finally, for the simplicity of later discussion, define

TABLE I. Summary of various quantities in the equal-time and light-cone limits.

	Equal time	Light cone
θ	π	$\pi/2$
c	1	0
s	0	1
x^+	x^0	$(x^0+x^1)/\sqrt{2}$
x^-	$-x^1$	$(x^0-x^1)/\sqrt{2}$
g^{++}	$\equiv g^{00}=1$	0
g^{--}	$\equiv g^{11}=-1$	0
g^{+-}	$\equiv g^{01}=0$	1

$$c\equiv-\cos\theta, \quad s\equiv\sin\theta \; . \tag{6}$$

Table I gives a summary of these quantities in both limits.

III. FREE SCALARS

At this point some simple systems with nontrivial but calculable vacua can be studied at arbitrary θ to see how they behave in the light-cone limit. A free massive scalar theory in $1+1$ dimensions provides a good initial example to show how this scheme works, as it contains all the essential features. The Lagrangian

$$\mathcal{L}=\tfrac{1}{2}\partial_\mu\phi\,\partial^\mu\phi-\tfrac{1}{2}m^2\phi^2 \tag{7}$$

becomes

$$\mathcal{L}=\tfrac{1}{2}c[(\partial_+\phi)^2-(\partial_-\phi)^2]+s\partial_+\phi\,\partial_-\phi-\tfrac{1}{2}m^2\phi^2 \; , \tag{8}$$

in these coordinates with the corresponding equation of motion,

$$[\partial^2+m^2]\phi=[c(\partial_+^2-\partial_-^2)+2s\,\partial_+\partial_-+m^2]\phi=0 \; . \tag{9}$$

The conjugate momentum is, as usual, the variation of $\mathcal{L}$ with respect to the time derivative of ϕ,

$$\pi(x)=\frac{\delta\mathcal{L}}{\delta(\partial_+\phi(x))}=\partial^+\phi=c\,\partial_+\phi+s\,\partial_-\phi \; . \tag{10}$$

For nonzero c the velocity $\partial_+\phi$ can be eliminated in favor of π by inverting Eq. (10),

$$\partial_+\phi=\frac{1}{c}(\pi-s\partial_-\phi) \; , \tag{11}$$

while in the light-cone limit, $c\to0$, Eq. (11) evolves into the usual light-cone constraint equation $\pi=\partial_-\phi$. Away from that limit the system is canonical, as opposed to constrained, with the usual equal-x^+ commutation relation

$$[\pi(x),\phi(y)]_{x^+=y^+}=-i\delta(x^--y^-) \; . \tag{12}$$

The Hamiltonian, which is conjugate to the time x^+, is

$$P_+=\int dx^-(\pi\partial_+\phi-\mathcal{L}) \; . \tag{13}$$

Plane-wave solutions of Eq. (9) are $\exp\{\pm i(p_+x^+ + p_-x^-)\}$, with the energy p_+ given by

$$p_+ = \frac{\omega_p - sp_-}{c} , \tag{14}$$

with

$$\omega_p \equiv (p_-^2 + cm^2)^{1/2} . \tag{15}$$

Expanded in these solutions,

$$\phi(x) = \int_{-\infty}^{\infty} \frac{dp_-}{(4\pi\omega_p)^{1/2}} [a(p_-)e^{-i(p_+x^+ + p_-x^-)} + a^\dagger(p_-)e^{i(p_+x^+ + p_-x^-)}] . \tag{16}$$

Imposing

$$[a(p_-), a^\dagger(q_-)] = \delta(p_- - q_-) \tag{17}$$

satisfies Eq. (12). The normalization in Eq. (17) is chosen to be independent of quantization angle. All dependence on c or s appears explicitly in coefficients in the fields, with no implicit dependence in a or $a^\dagger$. As a result, it is simple to trace the c dependence of states or fields. In particular, in the light-cone limit, $a, a^\dagger$ and basis Fock states are always of order 1.

The Hamiltonian in terms of these creation and annihilation operators is

$$P_+ = \int_{-\infty}^{\infty} dp_- \left[\frac{\omega_p - sp_-}{c} \right] a^\dagger(p_-)a(p_-) , \tag{18}$$

while the momentum

$$P_- = \int_{-\infty}^{\infty} dp_- [p_-] a^\dagger(p_-)a(p_-) . \tag{19}$$

Equation (14) is illustrated in Fig. 1. For positive-definite p_- near the light cone, the light-cone energy reduces to the conventional value

$$p_+ \sim \frac{m^2}{2p_-} + O(c) , \tag{20}$$

while negative-definite p_- particles have divergent energy,

$$p_+ \sim \frac{2|p_-|}{c} + O(1) , \tag{21}$$

in this limit. At $p_- = 0$, $p_+ = m/c^{1/2}$, while at $p_- = m/c^{1/2}$, p_+ has its minimum value of $c^{1/2}/m$. For the special case when $m = 0$, $p_+ = [(1+s)/c]|p_-|$ for $p_- < 0$ and $p_+ = [(1-s)/c]|p_-|$ for $p_- > 0$, with respective leading behaviors of $p_+ = (2/c)|p_-|$ and $p_+ = 0$ near the light cone.

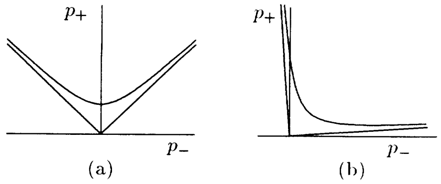

FIG. 1. Energy vs momentum at (a) equal time and (b) near the light cone for both massive and massless particles.

A simple example which nevertheless contains most of the interesting results that appear in spontaneous symmetry breaking is that of a scalar field coupled to a constant source [16]. Specifically, take Eq. (7) as the free Lagrangian $\mathcal{L}_0$. Imposing periodic boundary conditions so that $p_- = \pi n/L$ and expanding ϕ at $x^+ = 0$ yields

$$\phi(x^-) = \sum_{n=-\infty}^{\infty} \frac{1}{\sqrt{4\pi\omega_n}} [a_n e^{-i(n\pi/L)x^-} + a_n^\dagger e^{i(n\pi/L)x^-}] . \tag{22}$$

Here

$$\omega_n \equiv (n^2 + c\hat{m}^2)^{1/2} , \tag{23}$$

with $\hat{m}$ the dimensionless mass $\hat{m} \equiv mL/\pi$. Equation (12) implies

$$[a_n, a_l^\dagger] = \delta_{n,l} . \tag{24}$$

This $\mathcal{L}$ is symmetric under $\phi \to -\phi$. The mass term $m^2\phi^2/2$ may be thought of as a quadratic potential with minimum at $\phi = 0$, and a simple way to generate symmetry breaking is to shift the minimum to $-v$ by replacing $\phi(x)$ with $\phi(x) + v$ in Eq. (7). The effect is to add to $\mathcal{L}_0$ an interaction plus a constant piece,

$$\mathcal{L} = \mathcal{L}_0 - m^2 v\phi - \tfrac{1}{2}m^2v^2 . \tag{25}$$

The sane approach, given this Lagrangian, would be to first shift ϕ back to its minimum and then quantize, as in treatments of the Higgs mechanism. The intention here is to use it as a model for more complicated theories such as QCD, where the solution is not known, and so I will feign ignorance and quantize as is. However, this Lagrangian remains quadratic and soluble. If it is quantized first and the ground state solved, ϕ must find its way back to $-v$, and the vacuum must be complicated. Once obtained at arbitrary quantization angle, the object of this exercise is to see what happens near the light cone.

Proceeding as usual, the constant term is dropped, and $\phi(x)$ is quantized and expanded as above. The Hamiltonian P_+ has the usual free part,

$$P_+^0 = \left[\frac{\pi}{L}\right] \sum_n \left[\frac{\omega_n - sn}{c} \right] a_n^\dagger a_n , \tag{26}$$

plus an interaction

$$P_+^I = \int_{-L}^{L} dx^- m^2 v \phi(x^-) = \left[\frac{\pi}{L}\right] \left[\frac{\hat{m}^{3/2}\pi^{1/2}v}{c^{1/4}} \right] [a_0 + a_0^\dagger] . \tag{27}$$

The result of solving $P_+|\Omega\rangle = E_\Omega|\Omega\rangle$ is not surprising. The ground state

$$|\Omega\rangle = U|0\rangle = \exp\{-(c^{1/2}\pi\hat{m})^{1/2}v(a_0^\dagger - a_0)\}|0\rangle$$
$$= \exp\{-(c^{1/2}\pi\hat{m})v^2/2\}\exp\{-(c^{1/2}\pi\hat{m})^{1/2}va_0^\dagger\}|0\rangle \tag{28}$$

is a coherent state of zero-momentum particles, and its energy

$$E_\Omega = -\tfrac{1}{2}m^2v^2(2L) \tag{29}$$

is just the discarded constant piece of $\mathcal{L}$ times the volume of space. The unitary operator U is the exponential of $\int dx^- v\pi(x^-)$, with π the momentum conjugate to ϕ, and so shifts ϕ by the constant v [17].

Furthermore, the vacuum expectation value of $\phi(x)$ is also calculable. It involves only the constant part of $\phi(x)$, and

$$\langle\Omega|\phi(x)|\Omega\rangle = \langle 0|\exp\{(c^{1/2}\pi\hat{m})^{1/2}v(a_0^\dagger - a_0)\}\left[\frac{(a_0+a_0^\dagger)}{2(c^{1/2}\pi\hat{m})^{1/2}}\right]\exp\{-(c^{1/2}\pi\hat{m})^{1/2}v(a_0^\dagger - a_0)\}|0\rangle$$
$$= -v\ , \tag{30}$$

as it must.

Because these results are valid at all quantization angles θ, it is possible to examine them near the light cone, that is, as c vanishes. In particular, the full nonperturbative vacuum given by Eq. (28),

$$|\Omega\rangle \sim [1-(c^{1/2}\pi\hat{m})^{1/2}va_0^\dagger + \cdots\ ]|0\rangle\ , \tag{31}$$

reduces to the perturbative vacuum $|0\rangle$ in this limit. This is in accord with the conventional result that in light-cone quantization the perturbative vacuum is the full vacuum. Nevertheless, the ground-state energy $E_\Omega = -(1/2)m^2v^2(2L)$ and the vacuum expectation value $\langle\Omega|\phi|\Omega\rangle$ are independent of c and so persist in this limit. While $|\Omega\rangle$ becomes trivial as $c\to 0$, the constant, zero-momentum part of ϕ is singular and diverges as $c^{-1/4}$. This is just sufficient to extract the leading correction to $|\Omega\rangle$ such that $\langle\Omega|\phi|\Omega\rangle$ is constant. E_Ω may be thought of similarly. P_+ diverges as $m/c^{1/2}$ near $p_-=0$ and picks out the first relevant correction in $\langle\Omega|P_+|\Omega\rangle$. This singularity in c is the analogue of the $1/k_-$ singularity which plagues the conventional light-cone approach. In this case, however, this singularity is fully controllable so long as c is not identically zero, whereas it is unclear how $1/k_-$ should be handled near $k_-=0$.

One last observation from this simple model is that knowing only the first correction to $|\Omega\rangle$ in c is enough to determine $\langle\Omega|\phi|\Omega\rangle$ exactly as c vanishes. One might hope that in a real problem, such as QCD, this procedure might allow similar, especially Lorentz-invariant quantities to be calculated as an expansion in c.

IV. FREE FERMIONS

In this second simple example, based on free fermions, the full vacuum will be built of particles with all momenta rather than one of zero momentum as above. It will also provide an opportunity to establish some conventions and simple results for fermions in two dimensions, which will be useful in somewhat less trivial examples later.

The field $\psi=(\psi_L,\psi_R)$ in the free Lagrangian

$$\mathcal{L}_0 = \bar{\psi}(i\partial\!\!\!/ - m)\psi = \bar{\psi}(i\gamma^+\partial_+ + i\gamma^-\partial_- - m)\psi \tag{32}$$

has single components of left and right chirality. In two dimensions there is no spin, and chirality for massless fermions indicates only the direction of motion.

I will use the chiral representation, with

$$\gamma^0 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix},\quad \gamma^5 = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \tag{33}$$

and

$$\gamma^+ = \begin{bmatrix} 0 & (1+s)^{1/2} \\ (1-s)^{1/2} & 0 \end{bmatrix},\quad \gamma^- = \begin{bmatrix} 0 & -(1-s)^{1/2} \\ (1+s)^{1/2} & 0 \end{bmatrix}. \tag{34}$$

These satisfy

$$\{\gamma^\mu,\gamma^\nu\} = 2g^{\mu\nu}\ , \tag{35}$$

for all θ (or c). In the light-cone limit,

$$\gamma^+ \to \begin{bmatrix} 0 & \sqrt{2} \\ 0 & 0 \end{bmatrix},\quad \gamma^- \to \begin{bmatrix} 0 & 0 \\ \sqrt{2} & 0 \end{bmatrix}. \tag{36}$$

Quantization follows as usual, with

$$\pi(x) = \frac{\delta\mathcal{L}}{\delta(\partial_+\psi(x))} = i\bar{\psi}\gamma^+ \tag{37}$$

and

$$\{\pi(x),\psi(y)\}_{x^+=y^+} = i\delta(x^- - y^-)\ . \tag{38}$$

Imposing antiperiodic boundary conditions and expanding $\psi(x)$ gives

$$\psi(x^-,x^+=0) = \frac{1}{\sqrt{2L}}\sum_{n=\pm 1/2,\pm 3/2,\ \cdots}[u_n b_n e^{-i(n\pi/L)x^-} + v_n d_n^\dagger e^{i(n\pi/L)x^-}]\ . \tag{39}$$

The spinor

$$u_n = \frac{1}{(2\omega_n)^{1/2}}\begin{bmatrix} (1-s)^{-1/4}(\omega_n - n)^{1/2} \\ (1+s)^{-1/4}(\omega_n + n)^{1/2} \end{bmatrix}, \tag{40}$$

while v_n differs in the sign of the top entry. The upper component vanishes as $c^{1/2}$ for $n>0$ and diverges as $c^{-1/2}$ for $n<0$; this divergence prevents $n<0$ particles from decoupling in Q, for example. The anticommutation relations are

$$\{b_n, b_l^\dagger\} = \{d_n, d_l^\dagger\} = \delta_{n,l} \ , \tag{41}$$

as usual.

