Symmetry and combinatorics in the δ expansion for QED

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Combinatorics and symmetry are used to solve calculational problems in applying the δ-expansion method to QED. Adequate Feynman rules are used to simplify previous calculations and Ward identities are obtained. In the case with no external fermions an equivalence to the conventional loop expansion is established, both through the use of path integrals and through explicit calculations of the effective vertices. The Ward identities are reduced in this case to the conventional ones implied by gauge invariance.

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I. INTRODUCTION

The δ expansion [1] is an analytical calculational scheme which provides a nonperturbative approach to nonlinear theories. A dimensionless parameter δ is introduced that interpolates between a linear theory at $\delta=0$ and the nonlinear theory at $\delta=1$ which is the one to be studied. δ is treated as a small parameter and perturbative techniques are used to calculate physical quantities as power series in δ.

In a recent series of papers the δ expansion was applied to gauge theories [2–4]. The Lagrangian for QED in the δ expansion is

$$\mathcal{L}_{\delta\,\mathrm{QED}}=-\tfrac{1}{4}(F_{\mu\nu})^2+M\bar{\psi}\left[\frac{i\partial\!\!\!/-eA\!\!\!/}{M}\right]^{\delta}\psi\ . \tag{1.1}$$

M is a parameter with mass dimension. At $\delta=0$ one has a free theory with an infinitely heavy fermion and at $\delta=1$ conventional QED. Expanding Eq. (1.1) to first order in δ gives

$$\mathcal{L}_{\delta\,\mathrm{QED}}=-\tfrac{1}{4}(F_{\mu\nu})^2+M\bar{\psi}\psi+\delta M\bar{\psi}\ln\left[\frac{i\partial\!\!\!/-eA\!\!\!/}{M}\right]\psi+O(\delta^2)\ . \tag{1.2}$$

Since this is a nonpolynomial Langrangian a provisional Lagrangian was introduced [1,3]:

$$\mathcal{L}_{N}=-\tfrac{1}{4}(F_{\mu\nu})^2+M\bar{\psi}\psi+\delta M\bar{\psi}\left[\frac{i\partial\!\!\!/-eA\!\!\!/}{M}\right]^{N}\psi\ . \tag{1.3}$$

This Lagrangian defines vertices and diagrams and the Green's functions of QED [$\delta=1$ in Eq. (1.1)] are calculated in two steps. In step (a) ordinary Feynman perturbation theory is used to obtain Green's functions from the provisional Lagrangian in Eq. (1.3) while N is regarded as a positive integer. In step (b) the Green's functions of the first step are differentiated with respect to N and then N is set to zero. N is regarded here as a real number [5] and higher orders in δ require generalizations of this procedure [1].

In this paper it is shown that symmetry and several combinatoric results can help to overcome many of the difficulties in treating the Lagrangian $\mathcal{L}_N$ in perturbation theory for arbitrary N. In Sec. II we introduce an improved combinatorial approach in order to calculate the free energy for a one-dimensional model. In Sec. III the combinatorial approach is adjusted to deal with space-time dimensions $d\geq 2$ which present additional technical problems since Dirac γ matrices are involved in the calculations and infinities are encountered. Diagrammatics for the δ expansion for QED (QED_δ) in the combinatorial approach are presented, and applied to the Schwinger model (2-dimensional QED) [6]. The photon propagator is calculated to order e^2 and first order in δ. In Sec. IV path integrals are used to establish the equivalence between QED_δ and the conventional perturbative expansion in the coupling constant, for Green's functions with no external fermions [7]. In Sec. V we show that, due to the closed fermionic loop in this case, the effective Lagrangian obtains a "cyclic symmetry" which implies the effective Lagrangian contains an explicit factor N, namely, $\mathcal{L}_{\mathrm{eff}}=NL_{\mathrm{eff}}$. This is important since in the δ expansion one differentiates the results with respect to N and then sets N to zero so $(\partial/\partial N)|_{N=0}\mathcal{L}_{\mathrm{eff}}=L_{\mathrm{eff}}(N=0)$. The photon propagator in the Schwinger model is calculated to first order in δ and order e^4. It is explicitly shown that $L(N=0)$ reduces in this case to conventional perturbation theory, reinforcing the conclusion of Sec. IV. In the conventional loop expansion for QED a major role is played by symmetries presented in the form of Ward identities [6]. In Sec. VI analogous identities for QED_δ are obtained.

II. A NEW COMBINATORIAL APPROACH APPLIED TO THE CALCULATION OF THE FREE ENERGY IN A ONE-DIMENSIONAL MODEL

Bender *et al.* [2] used the δ expansion to calculate the free energy of a quantum-mechanical system defined by the vacuum partition function:

$$\frac{Z(g)}{Z(g=0)}=\frac{1}{Z(g=0)}\int\mathcal{D}\phi\,\mathcal{D}\bar{\psi}\,\mathcal{D}\psi\exp\left(-\int_0^T dt\,\mathcal{L}_1\right),$$
$$\mathcal{L}_1\equiv\tfrac{1}{2}\dot{\phi}^2+\frac{m^2}{2}\phi^2-M\bar{\psi}\left[\frac{\partial_t-g\phi(t)}{M}\right]^{\delta}\psi\ . \tag{2.1}$$

The free energy E is defined by

$$E=-\frac{1}{T}\ln\frac{Z(g)}{Z(g=0)} . \tag{2.2}$$

This one-dimensional problem is used here as our first example in order to introduce a new combinatorial approach which gives promptly the results of Ref. [2].

In step (a) one calculates the ground-state free energy E_N of the system defined by the provisional Lagrangian

$$\mathcal{L}_N=\tfrac{1}{2}\dot{\phi}^2+\frac{m^2}{2}\phi^2-M\bar{\psi}\psi-\delta M^{1-N}\bar{\psi}[\partial_t-g\phi(t)]^N\psi . \tag{2.3}$$

$[\partial_t-g\phi(t)]^N\psi(t)$ will be written now as a sum over powers of g. In order to calculate

$$[\partial_t-g\phi(t)]^N\psi(t)=\mathcal{D}_N\mathcal{D}_{N-1}\cdots\mathcal{D}_3\mathcal{D}_2\mathcal{D}_1\psi(t) \tag{2.4}$$

we define N operators $\mathcal{D}_i$, $i=1,\dots,N$. All are equal to $[\partial_t-g\phi(t)]$ but differ by their location in the chain operating on ψ. Equation (2.4) can be written as

$$[\partial_t-g\phi(t)]^N\psi(t)=\sum_{\lambda=0}^{N}(-g)^\lambda\frac{1}{\lambda!}\sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty}\cdots\sum_{k_\lambda=0}^{\infty}B(N,\lambda,\{k_i\})\phi_{1,k_1}\phi_{2,k_2}\cdots\phi_{\lambda,k_\lambda}\psi_{,N-\lambda-\sum_i k_i} \tag{2.5}$$

where

$$\phi_{j,k_j}=\frac{\partial^{k_j}}{\partial t^{k_j}}\phi_j . \tag{2.6}$$

The scalar fields were arbitrarily labeled ϕ_i, $i=1,\dots,\lambda$. In order to sum over all different ways of labeling the fields in a given term one has to divide by $\lambda!$. The sums over k_i were taken to infinity although clearly $\sum_{i=1}^{\lambda}k_i\le N-\lambda$ since $B(\sum_{i=1}^{N}k_i>N-\lambda)=0$ as will be seen below. $B(N,\lambda,\{k_i\})$ is the combinatorial weight of each term in the sum. It counts the number of ways to distribute N operators $\mathcal{D}_i$, $i=1,\dots,N$ into $\lambda+1$ cells marked by $\phi_1,\phi_2,\dots,\phi_\lambda$ and ψ, the number of operators in each cell being $k_1+1,k_2+1,\dots,k_\lambda+1$ and $N-\lambda-\sum_{i=1}^{\lambda}k_i$, respectively. The first operator (the one with the smallest index) placed in the cell labeled ϕ_j give $(-g\phi_j)$. All the others give derivatives acting on it. All the operators placed in the cell labeled ψ give derivatives acting on ψ. Distributing the operators in this manner we indeed get immediately Eq. (2.5) with

$$B(N,\lambda,\{k_i\})=\frac{N!}{(k_1+1)!(k_2+1)!\cdots(k_\lambda+1)!\left[N-\lambda-\sum_{i=1}^{\lambda}k_i\right]!}$$

$$=\binom{N}{\lambda+\sum_{i=1}^{\lambda}k_i}\frac{\left[\sum_{i=1}^{\lambda}k_i+\lambda\right]!}{\prod_{i=1}^{\lambda}(k_i+1)!} \tag{2.7}$$

We will show that this result shortens considerably the derivation of Eq. (2.32) in Ref. [2]. The interaction term of the provisional Langrangian

$$I_N\equiv-\delta M^{1-N}\int_0^T dt\,\bar{\psi}(t)[\partial_t-g\phi(t)]^N\psi(t) \tag{2.8}$$

is most easily calculated in momentum space (in fact, energy space):

$$\begin{gathered}\psi(t)=\sum_{n=-\infty}^{\infty}\psi(p_n)e^{ip_nt},\quad \psi(p_n)=\frac{1}{T}\int_0^T\psi(t)e^{-ip_nt},\quad p_n=\frac{\pi}{T}(2n-1) ,\\ \phi(t)=\sum_{s=-\infty}^{\infty}\phi(l_s)e^{il_st},\quad \phi(l_s)=\frac{1}{T}\int_0^T\phi(t)e^{-il_st},\quad l_s=\frac{\pi}{T}2s .\end{gathered} \tag{2.9}$$

Differentiating with respect to t, using $\int_0^T dt\, e^{i(a-b)t}=T\delta(a-b)$ and Eqs. (2.5), (2.7) we get

$$I_N=-\delta M^{1-N}T\sum_{\lambda=0}^{N}\frac{(-g)^{\lambda}}{\lambda!}(i)^{N-\lambda}$$
$$\times\sum_{n=-\infty}^{\infty}\;\sum_{s_1,s_2,\dots,s_\lambda=-\infty}^{\infty}\bar{\psi}\left[-p_n-\sum_{j=1}^{\lambda}l_{s_j}\right]\psi(p_n)\phi(l_{s_1})\phi(l_{s_2})\cdots\phi(l_{s_\lambda})$$
$$\times\sum_{k_1,k_2,\dots,k_\lambda=0}^{\infty}\left[\begin{matrix}N\\ \sum_{i=1}^{\lambda}k_i+\lambda\end{matrix}\right]\frac{\left[\sum_{i=1}^{\lambda}k_i+\lambda\right]!}{\prod_{i=1}^{\lambda}(k_i+1)!}p_n^{(N-\lambda-\sum_{j=1}^{\lambda}k_j)}l_{s_1}^{k_1}l_{s_2}^{k_2}\cdots l_{s_\lambda}^{k_\lambda}\,. \tag{2.10}$$

This is a sum over vertices with two fermion legs and λ bosons (Fig. 1). In order to calculate the free energy E_N one has to sum over Feynman diagrams (Fig. 2) obtained from these vertices by contracting $\bar{\psi}$ with ψ and pairs of ϕ's in all possible ways. We will first evaluate the contributions of these contractions to the diagrams. The different ways to contract K pairs from $\lambda=2K$ bosonic fields give $(2K)!/K!2^K$ identical terms. In every one of these terms the indices are renamed so $\phi(l_{s_j})$ is contracted with $\phi(l_{s_{K+j}})$ $j=1,\dots,K$. In the expression for E_N each contraction of bosonic fields gives a factor of

$$\sum_{s_j=-\infty}^{\infty}\;\sum_{s_{K+j}=-\infty}^{\infty}\;\sum_{k_j=0}^{\infty}\;\sum_{k_{K+j}=0}^{\infty}\overline{\phi(l_{s_j})\phi(l_{s_{K+j}})}\,l_{s_j}^{k_j}l_{s_{K+j}}^{k_{K+j}}\frac{1}{(k_j+1)!}\frac{1}{(k_{K+j}+1)!} \tag{2.11}$$

substituting

$$\sum_{s_j=-\infty}^{\infty}\;\sum_{s_{K+j}=-\infty}^{\infty}\overline{\phi(l_{s_j})\phi(l_{s_{K+j}})}=\sum_{s_j=-\infty}^{\infty}\frac{1}{T}\frac{1}{l_{s_j}^2+m^2} \tag{2.12}$$

and defining $r_j\equiv k_j+k_{K+j}$, which counts the number of derivatives (momenta factors) on the bosonic loop (thus $\sum_{j=1}^{\lambda}k_j=\sum_{j=1}^{K}r_j$), we get, for Eq. (2.11),

$$\sum_{s_j=-\infty}^{\infty}\;\sum_{r_j=0}^{\infty}\frac{1}{T}\frac{l_{s_j}^{r_j}}{l_{s_j}^2+m^2}\frac{1}{(r_j+2)!}\sum_{k_{K+j}=0}^{r_j}(-1)^{k_{K+j}}\left[\begin{matrix}r_j+2\\ k_{K+j}+1\end{matrix}\right]\,. \tag{2.13}$$

Since

$$\sum_{k=0}^{r}(-1)^k\left[\begin{matrix}r+2\\ k+1\end{matrix}\right]=1+(-1)^r\,,$$

only even r contribute. Finally, contracting $\bar{\psi}$ with ψ gives the fermion propagator $\overline{\bar{\psi}(-p_n)\psi(p_n)}=(1/T)(1/M)$. Thus,

$$E_N=-\delta M^{1-N}\sum_{K=0}^{N/2}\frac{g^{2K}}{K!}\sum_{r_1,r_2\cdots r_K=0}^{\infty}{}'\left[\begin{matrix}N\\ \sum r_j+2K\end{matrix}\right]\frac{\left[\sum_{j=1}^{K}r_j+2K\right]!}{\prod_{j=1}^{K}(r_j+2)!}$$
$$\times\left[\sum_{n=-\infty}^{\infty}\frac{1}{T}\frac{(ip_n)^{N-2K-\sum_{j=1}^{\lambda}r_j}}{M}\right]\prod_{j=1}^{K}\left[\sum_{s_j=-\infty}^{\infty}\frac{1}{T}\frac{(il_{s_j})^{r_j}}{l_{s_j}^2+m^2}\right] \tag{2.14}$$

[In Eq. (2.14), and from here on, $\sum'$ denotes a sum on even integers only.]

This concludes step (a) (namely, the calculation of the free energy using the vacuum partition function built with the provisional Lagrangian $\mathcal{L}_N$). Equation (2.14) agrees with Eq. (2.32) in Ref. [2]. Step (b) of the calculation, differentiating with respect to N and setting N to zero is described in Appendix A. One finally gets

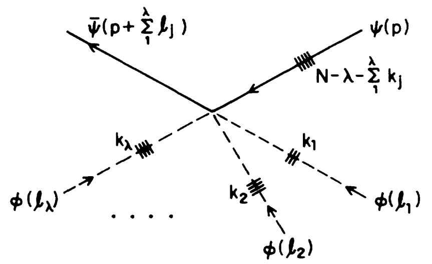

FIG. 1. g^λ vertex in the one-dimensional model of QED_δ. A tick symbolizes time differentiation or momentum in momentum space (energy factors l_s or p_n on boson or fermion lines, respectively).