The Hamiltonian, after discarding an infinite constant, is similar to that for the free scalar theory,

$$P_+ = \left[\frac{\pi}{L}\right] \sum_{n=\pm 1/2, \pm 3/2, \cdots} \left[\frac{\omega_n - sn}{c}\right] (b_n^\dagger b_n + d_n^\dagger d_n) \ . \tag{42}$$

Near the light cone, eigenstates may be divided into those with positive $p_- = n\pi/L$ of energy $p_+ \sim m^2/2p_-$ and negative p_- with energy $p_+ \sim 2|p_-|/c$. As $c \to 0$, those with negative p_- possess infinite energy, and if states are restricted to those of finite energy, these may be excluded by fiat, leaving the usual light-cone theory. Particles with $p_- \to 0$ have energy $m/c^{1/2}$, and so the usual light-cone energy singularity at $k_- = 0$ is cut off for finite c.

To modify $\mathcal{L}_0$ in Eq. (32) such that the theory is still trivially soluble but possesses vacuum condensation, add a term

$$\mathcal{L}_I = -\mu \bar{\psi} \psi \ . \tag{43}$$

This is simply a mass shift, and so the new eigenstates are obviously known. Here, however, it will be treated as though it were an interesting interaction, and its vacuum and eigenstates will be solved for in terms of those of $\mathcal{L}_0$. (This example is something of a cheat when used as a comparison with conventional light-cone quantization, since in that case this shift in mass would automatically be incorporated in the solution of the constraint equation for ψ_L. It will have a more interesting application in the Gross-Neveu model discussed later.)

The interacting Hamiltonian is

$$P_+^I = \mu \int dx^- :\bar{\psi}\psi(x): \tag{44}$$

and the full Hamiltonian

$$P_+^0 + P_+^I = \left[\frac{\pi}{L}\right] \sum_{n=\pm 1/2, \pm 3/2, \cdots} \left\{ \left[\frac{\omega_n - sn}{c} + \frac{\hat{m}\hat{\mu}}{\omega_n}\right] (b_n^\dagger b_n + d_n^\dagger d_n) - \frac{\hat{\mu} n}{c^{1/2}\omega_n} (b_n^\dagger d_{-n}^\dagger + d_{-n} b_n) \right\} . \tag{45}$$

At this stage it is not possible to take a light-cone limit by letting c vanish and discarding states of infinite energy. The last term in P_+, which mixes particles of positive and negative momenta, becomes infinitely strong in this limit. That is, states with divergent energy couple to finite-energy states strongly and cannot be ignored. Before infinite-energy states can be discarded, then, P_+ must be diagonalized, at least up to an appropriate power in c.

Of course, because $\mathcal{L}$ is free, this is possible and P_+ may be diagonalized by a Bogoliubov transformation:

$$\begin{aligned} B_n &\equiv G_+(n) b_n - \epsilon(n) G_-(n) d_{-n}^\dagger \ , \\ D_n &\equiv G_+(n) d_n - \epsilon(n) G_-(n) b_{-n}^\dagger \ , \end{aligned} \tag{46}$$

where

$$G_\pm(n) \equiv \frac{1}{(2\omega_n \tilde{\omega}_n)^{1/2}} [\omega_n(\tilde{\omega}_n \pm \omega_n) \pm c\hat{m}\hat{\mu}]^{1/2} \tag{47}$$

and

$$\omega_n \equiv (n^2 + c\hat{m}^2)^{1/2}, \quad \tilde{\omega}_n \equiv [n^2 + c(\hat{m} + \hat{\mu})^2]^{1/2} \ , \tag{48}$$

with $\epsilon(n)$ the antisymmetric step function. As before, $\hat{m}$ and $\hat{\mu}$ are dimensionless. Near $c \to 0$, $G_+(n) \to 1$ and $G_-(n) \to c^{1/2}\hat{\mu}/2|n|$, as long as $c^{1/2} \ll |n|/\hat{m}$ or $c^{1/2} \ll |p_-|/m$ in the continuum.

In terms of B_n and D_n,

$$P_+ = \left[\frac{\pi}{L}\right] \sum_{n=\pm 1/2, \pm 3/2, \cdots} \left[\frac{\tilde{\omega}_n - sn}{c}\right] (B_n^\dagger B_n + D_n^\dagger D_n) + E_\Omega \ , \tag{49}$$

which is the free Hamiltonian for a fermion of mass $m+\mu$.

The new vacuum should satisfy

$$B_n |\Omega\rangle = D_n |\Omega\rangle = 0 \ , \tag{50}$$

and so may be constructed in terms of the particles of $\mathcal{L}_0$ by

$$|\Omega\rangle = \left\{ \prod_{n>0} B_n B_{-n} D_n D_{-n} \right\} |0\rangle \ , \tag{51}$$

using Eq. (46). For a particular momentum n,

$$\begin{aligned} B_n B_{-n} D_n D_{-n} |0\rangle &= [G_+^2(n) + \epsilon(n) G_+(n) G_-(n) (d_n^\dagger b_{-n}^\dagger - d_{-n}^\dagger b_n^\dagger) \\ &\quad - G_-^2(n) d_{-n}^\dagger d_n^\dagger b_{-n}^\dagger b_n^\dagger] |0\rangle \ . \end{aligned} \tag{52}$$

Near the light cone, with $c^{1/2} \ll |n|/\hat{m}$, this becomes

$$\left[1 + \frac{c^{1/2}\hat{\mu}}{2n} (d_n^\dagger b_{-n}^\dagger - d_{-n}^\dagger b_n^\dagger) - \frac{c\hat{\mu}^2}{4n^2} d_{-n}^\dagger d_n^\dagger b_{-n}^\dagger b_n^\dagger \right] |0\rangle \ . \tag{53}$$

The vacuum energy of $|\Omega\rangle$ relative to $|0\rangle$ is

$$E_\Omega = -\left[\frac{\pi}{L}\right] \sum_n \left[\frac{\tilde{\omega}_n - \omega_n}{c} - \frac{\hat{m}\hat{\mu}}{\omega_n}\right] , \tag{54}$$

which becomes

$$E_\Omega \sim -\left[\frac{\pi}{L}\right] \sum_n \frac{\hat{\mu}^2}{2|n|} = -\left[\frac{L}{\pi}\right] \sum_n \frac{\mu^2}{2|n|} \ , \tag{55}$$

near the light cone. The chiral condensate in this new vacuum is also calculable, with

$$\langle \Omega | :\bar{\psi}\psi: | \Omega \rangle = \frac{1}{2L} \sum_n \left[\frac{\hat{m}}{\omega_n} - \frac{\hat{m}+\hat{\mu}}{\tilde{\omega}_n} \right] \underset{c\to 0}{\sim} -\frac{1}{2L} \sum_n \frac{\hat{\mu}}{|n|} . \quad (56)$$

The normal ordering for $\bar{\psi}\psi$ is with respect to the original b_n and d_n.

It was initially observed in the discretized Hamiltonian of [18] that the only appearance of the box length L was as a factor in front and only as L^{-1} in front of P_-. Under a longitudinal boost with rapidity α, the light-cone Hamiltonian P_+ is simply rescaled by $e^{-\alpha}$, as is x^- and therefore L. On the light cone, boosts are kinematic symmetries and only rescale the arguments of operators such as $b(p_-)$ [19,4]; this is also true when discretized. As a result and with the dimensionless normalization of Eq. (41), L is the only part of P_+ which scales, and any appearance within P_+ apart from an overall coefficient must vanish in this limit, as was observed in [15] and is evident in Eq. (55).

By restricting the system to a box, Lorentz invariance is explicitly broken, and it is instructive to see how it is recovered in the continuum limit. As $L\to\infty$, Eq. (56) becomes

$$\frac{1}{\pi}\int_0^\infty dk \left[\frac{m}{(k^2+cm^2)^{1/2}} - \frac{m+\mu}{[k^2+c(m+\mu)^2]^{1/2}} \right] . \quad (57)$$

One way to obtain a manifestly Lorentz-invariant Pauli-Villars regulation of this integral is to subtract the same integral with m replaced by Λ and $\Lambda \gg m,\mu$. This yields the continuum result

$$\langle \Omega | :\bar{\psi}\psi: | \Omega \rangle = -\frac{1}{\pi}\left[m \ln\left[\frac{m}{m+\mu}\right] + \mu \ln\left[\frac{\Lambda}{m+\mu}\right] + \mu \right] , \quad (58)$$

to $O(m/\Lambda)$. In the special case where the fermions are originally massless ($m=0$) and a mass term is introduced,

$$\langle \Omega | :\bar{\psi}\psi: | \Omega \rangle \underset{m\to 0}{\longrightarrow} -\frac{\mu}{2\pi}\ln\left[\frac{\Lambda^2}{\mu^2}\right] , \quad (59)$$

to leading order in μ/Λ; this result will be useful later.

The parameter c has entirely dropped out of Eq. (58). This can be directly observed in Eq. (57), through a change of variables, as discussed below. Had a sharp cutoff in k at Λ been introduced, which is not Lorentz invariant, the result would have been

$$\langle \Omega | :\bar{\psi}\psi: | \Omega \rangle = -\frac{1}{\pi}\left[m \ln\left[\frac{m}{m+\mu}\right] + \mu \ln\left[\frac{2\Lambda}{c^{1/2}(m+\mu)}\right] + O\left[\frac{m}{\Lambda}\right]\right] . \quad (60)$$

The appearance of c in what should have been a scalar explicitly indicates the violation of Lorentz invariance by the regulator. The parameter c acts in a way analogous to the gauge parameter in generalized covariant gauges or the four-vector in axial gauges whose disappearance provides a check of gauge invariance.

Integrals of the form

$$I = \int_{-\infty}^{\infty} dp \left[\frac{1}{(p^2+cm^2)^{1/2}} - \frac{1}{(p^2+c\Lambda^2)^{1/2}} \right] \quad (61)$$

are typical of those which will appear repeatedly throughout these examples, especially in regard to vacuum properties. The first term is logarithmically divergent, with the second term a Pauli-Villars regulator. It is not necessary that $c\Lambda^2$ be large, but only that Λ^2 be much larger than physical scales represented by m^2.

In the light-cone limit, $c\to 0$, and the integrand vanishes for any fixed value of p. However, the change of variables $p\to c^{1/2}p$ completely removes c, and the integral is c independent. It is clear that as $c\to 0$, the integral gets its contribution from an increasingly narrow region around $|p|\sim 0$ [20]; specifically, from $|p|\sim c^{1/2}m$ to $c^{1/2}\Lambda$. The support is pushed to the region with the lowest kinetic energy that can still participate in the vacuum. As c is made to vanish, the integrand may be replaced by $I\delta(p)$. In usual light-cone language, the integrand at finite p contributes the usual light-cone result for this vacuum quantity, that is, zero, while $I\delta(p)$ gives the unique correct interpretation of the light-cone singularity at $1/p_-$.

The end result of this exercise is that, except for a small region around $p\sim[cm(m+\mu)]^{1/2}$ where $|G_\pm(p)|$ are maximum, the mixing in Eq. (46) vanishes and the full vacuum reduces to the perturbative one as c vanishes. Nevertheless, $:\bar{\psi}\psi:$ possesses a $c^{-1/2}$ singularity, as is evident from P_+^I in Eq. (45), which preserves a finite value for its expectation value. In the continuum this value was independent of c, as it must be.

As a final aside, note that $\langle \Omega | :\bar{\psi}\psi: | \Omega \rangle$ could have been computed directly from the vacuum energy,

$$\langle \Omega | :\bar{\psi}\psi: | \Omega \rangle = \frac{1}{2L}\partial_\mu E_\Omega , \quad (62)$$

which follows from the form of the interaction $\mu\bar{\psi}\psi$ in P_+ and the Feynman-Hellman theorem. Curiously, in the continuum limit, this energy

$$E_\Omega = -\left[\frac{L}{\pi}\right]\int_{-\infty}^{\infty} dp \left[\frac{\tilde{\omega}_p - \omega_p}{c} - \frac{m\mu}{\omega_p} \right] - (m\to\Lambda) \quad (63)$$

is also independent of c, as is evident by the previous change of variables. It is not clear why this should be the case; however, since $P_-=0$, this leads to a c-dependent mass $M_\Omega^2 = cE_\Omega^2$ for $|\Omega\rangle$ and indicates that this constant should be subtracted from P_+ to maintain covariance.