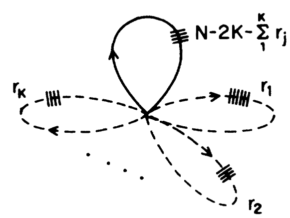

FIG. 2. g^λ diagram for the free energy in the one-dimensional model of QED_δ ($\lambda=2K$). This figure is obtained from Fig. 1 by closing the loops. The fermion propagator in momentum space is $(1/T)(1/M)$. The boson propagator in momentum space is $(1/T)[1/(l^2+m^2)]$.

$$E=\frac{\partial}{\partial N}\bigg|_{N=0} E_N=\frac{\delta}{T}\sum_{K=1}^{\infty}\frac{(-1)^K}{K}(1-2^{-2K})(2K-1)!!\frac{\zeta(2K)}{\pi^{2K}}\left[\frac{g^2T}{m^2}\right]^K+O(\delta^2)$$
$$=\delta\left[-\frac{g^2}{8m^2}+\frac{g^4T}{64m^4}-\frac{g^6T^2}{192m^6}+\cdots\right]+O(\delta^2)\ , \tag{2.15}$$

where $(2K-1)!!\equiv(2K-1)(2K-3)\cdots 3\times 1$ and $\zeta(2K)$ is Riemann's zeta function.

A further simplification in the derivation of Eq. (2.14) is obtained by using the combinatorial approach directly for the contracted graphs. Instead of distributing the N operators $\mathcal{D}_i$, $i=1,\ldots,N$ into $\lambda+1$ cells labeled ϕ_j, $j=1,\ldots,\lambda$ and ψ, they are distributed now into $K+1$ cells ($K=\lambda/2$) labeled by pairs of contracted fields: $(\phi\phi)_i$, $i=1,\ldots,K$ and $(\bar\psi\psi)$. The number of operators in each cell is r_i+2 and $N-\lambda-\sum_{i=1}^K r_i$ respectively. r_i is the number of derivatives operating on any of the two fields constituting the contracted pair $(\phi\phi)_i$. Contracted pairs $\bar\psi(-p)\psi(p)$ and $\phi(l_{S_i})\phi(-l_{S_i})$, $i=1,\ldots,K$ contribute $\sum_{n=-\infty}^{\infty}(1/T)(1/M)$ and $\sum_{S_i=-\infty}^{\infty}(1/T)[1/(l_{S_i}^2+m^2)]$, respectively. Each derivative operating on $\psi(p),\phi(l)$, or $\phi(-l)$ give $(ip),(il)$, and $(-il)$, respectively. There are now only K arbitrary labels and thus one has to divide by $K!$. The number of different ways to distribute the N operators into these cells is

$$\frac{N!}{(r_1+2)!(r_2+2)!\cdots(r_K+2)!\left[N-\lambda-\sum_{i=1}^K r_i\right]}=\left[\begin{matrix}N\\ \sum_{i=1}^K r_i+2K\end{matrix}\right]\frac{\left[\sum_{i=1}^K r_i+2K\right]!}{\prod_{i=1}^K(r_i+2)!} \tag{2.16}$$

which is the factor appearing in Eq. (2.14). Note that in a given cell the first operator ($\mathcal{D}_i$ with the smallest index) must be $(-g)\phi(l)$. The second field in the contracted pair, $(-g)\phi(-l)$, must come from the last operator ($\mathcal{D}_i$ with the largest index), otherwise there would be at least one derivative operating on both $\phi(l)$ and $\phi(-l)$ and the two resulting contributions to the graph will cancel. All the intermediate $\mathcal{D}_i$ must therefore be r derivatives operating on $\phi(l)$, so we shall have $(il)^r$ multiplying the propagator. Since odd r_i cause cancellation in pairs between $l_{s_j}^{\ r_j}$ and $l_{-sj}^{\ r_j}$ one should sum only over even r's. Putting all this together gives immediately Eq. (2.14).

III. THE COMBINATORIAL APPROACH APPLIED TO THE CALCULATION OF THE PHOTON PROPAGATOR IN THE SCHWINGER MODEL

In the second paper [3] which established the rules for applying the δ expansion to gauge theories, the photon propagator in two-dimensional QED (to first order in δ and second order in e) has been calculated. We present here a different strategy for this calculation based on our combinatorial approach. This enables one to obtain promptly the results of Ref. [3] and improve the efficiency of the calculation rules.

The Lagrangian for QED in the δ expansion is given by Eq. (1.1) for any spacetime dimension d and the provisional Lagrangian is given by Eq. (1.3):

$$\mathcal{L}_N=-\tfrac{1}{4}(F_{\mu\nu})^2+M\bar\psi\psi+\delta M^{1-N}\bar\psi(i\partial\!\!\!/-eA\!\!\!/)^N\psi\ . \tag{3.1}$$

$(i\partial\!\!\!/-eA\!\!\!/)^N\psi$ will be written now as a sum over powers of e.

A. The combinatorial approach in the presence of γ matrices

When the space-time dimension is $d\geq 2$ the covariant derivative contains a γ matrix and Eq. (2.4) is replaced by

$$(\partial\!\!\!/+gA\!\!\!/)^N\psi=\Gamma D\psi\ , \tag{3.2}$$

$$\Gamma\equiv\gamma_{\nu_N}\gamma_{\nu_{N-1}}\cdots\gamma_{\nu_i}\cdots\gamma_{\nu_2}\gamma_{\nu_1},\quad \nu_i=0,1,\ldots,d-1\ , \tag{3.3}$$

$$D\equiv\mathcal{D}_N^{\nu_N}\mathcal{D}_{N-1}^{\nu_{N-1}}\cdots\mathcal{D}_i^{\nu_i}\cdots\mathcal{D}_2^{\nu_2}\mathcal{D}_1^{\nu_1},\quad \mathcal{D}_i^{\nu_i}\equiv(\partial^{\nu_i}+gA^{\nu_i})\ . \tag{3.4}$$

$\mathcal{D}_i^{\nu_i}$ is the ith operator in the chain operating on ψ:

$$(\partial\!\!\!/+gA\!\!\!/)^N\psi=\sum_{\lambda=0}^{N}g^\lambda I_\lambda\ . \tag{3.5}$$

Following the one-dimensional example, I_λ can be written as a sum of terms where in every term each operator $\mathcal{D}_i$ $(i=1,\ldots,N)$ gives only one contribution. This contribution can be either a derivative operating on one of the fields or a new scalar field A^ν.

I_λ can be expressed in momentum space where the momentum of the fermionic field is labeled by p and the momentum of the scalar field A_j by l_j $(j=1,\ldots,\lambda)$. Arbitrary labeling the scalar fields by j and summing over different labels is corrected by a $1/\lambda!$ factor.

Similarly to Eq. (2.5) we have here

$$I_\lambda=\frac{i^{N-\lambda}}{\lambda!}\sum_{k_1,k_2,\ldots,k_\lambda=0}^{\infty}B(N,\lambda,\{k_j\})^{(\alpha_{k_1}\cdots\alpha_1\alpha_0)(\beta_{k_2}\cdots\beta_1\beta_0)\cdots(\zeta_{k_\lambda}\cdots\zeta_1\zeta_0)(\sigma_{(N-\lambda-\sum k)}\cdots\sigma_2\sigma_1)}$$
$$\times[l_{1_{\alpha_{k_1}}}\cdots l_{1_{\alpha_2}}l_{1_{\alpha_1}}A_{\alpha_0}(l_1)][l_{2_{\beta_{k_2}}}\cdots l_{2_{\beta_2}}l_{2_{\beta_1}}A_{\beta_0}(l_2)]\cdots$$
$$\times\cdots[l_{\lambda_{\zeta_{k_\lambda}}}\cdots l_{\lambda_{\zeta_2}}l_{\lambda_{\zeta_1}}A_{\zeta_0}(l_\lambda)][p_{\sigma_{(N-\lambda-\sum k)}}\cdots p_{\sigma_2}p_{\sigma_1}\psi(p)]\ , \tag{3.6}$$

where $\alpha_i,\beta_i,\ldots,\zeta_i$ and σ_i are all Lorentz indices. In complete analogy with the one-dimensional case, getting Eq. (3.6) from Eqs. (3.2)–(3.4) is basically the distribution of N operators $\mathcal{D}_i$ into $\lambda+1$ cells, but here each term carries with it a chain of γ matrices. Since γ matrices do not commute this chain retains the original order of the N operators $\mathcal{D}_i$, $i=1,\ldots,N$. In Eq. (3.6) the summation indices ν_i were renamed into $\alpha_i,\beta_i,\ldots,\zeta_i$ or σ_i according to the specific contribution of $\mathcal{D}_i$. To get B we have to sum over all possible orders of appearance of a given set of fields and momenta, namely, over different chains of γ matrices. One gets

$$B(N,\lambda,\{k_j\})^{(\alpha_{k_1}\cdots\alpha_1\alpha_0)(\beta_{k_2}\cdots\beta_1\beta_0)\cdots(\zeta_{k_\lambda}\cdots\zeta_1\zeta_0)(\sigma_{(N-\lambda-\sum k)}\cdots\sigma_2\sigma_1)}$$
$$=\tilde{Q}\{\gamma^{\alpha_{k_1}}\cdots\gamma^{\alpha_1}\gamma^{\alpha_0};\gamma^{\beta_{k_2}}\cdots\gamma^{\beta_1}\gamma^{\beta_0};\ldots;\gamma^{\zeta_{k_\lambda}}\cdots\gamma^{\zeta_1}\gamma^{\zeta_0};\gamma^{\sigma_{(N-\lambda-\sum k)}}\cdots\gamma^{\sigma_2}\gamma^{\sigma_1}\}\ , \tag{3.7}$$

$\tilde{Q}\{X\}\equiv\{X+$ all permutations of indices that keep the order between indices marked with the same greek letters. (If $m>n$ α_m must be to the left of α_n, β_m to the left of β_n, etc.)$\}$. (3.8)

$\tilde{Q}$ maps a product of N matrices γ^μ, each labeled by one Lorentz index, into one matrix B labeled by N indices. Note that the number of terms in Eq. (3.7) for a given set of parameters $(N,\lambda,\{k_j\})$ is

$$\frac{N!}{\left[N-\lambda-\sum_{j=1}^{\lambda}k_j\right]!\prod_{j=1}^{\lambda}(k_j+1)!}\ . \tag{3.9}$$

In the one-dimensional case when $\Gamma=1$ all these terms were identical and the factor in Eq. (3.9) showed up explicitly in Eq. (2.7). Here these terms differ by the ordering of the γ matrices.

Taking N to be even (generalization to odd N is easy), it is useful to define

$$B_1^{\alpha_{k_1}\cdots\sigma_1}\equiv\tilde{P}\{g^{\alpha_{k_1}\alpha_{k_1-1}}\cdots g^{\sigma_4\sigma_3}g^{\sigma_2\sigma_1}\}\ . \tag{3.10}$$

$\tilde{P}$ is defined by the same rule as $\tilde{Q}$ in Eq. (3.8) but maps a product of $N/2$ unit matrices $(\hat{1}g^{\nu\mu})$ each labeled by two Lorentz indices into one unit matrix B_1 labeled by N indices. With an adequate correction term, labeled B_2, that will be calculated and added later, one can replace B in Eq. (3.6) by B_1 and get a simple sum of scalar products among fields and momenta. B_1 has the same number of terms as B. It is formally obtained from B by dividing each chain of γ matrices into subsequent pairs and replacing every pair by the metric tensor: namely, $\gamma^\nu\gamma^\mu\rightarrow\hat{1}g^{\nu\mu}$. To every pair of γ matrices so defined corresponds a pair of operators $\mathcal{D}$'s. There are three cases. (1) If this pair is $(l_{i_\nu}l_{i_\mu})$ or $(p_\nu p_\mu)$ the identity $a\!\!\!/a\!\!\!/=a^2$ justifies the replacement and no correction term B_2 is needed. (2) If the pair contains two operators belonging to different fields, there exists in B a term with the order inside this pair reversed but otherwise identical. The replacement is justified now by the identity $\gamma^a\gamma^b+\gamma^b\gamma^a=g^{ab}+g^{ba}$. (3) Only if the pair is (l_iA_i) is a correction term needed. We therefore have

$$B_2=\tilde{P}\{G(\alpha_{k_1},\alpha_{k_1-1})\cdots G(\sigma_2\sigma_1)| \text{ must contain } G(\alpha_1\alpha_0) \text{ or/and } G(\beta_1,\beta_0), \text{ etc.}\} , \tag{3.11}$$

where

$$G(a,b)\equiv\begin{cases}\gamma^a\gamma^b-g^{ab}, & (a,b)=(\alpha_1,\alpha_0) \text{ or } (\beta_1,\beta_0), \text{ etc.,}\\ g^{ab}, & \text{otherwise.}\end{cases} \tag{3.12}$$

These results can be used to write the interaction term of the provisional Lagrangian in momentum space:

$$\begin{aligned}\delta M^{1-N}\bar{\psi}(p')(-1)^N\sum_{\lambda=0}^{N}\frac{e^\lambda}{\lambda!}\sum_{k_1\cdots k_\lambda=0}^{\infty}&(B_1+B_2)^{\alpha_{k_1}\cdots\sigma_1}\\ &\times[l_{1_{\alpha_{k_1}}}\cdots l_{1_{\alpha_2}}l_{1_{\alpha_1}}A_{\alpha_0}(l_1)][l_{2_{\beta_{k_2}}}\cdots l_{2_{\beta_2}}l_{2_{\beta_1}}A_{\beta_0}(l_2)]\\ &\vdots\\ &\times[l_{\lambda_{\zeta_{k_\lambda}}}\cdots l_{\lambda_{\zeta_2}}l_{\lambda_{\zeta_1}}A_{\zeta_0}(l_\lambda)][p_{\sigma_{(N-\lambda-\sum_{j=1}^{\lambda}k_j)}}\cdots p_{\sigma_2}p_{\sigma_1}\psi(p)] .\end{aligned} \tag{3.13}$$

B_1 and B_2 were defined above. This is a sum of vertices with one incoming and one outgoing fermions and λ bosons (Fig. 3).

For a given λ, one has sums over k_j, $j=1,\ldots,\lambda$ and internal sums included in B_1 and B_2. Each term in this sum can be represented diagrammatically. An example for diagramatic representation of a single term is given in Fig. 4. We put k_j ticks on the line representing A_j, $j=1,\ldots,\lambda$ and $N-\lambda-\sum_{j=1}^{\lambda}k_j$ ticks on the incoming fermion line. Each tick that represented a time derivative in the one-dimensional case, represents here d-dimensional momentum and carries a space-time index. In B_1 and B_2 one sums over the different ways to contract these indices. Diagramatically, this can be represented by $N/2$ thin lines that form scalar products from the fields and momenta they connect. The question of how many different terms in B_1 or B_2 give the same contribution reduces again to a combinatorial question of the distribution of these thin lines.

We will discuss now the effective vertices for diagrams with no external fermions. Closing the fermionic loop in the vertex of Fig. 3 gives the "octopus" diagram in Fig. 5(a). As a result of closing the fermionic loop we have (a) momentum conservation at the vertex gives $l_1+l_2+\cdots l_\lambda=0$, (b) there is a trace over B, and (c) there is an integral over the fermion momentum p. The integration over p and the identity [3]

$$\int f(p^2)p_{\sigma_h}\cdots p_{\sigma_2}p_{\sigma_1}d^dp=\frac{\Gamma\left[\frac{d}{2}\right]}{2^{h/2}\Gamma\left[\frac{d+h}{2}\right]}\left[\int f(p^2)(p^2)^{h/2}d^dp\right]P\{g_{\sigma_h\sigma_{h-1}}\cdots g_{\sigma_2\sigma_1}\} , \tag{3.14}$$

where P stands for all permutations of indices, enables us to replace in each term of Eq. (3.13) the mixed scalar products (such as $g^{\sigma\alpha}p_\sigma l_\alpha$ and $g^{\sigma\alpha_0}p_\sigma A_{\alpha_0}$) with powers of p^2 multiplied by scalar products of bosonic fields and their momenta, namely

$$g^{\sigma_1\rho_1}\cdots g^{\sigma_h\rho_h}p_{\sigma_1}\cdots p_{\sigma_h}\to\frac{\Gamma\left[\frac{d}{2}\right]}{2^{h/2}\Gamma\left[\frac{d+h}{2}\right]}(p^2)^{h/2}P\{g^{\rho_1\rho_2}\cdots g^{\rho_{h-1}\rho_h}\} \tag{3.15}$$

Diagrammatically (see Fig. 6 for an example), the replacement in Eq. (3.15) can be viewed as "isolating" the loop: removing all thin lines that connect the loop to the bosonic lines, reconnecting ticks in the loop with each other and forming new pairs from their partners in all possible ways.

B. Calculation of the photon propagator in the Schwinger model

The photon propagator to first order in δ is given by a sum of diagrams with one vertex and two uncontracted boson legs, given in Fig 7. To order e^2 it is the octopus diagram with two bosonic legs. The relevant interaction term is given by Eq. (3.13) with $\lambda=2$:

$$\begin{aligned}&\delta M^{1-N}\bar{\psi}(p')\frac{e^2}{2!}\sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty}(B_1+B_2)^{\alpha_{k_1}\cdots\sigma_1}\\ &\times[l_{1_{\alpha_{k_1}}}\cdots l_{1_{\alpha_2}}l_{1_{\alpha_1}}A_{\alpha_0}(l_1)][l_{2_{\beta_{k_2}}}\cdots l_{2_{\beta_2}}l_{2_{\beta_1}}A_{\beta_0}(l_2)][p_{\sigma_{(N-2-k_1-k_2)}}\cdots p_{\sigma_2}p_{\sigma_1}\psi(p)] .\end{aligned} \tag{3.16}$$

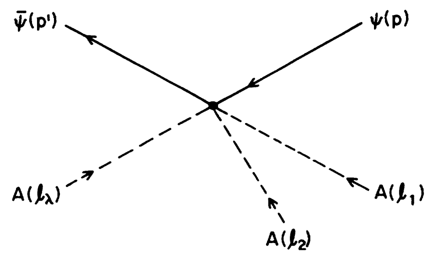

FIG. 3. e^{λ} vertex in d-dimensional QED$_\delta$, denoted as $\delta M e^{\lambda} G_{\lambda}^{\nu_1,\nu_2 \cdots \nu_\lambda}(p,l_1,l_2,\ldots,l_\lambda)$.

The relevant diagram (first in Fig. 7) is obtained by closing the fermionic loop. Following Sec. III A one finds the following.

(a) Since $l_1+l_2=0$, there will be a cancellation between any two terms in B_2 obtained from each other by one replacement $l_1 \leftrightarrow l_2$. The only terms that survive are those where the replacement is not allowed because l_i can never appear to the right of $A(l_i)$: In B_1 these have $k_2=0$ $k_1=r$ with all α_j, $j=1,\ldots,r$ between α_0 on their right and β_0 on their left or $k_1=0$ $k_2=r$ and all β_j, $j=1,\ldots,r$ between β_0 on their right and α_0 on their left. We drop the 1/2! and take only the first case. In B_2 the relevant terms have $k_2=1$ and $k_1=r-1$ with all the α_j, $j=2,\ldots,r-1$ between $(\alpha_1\alpha_0)$ on their right and (β_1,β_0) on their left. 1/2! is likewise dropped. As in the one-dimensional case (Sec. II) the double sum over k_1 and k_2 was replaced by one sum over r.

(b) There is a trace over B_1 and B_2. Since $\mathrm{Tr}(\gamma^a\gamma^b-\hat{1}g^{ab})=0$, all nonvanishing chains of B_2 must contain

$$\mathrm{Tr}\{(\gamma^{\alpha_1}\gamma^{\alpha_0}-\hat{1}g^{\alpha_1\alpha_0})(\gamma^{\beta_1}\gamma^{\beta_0}-\hat{1}g^{\beta_1\beta_0})\} = \mathrm{Tr}\hat{1}(g^{\alpha_1\beta_0}g^{\beta_1\alpha_0}-g^{\alpha_1\beta_1}g^{\beta_0\alpha_0}) . \quad (3.17)$$

(c) There is an integral over the fermion momentum p. The contraction of $\bar{\psi}(p)$ with $\psi(p)$ gives the fermion propagator $1/M$.

Finally, the e^2 octopus diagram in Fig. 5(a) reduces to

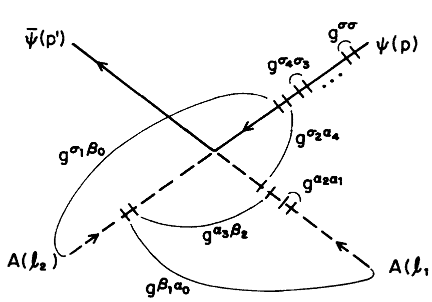

FIG. 4. An example for the diagrams summed in the combinatorial approach in order to get the vertex of Fig. 3. Contractions of Lorentz indices are represented by thin lines. In this example, $\lambda=2$; $k_1=4$; $k_2=2$; $N-\lambda-\sum_{j=1}^{\lambda}k_j=N-8$ and one element of B_1 is drawn, namely, $g^{\sigma_{N-8}\sigma_{N-9}}\cdots g^{\sigma_4\sigma_3}g^{\sigma_2\alpha_4}g^{\alpha_3\beta_2}g^{\alpha_2\alpha_1}g^{\beta_1\alpha_1}g^{\sigma_1\beta_0}$.

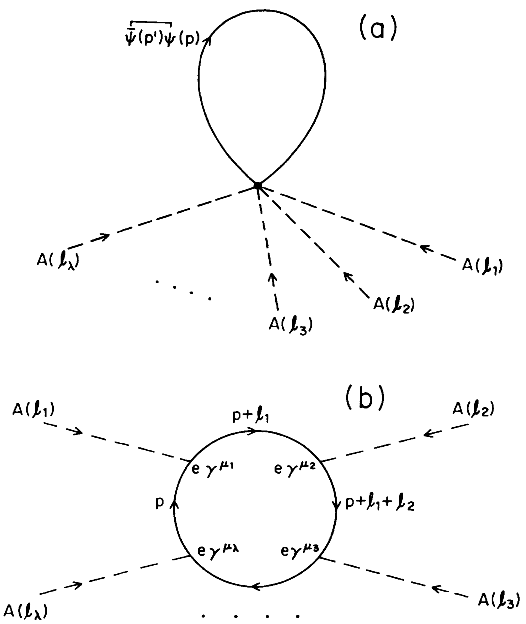

FIG. 5. (a) An "octopus" diagram of order e^{λ} for QED$_\delta$ is obtained from closing the fermionic loop in Fig. 3. (b) A fermion loop with λ attached bosons in conventional perturbation theory for QED [6].

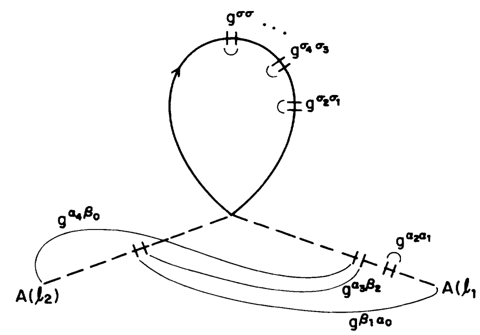

FIG. 6. Closing the fermionic loop and reconnecting ticks in the example of Fig. 4. $(g^{\sigma_2\alpha_4}g^{\sigma_1\beta_0}\to d^{-1}g^{\sigma_2\sigma_1}g^{\alpha_4\beta_0})$.

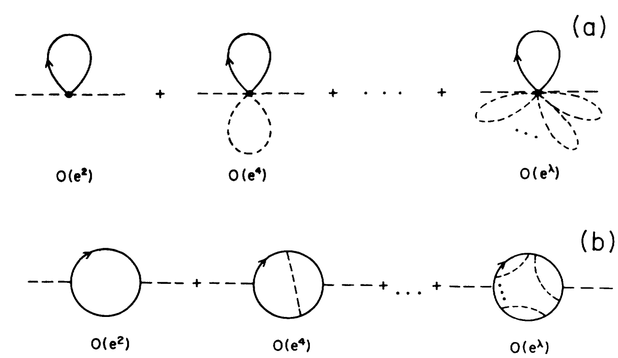

FIG. 7. (a) The photon propagator in QED$_\delta$ to first order in δ. (The diagram of order e^{λ} has $(\lambda-2)/2$ bosonic loops.) (b) Diagrams proportional to δ that contribute to the photon propagator in conventional perturbation theory for QED with δ fermion species. (The diagram of order e^{λ} has $(\lambda-2)/2$ internal bosonic lines.)

$$\delta M^{-N} e^2 \sum_{r=0}^{\infty}{}' \int \frac{d^d p}{(2\pi)^d} p_{\sigma_{N-2-r}} \cdots p_{\sigma_2} p_{\sigma_1}$$
$$\times \mathrm{Tr}[\tilde{B}_1^{\sigma_{(N-2-r)} \cdots \sigma_1 \alpha_{r+1} \alpha_r \alpha_{r-1} \cdots \alpha_0}(r) A_{\alpha_{r+1}}(-l) l_{\alpha_r} \cdots l_{\alpha_1} A_{\alpha_0}(l)$$
$$+\tilde{B}_2^{\sigma_{(N-2-r)} \cdots \sigma_1 \beta_1 \beta_0 \alpha_{r-1} \cdots \alpha_0}(r)(-l)_{\beta_1} A_{\beta_0}(-l) l_{\alpha_{r-1}} \cdots l_{\alpha_1} A_{\alpha_0}(l)] \ . \tag{3.18}$$

Note that in the $\tilde{B}_1$ term β_0 was renamed α_{r+1}:

$$\tilde{B}_1^{\sigma_{(N-2-r)} \cdots \sigma_2 \sigma_1 \alpha_{r+1} \alpha_r \alpha_{r-1} \cdots \alpha_0}(r) = \tilde{P}\{g^{\sigma_{N-2-r}\sigma_{N-3-r}} \cdots g^{\sigma_2\sigma_1} g^{\alpha_{r+1}\alpha_r} \cdots g^{\alpha_3\alpha_2} g^{\alpha_1\alpha_0}\} \ , \tag{3.19}$$

$$\tilde{B}_2^{\sigma_{(N-2-r)} \cdots \sigma_2 \sigma_1 \beta_1 \beta_0 \alpha_{r-1} \cdots \alpha_0}(r) = \tilde{P}\{G(\sigma_{N-2-r}\sigma_{N-3-r}) \cdots G(\sigma_2\sigma_1) G(\beta_1\beta_0) G(\alpha_{r-1},\alpha_{r-2}) \cdots G(\alpha_3\alpha_2) G(\alpha_1\alpha_0)$$
$$|(\beta_1\beta_0) \text{ and } (\alpha_1\alpha_0) \text{ are kept as pairs to the left and right of other } \alpha\text{'s}\} \ . \tag{3.20}$$

Notice, although the sum over r was formally written from zero to infinity $\tilde{B}_1$ differs from zero only for $0 \le r \le N-2$ and $\tilde{B}_2$ for $2 \le r \le N-2$. Furthermore, the sum is only over even r's since for odd r's the integral over p vanishes.

In a general permutation in $\tilde{B}_1$ or $\tilde{B}_2$ out of $N/2$, g's there are h mixed pairs $g^{\sigma\alpha}$. $[0 \le h \le \min(r+2, N-r-2)$; h is even since r is even$]$. Equation (3.15) can be used to replace

$$g^{\alpha\sigma} \cdots g^{\alpha\sigma} p_\sigma \cdots p_\sigma$$
$$\to \frac{\Gamma\left[\frac{d}{2}\right]}{\Gamma\left[\frac{d+h}{2}\right]} 2^{-h/2} (p^2)^{h/2} P\{g^{\alpha\alpha} \cdots g^{\alpha\alpha}\} \ . \tag{3.21}$$

Each permutation gives a sum over

$$\frac{h!}{\left[\frac{h}{2}\right]!} 2^{-h/2}$$

terms (the number of different ways to form pairs from $h\,\alpha$'s).