V. FINITE BOXES AND ORDERING OF LIMITS

There are some subtleties in the ordering of the continuum, light-cone, and massless limits when the system is discretized, as is evident by the appearance of combinations such as $c^{1/2}mL/\pi$ [15]. These limits are clarified by examining how particles intercept the initial surface $x^+=0$ as the limit $c\to 0$ is taken.

One way of stating that an initial surface is sufficient to specify a system is that all independent degrees of freedom intercept it at some point. In particular, the surface $x^0+x^1=0$ is insufficient for massless particles because those which are left moving run parallel to it. However, these particles will intercept the spacelike surface $x^+=0$ when $c\neq 0$ regardless of how small c becomes, provided the system's extent is infinite. Massive particles of finite energy, on the other hand, intercept the surface for any c, including zero.

However, when the system is confined to a box, even for nonzero c, both massless and very energetic massive particles can escape or enter through the sides of the box without hitting the initial surface. This becomes a problem particularly as c becomes small. It is possible to measure the adequacy of a box length by estimating when boundary effects could become important for a system localized around the origin. For example, information from the boundary (traveling at the speed of light) can first reach the point $x^-=0$ at the time

$$x^+_{\text{bound}}=\left[\frac{1+c-s}{1+c+s}\right]L\ . \tag{64}$$

For equal-time quantization, $x^+_{\text{bound}}\sim L$, while near the light cone, $x^+_{\text{bound}}\sim cL/2$. Thus a physical system contained in a region much smaller than L in an equal-time box can propagate a long time (of order L) before knowing about its edges, whereas near the light cone, boundary effects are felt at the much smaller time $cL/2$. L must be large enough that cL is much greater than times of physical interest.

If only massive excitations exist with m the lowest mass, the restriction is less severe. Suppose that a typical energy scale E_{phys} can be assigned a system and an energy cutoff Λ is imposed on particles, with $\Lambda\gg E_{\text{phys}}$. This imposes a maximum velocity, and the smallest time x^+_{bound} for which information at the boundary can reach the region near $x^-=0$ when $c\sim 0$ is

$$x^+_{\text{bound}}\sim\tfrac{1}{2}[c+\tfrac{1}{2}(m/\Lambda)^2]L\ . \tag{65}$$

The requirement now is only that either $(cL)^{-1}$ or $(m^2L/\Lambda^2)^{-1}$ be much less than E_{phys}.

Another way to see the interplay of these quantities near the light cone is to examine the dispersion relation illustrated in Fig. 1(b). If periodic boundary conditions are imposed in x^-, the lowest-energy, negative-p_- massless state has

$$p_-=-\frac{n\pi}{L},\ \ p_+=\left[\frac{1+s}{c}\right]\frac{n\pi}{L}\ , \tag{66}$$

with $n=1$. For L held fixed, as c vanishes, this energy diverges as $(cL)^{-1}$. The temptation is then to discard all negative-p_- particles, since p_+ of the least energetic one diverges. However, there still exist in the continuum low-energy, negative-p_- states with energies arbitrarily close to zero which certainly play a role in finite-energy systems; the grid is simply too coarse to intercept them. This will be evident in the Schwinger model, where states in this region build up to the degenerate vacua. To ensure that these are not neglected as $c\sim 0$ requires that

$$\frac{\pi}{cL}\ll E_{\text{phys}}\ . \tag{67}$$

Again, the restriction is less severe when $m\neq 0$ and the lowest energy for $p_-\le 0$ is $p_+=m/c^{1/2}$. Regardless of the size of L, negative-p_- states can probably be neglected so long as

$$m/c^{1/2}\gg E_{\text{phys}}\ . \tag{68}$$

In this case the grid is still coarse for p_- positive but small. That is, the difference in p_+ when $p_-=n\pi/L$ versus $(n+1)\pi/L$ can be much larger than E_{phys} when n is small. This should not be a problem so long as these states are energetic enough to neglect. Requiring

$$p_+(n\pi/L)\gg E_{\text{phys}}\ , \tag{69}$$

when the grid becomes coarse, such that

$$p_+(n\pi/L)-p_+[(n+1)\pi/L]\gtrsim E_{\text{phys}}\ , \tag{70}$$

imposes

$$m^2L\gg 2\pi E_{\text{phys}}\ . \tag{71}$$

So, when massless excitations exist, it is necessary to retain some states for which $p_-<0$ and to require $(cL)^{-1}\ll E_{\text{phys}}$. When only massive excitations occur, retaining only positive-p_- particles and a finite box should produce accurate results if Eqs. (68) and (71) are respected.

Evidently, it becomes difficult to obtain accuracy when discretizing this formalism near the light cone if light particles are present or when studying processes sensitive to high energies, such as the axial anomaly. An alternative and complementary formulation exists which also addresses the deficiencies of conventional light-cone quantization while avoiding some of these difficulties. In it the initial surface includes both the conventional one $x^0+x^1=0$ as well as the boundaries along $x^0-x^1=\pm L$ [8]. The boundaries are essential to initialize massless left-moving particles, which would otherwise be lost. In fact, it is straightforward to show from the equations of motion that for a finite box the boundaries are necessary even for massive particles [21].

Such a procedure has several advantages. It avoids the necessity of taking a limit, and it treats left and right movers symmetrically rather than relying on cancellations of singularities in c (or at $k_-\sim 0$) to retain small-mass or very energetic left movers. As a result, it leads to a more uniform discretization of these left-moving states. Because the boundaries directly intercept massless or energetic left movers, quantities sensitive to large energies or short times in x^+ are naturally incorporated even for finite boxes. For example, the small-x singularity in $\psi_L^\dagger(x)\psi_L(0)\sim 1/x^+$ which picks up half the axial anomaly in QED_{1+1} is present directly in the boundary degrees of freedom.

VI. EFFECTIVE LIGHT-CONE HAMILTONIAN

Given the subtlety of the vacuum in the light-cone limit and the importance of negative-momentum, high-energy quanta in its construction, an obvious question is whether these effects can be incorporated into the conventional light-cone approach. Fortunately, a formalism exists which is ideally suited to this problem and in which these effects can be systematically computed [22].

The conventional light-cone Hamiltonian includes interactions among quanta whose momenta are positive definite. The Hamiltonians discussed thus far include particles of all momenta. In the light-cone limit, however, the energies of particles with negative momentum typically become very large, and one might imagine that these are irrelevant to states of low energy in which positive-momentum quanta should predominate. An exception occurs when these energetic particles have couplings to positive-momentum quanta which also become large near the light cone. In this case an effective Hamiltonian can be defined which acts only on positive-momentum states, but with extra interactions added to account for the effect of excluded states. Because these excluded states are very energetic, their interactions with low-energy states occur over small (light-cone) times and these additional interactions will be local in x^+.

To see how these additional interactions can be computed, consider an operator $\mathcal{P}$, which projects out the subspace of states which contain only positive-momentum particles, while $\mathcal{Q}=1-\mathcal{P}$ projects out states in which at least one particle has negative or zero momentum. Then, if in the full state space

$$H|\psi\rangle=E|\psi\rangle \ , \tag{72}$$

the effective Hamiltonian

$$H_{\rm eff}(E)=H_{\mathcal{PP}}+H_{\mathcal{PQ}}\frac{1}{E-H_{\mathcal{QQ}}}H_{\mathcal{QP}} \tag{73}$$

acts within the projected subspace and so may be thought of as a conventional light-cone Hamiltonian. It satisfies

$$H_{\rm eff}(E)\mathcal{P}|\psi\rangle=E\mathcal{P}|\psi\rangle \ . \tag{74}$$

Here

$$H_{\mathcal{PQ}}\equiv\mathcal{P}H\mathcal{Q} \ , \tag{75}$$

and so on.

If the states removed by $\mathcal{P}$ are at much higher energy than the energy E for states of interest, the denominator in Eq. (73) may be expanded in $E/H_{\mathcal{QQ}}$. As a result, $H_{\rm eff}(E)$ is given as a series of effective interactions of decreasing importance which are polynomial in E:

$$H_{\rm eff}(E)\sim H_{\mathcal{PP}}-H_{\mathcal{PQ}}\left[\frac{1}{H_{\mathcal{QQ}}}+\frac{E}{H^2_{\mathcal{QQ}}}+\cdots\right]H_{\mathcal{QP}} \ . \tag{76}$$

These powers of E correspond to derivatives in time and so represent local interactions. This expansion works so long as $E/H_{\mathcal{QQ}}$ is small. However, in cases where $H_{\mathcal{PQ}}$ is large near the light-cone, $H_{\rm eff}$ may acquire terms in addition to the conventional light-cone Hamiltonian $H_{\mathcal{PP}}$.

A. Discrete free fermions

Consider the example of Sec. IV when c becomes small, first in the discretized version. The dominant terms are

$$\begin{aligned}
H_{\mathcal{PP}}&=\left[\frac{\pi}{L}\right]\sum_{n>0}\left[\frac{\hat{m}^2}{2n}+\frac{\hat{m}\hat{\mu}}{n}\right](b_n^\dagger b_n+d_n^\dagger d_n) \ ,\\
H_{\mathcal{PQ}}&\sim\mathcal{P}\left[\frac{\pi}{L}\right]\sum_{n}\left[\frac{-\hat{\mu}\epsilon(n)}{c^{1/2}}\right](b_n^\dagger d_{-n}^\dagger+d_{-n}b_n)\mathcal{Q} \ ,\\
H_{\mathcal{QQ}}&\sim\left[\frac{\pi}{L}\right]\sum_{n<0}\left[\frac{2|n|}{c}\right](b_n^\dagger b_n+d_n^\dagger d_n) \ .
\end{aligned} \tag{77}$$

Because the interactions $H_{\mathcal{PQ}}$ which couple states in $\mathcal{Q}$ to $\mathcal{P}$ are of order $c^{-1/2}$, while those of $H_{\mathcal{QQ}}$ are order c^{-1},

$$H_{\rm eff}\sim H_{\mathcal{PP}}-H_{\mathcal{PQ}}\frac{1}{H_{\mathcal{QQ}}}H_{\mathcal{QP}}+O(c) \ . \tag{78}$$

The correction to $H_{\mathcal{PP}}$ in the subspace $\mathcal{P}$ is simple to evaluate. Each vertex in $H_{\mathcal{PQ}}$ contributes a factor $-\hat{\mu}\epsilon(n)\pi/c^{1/2}L$ with vacuum pair production or annihilation, while the denominator from $H_{\mathcal{QQ}}$ contributes, to leading order, an energy $2\pi|n|/cL$.

Acting on the perturbative vacuum, which is included in $\mathcal{P}$, with $H_{\rm eff}$ gives zero for $H_{\mathcal{PP}}$, while the correction, illustrated in Fig. 2(a), is

$$\begin{aligned}
H_{\rm eff}|0\rangle&\sim -H_{\mathcal{PQ}}\frac{1}{H_{\mathcal{QQ}}}H_{\mathcal{QP}}|0\rangle\\
&=\left[-\left[\frac{\pi}{L}\right]\frac{\hat{\mu}^2}{2}\sum_n\frac{1}{|n|}\right]|0\rangle=E_\Omega|0\rangle \ ,
\end{aligned} \tag{79}$$

as c vanishes. So $|0\rangle$ is an eigenstate of $H_{\rm eff}$, but with the same vacuum energy computed by diagonalizing H in the full space. Likewise, $H_{\rm eff}$ on the one-particle state of momentum $k>0$ is illustrated in Fig. 2(b). $H_{\mathcal{PP}}$ assigns an energy $(\pi/L)[\hat{m}^2/2k+\hat{m}\hat{\mu}/k]$, while the correction reproduces both the vacuum energy and a term $(\pi/L)[\hat{\mu}^2/2k]$. As a result,

$$H_{\rm eff}b_k^\dagger|0\rangle=\left[E_\Omega+\frac{\pi}{L}\frac{(\hat{m}+\hat{\mu})^2}{2k}\right]b_k^\dagger|0\rangle \ . \tag{80}$$

So $b_k^\dagger|0\rangle$ is an eigenstate of $H_{\rm eff}$, but with both the vacuum energy and mass shift $m\to m+\mu$ correctly accounted for.

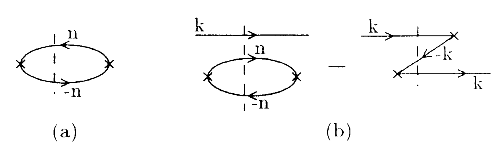

FIG. 2. Diagrams contributing to $H_{\rm eff}$ on (a) the vacuum and (b) the one-particle state.

This is a typical renormalization-group result. High-energy states are removed through a cutoff, but their effect, which can be large, can be entirely incorporated into a series of extra terms in the Hamiltonian. In particular, the energy of the fermion external to the vacuum bubble in the first diagram of Fig. 2(b) is negligible relative to that in the loop, and so it disappears from the energy denominator. The large energy associated with this vacuum process makes it essentially instantaneous, unaffected by the propagation of the external fermion. As a result, the vacuum energy due to this fluctuation is disentangled from the fermion, and its effect is to contribute a particularly simple effective interaction, the universal constant E_Ω, to H_{eff}.

An alternative to computing these extra terms would be to enumerate the possible interactions consistent with the symmetries of the QCD light-cone Hamiltonian and match them to experiment [23]. Examples of such an approach are [24,25], which graft phenomenological results from QCD sum rules to a light-cone formalism.

The effective Hamiltonian may be useful for more than simply incorporating negative-momentum particles to define a conventional light-cone Hamiltonian. In many cases it would be calculationally advantageous to include the effect of states into H_{eff} with positive but small, as well as negative, p_-. For example, in $(1+1)$-dimensional QED and QCD, the Hamiltonian at small p_- is singular. Wave functions are badly behaved in this region, especially when the electron or quark mass m is small. While this singularity leads to interesting analytic results, it makes numerical calculations difficult [21,26–28]. However, fermions at small p_- have large kinetic energies $m^2/2p_-$ and should probably be removed from the problem and incorporated into H_{eff}. Analytic results which depend on the singularity of this region would then reappear in the form of corrections to a better behaved H_{eff} with this region excluded.

One such analytic result is that in $(1+1)$-dimensional QCD at large N_c, the meson mass squared vanishes as a single power of the quark mass m [29]. Presumably, this would be accounted for in H_{eff} by a new, less singular term proportional to $|m|$. It is perhaps also worth mentioning in the context of this model that its chiral condensate [3] and even current conservation [30] can be computed only indirectly, by examining operators within matrix elements of low-energy eigenstates. That such a procedure is frequently necessary further indicates that the conventional light-cone P_+ is best treated as an effective Hamiltonian.

B. Continuum free fermions

To obtain the correct continuum result for E_Ω in this example requires a bit more care in counting powers of c

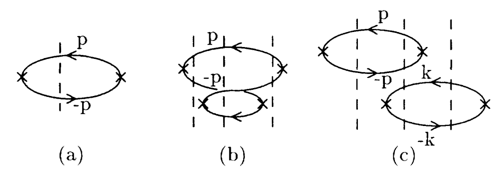

FIG. 3. Diagrams contributing to H_{eff} on the vacuum in the continuum.

to ensure that important contributions are not discarded. As c vanishes, the dominant term in H_{QQ} is diagonal and of order $|p_-|/c$ for particles of fixed and negative p_-. However, particles in the region $|p_-|\lesssim c^{1/2}m$, which are present in the continuum limit, have a diagonal term for p_+ which only diverges as $c^{-1/2}$; in fact, $p_+=m/c^{1/2}$ when $p_-=0$. As a result, the least divergent piece of H_{QQ} is of order $c^{-1/2}$. To be consistent the nondiagonal term in H_{QQ}, which is also of this order, must be retained.

In the expansion of $(E_\Omega-H_{QQ})^{-1}$ in

$$H_{\mathcal{P}Q}\frac{1}{E_\Omega-H_{QQ}}H_{Q\mathcal{P}}|\Omega\rangle_{\mathcal{P}}=E_\Omega|\Omega\rangle_{\mathcal{P}}\ , \tag{81}$$

it is still only necessary to keep the leading term in E_Ω/H_{QQ}. Although $H_{\mathcal{P}Q}\sim c^{-1/2}$ and $H_{QQ}\gtrsim c^{-1/2}$, H_{QQ} is of order $c^{-1/2}$ only in the region $|p_-|\lesssim c^{1/2}m$, introducing another factor of $c^{1/2}$ when summing over states. Consequently, $-H_{\mathcal{P}Q}H_{QQ}^{-1}H_{Q\mathcal{P}}$ is still of order 1.

The power counting in this discussion depends on the mass m not vanishing, so that particles with $p_-\le 0$ have p_+ of at least $m/c^{1/2}$, making their exclusion sensible. For $m=0$ the p_- chosen as a cutoff would have to be at least slightly negative so as to include low-energy states with p_- small but negative. Because negative-p_- particles would now also appear in the projected ($\mathcal{P}$) space, it is possible that the perturbative vacuum would not be the ground state even for the effective Hamiltonian. In this discussion m will be kept nonzero.

To compute E_Ω in the continuum limit, consider Eq. (81) with only the leading term in E_Ω/H_{QQ} retained on the left. Because $\mathcal{P}$ excludes negative-p_- states, $|\Omega\rangle_{\mathcal{P}}\propto|0\rangle$. Then

$$-\langle 0|H_{\mathcal{P}Q}\frac{1}{H_{QQ}}H_{Q\mathcal{P}}|0\rangle=E_\Omega\ . \tag{82}$$

As discussed, the nondiagonal terms in H_{QQ} must be retained, and H_{QQ}^{-1} may be treated as a series in the interaction

$$H_{QQ}^I\sim Q\int_{-\infty}^{\infty}dp_-\left[\frac{-\mu p_-}{c^{1/2}\omega_p}\right][d(-p_-)b(p_-)+b^\dagger(p_-)d^\dagger(-p_-)]Q\ , \tag{83}$$

over the free part H_{QQ}^0, which contributes a denominator $[(\omega_p-sp_-)/c+m\mu/\omega_p]$ for each particle.

The first term, represented by Fig. 3(a), is

$$-\langle 0|H_{\mathcal{P}Q}\frac{1}{H^0_{QQ}}H_{Q\mathcal{P}}|0\rangle=\left[\frac{L}{\pi}\right]\left[-\frac{1}{2}\right]\int_{-\infty}^{\infty}dp_-\left[\frac{\mu p_-}{c^{1/2}\omega_p}\right]^2\left[\frac{\omega_p}{c}+\frac{m\mu}{\omega_p}\right]^{-1}-(m\to\Lambda)\ , \tag{84}$$

which is the continuum limit of the discretized term discussed in the previous subsection. That this diagram is of order 1 can be seen by the change of variables $p_-\to c^{1/2}p_-$, such that c disappears. The next term which also contributes to order 1 in c,

$$-\langle 0|H_{\mathcal{P}Q}\frac{(H^I_{QQ})^2}{(H^0_{QQ})^3}H_{Q\mathcal{P}}|0\rangle=\left[\frac{L}{\pi}\right]\left[\frac{1}{8}\right]\int_{-\infty}^{\infty}dp_-\left[\frac{\mu p_-}{c^{1/2}\omega_p}\right]^4\left[\frac{\omega_p}{c}+\frac{m\mu}{\omega_p}\right]^{-3}-(m\to\Lambda) \tag{85}$$

is shown in Fig. 3(b). As before, c may be completely scaled out. To this order in H^I_{QQ}/H^0_{QQ}, terms such as Fig. 3(c) also appear, but these vanish as $c^{1/2}$. In general, the order for any diagram in this example is determined by assigning $c^{1/2}$ for each loop, $c^{-1/2}$ for each vertex, and $c^{1/2}$ for each denominator. Then, for example, diagrams with more than the minimum possible number of loops vanish. A large number of diagrams disappears as a result. The power counting is completely analogous to infinite-momentum-frame calculations, as in [2,31–33], apart from the need to choose a particular frame. In [32,33] an effective interaction in QED is introduced to incorporate backward fermions in Z graphs, much as in this example.

The contributions from the terms which survive as $c\to 0$ are relatively easy to sum, with

$$-\left[\frac{L}{\pi}\right]\int_{-\infty}^{\infty}dp_-\left\{\left[\frac{1}{2}\right]\left[\frac{\mu p_-}{c^{1/2}\omega_p}\right]^2\left[\frac{\omega_p}{c}+\frac{m\mu}{\omega_p}\right]^{-1}-\left[\frac{1}{8}\right]\left[\frac{\mu p_-}{c^{1/2}\omega_p}\right]^4\left[\frac{\omega_p}{c}+\frac{m\mu}{\omega_p}\right]^{-3}+\cdots\right\}$$
$$=-\left[\frac{L}{\pi}\right]\int_{-\infty}^{\infty}dp_-\left\{\frac{\tilde{\omega}_p}{c}-\left[\frac{\omega_p}{c}+\frac{m\mu}{\omega_p}\right]\right\}, \tag{86}$$

and reproduce the continuum expression for E_Ω obtained directly from the full solution for $|\Omega\rangle$. The continuum expression for $\langle\Omega|:\bar{\psi}\psi:|\Omega\rangle$ may also be obtained directly from Eq. (86) by computing $\partial_\mu E_\Omega$.

One special case should be mentioned. If $\mu=-m$, the massive free fermions are converted to massless fermions. It would not be sensible to incorporate all $p_-<0$ fermions into H_{eff}, even though the free Hamiltonian with massive fermions suggests otherwise. This is reflected in the denominator, which becomes

$$\frac{\omega_p}{c}-\frac{m^2}{\omega_p}=\frac{p_-^2}{c\omega_p}\ . \tag{87}$$

This may become arbitrarily small as p_- vanishes, and the suppression at small c is lost.

The lesson from this subsection is that to accurately reproduce the continuum value for the vacuum energy requires power counting in c that is slightly more subtle than straightforward light-cone arguments might suggest, because of the importance of the region $|p_-|\sim c^{1/2}m$. Nevertheless, the light-cone limit eliminates a large number of diagrams and makes the calculation of H_{eff} on $|\Omega\rangle_{\mathcal{P}}$ much simpler, if not trivial. Furthermore, once the vacuum energy is removed, computing H_{eff} on excited states such as $b^\dagger(k_-)|\Omega\rangle_{\mathcal{P}}$, with k_- finite, follows the earlier discrete discussion and is essentially trivial. In particular, while in the continuum this full set of vacuum diagrams will appear in Fig. 2(a), no continuum vacuum diagrams become important between the interactions in Fig. 2(b). In that case the denominator would remain of order c^{-1} rather than $c^{-1/2}$ because of the presence of an external particle of fixed momentum $-k_-$, and such diagrams would vanish as $c^{1/2}$ or faster.

VII. LIGHT-CONE CONSTRAINT EQUATIONS

The equation of motion for a two-dimensional free fermion of mass m,

$$(i\partial\!\!\!/-m)\psi=0\ , \tag{88}$$

on the light cone becomes

$$i\partial_+\psi_R=\frac{m}{\sqrt{2}}\psi_L\ , \tag{89}$$

$$i\partial_-\psi_L=\frac{m}{\sqrt{2}}\psi_R\ . \tag{90}$$

In particular, Eq. (90) is a constraint equation, as it includes only a spatial derivative. In conventional light-cone quantization, one solves for the dependent field ψ_L in terms of ψ_R and quantizes only the independent field ψ_R. For nonzero c the fields do not separate in this manner; both are independent and so are quantized. It might be useful to see how the constraint Eq. (90) emerges in this picture as $c\to 0$.

Consider the previous example with free fermions but with zero initial mass and a mass shift of $\mu=m$. Then the two sides of Eq. (90) as $c\to 0$ (and at $x^+=0$) are

$$i\partial_-\psi_L\sim\left[\frac{m}{2^{5/4}L^{1/2}}\right]\sum_{n>0}\left[\left[B_n-\frac{2n}{c^{1/2}\hat{m}}D^\dagger_{-n}\right]e^{-i(n\pi/L)x^-}+\left[D^\dagger_n-\frac{2n}{c^{1/2}\hat{m}}B_{-n}\right]e^{i(n\pi/L)x^-}\right]+O(c^{1/2}) \tag{91}$$

and

$$\frac{m}{\sqrt{2}}\psi_R \sim \left[\frac{m}{2^{5/4}L^{1/2}}\right]\sum_{n>0}[B_n e^{-i(n\pi/L)x^-}+D_n^\dagger e^{i(n\pi/L)x^-}]+O(c^{1/2}) . \tag{92}$$

The only difference arises from operators which involve particles of negative momenta and therefore divergent energies. If these can be excluded by hand as physically unimportant, then the constraint equation is satisfied. Consequently, constraints which appear in the equations of motion in conventional light-cone quantization appear in this approach as dynamical results. That is, the constraint equations are satisfied on the subspace of states with finite energy as c vanishes.

VIII. RELATIONS FROM LORENTZ INVARIANCE

A possible benefit of this interpolating quantization is that it preserves something more of Lorentz invariance than a usual Hamiltonian approach. Quantities such as masses which are known to be Lorentz-invariant must be independent of the quantization surface. However, the parameter c which identifies this surface appears explicitly in the Hamiltonian P_+. Consequently, relations can be derived which express this invariance and provide nontrivial constraints on energies and wavefunctions.

For example, the mass M_ψ of an eigenstate $|\psi\rangle$ must be independent of c. The mass-squared operator is

$$M^2=P^\mu P_\mu=c(P_+^2-P_-^2)+2sP_+P_- . \tag{93}$$

While the operator P_+ depends explicitly on c, P_- is just the spatial momentum and does not. Requiring $\partial_c M_\psi^2=0$ in a state $|\psi\rangle$ of momentum p_- and energy p_+ and using the Feynman-Hellman theorem leads to

$$\begin{aligned}\langle\psi|\partial_c P_+|\psi\rangle&=\partial_c\langle\psi|P_+|\psi\rangle\\&=\partial_c E_\Omega-\left[\frac{p_+^2-p_-^2-(2c/s)p_-p_+}{2(cp_++sp_-)}\right] .\end{aligned} \tag{94}$$

The first term on the far right allows for a variation in the vacuum energy.