For given r and h the number of permutations in $\tilde{B}_1(r)$ is

$$\begin{pmatrix} \frac{N}{2} \\ \frac{r+h+2}{2} \end{pmatrix} \begin{pmatrix} \frac{r+h+2}{2} \\ h \end{pmatrix} 2^h \ . \tag{3.22}$$

Namely, out of $N/2$ g's choose $(N-2-r-h)/2$ to be $g^{\sigma\sigma}$. Out of $(r+2+h)/2$ g's with α choose h mixed pairs. Each mixed pair can be $g(\sigma\alpha)$ or $g(\alpha\sigma)$ giving a combinatorial factor of 2^h. The order among the α's and among the σ's is kept. For $r=0$ and for each of these permutations, all the terms obtained by (3.21) are identical. Thus each permutation gives

$$\delta M^{-N} e^2 \int \frac{d^d p}{(2\pi)^d} \frac{\Gamma\left[\frac{d}{2}\right]}{\Gamma\left[\frac{d+h}{2}\right]} \frac{h!}{\left[\frac{h}{2}\right]!} 2^{-h} (p^2)^{(N-2)/2} \mathrm{Tr}\hat{1} g^{\mu\nu} A_\mu A_\nu \ . \tag{3.23}$$

For $r \ge 2$ each permutation gives a sum of

$$\frac{h!}{\left[\frac{h}{2}\right]!} 2^{-h/2}$$

terms. Each permutation that does not contain both $g^{\sigma\alpha_0}$ and $g^{\sigma\alpha_{r+1}}$ gives

$$\delta M^{-N} e^2 \int \frac{d^d p}{(2\pi)^d} \frac{\Gamma\left[\frac{d}{2}\right]}{\Gamma\left[\frac{d+h}{2}\right]} \frac{h!}{\left[\frac{h}{2}\right]!} 2^{-h} (p^2)^{(N-2-r)/2} (l^2)^{(r-2)/2} \mathrm{Tr}\hat{1} l^\mu l^\nu A_\mu A_\nu \ , \tag{3.24}$$

but permutations in which both $g^{\sigma\alpha_0}$ and $g^{\sigma\alpha_{r+1}}$ appear give in addition

$$\delta M^{-N}e^2\int\frac{d^dp}{(2\pi)^d}\frac{2^{[d/2]-h}}{h-1}\frac{\Gamma\left[\frac{d}{2}\right]}{\Gamma\left[\frac{d+h}{2}\right]}\frac{h!}{\left[\frac{h}{2}\right]!}(p^2)^{(N-2-r)/2}(l^2)^{(r-2)/2}(l^2g^{\mu\nu}-l^\mu l^\nu)A_\mu A_\nu \tag{3.25}$$

since under the action of (3.21), α_0 and α_{r+1} form a pair in one out every $(h-1)$ terms. The number of permutations in $\tilde{B}_1(r\geq 2)$ when α_0 and α_{r+1} are paired with σ's, namely both $g^{\sigma\alpha_0}$ and $g^{\sigma\alpha_{r+1}}$ appear, is

$$\frac{4h(h-1)}{(r+h+2)(r+h)}\begin{bmatrix}\frac{N}{2}\\ \frac{r+h+2}{2}\end{bmatrix}\begin{bmatrix}\frac{r+h+2}{2}\\ h\end{bmatrix}2^h \tag{3.26}$$

because from $(r+2+h)/2$ g's the first one must be $(\sigma\alpha_0)$ and the last one must be $(\sigma\alpha_{r+1})$. One must therefore replace

$$\begin{bmatrix}\frac{r+h+2}{2}\\ h\end{bmatrix}$$

in (3.22) by

$$\begin{bmatrix}\frac{r+h+2}{2}-2\\ h-2\end{bmatrix}.$$

The number of permutations in $\tilde{B}_2(r)$ is

$$\left[1-\frac{4h(r+1)}{(r+h+2)(r+h)}\right]\begin{bmatrix}\frac{N}{2}\\ \frac{r+h+2}{2}\end{bmatrix}\begin{bmatrix}\frac{r+h+2}{2}\\ h\end{bmatrix}2^h . \tag{3.27}$$

Here, from $(r+2+h)/2$ g's the first one must be $(\alpha_1\alpha_0)$ and the last one must be $(\beta_1\beta_0)$, so one must replace

$$\begin{bmatrix}\frac{r+h+2}{2}\\ h\end{bmatrix}$$

in (3.22) by

$$\begin{bmatrix}\frac{r+h+2}{2}-2\\ h\end{bmatrix}.$$

Using Eqs. (3.17) and (3.21) each permutation gives here

$$\delta M^{-N}e^2\int\frac{d^dp}{(2\pi)^d}\frac{\Gamma\left[\frac{d}{2}\right]}{\Gamma\left[\frac{d+h}{2}\right]}\frac{h!}{\left[\frac{h}{2}\right]!}2^{-h}(p^2)^{(N-2-r)/2}(l^2)^{(r-2)/2}\mathrm{Tr}(l^2g^{\mu\nu}-l^\mu l^\nu)A_\mu A_\nu . \tag{3.28}$$

Finally, the octopus diagram of Eq. (3.18) is given by summing over all the permutations.

For $r=0$ we get

$$\delta M^{-N}e^2\left[\int\frac{d^dp}{(2\pi)^d}(p^2)^{(N-2)/2}\right]2^{d/2}\left[\sum_{h=0,2}{}'\omega(N,r=0,h)\right]g^{\mu\nu}A_\mu A_\nu . \tag{3.29}$$

For $r\geq 2$ we get

$$\begin{aligned}\delta M^{-N}e^2&\left[\int\frac{d^dp}{(2\pi)^d}(p^2)^{(N-2-r)/2}\right]2^{d/2}(l^2)^{(r-2)/2}A_\mu A_\nu\\ &\times\left\{\left[\sum_{h=0}^{\infty}{}'\omega(N,r,h)\right]l^\mu l^\nu+\left[\sum_{h=0}^{\infty}{}'\frac{4h}{(r+h+2)(r+h)}\omega(N,r,h)\right](l^2g^{\mu\nu}-l^\mu l^\nu)\right.\\ &\left.+\left[\sum_{h=0}^{\infty}{}'\left[1-\frac{4h(r+1)}{(r+h+2)(r+h)}\right]\omega(N,r,h)\right](l^2g^{\mu\nu}-l^\mu l^\nu)\right\},\end{aligned} \tag{3.30}$$

where

$$\omega(N,r,h)\equiv\begin{bmatrix}\frac{N}{2}\\ \frac{r+h+2}{2}\end{bmatrix}\begin{bmatrix}\frac{r+h+2}{2}\\ h\end{bmatrix}\frac{h!}{\left[\frac{h}{2}\right]!}\frac{\Gamma\left[\frac{d}{2}\right]}{\Gamma\left[\frac{d+h}{2}\right]} \tag{3.31}$$

[$\omega \neq 0$ only for $0 \le r \le N-2$ and $0 \le h \le \min(r+2, N-r-2)$].

One should now differentiate Eqs. (3.29) and (3.30) with respect to N and then set N to zero. Since $\omega(N=0)=0$ the differentiation must operate on w. In Appendix B we prove [5]

$$\frac{\partial}{\partial N}\bigg|_{N=0,d=2}\left[\sum_{h=0}^{\infty}{}' \omega(N,r,h)\right]=0 , \tag{3.32}$$

$$\frac{\partial}{\partial N}\bigg|_{N=0,d=2}\left[\sum_{h=0}^{\infty}{}' \frac{4hr}{(r+h+2)(r+h)}\omega(N,r,h)\right]=\frac{r}{2}\frac{\left[\frac{r}{2}\right]!\left[\frac{r}{2}\right]!}{(r+1)!} . \tag{3.33}$$

Notice that all the nonvanishing terms are explicitly gauge invariant except for the term with $r=0$ that is set to zero by gauge-invariant renormalization (see Appendix B). We get for $d=2$ the e^2 octopus diagram

$$-\delta 2e^2\left[g^{\mu\nu}-\frac{l^\mu l^\nu}{l^2}\right]A_\mu A_\nu \sum_{r=0}^{\infty}{}' \frac{r}{2}\frac{\left[\frac{r}{2}\right]!\left[\frac{r}{2}\right]!}{(r+1)!}(l^2)^{r/2}\int\frac{d^dp}{(2\pi)^2}(p^2)^{(-r-2)/2} . \tag{3.34}$$

This has to be multiplied by a factor of 2 which is a symmetry factor for having two ways to choose which of the octopus legs is the incoming and the outgoing photon. After some calculations (see Appendix C) we get, from Eq. (3.34) for $\delta=1$,

$$\left[\frac{l^\mu l^\nu}{l^2}-g^{\mu\nu}\right]\frac{e^2}{\pi} . \tag{3.35}$$

This is indeed the photon propagator in the Schwinger model [3,6].

IV. THE RELATION TO CONVENTIONAL PERTURBATION THEORY

In ordinary perturbation theory in QED, the photon propagator for the Schwinger model to order e^2 is calculated from the first diagram in Fig. 7(b). In Sec. III we calculated the octopus diagram of order e^2 [the first diagram in Fig. 7(a)] in the δ expansion, and reproduced in two dimensions the conventional perturbation theory result multiplied by δ. This is an example of a general equivalence that will be discussed now. The octopus diagram of Fig. 5(a) and the diagram of Fig. 5(b) which is a diagram in conventional QED with δ fermion species have in common one fermion loop with λ attached bosons and a factor of δe^λ. In the following we will argue that they are equivalent. Namely, an octopus diagram calculated within the δ expansion (the fermion propagator being $1/M$, etc.) and the corresponding diagram calculated within conventional QED with δ fermion species (the fermion propagator now being $i\not{p}^{-1}$, etc.) give the same result [7].

The simplest way to show the equivalence is using path integrals. Green's functions with no external fermionic fields are derived from

$$Z_{\text{eff}}(J_\mu)=\int\mathcal{D}A_\mu\mathcal{D}\bar\psi\mathcal{D}\psi \times\exp\left[i\int d^dx[\mathcal{L}(A_\mu,\bar\psi,\psi)+J^\mu A_\mu]\right] . \tag{4.1}$$

The δ expansion for QED uses

$$\mathcal{L}=\mathcal{L}_\delta=M^{1-\delta}\bar\psi\mathcal{D}^\delta\psi-\tfrac{1}{4}(F_{\mu\nu})^2 . \tag{4.2}$$

Integrating out the fermionic fields gives

$$Z_{\text{eff }\delta}(J_\mu)=\int\mathcal{D}A_\mu\det(\mathcal{D}^\delta) \times\exp\left[i\int d^dx[-\tfrac{1}{4}(F_{\mu\nu})^2+J^\mu A_\mu]\right] . \tag{4.3}$$

On the other hand, in QED with δ massless fermion species one has

$$\mathcal{L}=\mathcal{L}_{\text{QED}}=\sum_{i=1}^{\delta}\bar\psi_i\mathcal{D}\psi_i-\tfrac{1}{4}F_{\mu\nu} . \tag{4.4}$$

The generating functional for Green's functions with no external fermions is here

$$Z_{\text{eff QED}}(J_\mu)=\int\mathcal{D}A_\mu\prod_i\mathcal{D}\bar\psi_i\mathcal{D}\psi_i\exp\left[i\int d^dx\left[\sum_{i=1}^{\delta}\bar\psi_i\mathcal{D}\psi_i-\tfrac{1}{4}(F_{\mu\nu})^2+J^\mu A_\mu\right]\right]$$

$$=\int\mathcal{D}A_\mu\left\{\prod_{i=1}^{\delta}\left[\int\mathcal{D}\bar\psi_i\mathcal{D}\psi_i\exp\left[i\int d^dx\bar\psi\mathcal{D}\psi_i\right]\right]\right\}\exp\left[i\int d^dx[-\tfrac{1}{4}(F_{\mu\nu})^2+J^\mu A_\mu]\right] . \tag{4.5}$$

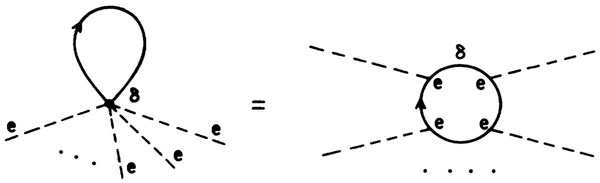

FIG. 8. The equivalence between octopus diagrams in QED_δ and diagrams obtained from conventional perturbation theory for QED with δ fermion species when there are no external fermions (here, to first order in δ). QED_δ: vertex→δ, each boson→e; QED with δ fermions: fermion loop→δ, vertex→e.

Integrating out now the fermionic fields gives

$$Z_{\mathrm{eff\ QED}}(J_\mu) = \int \mathcal{D}A_\mu[\det\mathcal{D}]^\delta \exp\left[i\int d^dx[-\tfrac{1}{4}(F_{\mu\nu})^2+J^\mu A_\mu\right]. \quad (4.6)$$

Since $\det(\mathcal{D}^\delta)=(\det\mathcal{D})^\delta$ Green's functions with only bosonic external fields calculated with Eq. (4.3) equal those calculated with Eq. (4.6). Expanding in powers of δ and e one gets the equality between the diagrams of Figs. 5(a) and 5(b) and of Figs. 7(a) and 7(b). The general equivalence, for first order in δ, is diagramatically represented in Fig. 8. Note, however, that this exact equivalence is restricted to Green's functions with no external fermions.

Some care must be taken as to the interpretation of this equivalence [5]. δ is a continuous parameter $0\le\delta\le1$ in QED_δ while for QED with δ fermions species δ is an integer. The only common acceptable δ is $\delta=1$ where indeed the δ expansion *is* conventional QED. We can treat the equivalence as a formal one, enabling us to obtain results in QED from calculations performed in QED_δ.

V. VERTICES AND DIAGRAMS IN QED_δ IN THE SYMMETRY APPROACH

In this section we will reinforce the equivalence established in Sec. IV by using the fact that in QED_δ one does not need the full N dependence but rather the value of the first derivative with respect to N at $N=0$. We will therefore take a different look at the interaction term $\bar\psi(i\partial\!\!\!/-eA\!\!\!/)^N\psi$. In momentum space it is a sum over elements of the form

$$(-1)^N e^\lambda\bar\psi(p')(p\!\!\!/+l\!\!\!/_1+\cdots+l\!\!\!/_\lambda)^{n_\lambda} \times A\!\!\!/(l_\lambda)\cdots(p\!\!\!/+l\!\!\!/_1+l\!\!\!/_2)^{n_2}A\!\!\!/(l_2)(p\!\!\!/+l\!\!\!/_1)^{n_1} \times A\!\!\!/(l_1)p\!\!\!/^{n_0}\psi(p)\,. \quad (5.1)$$

The sum is over all integers n_j $j=0,\ldots,\lambda$ provided $\sum_{j=0}^{\lambda}n_j=N-\lambda$. We will also sum over all permutations of the indices $1,2,\ldots,\lambda$ and therefore have to divide by $\lambda!$. In the combinatorial approach we have written Eq. (5.1) as a sum given in Sec. III by Eqs. (3.5) and (3.6). Though that form was useful for calculations, the form in Eq. (5.1) that will be treated now enables a better look at the symmetries of the theory.