Further relations may be generated with higher derivatives. As a result, expectation values of various derivatives of the operator P_+ with respect to c can be related to the energies of eigenstates. These relations hold for arbitrary values of c, as well as $|\psi\rangle$ and p_-, and so may be examined in both the equal-time and light-cone limits. As may be checked, the ground states in the previous examples satisfy this.

It is not clear yet how useful these relations might prove. Perhaps a judicious choice of combinations of derivatives with respect to c can be used to isolate important parts of P_+ or to focus on particular regions of wave functions. In the massless Schwinger model, for example, expectation values of the free Hamiltonian alone can be related to the eigenvalues p_+ and p_- of the full theory, as will be shown. In general P_+ tends to have a simple dependence on c in the light-cone limit and these relations may be most useful there.

Also, with regard to Lorentz invariance, it should be noted that for the quantization surface away from $t=0$, there will be c-dependent Poincaré generators which leave the surface invariant but affect the momentum of states. As a result, the ease of boosting states on the light cone [19] should be preserved for arbitrary values of c.

IX. GROSS-NEVEU MODEL

A less trivial two-dimensional model than the free theories considered thus far is that of Gross and Neveu [34,35]. One version is given by the Lagrangian

$$\mathcal{L}=\sum_j^{N_f}\left\{\bar\psi^j(i\partial\!\!\!/-m)\psi_j+\frac{g^2}{2}(\bar\psi^j\psi_j)^2\right\} , \tag{95}$$

with the index j identifying flavor. When $m=0$ it possesses the discrete symmetry $\psi\to\gamma^5\psi$, so that $\bar\psi\psi\to-\bar\psi\psi$, which would seem to preclude mass generation. (Other versions of this model possess a continuous chiral symmetry. This particular version is sufficient for what follows.) As discussed in [34], this model is soluble in the large-N_f limit and shares several features with (3+1)-dimensional QCD. It is asymptotically free and, at least at large N_f, displays chiral-symmetry breaking: $\langle\Omega|{:}\bar\psi\psi{:}|\Omega\rangle$ is not zero, and a fermion mass is generated spontaneously even when m vanishes. Following [34], m initially will be kept finite to induce breaking in a particular direction, then taken to zero.

This model will serve to illustrate two useful features of this quantization procedure. The first is that it allows models to be quantized near the light cone which would be difficult to do in the conventional light-cone approach. The second is that results associated with a nontrivial vacuum can be reproduced.

In conventional light-cone quantization, dependent fields are identified in the equations of motion. These are

$$i\partial_+\psi_{Rj}=\frac{m}{\sqrt{2}}\psi_{Lj}+\frac{g^2}{2\sqrt{2}}[\psi_{Lj}(\psi_L^\dagger\psi_R+\psi_R^\dagger\psi_L)+(\psi_L^\dagger\psi_R+\psi_R^\dagger\psi_L)\psi_{Lj}] \tag{96}$$

and

$$i\partial_-\psi_{Lj}=\frac{m}{\sqrt{2}}\psi_{Rj}+\frac{g^2}{2\sqrt{2}}[\psi_{Rj}(\psi_L^\dagger\psi_R+\psi_R^\dagger\psi_L)+(\psi_L^\dagger\psi_R+\psi_R^\dagger\psi_L)\psi_{Rj}] . \tag{97}$$

The second equation relates ψ_L to ψ_R by a spatial derivative and so is a constraint. As the only independent field, ψ_R should be quantized and ψ_L solved in terms of ψ_R. Unfortunately, that Eq. (97) is nonlinear makes this difficult. This could be made easier by introducing an auxiliary scalar field σ coupled to $\bar{\psi}\psi$, as in [34], and then perhaps giving σ dynamics but a large mass. Also, [36] is able to treat a similar model on the light cone in a mean-field approximation and reproduce spontaneous symmetry breaking. That will be unnecessary in this discussion.

In contrast, quantization at arbitrary c is canonical and straightforward. The fermion fields satisfy the usual commutation relations given in Eq. (38). The Hamiltonian has the usual free part

$$P_+^0=\int_{-\infty}^{\infty}dp_-\left[\frac{\omega_p-sp_-}{c}\right][b^\dagger(p_-)b(p_-)+d^\dagger(p_-)d(p_-)] \tag{98}$$

and an interaction

$$\begin{aligned}P_+^I=\left[\frac{-g^2}{4\pi}\right]\int_{-\infty}^{\infty}&dp_-^a dp_-^b dp_-^c dp_-^d\,\delta\left[\sum p_-\right]\\ &\times\{[\bar{u}(p_-^a)u(p_-^b)\bar{u}(p_-^c)u(p_-^d)]b^{\dagger i}(p_-^a)b_i(p_-^b)b^{\dagger j}(p_-^c)b_j(p_-^d)\\ &\quad+[\bar{u}(p_-^a)u(p_-^b)\bar{u}(p_-^c)v(p_-^d)]b^{\dagger i}(p_-^a)b_i(p_-^b)b^{\dagger j}(p_-^c)d_j^\dagger(p_-^d)+\cdots\}\,,\end{aligned} \tag{99}$$

which includes all flavor-conserving four-fermion vertices.

As usual, the spinors u and $\bar{v}$ of Eq. (40) appear for incoming particles and antiparticles and $\bar{u}$ and v for those outgoing. These satisfy

$$\begin{aligned}\bar{u}(p)u(q)&=-\bar{v}(p)v(q)=W_{pq}^{+}+W_{pq}^{-}\,,\\ \bar{u}(p)v(q)&=-\bar{v}(p)u(q)=-W_{pq}^{+}+W_{pq}^{-}\,,\end{aligned} \tag{100}$$

with

$$W_{pq}^{\pm}\equiv\frac{(\omega_p\pm p)^{1/2}(\omega_q\mp q)^{1/2}}{2(c\omega_p\omega_q)^{1/2}}\,. \tag{101}$$

In the light-cone limit, if both p and q are positive or both negative, with $|p|,|q|\gg c^{1/2}m$,

$$W_{pq}^{+}+W_{pq}^{-}\simeq\frac{m}{2}\left[\frac{1}{|p|}+\frac{1}{|q|}\right],\quad W_{pq}^{+}-W_{pq}^{-}\simeq-\frac{m}{2}\left[\frac{1}{p}-\frac{1}{q}\right], \tag{102}$$

but

$$W_{pq}^{+}+W_{pq}^{-}\sim c^{-1/2},\quad W_{pq}^{+}-W_{pq}^{-}\sim\epsilon(p)c^{-1/2}\,, \tag{103}$$

when p and q have opposite sign. Equation (103) also holds when $|p|$ or $|q|$ or both are of order $c^{1/2}m$, for either relative sign.

This behavior with respect to c is typical of scalar couplings of fermions in two dimensions. It is the job of a scalar interaction to couple left- to right-handed particles, and it must become strong as c disappears to keep them coupled. This is not true for vector interactions, as in QED and QCD, which preserve chirality.

If the quartic terms in P_+^I are put into normal order, extra quadratic terms are generated. The complete quadratic part of P_+, including P_+^0, becomes

$$\begin{aligned}P_+^{\text{quad}}=\int_{-\infty}^{\infty}dp_-\Bigg\{&\left[\frac{\omega_p-sp_-}{c}+\frac{g_N^2m^2\ln(\Lambda^2/m^2)}{2\pi\omega_p}\right][b^\dagger(p_-)b(p_-)+d^\dagger(p_-)d(p_-)]\\ &-\left[\frac{g_N^2m\ln(\Lambda^2/m^2)p_-}{2\pi c^{1/2}\omega_p}\right][b^\dagger(p_-)d^\dagger(-p_-)+d(p_-)b(-p_-)]\Bigg\}\,.\end{aligned} \tag{104}$$

The parameter Λ is a Pauli-Villars regulator mass, and $g_N^2\equiv g^2(N_f-\frac{1}{2})$.

P_+^{quad} is identical to P_+ of Sec. IV, where the fermion mass in P_+^0 was shifted by an interaction of the form $P_+^I=\mu\int dx^-:\bar{\psi}\psi:$. Here μ corresponds to

$$\frac{g_N^2m\ln(\Lambda^2/m^2)}{2\pi}\equiv\delta m\,. \tag{105}$$

As in that example, P_+^{quad} can be diagonalized by the transformation in Eq. (46). In terms of $B(p_-)$ and

$D(p_-)$ from that equation, $P_+^{\rm quad}$ is a free Hamiltonian with mass

$$m+\delta m = m\left[1+\frac{g_N^2 \ln(\Lambda^2/m^2)}{2\pi}\right] . \qquad (106)$$

Because of the discrete symmetry, δm is proportional to m. It is also independent of c, as would be expected for a mass.

Thus far P_+^I has been normal ordered in terms of $b(p_-)$ and $d(p_-)$, that is, with respect to the original mass. Replacing b and d in P_+^I with B and D and normal ordering again produces additional quadratic terms, and the process iterates. The new fermion mass, after $n+1$ iterations, is

$$m(n+1)=m+\frac{g_N^2}{2\pi} m(n)\ln[\Lambda^2/m^2(n)] , \qquad (107)$$

with m the original mass appearing in $\mathcal{L}$. For $m<\Lambda$, Eq. (107) converges so that

$$m(n+1)/m(n) \underset{n\to\infty}{\sim} 1 \qquad (108)$$

and

$$m/m(\infty)=1+\frac{g_N^2}{\pi}\ln[m(\infty)/\Lambda] , \qquad (109)$$

which may be solved self-consistently. A particular case is illustrated in Fig. 4. For the special case when the Lagrangian mass m approaches zero, the left side of Eq. (109) vanishes and

$$m_{\rm phys}=\Lambda \exp(-\pi/g_N^2) . \qquad (110)$$

This is the final, physical, mass which appears in the quadratic part of P_+ when the original mass m is finite but taken to zero. Inverting this gives the running coupling

$$g_N^2=\frac{2\pi}{\ln(\Lambda^2/m_{\rm phys}^2)} . \qquad (111)$$

Results for the vacuum wave function and expectation values can also be translated directly from Eqs. (51), (58), and (59) of Sec. IV, with the shifted mass μ replaced by $m_{\rm phys}$. In particular, when m vanishes

$$\langle \Omega | :\bar\psi\psi: |\Omega\rangle = (-\Lambda/g_N^2)\exp(-\pi/g_N^2) . \qquad (112)$$

For $N_f \gg 1$ these reproduce the large-N_f results deduced diagrammatically in [34], since diagrams which correspond to normal-ordering the interaction include those which dominate as $N_f\to\infty$ with g^2N_f fixed. These same results could also have been obtained from a covariant Schwinger-Dyson equation or its Hamiltonian equivalent. All dependence on c has vanished in the expressions for scalars such as $\langle :\bar\psi\psi: \rangle$ and the running coupling. While these are only leading order expressions in $1/N_f$, this expansion is Lorentz invariant and results at each order should have a sensible light-cone limit.

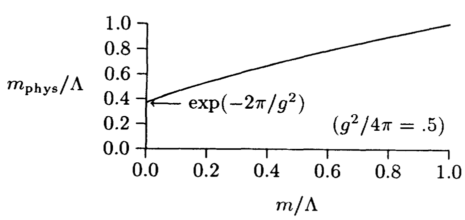

FIG. 4. Physical mass as a function of Lagrangian mass m for $g_N^2/2\pi = 0.5$.

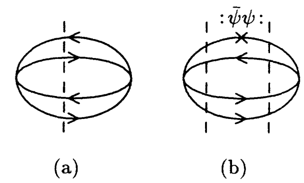

FIG. 5. Higher-order (in $1/N_f$) contributions to (a) the vacuum energy and (b) the chiral condensate.

Before the full Hamiltonian to all orders in $1/N_f$ makes sense near the light cone, much more needs to be done. For example, the leading terms as $c\to 0$ in the now normal-ordered interaction have the form

$$P_+^I \sim \left[\frac{-g^2}{4\pi}\right]\int_{-\infty}^{\infty} dp_-^a dp_-^b dp_-^c dp_-^d\, \delta\left[\sum p_-\right] \times\left\{\left[\frac{\Theta(-p_-^a p_-^b)\Theta(-p_-^c p_-^d)}{c}\right] \times b^{\dagger i}(p_-^a) b^{\dagger j}(p_-^c) b_i(p_-^b) b_j(p_-^d) + \cdots \right\} . \qquad (113)$$

Divergent terms in c involve couplings to negative-momentum particles, which could be incorporated into an effective Hamiltonian. A related alternative might be to diagonalize divergent terms in P_+ order by order in $1/c$ and then to restrict the full P_+ to the lowest of these, that is, to define physical states as the subspace of states for which the divergent part of P_+ vanishes [14]. The simple, nearly structureless, form of the divergent terms in P_+ in Eq. (113) suggests that this might not be intractable. Moreover, this procedure could be further simplified by performing it as an expansion in $1/N_f$.

That terms higher order in $1/N_f$ near the light cone contribute to the vacuum energy of an effective Hamiltonian and to $\langle :\bar\psi\psi: \rangle$ is evident from diagrams such as those of Fig. 5. As before, $c^{1/2}$ is associated with each denominator, $c^{1/2}$ with each loop, and c^{-1} for each vertex in this model, and Fig. 5(a) is therefore finite near the light cone. Related contributions to the condensate can be computed from $\partial_m E_\Omega$ or directly, as in Fig. 5(b). There, the extra $c^{1/2}$ from a second denominator is canceled by the $c^{-1/2}$ singularity in $:\bar\psi\psi:$.