A general vertex of order e^λ (Fig. 3) is obtained from Eqs. (3.1) and (5.1):

$$\delta M\bar\psi(p')\psi(p)A_{\nu_\lambda}(l_\lambda)\cdots A_{\nu_2}(l_2)A_{\nu_1}(l_1)G_\lambda^{\nu_1\nu_2\cdots\nu_\lambda}(p,l_1,l_2,\ldots,l_\lambda)\,, \quad (5.2)$$

$$G_\lambda^{\nu_1\nu_2\cdots\nu_\lambda}(p,l_1,l_2,\ldots,l_\lambda)=\frac{e^\lambda}{\lambda!}\frac{\partial}{\partial N}\bigg|_{N=0}\{g_\lambda^{\nu_1\nu_2\cdots\nu_\lambda}+\text{all permutations of }1,\ldots,\lambda\}\,, \quad (5.3)$$

where

$$g_\lambda^{\nu_1\nu_2\cdots\nu_\lambda}(p,l_1,l_2,\ldots,l_\lambda)=\sum_{n_0,\ldots,n_\lambda=0}^{\infty}\delta\left[N-\lambda-\sum_{j=0}^{\lambda}n_j\right](p\!\!\!/+l\!\!\!/_1+\cdots+l\!\!\!/_\lambda)^{n_\lambda}\gamma^{\nu_\lambda}\cdots(p\!\!\!/+l\!\!\!/_1+l\!\!\!/_2)^{n_2}\gamma^{\nu_2}(p\!\!\!/+l\!\!\!/_1)^{n_1}\gamma^{\nu_1}p\!\!\!/^{n_0}\,. \quad (5.4)$$

The octopus diagrams of order e^λ [Fig 5(a)] are the effective vertices for QED_δ with no external fermions. Closing the fermionic loop one has (as explained in Sec. III A)

$$\delta A_{\nu_\lambda}(l_\lambda)\cdots A_{\nu_2}(l_2)A_{\nu_1}(l_1)F_\lambda^{\nu_1\nu_2\cdots\nu_\lambda}(l_1,l_2,\ldots,l_\lambda)\,, \quad (5.5)$$

$$F_\lambda^{\nu_1\nu_2\cdots\nu_\lambda}(l_1,l_2,\ldots,l_\lambda)=\int\frac{d^dp}{(2\pi)^d}\frac{e^\lambda}{\lambda!}\frac{\partial}{\partial N}\bigg|_{N=0}\{f_\lambda^{\nu_1\cdots\nu_\lambda}+\text{all permutations of }1,\ldots,\lambda\}\,, \quad (5.6)$$

where

$$f_\lambda^{\nu_1\nu_2\cdots\nu_\lambda}(l_1,l_2,\ldots,l_\lambda)=\sum_{m_1,\ldots,m_\lambda=0}^{\infty}\delta\left[N-\lambda-\sum_{\alpha=1}^{\lambda}m_\alpha\right](m_\lambda+1)T_r[p\!\!\!/^{m_\lambda}\gamma^{\nu_\lambda}\cdots(p\!\!\!/+l\!\!\!/_1+l\!\!\!/_2)^{m_2}\gamma^{\nu_2}(p\!\!\!/+l\!\!\!/_1)^{m_1}\gamma^{\nu_1}]\,. \quad (5.7)$$

In the derivation of Eq. (5.7) from Eq. (5.4) we used $l_1+l_2+\cdots+l_\lambda=0$ and the fact that the trace allows cyclic permutations, and renamed summation indices $m_\alpha=n_\alpha$ $\alpha=1,\ldots,\lambda-1$; $m_\lambda=n_\lambda+n_0$. $(m_\lambda+1)$ different terms in f_λ

(those with the same value for $m_\lambda = n_0 + n_\lambda$ but different $n_0 = 0,1,2,\ldots,m_\lambda$) are shown to be identical by cyclic permutations.

We will show now that F_λ contains sets of N terms which are N cyclic permutations of the same term up to a shift in the loop momentum p and are therefore identical. (In Appendix D it is shown that for $\lambda < d$ the equality holds for the most divergent part and for $\lambda \geq d$ it is exact.) We call this the "cyclic symmetry" of the effective Lagrangian in QED$_\delta$ with no external fermions. In order to get F from f one has to sum in Eq. (5.6) over all permutations of the indices $1,\ldots,\lambda$. We will first sum the cyclic permutations:

$$\{f_\lambda^{\nu_1 \cdots \nu_\lambda} + \text{all cyclic permutations of the indices } 1,\ldots,\lambda\}$$
$$= \mathrm{Tr} \sum_{m_1,m_2,\ldots,m_\lambda=0}^{\infty} \delta\left[N-\lambda-\sum_{\alpha=0}^{\lambda} m_\alpha\right]$$
$$\times [\not{p}^{m_\lambda}\gamma^{\nu_\lambda}\cdots(\not{p}+\not{l}_1+\not{l}_2)^{m_2}\gamma^{\nu_2}(\not{p}+\not{l}_1)^{m_1}\gamma^{\nu_1}(m_\lambda+1)$$
$$+\not{p}^{m_1}\gamma^{\nu_1}\cdots(\not{p}+\not{l}_2+\not{l}_3)^{m_3}\gamma^{\nu_3}(\not{p}+\not{l}_2)^{m_2}\gamma^{\nu_2}(m_1+1)$$
$$+\not{p}^{m_2}\gamma^{\nu_2}\cdots(\not{p}+\not{l}_3+\not{l}_4)^{m_4}\gamma^{\nu_4}(\not{p}+\not{l}_3)^{m_3}\gamma^{\nu_3}(m_2+1)$$
$$\vdots$$
$$+\not{p}^{m_{\lambda-1}}\gamma^{\nu_{\lambda-1}}\cdots(\not{p}+\not{l}_\lambda+\not{l}_1)^{m_1}\gamma^{\nu_1}(\not{p}+\not{l}_\lambda)^{m_\lambda}\gamma^{\nu_\lambda}(m_{\lambda-1}+1)] \,. \tag{5.8}$$

Apart from the factors (m_k+1) for the kth line all the lines in Eq. (5.8) are cyclic permutations (under the trace of the matrices) of the first line with the loop momentum shift $p+l_1+l_2+\cdots+l_{k-1}\to p$. A single shift properly arranges all the line because the elements to the left of γ^{ν_j} equal the elements to its right plus l_j. Summing all λ lines in Eq. (5.8) we have $\sum_{k=1}^{\lambda}(m_k+1)=N$ identical elements:

$$\{f_\lambda^{\nu_1 \cdots \nu_\lambda} + \text{all cyclic permutations of the indices } 1,\ldots,\lambda\}$$
$$= N \sum_{m_1,\ldots,m_\lambda=0}^{\infty} \delta\left[N-\lambda-\sum m\right] \mathrm{Tr}[\not{p}^{m_\lambda}\gamma^{\nu_\lambda}\cdots(\not{p}+\not{l}_1)^{m_1}\gamma^{\nu_1}] \,. \tag{5.9}$$

This is an important result, since we can use

$$\left.\frac{\partial}{\partial N}\right|_{N=0}[NX(N)] = X(N=0) \,.$$

Thus, after taking the derivative with respect to N and setting N to zero, we get, for the octopus diagram of Fig. 5(a),

$$\delta A_{\nu_\lambda}(l_\lambda)\cdots A_{\nu_2}(l_2)a_{\nu_1}(l_1)\int\frac{d^dp}{(2\pi)^d}\frac{e^\lambda}{\lambda!}\left[\sum_{m_1,m_2,\ldots,m_\lambda=0}^{\infty}\delta\left[N-\lambda-\sum_{\alpha=1}^{\lambda}m_\alpha\right]\right.$$
$$\left.\left.\times\mathrm{Tr}[\not{p}^{m_\lambda}\gamma^{\nu_\lambda}\cdots(\not{p}+\not{l}_1+\not{l}_2)^{m_2}\gamma^{\nu_2}(\not{p}+\not{l}_1)^{m_1}\gamma^{\nu_1}] + \text{noncyclic permutations of } 1,\ldots,\lambda\right]\right|_{N=0} \,. \tag{5.10}$$

Note that Eqs. (5.9) and (5.10) are exact for $\lambda \geq d$ but for $\lambda < d$ they apply only to the most divergent part (see Appendix D).

In Sec. IV we have established the equivalence between the octopus diagram in Fig. 5(a) and the diagram in Fig. 5(b). Conventional perturbation theory of QED gives for the diagram in Fig. 5(b) [3,6]

$$-\delta A_{\nu_\lambda}(l_\lambda)\cdots A_{\nu_2}(l_2)A_{\nu_1}(l_1)\int\frac{d^dp}{(2\pi)^d}\frac{(-e)^\lambda}{\lambda!}$$
$$\times\{\mathrm{Tr}[\not{p}^{-1}\gamma^{\nu_\lambda}\cdots(\not{p}+\not{l}_1+\not{l}_2)^{-1}\gamma^{\nu_2}(\not{p}+\not{l}_1)^{-1}\gamma^{\nu_1}] + \text{noncyclic permutations of } 1,\ldots,\lambda\} \,. \tag{5.11}$$

(In both cases, when calculating Green's functions, the symmetry factor cancels $\lambda!$.) The main difference between Eq. (5.10) and Eq. (5.11) is that instead of $m_\alpha=-1$, $\alpha=1,\ldots,\lambda$ in conventional QED we obtained for the δ expansion $\lambda-1$ sums over all positive integer values of m_α, $\alpha=1,\ldots,\lambda$ provided $\sum_{\alpha=1}^{\lambda}m_\alpha=N-\lambda$. Note that for $N=0$ the δ function in Eq. (5.10) is consistent with $m_\alpha=-1$ for each α, but this calls for a more rigorous treatment [8]. In Appendix E we give a complete calculation of Eq. (5.10) for $\lambda=4$ and indeed get Eq. (5.11). We prove there that

$$\left[\sum_{m_4,m_3,m_2,m_1=0}^{\infty}\delta(N-4-m_1-m_2-m_3-m_4)\right.$$
$$\left.\left.\times\mathrm{Tr}[\not p^{m_4}\gamma^{\nu_4}(\not p+\not l_1+\not l_2+\not l_3)^{m_3}\gamma^{\nu_3}(\not p+\not l_1+\not l_2)^{m_2}\gamma^{\nu_2}(\not p+\not l_1)^{m_1}\gamma^{\nu_1}]\right]\right|_{N=0}$$
$$=-\mathrm{Tr}[\not p^{-1}\gamma^{\nu_4}(\not p+\not l_1+\not l_2+\not l_3)^{-1}\gamma^{\nu_3}(\not p+\not l_1+\not l_2)^{-1}\gamma^{\nu_2}(\not p+\not l_1)^{-1}\gamma^{\nu_1}]\ . \tag{5.12}$$

This result completes the calculation of the photon propagator in the Schwinger model to order e^4. Since the right-hand-side (RHS) of Eq. (5.12) is shown to be zero in the context of conventional perturbation theory for QED for $d=2$ [6], the octopus diagram to order e^4 equals zero for $d=2$ and thus the contribution of order e^4 to the photon propagator vanishes. The significance of Eq. (5.12) goes beyond the context of this particular calculation—it reinforces the observation that was made in Sec. IV concerning the equivalence of QED_δ to conventional perturbation theory in QED for Green's functions with no external fermions.

VI. WARD IDENTITIES AND GAUGE INVARIANCE

In analogy to the well-known Feynman identity for conventional QED [6], we write an identity that can play a similar role for the δ expansion. In Appendix F we prove that

$$\sum_{n_1=0}^{\infty}\sum_{n_0=0}^{\infty}(\not p+\not l)^{n_1}\not l\not p^{n_0}=\sum_{\alpha=0}^{\infty}[(\not p+\not l)^{\alpha}-\not p^{\alpha}]\ . \tag{6.1}$$

The amputated Green's function $G_\lambda^{\nu_1\nu_2\cdots\nu_\lambda}(p,l_1,\ldots,l_\lambda)$ was given in Sec. V [Eqs. (5.2)–(5.4)]. In analogy with the Ward identities in conventional perturbative QED [6], we want to write the contraction of G_λ with the momentum of one of the bosons, as a sum of $G_{\lambda-1}$'s. Using Eq. (6.1) this can be easily done. In Appendix G we prove the main result of this section, the Ward identities for QED_δ:

$$l_{j_{\nu_j}}G_\lambda^{\nu_1\nu_2\cdots\nu_j\cdots\nu_\lambda}(p_\psi,l_1,\ldots,l_j,\ldots,l_\lambda)$$
$$=\frac{e}{\lambda}G_{\lambda-1}^{\nu_1\nu_2\cdots\nu_{j-1}\nu_{j+1}\cdots\nu_\lambda}(p_\psi+l_j,l_1,\ldots,l_{j-1},l_{j+1},\ldots,l_\lambda)-\frac{e}{\lambda}G_{\lambda-1}^{\nu_1\nu_2\cdots\nu_{j-1}\nu_{j+1}\cdots\nu_\lambda}(p_\psi,l_1,\ldots,l_{j-1},l_{j+1},\ldots,l_\lambda)\ . \tag{6.2}$$

Diagrammatic representation of Eq. (6.2) is given in Fig. 9.

In the special case, described in previous sections, when there are no external fermions the effective vertex (the octopus) is given by Eqs. (5.5)–(5.7). The amputated Green's functions in this case are

$$F_\lambda^{\nu_1\nu_2\cdots\nu_\lambda}(l_1,l_2,\ldots,l_\lambda)=\int\frac{d^dp}{(2\pi)^d}\mathrm{Tr}G_\lambda^{\nu_1\nu_2\cdots\nu_\lambda}(p,l_1,l_2,\ldots,l_\lambda)\ . \tag{6.3}$$

Using Eq. (6.2), the contraction of an octopus with l_j gives

$$l_{j_{\nu_j}}F_\lambda^{\nu_1\cdots\nu_j\cdots\nu_\lambda}(l_1,\ldots,l_\lambda)=\frac{e}{\lambda}\int\frac{d^dp}{(2\pi)^d}\mathrm{Tr}G_{\lambda-1}^{\nu_1\nu_2\cdots\nu_{j-1}\nu_{j+1}\cdots\nu_\lambda}(p_\psi+l_j,l_1,\ldots,l_{j-1},l_{j+1},\ldots,l_\lambda)$$
$$-\frac{e}{\lambda}\int\frac{d^dp}{(2\pi)^d}\mathrm{Tr}G_{\lambda-1}^{\nu_1\nu_2\cdots\nu_{j-1}\nu_{j+1}\cdots\nu_\lambda}(p_\psi,l_1,\ldots,l_{j-1},l_{j+1},\ldots,l_\lambda)\ . \tag{6.4}$$

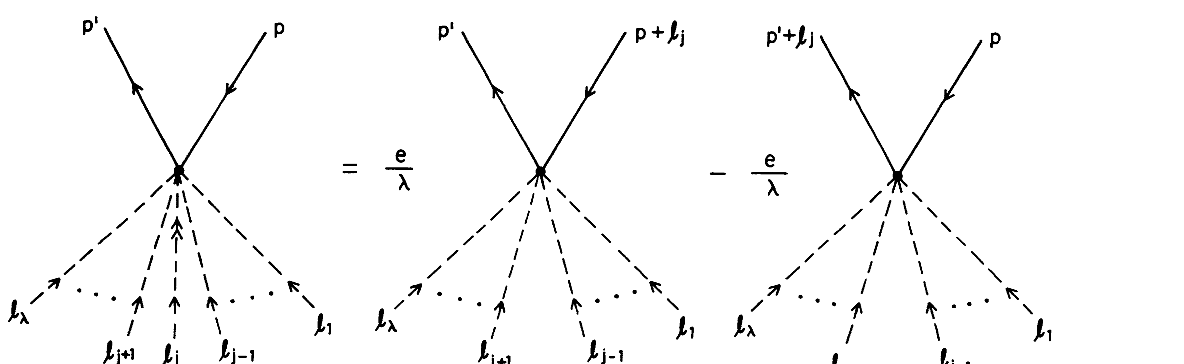

FIG. 9. Ward identities for amputated vertices in QED_δ. The double arrow on the l_j line indicates contraction with the boson's momentum.