X. TWO-DIMENSIONAL QED

As a final example, this section will present the results of applying this scheme to the Schwinger model [37] or two-dimensional electrodynamics. This model is interesting, especially for massless fermions, for several reasons. It is known to have degenerate θ vacua which break chiral symmetry, it has left-moving fermions which would appear to decouple in the usual light-cone approach, and it has been solved exactly using a wide variety of methods. In this discussion I will follow closely the treatment and notation of Nakawaki [38] especially as reviewed in [39], focusing on the aspects peculiar to this quantization procedure. These references should be consulted for a more detailed exposition, including discussions on subtleties such as point splitting and infrared regulation.

Nakawaki diagonalizes the equal-time Hamiltonian in a Fock-state basis and in Coulomb gauge, and this can be carried over directly into quantization at arbitrary angle. Most of the results in this section will be a straightforward generalization of the careful treatments of this model in the near-light-cone limit in [14] and [15], in that their results are extended continuously from near the light cone back to equal time. Related work appears in [40], where an operator solution on the light cone is discussed; also, nontrivial vacua in the bosonized version are treated in [41].

The conserved electric current for a free massive fermion in a box with antiperiodic boundary conditions at $x^+=0$ has components

$$\begin{aligned} J^+ &= :\bar{\psi}\gamma^+\psi: = (1-s)^{1/2}:\psi_L^\dagger\psi_L: + (1+s)^{1/2}:\psi_R^\dagger\psi_R: \\ &= \frac{1}{2L}\sum_{k,l}\{[V_{kl}^+ + V_{kl}^-](b_{k+1/2}^\dagger b_{l+1/2} - d_{k+1/2}^\dagger d_{l+1/2})e^{i(k-l)(\pi/L)x^-} \\ &\qquad + [V_{kl}^+ - V_{kl}^-](b_{k+1/2}^\dagger d_{l+1/2}^\dagger - d_{-l-1/2}b_{-k-1/2})e^{i(k+l+1)(\pi/L)x^-}\}\ , \end{aligned} \tag{114}$$

$$\begin{aligned} J^- &= :\bar{\psi}\gamma^-\psi: = (1+s)^{1/2}:\psi_L^\dagger\psi_L: - (1-s)^{1/2}:\psi_R^\dagger\psi_R: \\ &= -\frac{1}{2L}\sum_{k,l}\left\{\left[\frac{c}{1+s}V_{kl}^+ - \frac{1+s}{c}V_{kl}^-\right](b_{k+1/2}^\dagger b_{l+1/2} - d_{k+1/2}^\dagger d_{l+1/2})e^{i(k-l)(\pi/L)x^-}\right. \\ &\qquad \left. + \left[\frac{c}{1+s}V_{kl}^+ + \frac{1+s}{c}V_{kl}^-\right](b_{k+1/2}^\dagger d_{l+1/2}^\dagger - d_{-l-1/2}b_{-k-1/2})e^{i(k+l+1)(\pi/L)x^-}\right\}\ , \end{aligned} \tag{115}$$

with

$$V_{kl}^\pm \equiv \frac{[\omega_{k+1/2} \pm (k+\frac{1}{2})]^{1/2}[\omega_{l+1/2} \pm (l+\frac{1}{2})]^{1/2}}{2[\omega_{k+1/2}\omega_{l+1/2}]^{1/2}}\ . \tag{116}$$

When the electron mass vanishes,

$$V_{kl}^\pm = \Theta(\pm k \pm \tfrac{1}{2})\Theta(\pm l \pm \tfrac{1}{2})\ , \tag{117}$$

so that positive- and negative-momentum particles decouple.

For a massless electron, the Lagrangian, though not the ground state, is chirally symmetric. The components of the axial current $J_5^\mu = :\bar{\psi}\gamma^\mu\gamma_5\psi:$ are identical to those for the gauge current J^μ, but with the opposite sign in front of the left-moving, negative-momentum operators. In the light-cone limit, the coefficients $V_{kl}^\pm$ approach the form of Eq. (117), and J^+ for massive fermions reduces to that for the massless case. However, J^- is ill behaved in this limit. The offending term is composed entirely of negative-momentum operators, which are discarded in conventional light-cone quantization, but are necessary for current conservation. This problem has been typically circumvented by computing matrix elements of the well-behaved J^+ and inferring those of J^- by Lorentz invariance or by evaluating J^- in finite-energy eigenstates for which this term is suppressed [30].

The electric charge

$$Q = \int_{-L}^{L} dx^- J^+ = \sum_k (b_{k+1/2}^\dagger b_{k+1/2} - d_{k+1/2}^\dagger d_{k+1/2})\ , \tag{118}$$

for both massive and massless cases. The dependence on quantization angle has dropped out, as it must for the Lorentz-scalar charge so that it is the same at equal time and on the light cone. In fact, for massless electrons, not just Q, but the current J^+ at $x^+=0$ is c independent, which will allow for a simple transcription of the equal-time results in [38,39]. Throughout the remainder of this discussion, the electron will be massless. In this case the axial charge

$$Q_5 = \int_{-L}^{L} dx^- J_5^+ = \sum_k [\Theta(k+\tfrac{1}{2}) - \Theta(-k-\tfrac{1}{2})](b_{k+1/2}^\dagger b_{k+1/2} - d_{k+1/2}^\dagger d_{k+1/2}) \tag{119}$$

is conserved and is also c independent.

Given that, near the light cone, $s\to 1$ and, in Eq. (114), $J^+\to\sqrt{2}:\psi_R^\dagger\psi_R:$, it is surprising that operators for left-moving massless electrons appear in J^+ with equal weight as right movers. Their retention is due to singularities in the spinors. For example, in Eq. (40), the upper component of u_n diverges as $c^{-1/2}$ when $n<0$. In products such as $\bar{u}\gamma^+u$, this divergence cancels the vanishing coefficient of $\psi_L^\dagger\psi_L$. Such a cancellation must occur to reproduce expressions for Q and Q_5 which are the same as for equal time.

A. General solution

The Lagrangian for massless electrodynamics is given by

$$\mathcal{L}=-\frac{1}{4}F_{\mu\nu}F^{\mu\nu}+\tfrac{1}{2}\bar{\psi}\overleftrightarrow{\partial\!\!\!/}\psi-gA_\mu J'^{\mu} . \tag{120}$$

Following [39], the current J'^{μ} is defined such that the overall charge is removed, with

$$J'^{+}\equiv J^+-\frac{1}{2L}Q . \tag{121}$$

The equations of motion are then

$$\begin{aligned}&[c(i\partial_+-eA_+)+(1+s)(i\partial_--eA_-)]\psi_L=0 ,\\ &[(1+s)(i\partial_+-eA_+)-c(i\partial_--eA_-)]\psi_R=0 ,\end{aligned} \tag{122}$$

$$\begin{aligned}&\partial_+\partial_-A_+-\partial_+^2A_-=J^- ,\\ &\partial_+\partial_-A_--\partial_-^2A_+=J'^{+} .\end{aligned} \tag{123}$$

The gauge condition which interpolates between light-cone gauge, $A_0-A_1=0$, and axial gauge at equal time, $A_1=0$, is [9]

$$A_-\equiv\cos(\theta/2)A_0-\sin(\theta/2)A_1=0 , \tag{124}$$

which yields the constraint equation

$$-\partial_-^2A_+=J'^{+} . \tag{125}$$

In this discussion I will use a slightly different gauge, $\partial_-A_-=0$, which becomes the Coulomb gauge at equal time and requires the retention of an x^--independent field denoted $A_-^{(0)}(x^+)$. This choice is more convenient for a system in a box, as it permits A_+ to satisfy periodic boundary conditions and has proved useful in both equal-time and light-cone contexts [42,38,43,39,14,15]. The time derivative of $A_-^{(0)}$ produces an x^--independent electric field, whereas in the strict $A_-=0$ gauge this field is represented by $-\partial_-A_+$ on the boundaries which must be included when inverting Eq. (125) [21].

Products of fields are regulated by point splitting a distance ϵ along x^-, following [38,39]. Inserting an exponentiated gauge field preserves gauge invariance when splitting fermion fields. As the splitting is along x^-, this involves only the constant field $A_-^{(0)}$. As $\epsilon\to 0$, the singularity in $\psi_L^\dagger\psi_L$ goes as $i[2\pi(1-s)^{1/2}\epsilon]^{-1}$ and $\psi_R^\dagger\psi_R$ as $-i[2\pi(1+s)^{1/2}\epsilon]^{-1}$, so that each acquires a finite correction. This adds $(g/c\pi)A_-^{(0)}$ to J^-, while J^+ is unchanged, and the kinetic term $i\bar{\psi}\gamma^\mu\partial_\mu\psi$ picks up $(g^2/c\pi)(A_-^{(0)})^2$.

$A_-^{(0)}$ also appears explicitly in the Lagrangian in $-\frac{1}{4}F_{\mu\nu}F^{\mu\nu}$ and $-gA_\mu J'^{\mu}$. Because $A_-^{(0)}$ is constant, it couples only to the zero-momentum part of J^-,

$$\frac{1}{2L}\int_{-L}^{L}dx^-J^-=\frac{1}{2L}\left[\left[\frac{1+s}{c}\right]Q_L-\left[\frac{1-s}{c}\right]Q_R\right] . \tag{126}$$

Q_L and Q_R are the charges carried by left- and right-moving fermions, with $Q=Q_L+Q_R$ and $Q_5=Q_L-Q_R$.

The terms in $\mathcal{L}$ containing $A_-^{(0)}$ contribute

$$\frac{1}{4L}\Pi^2_{A_-^{(0)}}+gA_-^{(0)}\left[\frac{1}{c}Q_5+\frac{s}{c}Q\right]+\frac{g^2}{c\pi}L(A_-^{(0)})^2 , \tag{127}$$

to the Hamiltonian P_+. The momentum conjugate to $A_-^{(0)}$,

$$\Pi_{A_-^{(0)}}\equiv\frac{\delta\mathcal{L}}{\delta(\partial_+A_-^{(0)})}=2L\partial_+A_-^{(0)} , \tag{128}$$

satisfies the usual canonical relation

$$[\Pi_{A_-^{(0)}}(x^+),A_-^{(0)}(x^+)]=-i . \tag{129}$$

Apart from terms involving $A_-^{(0)}$, P_+ is most simply constructed from the fusion operators C_n, which are modes of the free charge density [44]

$$J'^{+}_{\text{free}}=\frac{1}{2L}\sum_{n\neq 0}[C_ne^{-ik_n\cdot x}+C_n^\dagger e^{ik_n\cdot x}] . \tag{130}$$

Here J'^{+}_{free} is periodic in x^-, with $k_{n^-}=n\pi/L$ and $k_{n^+}=(|n|-sn)\pi/cL$. By Eq. (121), J'^{+}_{free} has no mode constant in x^-. For $n>0$

$$\begin{aligned}C_n=&\sum_{l=1}^{\infty}(b^\dagger_{l+1/2}b_{n+l+1/2}-d^\dagger_{l+1/2}d_{n+l+1/2})\\ &+\sum_{l=1}^{n-1}d_{l+1/2}b_{n-l-1/2} ,\end{aligned} \tag{131}$$

while for $n<0$

$$\begin{aligned}C_n=&\sum_{l=-1}^{-\infty}(b^\dagger_{l+1/2}b_{n+l+1/2}-d^\dagger_{l+1/2}d_{n+l+1/2})\\ &-\sum_{l=-1}^{n}d_{l+1/2}b_{n-l-1/2} .\end{aligned} \tag{132}$$

Note that C_n is independent of quantization angle, as is J'^{+} at $x^+=0$.

Two properties of these fusion operators which will be useful later are that they satisfy bosonic commutation relations,

$$[C_n,C_m^\dagger]=|n|\delta_{n,m} , \tag{133}$$

and that they annihilate a large class of low-lying states. Let $|M,N\rangle$ denote the state of lowest kinetic energy with $|M|$ left-moving and $|N|$ right-moving fermions, with M and N positive for positrons, negative for electrons. Then

$$C_n|M,N\rangle=0\ , \tag{134}$$

for all M, N, and n. Of particular interest are the charge-singlet states $|N,-N\rangle$ for which $Q=0$ and $Q_5=-2N$. These states provide the basis for the θ vacua at equal time. As discussed in Sec. V, because the fermions are massless, the usual suppression of such states near the light cone is lost, and their importance in vacuum dynamics will persist. Their role may differ dramatically with massive fermions, however.

In terms of the fusion operators,

$$J^-_{\text{free}}=\frac{1}{2L}\left[\frac{1+s}{c}\right]\left[Q_L+\sum_{n<0}[C_ne^{-ik_n\cdot x}+C_n^\dagger e^{ik_n\cdot x}]\right]-\frac{1}{2L}\left[\frac{1-s}{c}\right]\left[Q_R+\sum_{n>0}[C_ne^{-ik_n\cdot x}+C_n^\dagger e^{ik_n\cdot x}]\right]\ , \tag{135}$$

and the momentum P_- and Hamiltonian P_+ are particularly simple. The momentum

$$P_-=-\frac{\pi}{2L}QQ_5-\frac{\pi}{L}\sum_{n\neq 0}\epsilon(n)C_n^\dagger C_n \tag{136}$$

has the same form for all c, while the free Hamiltonian

$$P_+^0=\frac{\pi}{2L}\left[\left[\frac{1+s}{c}\right]Q_L^2+\left[\frac{1-s}{c}\right]Q_R^2\right]+\frac{\pi}{L}\left[\left[\frac{1+s}{c}\right]\sum_{n<0}C_n^\dagger C_n+\left[\frac{1-s}{c}\right]\sum_{n>0}C_n^\dagger C_n\right] \tag{137}$$

explicitly depends on c.