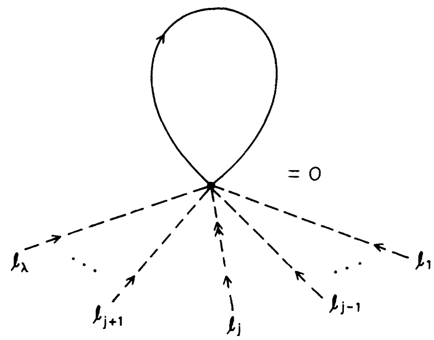

FIG. 10. Gauge invariance of the "octopus" diagram. (The A_j line, denoted with a double arrow, is contracted with the boson's momentum l_j.)

This is zero provided one can make a finite shift in the loop momentum (see discussion in Appendix D):

$$l_{j_{\nu_j}} F_\lambda^{\nu_1 \cdots \nu_j \cdots \nu_\lambda}(l_1,\ldots,l_\lambda)=0 \ . \tag{6.5}$$

Diagramatic representation of Eq. (6.5) is given in Fig. 10. Equation (6.5) has a well-known meaning: it expresses the gauge invariance of the octopus diagrams. In conventional QED gauge invariance of diagrams with one fermionic loop and λ attached bosons was discussed in a similar way in Ref. [6].

VII. CONCLUSIONS

Symmetry and several combinatorial results were used in this paper in order to formulate new and improved rules for the δ expansion and its application to QED. Recent results in the application of the δ expansion to systems with local gauge symmetry are promptly reproduced by using these rules. Ward identities are formulated for the first time in the δ expansion, reflecting the symmetry of the system to all orders in e [see, e.g., Eq. (6.2) and Fig. (9)].

The relation between the expansion in δ of QED with one fermion species [Eq. (4.2)] and QED with δ fermion species [Eq. (4.4)] has been demonstrated using a path integral in Sec. IV and explicit calculation to first order in δ and all orders in e. This gives an appealing presentation of the δ expansion as an expansion around $\delta=0$ fermion species in case of Green's functions with no external fermion lines. This equivalence to conventional perturbation theory is clearly demonstrated here.

The calculation rules, Ward identities, and the above observations are appropriate tools for future calculations in QED in four dimensions using the δ-expansion calculation method.

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APPENDIX A: DERIVATION OF E FROM E_N IN THE ONE-DIMENSIONAL MODEL [2]

In Sec. II B, E_N the ground-state energy for a one-dimensional model with the provisional Lagrangian was calculated [Eq. (2.14)]. Substituting Eq. (2.9) into Eq. (2.14) and taking the sum over K up to infinity, since

$$\begin{bmatrix} N \\ 2K+\cdots \end{bmatrix}=0 \quad \text{when } K>\frac{N}{2} \ ,$$

gives

$$E_N=-\delta M^{1-N}\sum_{K=1}^{\infty}\frac{g^{2K}}{K!}\left[\frac{i\pi}{T}\right]^{N-2K}\sum_{r_1,\ldots,r_K=0}^{\infty}{}' \begin{bmatrix} N \\ \sum_{j=1}^{K} r_j+2K \end{bmatrix} \frac{\left[\sum_{j=1}^{K} r_j+2K\right]!}{\prod_{j=1}^{K}(r_j+2)!}\left[\sum_{n=-\infty}^{\infty}\frac{1}{T}\frac{1}{M}(2n-1)^{N-2K-\sum_{j=1}^{K}r_j}\right]$$
$$\times\prod_{j=1}^{K}\left[\sum_{s_j=-\infty}^{\infty}\frac{1}{T}\frac{1}{\left[\frac{2\pi S_j}{T}\right]^2+m^2}(2S_j)^{r_j}\right] \ . \tag{A1}$$

In order to find the ground-state energy E of the one-dimensional model in the δ expansion to first order in δ, one has to differentiate (A1) with respect to N and set N to zero using the identity [5]

$$\left.\frac{\partial}{\partial N}\right|_{N=0}\begin{bmatrix} N \\ \alpha \end{bmatrix}=\frac{(-1)^{\alpha-1}}{\alpha} \ , \tag{A2}$$

$$E=\left.\frac{\partial}{\partial N}\right|_{N=0}E_N=\frac{\delta}{T}\sum_{K=1}^{\infty}\frac{g^{2K}}{K!}\left[\frac{-T}{\pi^2}\right]^{K}\sum_{s_1,\ldots,s_K=0}^{\infty}\left[\prod_{j=1}^{K}\left[\frac{1}{\left[\frac{2\pi s_j}{T}\right]^2+m^2}\right]\right]X_K(s_j|_{j=1,\ldots,K}) \ , \tag{A3}$$

$$X_K(s_j|_{j=1,\dots,K}) \equiv \sum_{n=-\infty}^{\infty} (2n-1)^{-2K} \sum_{r_1,\dots,r_K=0}^{\infty}{}' \frac{1}{\left[\sum_j r_j+2K\right]} \frac{\left[\sum_j r_j+2K\right]!}{\prod_{j=1}^{K}(r_j+2)!} \prod_{j=1}^{K}\left[\frac{2s_j}{2n-1}\right]^{r_j} . \quad \text{(A4)}$$

X_K vanishes unless all s_j are zero [2]. For $s_j=0$ we are left with the $r_j=0$ term only:

$$X_k(s_j=0|_{j=1,\dots,K}) = \sum_{n=-\infty}^{\infty} (2n-1)^{-2K}\frac{(2K-1)!}{2^K} . \quad \text{(A5)}$$

Substituting Eq. (A5) into Eq. (A3) and using $\sum_{n=-\infty}^{\infty}(2n-1)^{-2K}=2(1-2^{-2K})\zeta(2K)$ where ζ is Riemann's zeta function, one gets Eq. (2.15):

$$E=\frac{\delta}{T}\sum_{K=1}^{\infty}\frac{(-1)^K}{K}(1-2^{-2K})(2K-1)!!\frac{\zeta(2K)}{\pi^{2K}}\left[\frac{g^2T}{m^2}\right]^K . \quad \text{(A6)}$$

APPENDIX B: THE OCTOPUS DIAGRAM OF ORDER e^2

The first step in the δ-expansion method for calculating the octopus diagram of order e^2 gives Eq. (3.29) and (3.30). The second step consists of $\partial/\partial N|_{N=0}$. There are only two kinds of terms to differentiate and the differentation of both is given in this appendix, proving Eqs. (3.32) and (3.33).

First we prove

$$\left.\frac{\partial}{\partial N}\right|_{N=0,d=2}\left[\sum_{h=0}^{\infty}{}' \omega(N,r,h)\right]=0 , \quad \text{(B1)}$$

where

$$\omega(N,r,h)\equiv\begin{bmatrix}\frac{N}{2}\\ \frac{r+h+2}{2}\end{bmatrix}\begin{bmatrix}\frac{r+h+2}{2}\\ h\end{bmatrix}\frac{h!}{\left[\frac{h}{2}\right]!}\frac{\Gamma\left[\frac{d}{2}\right]}{\Gamma\left[\frac{d+h}{2}\right]} . \quad \text{(B2)}$$

Substituting Eq. (A2),

$$\left.\frac{\partial}{\partial N}\right|_{N=0}\left[\sum_{h=0}^{r+2}{}' \omega(N,r,h)\right]=(-1)^{r/2}\Gamma\left[\frac{d}{2}\right]\sum_{h=0}^{r+2}{}'(-1)^{h/2}\frac{1}{(r+h+2)}\begin{bmatrix}\frac{r+h+2}{2}\\ h\end{bmatrix}\frac{h!}{\left[\frac{h}{2}\right]!}\frac{1}{\Gamma\left[\frac{d+h}{2}\right]}$$

$$=(-1)^{m+1}\Gamma(D)\frac{1}{m!}\frac{1}{2}\sum_{\alpha=0}^{m}\begin{bmatrix}m\\ \alpha\end{bmatrix}(-1)^{\alpha}\frac{(m+\alpha-1)!}{(D+\alpha-1)!} , \quad \text{(B3)}$$

where $D\equiv d/2$, $\alpha\equiv h/2$ and $m\equiv(r+2)/2$ are all integers. Using the beta function [5] $B(x,y)=\Gamma(x)\Gamma(y)/\Gamma(x+y)=\int_0^1 dt\, t^{x-1}(1-t)^{y-1}$ we get

$$\sum_{\alpha=0}^{m}\begin{bmatrix}m\\ \alpha\end{bmatrix}(-1)^{\alpha}\frac{\Gamma(m+\alpha)}{\Gamma(D+\alpha)}=\frac{1}{\Gamma(D-m)}\sum_{\alpha=0}^{m}\begin{bmatrix}m\\ \alpha\end{bmatrix}(-1)^{\alpha}B(m+\alpha;D-m)$$

$$=\frac{1}{\Gamma(D-m)}\sum_{\alpha=0}^{m}\begin{bmatrix}m\\ \alpha\end{bmatrix}(-1)^{\alpha}\int dx\, x^{D-m-1}(1-x)^{m+\alpha-1}$$

$$=\frac{1}{\Gamma(D-m)}\int_0^1 dx\, x^{D-m-1}(1-x)^{m-1}\sum_{\alpha=0}^{m}\begin{bmatrix}m\\ \alpha\end{bmatrix}(x-1)^{\alpha}$$

$$=\frac{1}{\Gamma(D-m)}\int_0^1 dx\, x^{D-1}(1-x)^{m-1}=\frac{1}{\Gamma(D-m)}B(D,m)$$

$$=\frac{1}{\Gamma(D-m)}\frac{\Gamma(D)\Gamma(m)}{\Gamma(D+m)} . \quad \text{(B4)}$$

If $m\geq D\geq 1$ ($r\geq d-2\geq 0$) this is zero. (For $d=2$ it is zero for all $r\geq 0$ but for $d=4$ it is zero only for $r\geq 2$.) Thus we have proven Eq. (B1) for $d=2$. Notice that at $r=0$ the expression in Eq. (B3), which vanishes at $d=2$, multiplies a logarithmic divergent integral. The product is set to zero by gauge invariance [3].

Next we prove

$$\frac{\partial}{\partial N}\Bigg|_{N=0,d=2}\left[\sum_{h=0}^{\infty}{}' \frac{4hr}{(r+h+2)(r+h)}\omega(N,r,h)\right]=\frac{r}{2}\frac{\left[\frac{r}{2}\right]!\left[\frac{r}{2}\right]!}{(r+1)!} . \tag{B5}$$

Substituting again Eq. (A2),

$$\frac{\partial}{\partial N}\Bigg|_{N=0}\left[\sum_{h=0}^{\infty}{}' \frac{4hr}{(r+h+2)(r+h)}\omega(N,r,h)\right]$$

$$=(-1)^{r/2}\Gamma\left[\frac{d}{2}\right]r\sum_{h=0}^{r+2}{}'(-1)^{h/2}\frac{4h}{(r+h+2)^2(r+h)}\begin{bmatrix}\frac{r+h+2}{2}\\ h\end{bmatrix}\frac{h!}{\left[\frac{h}{2}\right]!}\frac{1}{\Gamma\left[\frac{d+h}{2}\right]}$$

$$=-2n(-1)^n\Gamma(D)\frac{1}{n!}\sum_{\beta=0}^{n}\begin{bmatrix}n\\ \beta\end{bmatrix}(-1)^\beta\frac{(n+\beta)!}{(D+\beta)!}\frac{1}{(n+\beta+2)} , \tag{B6}$$

where $\beta\equiv h/2-1$, $n\equiv r/2$ and $D\equiv d/2$ are all integers. We will now use $1/(n+\beta+2)=\int_0^\infty dt\, e^{-t(n+\beta+2)}$. For $n\geq D\geq 0$,

$$\sum_{\beta=0}^{n}\begin{bmatrix}n\\ \beta\end{bmatrix}(-1)^\beta\frac{(n+\beta)!}{(D+\beta)!}\frac{1}{(n+\beta+2)}$$

$$=\left[\frac{\partial}{\partial x}\right]^{n-D}\Bigg|_{x=1}\left[x^{-2}\int_0^\infty dt(xe^{-t})^{n+2}\sum_{\beta=0}^{n}\begin{bmatrix}n\\ \beta\end{bmatrix}(-xe^{-t})^\beta\right]$$

$$=\left[\frac{\partial}{\partial x}\right]^{n-D}\Bigg|_{x=1}\left[x^{-2}\int_0^x dz\, z^{n+1}(1-z)^n\right]$$

$$=\left[\left[\frac{\partial}{\partial x}\right]^{n-D}\Bigg|_{x=1}x^{-2}\right]\left[\int_0^1 dz\, z^{n+1}(1-z)^n\right]$$

$$=\frac{1}{2}(-1)^{n-D}(n-D+1)!\frac{n!n!}{(2n+1)!} , \tag{B7}$$

where we have substituted $z\equiv xe^{-t}$ and used $(n+\beta)!/(D+\beta)!=(\partial/\partial x)^{n-D}|_{x=1}x^{\beta+n}$ [5]. ($n=D$ implies no derivative.) Substituting this into Eq. (B6) one gets, for $r\geq d\geq 0$,

$$\frac{\partial}{\partial N}\Bigg|_{N=0}\left[\sum_{h=0}^{\infty}{}' \frac{4hr}{(r+h+2)(r+h)}\omega(N,r,h)\right]$$

$$=(-1)^{1-d/2}\Gamma\left[\frac{d}{2}\right]\frac{r}{2}\left[\frac{r}{2}-\frac{d}{2}+1\right]!\frac{\left[\frac{r}{2}\right]!}{(r+1)!} . \tag{B8}$$

For $d=2$ this completes the proof of Eq. (B5).

APPENDIX C: THE PHOTON PROPAGATOR IN THE SCHWINGER MODEL (COMPLETED)

In this appendix the calculation of the photon propagator in the Schwinger model is completed. The relevant diagram, namely the octopus diagram of order e^2 was calculated in Sec. III B [Eq. (3.34)]. Multiplying Eq. (3.34) by a symmetry factor 2 gives

$$\delta e^2\left[\frac{l^\mu l^\nu}{l^2}-g^{\mu\nu}\right]A_\mu A_\nu C , \tag{C1}$$

$$C=4\sum_{r=0}^{\infty}{}'\frac{r}{2}\frac{\left[\frac{r}{2}\right]!\left[\frac{r}{2}\right]!}{(r+1)!}(l^2)^{r/2}\int\frac{d^2p}{(2\pi)^2}(p^2)^{(-r-2)/2} . \tag{C2}$$

$d^2p=2\pi p\, dp$ where $p\equiv\sqrt{p_\mu p^\mu}$, $l\equiv\sqrt{l_\mu l^\mu}$. Each term ($r\geq 2$) in (C2) gives an integral that is infrared divergent. However, if one performs the sum over r first, one obtains a convergent integral [3]. Taking $r/2=n$, $(l/p)=z$ one gets

$$C=\frac{4}{2\pi}\int_0^\infty\frac{dz}{z}\sum_{n=0}^{\infty}z^{2n}n\frac{n!n!}{(2n+1)!} . \tag{C3}$$

Using again the beta function $n!n!/(2n+1)!=\int_0^1 x^n(1-x)^n dx$ [5] and taking $y=x(1-x)z^2-1$,

$$C=\frac{2}{\pi}\int_0^\infty \frac{dz}{z}\int_0^1 dx \sum_{n=0}^{\infty} n[x(1-x)z^2]^n$$
$$=\frac{2}{\pi}\int_0^\infty z\,dz\int_0^1 x(1-x)dx\frac{\partial^2}{\partial[x(1-x)z^2]^2}(-\ln\{1-[x(1-x)z^2]\})$$
$$=\frac{1}{\pi}\int_0^1 dx\int_{-1}^\infty \frac{dy}{y^2}=\frac{1}{\pi}\ . \tag{C4}$$

Substituting C back into (C1) one gets

$$\delta\frac{e^2}{\pi}\left[\frac{l^\mu l^\nu}{l^2}-g^{\mu\nu}\right]A_\mu A_\nu\ . \tag{C5}$$

For $\delta=1$ this is indeed the photon propagator in two-dimensional QED [Eq. (3.35)].

APPENDIX D: DISCUSSION OF THE SHIFT IN THE LOOP MOMENTUM INTEGRATION

In the proof of the cyclic symmetry in Sec. V and in the verification of gauge invariance in Sec. VI an assumption was made that one can shift the loop momentum p. In the following we will discuss the legitimacy of this shift taking into account ultraviolet and infrared divergences.

Each one of the integrals summed over is of the form [Eqs. (5.6) and (5.7)]

$$\frac{\partial}{\partial N}\Bigg|_{N=0}\sum_{m_1,\ldots,m_\lambda=0}^{\infty}\delta\left[N-\lambda-\sum_{\alpha=1}^{\lambda}m_\alpha\right]\int\frac{d^dp}{(2\pi)^d}$$
$$\mathrm{Tr}[\not p^{m_\lambda}\not A_\lambda\cdots(\not p+\not l_1+\not l_2)^{m_2}\not A_2(\not p+\not l_1)^{m_1}\not A_1\ . \tag{D1}$$

1. Infrared divergences

Equation (D1) has been calculated in this paper and in Ref. [3] for $d=2$ and $\lambda=2$. IR divergences were treated by summing over $\{m_j\}$ before integrating the momentum (see Appendix C and Ref. [3]). Increasing d can only help to soften the infrared divergences. In order to discuss the effect of increasing λ one can study a simplified from of Eq. (D1): namely

$$\sim\frac{\partial}{\partial N}\Bigg|_{N=0}\sum_m f(m,\lambda,N)\int\frac{d^dp}{(2\pi)^d}l^m p^{N-\lambda-m}$$
$$=\int\frac{d^dp}{(2\pi)^d}\sum_m l^m p^{-\lambda-m}\frac{\partial}{\partial N}\Bigg|_{N=0} f(m,\lambda,N)\ . \tag{D2}$$

The derivative must act on f since $f(m,\lambda,N=0)=0$ because f is basically, as was explained in Secs. II and III, a combinatorial weight for distributing N objects. Note that the continuation of N from the integers to the real axis is used here for regularization [5]. In the case of the two-point function it has been shown in Appendix C that, for $d=2$ [Eq. (C2)],

$$\frac{\partial}{\partial N}\Bigg|_{N=0} f(\text{even } m,\lambda=2,N)=\frac{m}{2}\frac{\left[\frac{m}{2}\right]!\left[\frac{m}{2}\right]!}{(m+1)!}$$

The infrared divergences in the general case might be overcome by first performing the sum over m, as was done in Appendix C. If the sum

$$\sum_m l^m p^{-\lambda-m}\frac{\partial}{\partial N}\Bigg|_{N=0} f(m,\lambda,N) \tag{D3}$$

converges in the limit of small p to $\sim p^z$ where $z>\lambda-d$ there are no IR divergences, and the shift was indeed proper. (For the previous example $z=2$ [Eq. (C4)] and this condition is indeed satisfied.) For higher Green's functions ($\lambda\geq d+z$) or for cases where the sum over m does not converge, IR divergences are unavoidable and should be treated, as they are usually treated in quantum field theory, by taking into account the experimental limit as an IR cutoff.

2. Ultraviolet divergences

The highest power of p in Eq. (D1) is $p^{N-\lambda}$. For the two-point function calculated in Sec. III the relevant term is given by Eq. (3.29). The leading UV divergences are taken care of by $\partial/\partial N|_{N=0}$ as was shown in Ref. [3] in the discussion preceding Eq. (4.9) there [5]. Setting N to zero before integrating gives for large p an integrand proportional to $p^{-\lambda}$. As long as $\lambda\geq d$ there are no ultraviolet divergences (for $\lambda=d$ there is a logarithmic divergence which still allows a shift in p). Since for $d=2$ this condition is always satisfied, a shift in the loop momentum gives the exact results. For $d>2$ this is also true for high Green's functions but not for low ones. For $\lambda<d$ UV divergences appear and a regularization scheme is needed. One can show that if an UV cutoff is used then for lower Green's functions in higher dimension the shift in the loop momentum results in adding a less divergent part to the integral. Thus, the equations that were proven by using the shift [Eqs. (5.9), (5.10), and (6.5)] hold in these cases only to the leading divergent order.

APPENDIX E: ORDER e^4 OF THE PHOTON PROPAGATOR (AN EXAMPLE FOR THE EQUIVALENCE TO CONVENTIONAL PERTURBATION THEORY)

In this appendix Eq. (5.12) is proven by calculating Eq. (5.10) for $\lambda=4$. We want to calculate

$$Z=\left[\sum_{m_3=0}^{N-4}\sum_{m_2=0}^{N-4-m_3}\sum_{m_1=0}^{N-4-m_3-m_2}\mathrm{Tr}(\not{q}_0^{N-4-m_3-m_2-m_1}\gamma^{\nu_4}\not{q}_3^{m_3}\gamma^{\nu_3}\not{q}_2^{m_2}\gamma^{\nu_2}\not{q}_1^{m_1}\gamma^{\nu_1})\right]\Bigg|_{N=0} . \tag{E1}$$

There are $2^3=8$ possibilities for the parities of m_3, m_2, and m_1 (N is even). We numbered them by i, and treated each one of them separately. All eight sums are of the same form with different parameters A_i, B_i, C_i, D_i and F_i. Table I gives for each choice of parity (e stands for even, o for odd) the relevant parameters ($m_0=N-4-m_3-m_2-m_1$), and we get

$$Z=\left[\sum_{i=1}^{8}\mathrm{Tr}F_i(q_0^2)^{D_i}\sum_{[m_3/2]=0}^{C_i}\left[\frac{q_3^2}{q_0^2}\right]^{[m_3/2]}\sum_{[m_2/2]=0}^{B_i-[m_3/2]}\left[\frac{q_2^2}{q_0^2}\right]^{[m_2/2]}\sum_{[m_1/2]=0}^{A_i-[m_3/2]-[m_2/2]}\left[\frac{q_1^2}{q_0^2}\right]^{[m_1/2]}\right]\Bigg|_{N=0} . \tag{E2}$$

To calculate Eq. (E1) we have to calculate eight sums of the form $S_3(z,y,x)$ for C_i, B_i, and A_i given in Table I where

$$S_3(z,y,x)\equiv\sum_{c=0}^{C_i}z^c\sum_{b=0}^{B_i-c}y^b\sum_{a=0}^{A_i-b-c}x^a=\frac{1}{1-x}S_2(z,y)-\frac{x^{A_i+1}}{1-x}S_2\left[\frac{z}{x},\frac{y}{x}\right] , \tag{E3}$$

$$S_2(z,y)\equiv\sum_{c=0}^{C_i}z^c\sum_{b=0}^{B_i-c}y^b=\frac{1}{1-y}s_1(z)-\frac{y^{B_i+1}}{1-y}S_1\left[\frac{z}{y}\right] , \tag{E4}$$

$$S_1(z)\equiv\sum_{c=0}^{C_i}z^c=\frac{1-z^{C_i+1}}{1-z} . \tag{E5}$$

When N is set to zero [5], the results are

$$S_1(z)=\begin{cases}-\frac{1}{z} & \text{for } C_i=-2,\\ -\frac{1}{z}\left[1+\frac{1}{z}\right] & \text{for } C_i=-3 ,\end{cases} \tag{E6}$$

$$S_2(z,y)=\begin{cases} & \text{for } C_i \quad B_i\\ 0 & \quad -2 \quad -2\\ \frac{1}{z}\frac{1}{1-y}\left[\frac{1}{y}-1\right] & \quad -2 \quad -3\\ \frac{1}{z}\frac{1}{1-y}\left[\frac{1}{y}-1\right] & \quad -3 \quad -3\end{cases} \tag{E7}$$

TABLE I. $2^3=8$ possibilities for the parities of m_3, m_2, and m_1 in Eq. (E1) (N is even) are numbered by i, and treated separately. All eight sums are of the same form given in Eq. (E2) with different parameters A_i, B_i, C_i, D_i, and F_i. (e stands for even, o for odd and $m_0=N-4-m_3-m_2-m_1$.)

m_3	m_2	m_1	m_0	F_i	C_i	B_i	A_i	D_i	i
e	e	e	e	$\gamma^{\nu_4}\gamma^{\nu_3}\gamma^{\nu_2}\gamma^{\nu_1}$	$\frac{N}{2}-2$	$\frac{N}{2}-2$	$\frac{N}{2}-2$	$\frac{N}{2}-2$	1
e	e	o	o	$\gamma^{\nu_4}\gamma^{\nu_3}\gamma^{\nu_2}\not{q}_1\gamma^{\nu_1}\not{q}_0$	$\frac{N}{2}-2$	$\frac{N}{2}-2$	$\frac{N}{2}-3$	$\frac{N}{2}-3$	2
e	o	e	o	$\gamma^{\nu_4}\gamma^{\nu_3}\not{q}_2\gamma^{\nu_2}\gamma^{\nu_1}\not{q}_0$	$\frac{N}{2}-2$	$\frac{N}{2}-3$	$\frac{N}{2}-3$	$\frac{N}{2}-3$	3
e	o	o	e	$\gamma^{\nu_4}\gamma^{\nu_3}\not{q}_2\gamma^{\nu_2}\not{q}_1\gamma^{\nu_1}$	$\frac{N}{2}-2$	$\frac{N}{2}-3$	$\frac{N}{2}-3$	$\frac{N}{2}-3$	4
o	e	e	o	$\gamma^{\nu_4}\not{q}_3\gamma^{\nu_3}\gamma^{\nu_2}\gamma^{\nu_1}\not{q}_0$	$\frac{N}{2}-3$	$\frac{N}{2}-3$	$\frac{N}{2}-3$	$\frac{N}{2}-3$	5
o	e	o	e	$\gamma^{\nu_4}\not{q}_3\gamma^{\nu_3}\gamma^{\nu_2}\not{q}_1\gamma^{\nu_1}$	$\frac{N}{2}-3$	$\frac{N}{2}-3$	$\frac{N}{2}-3$	$\frac{N}{2}-3$	6
o	o	e	e	$\gamma^{\nu_4}\not{q}_3\gamma^{\nu_3}\not{q}_2\gamma^{\nu_2}\gamma^{\nu_1}$	$\frac{N}{2}-3$	$\frac{N}{2}-3$	$\frac{N}{2}-3$	$\frac{N}{2}-3$	7
o	o	o	o	$\gamma^{\nu_4}\not{q}_3\gamma^{\nu_3}\not{q}_2\gamma^{\nu_2}\not{q}_1\gamma^{\nu_1}\not{q}_0$	$\frac{N}{2}-3$	$\frac{N}{2}-3$	$\frac{N}{2}-4$	$\frac{N}{2}-4$	8

$$S_3(z,y,x)=\begin{cases} & \text{for} \quad A_i \quad B_i \\ 0 & \quad\text{any} \quad -2 \\ 0 & \quad -3 \quad -3 \\ -\frac{1}{x}\frac{1}{y}\frac{1}{z} & \quad -4 \quad -3 \end{cases} \tag{E8}$$

Two examples of the explicit calculations will be given now. The other terms were obtained in a similar way:

$$S_3(z,y,x)(A_i=-3,B_i=-3)=\frac{1}{1-x}\frac{1}{z}\frac{1}{1-y}\left[\frac{1}{y}-1\right]-\frac{1}{x^2}\frac{1}{1-x}\frac{x}{z}\frac{1}{1-\frac{y}{x}}\left[\frac{x}{y}-1\right]=0\,,$$

$$S_3(z,y,x)(A_i=-4,B_i=-3)=\frac{1}{1-x}\frac{1}{z}\frac{1}{1-y}\left[\frac{1}{y}-1\right]-\frac{1}{x^3}\frac{1}{1-x}\frac{x}{z}\frac{1}{1-\frac{y}{x}}\left[\frac{x}{y}-1\right]$$

$$=\frac{1}{1-x}\frac{1}{z}\frac{1}{y}-\frac{1}{x}\frac{1}{1-x}-\frac{1}{z}\frac{1}{x-y}\frac{x-y}{y}=-\frac{1}{x}\frac{1}{y}\frac{1}{z}\,.$$

We found that $S_3(z,y,x)=0$ for all i except $i=8$. The only sum that does not vanish, the sum over elements with odd m_1, m_2 and m_3, gives $-(xyz)^{-1}$. In Eq. (E1),

$$x=\frac{q_1^2}{q_0^2},\quad y=\frac{q_2^2}{q_0^2},\quad z=\frac{q_3^2}{q_0^2}\,,$$

and we get

$$Z=-(q_0^2)^{-4}\frac{q_0^2}{q_1^2}\frac{q_0^2}{q_2^2}\frac{q_0^2}{q_3^2}\mathrm{Tr}(\gamma^{\nu_4}\not{q}_3\gamma^{\nu_3}\not{q}_2\gamma^{\nu_1}\not{q}_1\gamma^{\nu_0}\not{q}_0)$$

$$=-\mathrm{Tr}\left[\gamma^{\nu_4}\frac{\not{q}_3}{q_3^2}\gamma^{\nu_3}\frac{\not{q}_2}{q_2^2}\gamma^{\nu_2}\frac{\not{q}_1}{q_1^2}\gamma^{\nu_1}\frac{\not{q}_0}{q_0^2}\right]=-\mathrm{Tr}\left[\gamma^{\nu_4}\frac{1}{\not{q}_3}\gamma^{\nu_3}\frac{1}{\not{q}_2}\gamma^{\nu_2}\frac{1}{\not{q}_1}\gamma^{\nu_1}\frac{1}{\not{q}_0}\right]\,. \tag{E9}$$

If q_0, q_1, q_2, and q_3 are now properly substituted the proof for Eq. (5.12) is completed.