This dependence follows immediately from the form of the free equal-time Hamiltonian H_{et}^0 given in [38,39] and two observations. First, H_{et}^0 separates into a term with only left-moving operators plus one with only right movers, both when expressed in terms of $b_{n+1/2}$ and $d_{n+1/2}$ and in terms of C_n. Second, to convert the H_{et}^0 given in terms of fermion operators,

$$H_{\text{et}}^0=P_+^0(c=1)=\frac{\pi}{L}\sum_n|n+\tfrac{1}{2}|(b_{n+1/2}^\dagger b_{n+1/2}+d_{n+1/2}^\dagger d_{n+1/2})\ , \tag{138}$$

to P_+^0 at arbitrary c, one replaces the energy $|n+\frac{1}{2}|$ with $[|n+\frac{1}{2}|-s(n+\frac{1}{2})]/c$. This is equivalent to multiplying the left part of H_{et}^0 by $(1+s)/c$ and the right part by $(1-s)/c$. Therefore to convert H_{et}^0 expressed as a function of C_n to P_+^0 only requires separating H_{et}^0 into left and right halves and multiplying by $(1+s)/c$ and $(1-s)/c$, respectively.

The interaction is even simpler to convert from H_{et}^I. Because the current density J'^+ as well as the constraint Eq. (125) are independent of c, the part of P_+^I due to $gA_+J'^+$ is also independent of c and may be copied directly from the corresponding term at equal time. So

$$P_+^I=\frac{g^2}{4L}\sum_{n\neq 0}\left[\frac{L}{n\pi}\right]^2[C_n+C_{-n}^\dagger][C_{-n}+C_n^\dagger]\ , \tag{139}$$

plus the terms involving $A_-^{(0)}$ in Eq. (127). In position space this is the Coulombic term

$$g^2\int_{-L}^{L}dx^-dy^-J'^+(x^-)|x^--y^-|J'^+(y^-)\ , \tag{140}$$

which produces linear confinement.

The conversion from H_{et} to P_+ at arbitrary quantization angle has been remarkably simple. As a result, the eigenstates at all angles are also simple to convert. Ignoring for now the zero-momentum parts of P_+ containing $A_-^{(0)}$ and Q_5, which will be considered next, and restricting Q to zero, P_+ may be rewritten as

$$P_+=\frac{1}{c}\left\{\frac{\pi}{L}\sum_{n\neq 0}C_n^\dagger C_n+\frac{cg^2}{4L}\sum_{n\neq 0}\left[\frac{L}{n\pi}\right]^2[C_n+C_{-n}^\dagger][C_{-n}+C_n^\dagger]\right\}-\frac{s}{c}\left\{\frac{\pi}{L}\sum_{n\neq 0}\epsilon(n)C_n^\dagger C_n\right\}\ . \tag{141}$$

This has the form

$$P_+=\frac{1}{c}H_{\text{et}}(g^2\to cg^2)-\frac{s}{c}P_{\text{et}}\ , \tag{142}$$

where H_{et} and P_{et} are the equal-time Hamiltonian and momentum and the coupling g^2 has been replaced by cg^2. Consequently, the eigenstates of the equal-time Hamiltonian given in [38,39] may be immediately translated to those of P_+ by simply replacing the parameter g^2 by cg^2. As is known, the spectrum of H_{et} is that of a free boson of mass $g/\pi^{1/2}$, with energy $(k^2+g^2/\pi)^{1/2}$ when $P_{\text{et}}=k$. By Eq. (142) the eigenvalues of P_+ are therefore $[(k^2+cg^2/\pi)^{1/2}-sk]/c$, which are the appropriate energies for free particles of mass $g/\pi^{1/2}$ at arbitrary c when

$P_-=k$, as in Eq. (18).

The operators which diagonalize $H_{\rm et}(g^2\to cg^2)$ transcribed from [38,39] are

$$a(n)=-\frac{i\epsilon(n)}{|n|^{1/2}}[\cosh\phi(n)C_n+\sinh\phi(n)C^\dagger_{-n}]\ , \tag{143}$$

with

$$\phi(n)\equiv\frac{1}{2}\ln\left[\frac{\omega(p_n)}{|p_n|}\right]=\frac{1}{4}\ln\left[1+\frac{cm^2}{p_n^2}\right] \tag{144}$$

and

$$\omega(p)\equiv(p^2+cm^2)^{1/2},\quad m^2\equiv g^2/\pi\ , \tag{145}$$

while $P_{\rm et}$ is already diagonal. Then, apart from the vacuum energy and zero-momentum contributions,

$$P_+=\sum_{n\neq 0}\left[\frac{\omega(p_n)-sp_n}{c}\right]a^\dagger(n)a(n)\ . \tag{146}$$

The unitary operator

$$S=\exp\left\{\frac{1}{2}\sum_{n\neq 0}\frac{\phi(n)}{|n|}[C_{-n}C_n-C_n^\dagger C^\dagger_{-n}]\right\} \tag{147}$$

effects the transformation of Eq. (143), with

$$SC_nS^\dagger=i\epsilon(n)|n|^{1/2}a(n)\ . \tag{148}$$

Because C_n annihilates $|0\rangle$, the state $S|0\rangle$ is annihilated by $a(n)$,

$$i\epsilon(n)|n|^{1/2}a(n)S|0\rangle=SC_nS^\dagger S|0\rangle=0\ , \tag{149}$$

and so may serve as the full vacuum. The same is true of states $S|N,-N\rangle$. As will be seen shortly, diagonalization of the zero-momentum sector makes these degenerate with $S|0\rangle$.

For any fixed and finite p_n, $\phi(n)$ vanishes as $cm^2/4p_n^2$ in the light-cone limit, S approaches unity, and the ground state reduces to the perturbative vacuum or, more generally, to the states $|N,-N\rangle$. Furthermore, the creation operator $a^\dagger(n)$ for free bosons with $n>0$ reduces to the simple fusion operator $i|n|^{-1/2}C_n^\dagger$. This may also be deduced directly from the form of P_+ in Eq. (141). As c vanishes, there is no interaction which strongly couples negative- to positive-momentum operators. It is consequently sensible to simply discard the negative sector and take $c\to 0$. P_+ reduces immediately to the light-cone Hamiltonian for free bosons of mass $m=g/\pi^{1/2}$,

$$P_+\to\sum_{n>0}\frac{(g^2/\pi)}{2p_n}a^\dagger(n)a(n)\ , \tag{150}$$

as observed in [20,26].

This argument fails for p_n near zero. The large kinetic energy p_n/c responsible for the suppression of negative-momentum states becomes small relative to their coupling to positive states, g^2/p_n, when $p_n\ll c^{1/2}g$. Therefore, for the quanta in this region, there is a great deal of mixing, and their contribution to the operator S and therefore the vacuum does not vanish. This region will sustain the vacuum expectation value of $\bar\psi\psi$. Nevertheless, the spectrum for all finite energy states with $p_-\neq 0$ is given by the Hamiltonian of Eq. (150).

At this point all that remains is to diagonalize the zero-momentum part of the Hamiltonian, $P_+(n=0)$, given by Eq. (127) and the first term of Eq. (137) with Q restricted to zero. This part plays a negligible role in the continuum limit. In the absence of fermions, $Q_5=0$ and $P_+(n=0)$ is quadratic in $A_-^{(0)}$ and $\Pi_{A_-^{(0)}}$. It is therefore simply a quantum-mechanical harmonic oscillator, diagonalized by the annihilation operator

$$a=i\left[\frac{mL}{c^{1/2}}\right]^{1/2}A_-^{(0)}-\left[\frac{c^{1/2}}{4mL}\right]^{1/2}\Pi_{A_-^{(0)}}\ . \tag{151}$$

Then, for the general case when $Q_5\neq 0$,

$$P_+(n=0)=\frac{m}{c^{1/2}}a^\dagger a+i\left[\frac{m\pi}{4c^{3/2}L}\right]^{1/2}(a^\dagger-a)Q_5 +\frac{\pi}{4cL}Q_5^2\ ; \tag{152}$$

the constant photon field $A_-^{(0)}$ couples to the axial charge of fermions. For fixed L this coupling becomes large near the light cone, although it vanishes in the continuum limit.

On a state of fixed axial charge, this Hamiltonian is identical to that of the shifted scalar field of Sec. II with v replaced by Q_5. In fact, the shift

$$A_-^{(0)}\to A_-^{(0)}-\frac{\pi^{1/2}}{2mL}Q_5 \tag{153}$$

completely removes Q_5 from P_+ and leads to the exact degeneracy of the states $|N,-N\rangle$. As in Sec. II, acting on these states with the unitary operator S_0,

$$S_0|N,-N\rangle=\exp\left\{\frac{i\pi^{1/2}}{2(c^{1/2}mL)^{1/2}}Q_5(a^\dagger-a)\right\}|N,-N\rangle \tag{154}$$

gives the ground state for $P_+(n=0)$. The cost in kinetic energy of additional electron-positron pairs popped from the vacuum in the state $|N,-N\rangle$ is compensated by a coherent state of zero-momentum photons equivalent to a constant electric field.

The corresponding shift on a,

$$a(0)\equiv S_0aS_0^\dagger=a+\frac{i}{2}\left[\frac{\pi}{c^{1/2}mL}\right]^{1/2}Q_5\ , \tag{155}$$

diagonalizes $P_+(n=0)$ completely to

$$P_+(n=0)=[m/c^{1/2}]a^\dagger(0)a(0)\ , \tag{156}$$

equivalent to the Hamiltonian for a free particle of mass $m\equiv g/\pi^{1/2}$ and $P_-=0$. Finally, the full Hamiltonian for all momenta n, including zero,

$$P_+=\frac{\pi}{L}\sum_{n=-\infty}^{\infty}\left[\frac{\omega_n-sn}{c}\right]a^\dagger(n)a(n)\ , \tag{157}$$

is equivalent to that for a scalar of mass m. As a result, the states

$$|\Omega(N)\rangle = SS_0|N,-N\rangle \tag{158}$$

are degenerate, zero-energy and -momentum vacua.

B. Vacuum energy

After the diagonalization of P_+ by means of Eqs. (143) and (155), the energy of the ground state $|\Omega(N)\rangle$ is

$$E_\Omega = \left[\frac{\pi}{L}\right]\left\{\frac{1}{4c}\sum_{n\neq 0}|n|\left[\frac{\omega_n}{|n|}+\frac{|n|}{\omega_n}-2\right] + \frac{\hat{m}^2}{4}\sum_{n\neq 0}\frac{1}{\omega_n}\right\} . \tag{159}$$

The two terms on the right are the contributions of P^0_+ and P^I_+, respectively; there are no contributions from $P_+(n=0)$.

The term due to P^I_+ is ultraviolet divergent, while the other term is finite. If P_+ is normal ordered with respect to the perturbative vacuum, the second term

$$\langle\Omega(N)|P^I_+|\Omega(N)\rangle - \langle 0|P^I_+|0\rangle = \left[\frac{\pi}{L}\right]\frac{\hat{m}^2}{4}\sum_{n\neq 0}\left[\frac{1}{\omega_n}-\frac{1}{|n|}\right] \tag{160}$$

acquires a subtraction which renders it ultraviolet finite but infrared divergent in the continuum limit. As $L\to\infty$, the dominant part of this sum is clearly an infrared logarithm which is cut off when $\omega_n \sim n$ and so will go as $\ln(c^{1/2}mL/\pi)$. Evidently, the dominant contribution is from momenta $p_- \equiv n\pi/L$ in the region $\pi/L < |p_-| < c^{1/2}m$. As will be seen, this contribution will persist even if c is taken to zero, so that if these states are discarded, their effect must be incorporated in some other manner to retain vacuum dynamics. In the formalism of [8], they would be contributed by boundary degrees of freedom.

In fact, in the continuum limit [45],

$$\sum_{n=1}^{c^{1/2}mL/\pi}\frac{1}{n} \sim \gamma_E + \ln\left[\frac{c^{1/2}mL}{\pi}\right] + O(L^{-1}) , \tag{161}$$

with γ_E Euler's constant, and

$$\begin{aligned}\sum_{c^{1/2}mL/\pi}^{\infty}\frac{1}{n} - \sum_{n=1}^{\infty}\frac{1}{\omega_n} &\sim \int_{c^{1/2}m}^{\infty}dp\left[\frac{1}{p}-\frac{1}{(p^2+cm^2)^{1/2}}\right] - \int_0^{c^{1/2}m}dp\frac{1}{(p^2+cm^2)^{1/2}} + O(L^{-1}) \\ &= -\ln(2) + O(L^{-1}) ,\end{aligned} \tag{162}$$

so that

$$\exp\left\{\sum_{n=1}^{\infty}\left[\frac{1}{n}-\frac{1}{\omega_n}\right]\right\} \sim \left[\frac{c^{1/2}mL}{2\pi}\right]e^{\gamma_E} . \tag{163}$$

The continuum limit for the first term in E_Ω may be taken straightforwardly. It contributes

$$\langle\Omega(N)|P^0_+|\Omega(N)\rangle \underset{l\to\infty}{=} \left[\frac{L}{\pi}\right]\frac{1}{2c}\int_0^\infty dp\, p\left[\frac{\omega_p}{p}+\frac{p}{\omega_p}-2\right] = \left[\frac{L}{\pi}\right]\frac{m^2}{4} . \tag{164}$$

Then

$$E_\Omega - \langle 0|P_+|0\rangle \underset{L\to\infty}{=} \left[\frac{L}{\pi}\right]\frac{m^2}{4}\left[1-2\gamma_E - \ln\left[\frac{cm^2L^2}{4\pi^2}\right]\right] . \tag{165}$$

The final c-dependent logarithm contains the remains of the noncovariant regulation of the infrared divergence.

The vacuum state provides an interesting example of the simplest of the relations contained in Eq. (94). Because only the free Hamiltonian P^0_+ contains an explicit, and simple, dependence on c,

$$\begin{aligned}\partial_c E_\Omega &= \langle\Omega(N)|\partial_c P^0_+|\Omega(N)\rangle \\ &= -\frac{1}{c^2}\left[\frac{\pi}{L}\right]\langle\Omega(N)|\sum_n |n|(b_n^\dagger b_n + d_n^\dagger d_n)|\Omega(N)\rangle \\ &= -\frac{1}{c}\langle\Omega(N)|P^0_+|\Omega(N)\rangle .\end{aligned} \tag{166}$$

In the continuum,

$$\langle\Omega(N)|P^0_+|\Omega(N)\rangle = \left[\frac{L}{\pi}\right]\frac{m^2}{4} , \tag{167}$$

which is independent of c, while

$$\partial_c E_\Omega = \partial_c\left\{-\left[\frac{L}{\pi}\right]\frac{m^2}{4}\ln\left[\frac{cm^2L^2}{4\pi^2}\right]\right\} = -\left[\frac{L}{\pi}\right]\frac{m^2}{4c} , \tag{168}$$

and is due entirely to $\langle\Omega(N)|P^I_+|\Omega(N)\rangle$. So Eq. (166) relates the expectation value of a simple, diagonal operator to the infrared-divergent $\ln c$ dependence from the interaction. In the same way, for the one-boson state of momentum p_- and $p_+ = (\omega_p - sp_-)/c$,

$$-\frac{1}{c}\langle\Omega(N)|a(p_-)P^0_+ a^\dagger(p_-)|\Omega(N)\rangle = -\left[\frac{L}{\pi}\right]\frac{m^2}{4c} - \left[\frac{p_+^2 - p_-^2 - (2c/s)p_-p_+}{2(cp_+ + sp_-)}\right] , \tag{169}$$

and so on.

C. θ vacua

To preserve desired qualities such as cluster decomposition, orthogonal θ vacua

$$|\theta\rangle = \frac{1}{(2\pi)^{1/2}}\sum_{N=-\infty}^{\infty} e^{iN\theta}|\Omega(N)\rangle \tag{170}$$

may be built from the degenerate vacua $|\Omega(N)\rangle$ such that

$$\langle\theta'|\theta\rangle=\delta(\theta'-\theta)\ , \tag{171}$$

just as the instanton vacuum is defined for QCD. The θ vacua in general break chiral symmetry, and the axial charge rotates θ according to

$$e^{-i(\alpha/2)Q_5}|\theta\rangle=|\theta+\alpha\rangle\ . \tag{172}$$

Once a choice is made for θ, excited states, that is, those with free bosons, are built with products of $a^\dagger(n)$ acting on $|\theta\rangle$. The spectrum is entirely independent of that choice.

As in previous examples, the value of the condensate $\langle\bar\psi\psi\rangle$ can be computed in the θ vacua as a measure of chiral-symmetry breaking and studied as the quantization angle goes from equal time to the light cone. Consider first the part of $\langle\bar\psi\psi\rangle$ which raises chirality, $\langle\psi_L^\dagger\psi_R\rangle$, evaluated in the $|\Omega(N)\rangle$ vacua:

$$\langle\Omega(N')|\psi_L^\dagger\psi_R|\Omega(N)\rangle=\langle N',-N'|S_0^\dagger S^\dagger\psi_L^\dagger\psi_R SS_0|N,-N\rangle\ . \tag{173}$$

C_n annihilates the states $|N,-N\rangle$; to take advantage of this, the relations

$$e^{A}Be^{-A}=B+[A,B]+\frac{1}{2!}[A,[A,B]]+\cdots\ ,$$
$$e^{A+B}=e^{A}e^{B}e^{-(1/2)[A,B]+\cdots}\ , \tag{174}$$

and

$$[\psi_L^\dagger\psi_R(0),C_n]=[\psi_L^\dagger\psi_R(0),C_n^\dagger]=\epsilon(n)\psi_L^\dagger\psi_R(0) \tag{175}$$

may be used to rearrange terms such that

$$S^\dagger\psi_L^\dagger\psi_R S=\exp\left\{-\sum_{n\neq0}\frac{1}{|n|}[\tfrac{1}{2}(e^{-\phi_n}-1)^2+(e^{-\phi_n}-1)]\right\}$$
$$\times\exp\left\{-\sum_{n\neq0}\frac{1}{n}[e^{-\phi_n}-1]C_n^\dagger\right\}\psi_L^\dagger\psi_R\exp\left\{\sum_{n\neq0}\frac{1}{n}[e^{-\phi_n}-1]C_n\right\}\ . \tag{176}$$

Using Eq. (144), the term multiplying $\psi_L^\dagger\psi_R$ that survives in Eq. (173) is the same exponential evaluated in Eq. (163).

The contribution from the zero-momentum sector may be evaluated separately. Using $Q_5|N,-N\rangle=-2N|N,-N\rangle$ in Eq. (173), $S_0^\dagger\psi_L^\dagger\psi_R S_0$ may be rearranged such that

$$S_0^\dagger\psi_L^\dagger\psi_R S_0=\exp\left\{-\frac{\pi(N-N')^2}{2c^{1/2}mL}\right\}\exp\left\{-\left[\frac{\pi}{2c^{1/2}mL}\right]^{1/2}(N-N')a^\dagger\right\}\psi_L^\dagger\psi_R\exp\left\{-\left[\frac{\pi}{2c^{1/2}mL}\right]^{1/2}(N-N')a\right\}\ . \tag{177}$$

Finally,

$$\langle N',-N'|\psi_L^\dagger\psi_R|N,-N\rangle=\frac{1}{2c^{1/2}L}\delta_{N',N-1}\ , \tag{178}$$

which involves only free fields. Combining these results,

$$\langle\Omega(N')|\psi_L^\dagger\psi_R|\Omega(N)\rangle=\delta_{N',N-1}\frac{1}{2c^{1/2}L}\exp\left\{\sum_{n=1}^{\infty}\left[\frac{1}{n}-\frac{1}{\omega_n}\right]\right\}\exp\left\{\frac{-\pi}{2c^{1/2}mL}\right\} \tag{179}$$

and

$$\langle\theta'|\psi_L^\dagger\psi_R|\theta\rangle=e^{i\theta}\delta(\theta'-\theta)\frac{1}{2c^{1/2}L}\exp\left\{\sum_{n=1}^{\infty}\left[\frac{1}{n}-\frac{1}{\omega_n}\right]\right\}\exp\left\{\frac{-\pi}{2c^{1/2}mL}\right\}\ . \tag{180}$$

In the continuum, $L\to\infty$, the zero-momentum contribution becomes negligible, and that due to finite momentum has been computed above. The exponent of the sum over n gives a factor $(c^{1/2}mL/\pi)$. The L dependence cancels that due to Eq. (178), leaving a finite result. In addition, the c dependence also cancels, as it must for a Lorentz scalar. Near the light cone, this is the same story as for symmetry breaking in previous examples. The singularity in $\psi_L^\dagger\psi_R$ near the light cone is just enough to pick out corrections to the perturbative vacuum that would like to disappear in that limit. Were this singularity not present in $\psi_L^\dagger\psi_R$, the result would be zero, just as for the perturbative vacuum. Unlike previous examples, vacuum states containing negative-momentum fermions, $|N,-N\rangle$, are not suppressed, since the fermions are massless. However, the complicated vacua $|\Omega(N)\rangle$ are built on the $|N,-N\rangle$ with the operators C_n associated with massive excitations, so that $|\Omega(N)\rangle\to|N,-N\rangle$ as c vanishes. It is this vanishing difference that the singularity in Eq. (178) picks out. So the θ vacua persist, but are greatly simplified, in that limit [39].

It is necessary to take the continuum limit prior to the light-cone limit so as not to lose critical degrees of freedom, as discussed in Sec. V and [14,15]. In particular,

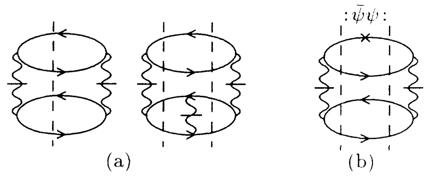

FIG. 6. Contributions to (a) the vacuum energy and (b) the chiral condensate in the massive Schwinger model.

the apparent exponential suppression in the last factor of Eq. (180) as $c \to 0$ is an artifact of the poor sampling a regular grid makes of massless, negative-p_- particles. In the continuum limit this factor goes to one rather than zero. Finally, including the contribution from the chiral lowering operator $\psi_R^\dagger \psi_L$, which is proportional to $e^{-i\theta}$, gives the correct total continuum result [46,14,15],

$$\langle \theta' | \bar{\psi}\psi | \theta \rangle = e^{\gamma_E} \frac{m}{2\pi} \cos\theta \, \delta(\theta' - \theta) \ . \tag{181}$$

While this section has focused on massless QED, where known equal-time results could be easily translated to those at arbitrary c, it would be an interesting exercise to study the more general massive case, where exact solutions are unavailable but numerical light-cone results are known [26]. In particular, it would be worth exploring how difficult it would be to construct an effective Hamiltonian and to what extent it would agree with [26]. That the vacuum is not trivial even for massive fermions is suggested by contributions to the vacuum energy in H_{eff} such as those in Fig. 6(a). The instantaneous photon propagator is $1/p_-^2$, and so each vertex introduces c^{-1}. This diagram and the related one for $\langle :\bar{\psi}\psi: \rangle$ are independent of c, just as those of Fig. 5, so that if these exist at equal time, they also survive the light-cone limit. As a result, light-cone quantization of massive fermions does not appear to preclude a vacuum expectation value for $:\bar{\psi}\psi:$.

XI. CONCLUSIONS

The vacuum wave functions and energies for a variety of two-dimensional models have been presented explicitly as a continuous function of the angle of the initial quantization surface. Near the light cone, these wave functions do in fact evolve to the perturbative vacuum, as conventional light-cone arguments dictate for massive excitations. Nevertheless, operators with nonzero vacuum expectation values possess singularities in the region $k_- = 0$ which are sufficient to extract finite contributions from the vanishing vacuum corrections. As a result, it is possible to maintain the effects of an involved vacuum on the light cone. This is important, especially in the context of QCD, as the goal of quantizing on the light cone is certainly to employ a formalism which is as simple as possible while still giving correct results.

In general, whether conventional light-cone arguments hold and the vacuum is trivial depends on what is being measured. The low-energy spectrum generally follows from conventional arguments, while quantities sensitive to high energies, large distances in x^-, or small k_- may be singular enough to probe nontrivial corrections. In certain cases it is possible to take advantage of the simplicity of the light cone by examining matrix elements which are not sensitive to this region and using dispersion relations or Lorentz invariance to relate them to those which are [47].

Keeping the quantization angle as a parameter has several advantages: It controls singularities in k_- and permits a simple power counting to determine when conventional arguments about the vacuum must be modified; it retains massless left-moving particles; it allows nontrivial relations from Lorentz invariance; and finally, it makes quantization simpler in certain cases by avoiding constraints.

The connection between short times and the long distances of the vacuum near the light cone suggests that computing an effective Hamiltonian is an appropriate way to include vacuum effects in the light-cone limit. Such a procedure was illustrated by a simple example here. It is worth exploring whether this approach is viable for more interesting models such as QCD.

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[46] For a calculation of this quantity, as well as general discussions of various aspects of the Schwinger model, see J. H. Lowenstein and J. A. Swieca, Ann. Phys. (N.Y.) **68**, 172 (1971); S. Coleman, R. Jackiw, and L. Susskind, *ibid.* **93**, 267 (1975); S. Coleman, *ibid.* **101**, 239 (1976); C. J. Hamer, J. Kogut, D. P. Crewther, and M. M. Mazzolini, Nucl. Phys. **B208**, 413 (1982); B. E. Baaquie, J. Phys. G **8**, 1621 (1982); G. T. Bodwin and E. V. Kovacs, Phys. Rev. D **35**, 3198 (1987); **36**, 1281 (1987); C. Jayewardena, Helv. Phys. Acta **61**, 636 (1988).
[47] This has long been a standard procedure. For recent examples, see [3,15].