APPENDIX F: AN ANALOG IN QED$_\delta$ TO THE FEYNMAN IDENTITY

In conventional perturbation theory for QED the well-known Feynman identity

$$(\not{p}+\not{l})^{-1}\not{l}\not{p}^{-1}=(\not{p}+\not{l})^{-1}-\not{p}^{-1}$$

was obtained by rewriting l as $(p+l)-p$ [6]. In QED$_\delta$ we have (with a general weight function w)

$$\sum_{n_1=0}^{\infty}\sum_{n_0=0}^{\infty} w(n_1+n_0)(\not{p}+\not{l})^{n_1}\not{l}\not{p}^{n_0}=\sum_{m=0}^{\infty}w(m)\sum_{n_1=0}^{m}(\not{p}+\not{l})^{n_1}\not{l}\not{p}^{m-n_1}$$

$$=\sum_{m=0}^{\infty}w(m)\sum_{n=0}^{m}(\not{p}+\not{l})^{n}(\not{p}+\not{l}-\not{p})\not{p}^{m-n}$$

$$=\sum_{m=0}^{\infty}w(m)\sum_{n=0}^{m}[(\not{p}+\not{l})^{n+1}\not{p}^{m-n}-(\not{p}+\not{l})^{n}\not{p}^{m-n+1}]$$

$$=\sum_{m=0}^{\infty}w(m)\left[\sum_{\bar{n}=1}^{m+1}(\not{p}+\not{l})^{\bar{n}}\not{p}^{m-\bar{n}+1}-\sum_{n=0}^{m}(\not{p}+\not{l})^{n}\not{p}^{m-n+1}\right]$$

$$=\sum_{m=0}^{\infty}w(m)[(\not{p}+\not{l})^{m+1}-\not{p}^{m+1}]=\sum_{\alpha=0}^{\infty}w(\alpha-1)[(\not{p}+\not{l})^{\alpha}-\not{p}^{\alpha}]\,, \tag{F1}$$

where $\alpha=m+1$, but although $m\geq 0$ α is taken from zero since the $\alpha=0$ term cancels. For $w\equiv 1$ this is Eq. (6.1).

APPENDIX G: WARD IDENTITIES FOR QED$_\delta$

The contraction of the amputated Green's function $G_\lambda^{\nu_1\nu_2\cdots\nu_\lambda}(p,l_1,\ldots,l_\lambda)$ with the momentum of one of the bosons will be calculated now. In order to get the results in a simple form we label the incoming fermion momentum $p_\psi\equiv l_0$

and the outgoing fermion momentum $p_{\bar{\psi}}\equiv l_{\lambda+1}$. We denote $G_\lambda^{\nu_1\nu_2\cdots\nu_\lambda}=G_\lambda^{\nu_1\nu_2\cdots\nu_\lambda}(l_0,l_1,l_2,\ldots,l_\lambda,l_{\lambda+1})$. Notice that momentum conservation at the vertex gives $l_{\lambda+1}=-\sum_{\alpha=0}^{\lambda}l_\alpha$ but although the outgoing fermion momentum is not an independent variable it is written explicitly. From Eq. (5.4),

$$l_{j_{\nu_j}}g_\lambda^{\nu_1\nu_2\cdots\nu_\lambda}(l_0,l_1,\ldots,l_\lambda,l_{\lambda+1})=\sum_{n_0,n_1,\ldots,n_\lambda=0}^{\infty}\delta\left[N-\lambda-\sum_{i=0}^{\lambda}n_i\right](\not{l}_0+\not{l}_1+\cdots+\not{l}_\lambda)^{n_\lambda}$$
$$\times\gamma^{\nu_\lambda}\cdots(\not{l}_0+\not{l}_1+\cdots+\not{l}_{j-1}+\not{l}_j)^{n_j}$$
$$\times\not{l}_j(\not{l}_0+\not{l}_1+\cdots+\not{l}_{j-1})^{n_{j-1}}\gamma^{\nu_{j-1}}\cdots\gamma^{\nu_1}\not{l}_0^{n_0}\ . \tag{G1}$$

Using Eq. (F1),

$$\sum_{n_j=0}^{\infty}\sum_{n_{j-1}=0}^{\infty}\delta\left[N-\lambda-\sum_{i=0}^{\lambda}n_i\right](\not{l}_0+\not{l}_1+\cdots+\not{l}_{j-1}+\not{l}_j)^{n_j}\not{l}_j(\not{l}_0+\not{l}_1+\cdots+\not{l}_{j-1})^{n_{j-1}}$$
$$=\sum_{\alpha=0}^{\infty}\delta(N-\lambda-[n_\lambda+\cdots+n_{j+1}+(\alpha-1)+n_{j-2}+\cdots+n_0])$$
$$\times[(\not{l}_0+\not{l}_1+\cdots+\not{l}_{j-1}+\not{l}_j)^\alpha-(\not{l}_0+\not{l}_1+\cdots+\not{l}_{j-1})^\alpha]$$

one gets

$$l_{j_{\nu_j}}g_\lambda^{\nu_1\nu_2\cdots\nu_\lambda}(l_0,l_1,\ldots,l_\lambda,l_{\lambda+1})$$
$$=\underbrace{\sum_{n_\lambda=0}^{\infty}\cdots\sum_{n_{j+1}=0}^{\infty}\sum_{\alpha=0}^{\infty}\sum_{n_{j-2}=0}^{\infty}\cdots\sum_{n_0=0}^{\infty}}_{(\lambda-1)\text{ sums}}\delta(N-(\lambda-1)-\underbrace{(n_\lambda+\cdots+n_{j+1}+\alpha+n_{j-2}+\cdots+n_0)}_{(\lambda-1)+1\text{ integers}})$$
$$\times(\not{l}_0+\not{l}_1+\not{l}_\lambda)^{n_\lambda}\gamma^{\nu_\lambda}\cdots(\not{l}_0+\not{l}_1+\cdots+\not{l}_{j-1}+\not{l}_j+\not{l}_{j+1})^{n_{j+1}}\gamma^{\nu_{j+1}}$$
$$\times[(\not{l}_0+\not{l}_1+\cdots+\not{l}_{j-1}+\not{l}_j)^\alpha-(\not{l}_0+\not{l}_1+\cdots+\not{l}_{j-1})^\alpha]$$
$$\times\gamma^{\nu_{j-1}}(\not{l}_0+\not{l}_1+\cdots+\not{l}_{j-2})^{n_{j-2}}\cdots\gamma^{\nu_1}\not{l}_0^{n_0}$$
$$=g_{\lambda-1}{}^{\nu_1\nu_2\cdots\nu_{j-1}\nu_{j+1}\cdots\nu_\lambda}(l_0,l_1,\ldots,l_{j-2},(l_{j-1}+l_j),l_{j+1},l_{j+2},\ldots,l_\lambda,l_{\lambda+1})$$
$$-g_{\lambda-1}{}^{\nu_1\nu_2\cdots\nu_{j-1}\nu_{j+1}\cdots\nu_\lambda}(l_0,l_1,\ldots,l_{j-2},l_{j-1},(l_{j+1}+l_j),l_{j+2},\ldots,l_\lambda,l_{\lambda+1})\ . \tag{G2}$$

The notation we used (namely, $p_\psi\equiv l_0$ and $p_{\bar{\psi}}\equiv l_{\lambda+1}$) enables us to write Eq. (G2) for contraction with any l_j, $j=1,\ldots,\lambda$ (including $j=\lambda$ and $j=1$). Thus we have proven

$$l_jg_\lambda(l_0,\ldots,l_{j-1},l_j,l_{j+1},\ldots,l_{\lambda+1})=g_{\lambda-1}(l_0,\ldots,(l_{j-1}+l_j),l_{j+1},\ldots,l_{\lambda+1})$$
$$-g_{\lambda-1}(l_0,\ldots,l_{j-1},(l_{j+1}+l_j),\ldots,l_{\lambda+1})\ ,$$

where Lorentz indices were suppressed.

In order to get the Ward identities for G_λ we use Eq. (5.3):

$$l_{j_{\nu_j}}G_\lambda^{\nu_1,\nu_2\cdots\nu_j\cdots\nu_\lambda}(l_0,l_1,\ldots,l_j,\ldots,l_\lambda,l_{\lambda+1})$$
$$=\frac{e^\lambda}{\lambda!}\frac{\partial}{\partial N}\bigg|_{N=0}\sum_\pi l_{j_{\nu_j}}g_\lambda{}^{\nu_{\pi(1)}\nu_{\pi(2)}\cdots\nu_{\pi(\lambda)}}(l_0,l_{\pi(1)},\ldots,l_{\pi(\alpha)},\ldots,l_{\pi(\lambda)},l_{\lambda+1})\ , \tag{G3}$$

where π is any permutation of the indices $1,\ldots,\lambda$. Notice that G does not depend on the order of momenta but g does: $g(\ldots,l_1,l_2,\ldots)\neq g(\ldots,l_2,l_1,\ldots)$. There always exist ζ for which $\pi(\zeta)=j$. We rename the momenta $q_\lambda\equiv l_j,q_\psi\equiv l_0,q_{\bar{\psi}}\equiv l_{\lambda+1},q_i\equiv l_{\pi(i)}$ for $i=1,2,\ldots,\zeta-1$ and $q_i\equiv l_{\pi(i+1)}$ for $i=\zeta,\ldots,\lambda-1$. The summation over all the permutations is done in two steps. In the first step, for a given ordering of all l_i excluding l_j we sum over λ insertions of l_j among them (q_λ is dotted into g_λ and we are still suppressing Lorentz indices):

$$q_\lambda[g_\lambda(q_\psi,q_\lambda,q_1,\ldots,q_{\lambda-1},q_{\bar{\psi}})+g_\lambda(q_\psi,q_1,q_\lambda,q_2,\ldots,q_{\lambda-1},q_{\bar{\psi}})+\cdots$$
$$+g_\lambda(q_\psi,q_1,\ldots,q_j,q_\lambda,q_{j+1},\ldots,q_{\lambda-1},q_{\bar{\psi}})+\cdots+g_\lambda(q_\psi,q_1,\ldots,q_{\lambda-1},q_\lambda,q_{\bar{\psi}})]$$
$$=g_{\lambda-1}(q_\psi+q_\lambda,q_1,\ldots,q_{\lambda-1},q_{\bar{\psi}})-g_{\lambda-1}(q_\psi,q_1,\ldots,q_{\lambda-1},q_{\bar{\psi}}+q_\lambda)\ , \tag{G4}$$

where Eq. (G2) was used to cancel all but two terms, each of lower order than the contracted one. Substituting Eq. (G4) into Eq. (G3) and returning to the notation with l, we are left with the second step in $\sum_\pi$, namely, to sum over permutations of the indices $1, \ldots, \lambda$ excluding j. We denote these permutations by π':

$$l_{j_{\nu_j}} G_\lambda^{\nu_1,\nu_2 \cdots \nu_j \cdots \nu_\lambda}(p_\psi, l_1, \ldots, l_j, \ldots, l_\lambda) = \frac{e^\lambda}{\lambda!} \frac{\partial}{\partial N}\bigg|_{N=0} \sum_{\pi'} [g_{\lambda-1}^{\nu_1,\nu_2 \cdots \nu_{j-1},\nu_{j+1} \cdots \nu_\lambda}(p_\psi + l_j, l_1, \ldots, l_{j-1}, l_{j+1}, \ldots, l_\lambda) - g_{\lambda-1}^{\nu_1,\nu_2 \cdots \nu_{j-1},\nu_{j+1} \cdots \nu_\lambda}(p_\psi, l_1, \ldots, l_{j-1}, l_{j+1}, \ldots, l_\lambda)] \ . \quad \text{(G5)}$$

The RHS of Eq. (G5) is easy to recognize as Eq. (5.3) for $(\lambda-1)$ instead of λ and we get the Ward identities [9]

$$l_j G_\lambda(p_\psi, l_1, \ldots, l_j, \ldots, l_1, p_{\bar\psi}) = \frac{e}{\lambda} G_{\lambda-1}(p_\psi + l_j, l_1, \ldots, l_{j-1}, l_{j+1}, \ldots, l_\lambda, p_{\bar\psi}) - \frac{e}{\lambda} G_{\lambda-1}(p_\psi, l_1, \ldots, l_{j-1}, l_{j+1}, \ldots, l_\lambda, p_{\bar\psi} + l_j) \ . \quad \text{(G6)}$$

[1] C. M. Bender, K. A. Milton, M. Moshe, S. S. Pinsky, and L. M. Simmons, Jr., Phys. Rev. Lett. **58**, 2615 (1987); Phys. Rev. D **37**, 1472 (1988).

[2] C. M. Bender, F. Cooper, K. A. Milton, M. Moshe, S. Pinsky, and L. M. Simmons, Jr., Phys. Rev. D **45**, 1248 (1992).

[3] C. M. Bender, K. A. Milton, and M. Moshe, Phys. Rev. D **45**, 1261 (1992).

[4] C. M. Bender, S. Boettcher, and K. A. Milton, Phys. Rev. D **45**, 639 (1992).

[5] An important problem concerning the rigor and validity of the δ expansion is continuing a function from the integers into the real axis. This, of course, is not a unique procedure. The choice advocated by the δ expansion results in reproducing the correct answers whenever a comparison with the exact results can be made [10]. One assumes that this procedure is valid and provides a good approximation also for the cases when no direct verification is available.

[6] See, for example, G. T. Bodwin and E. V. Kovacs, Phys. Rev. D **35**, 3198 (1987), and references therein.

[7] The equivalence is mentioned in the Introduction of Ref. [2] but the complete proof is given here for the first time.

[8] After obtaining this result we received Ref. [4] where it is shown that, for any number $n > 1$ and for any function f of $n-1$ integer variables,

$$\sum_{0 \le x_2 + \cdots + x_n \le N-n} f(x_2, \ldots, x_n)|_{N=0} = (-1)^{n-1} f(-1, \ldots, -1) \ .$$

[9] Since no use was made of $\partial/\partial N|_{N=0}$, the Ward identities of Eq. (6.2) and thus of Eq. (6.5) hold for Green's functions of the provisional Lagrangian as well as for the δ expansion.

[10] Several of the main applications of the δ expansion are as follows: to self-interacting scalar fields in Ref. [1] and by N. Brown, Phys. Rev. D **38**, 723 (1988); S. S. Pinsky and L. M. Simmons, Jr., *ibid.* **38**, 2518 (1988); I. Yotsuyanagi, *ibid.* **39**, 485 (1989); J. Cline, S. S. Pinsky, and L. M. Simmons, Jr., *ibid.* **39**, 3043 (1989); M. Monoyios, Z. Phys. C **42**, 325 (1989); to lattice gauge theory by A. Duncan and H. F. Jones, Nucl. Phys. **B320**, 189 (1989); A. Duncan and M. Moshe, Phys. Lett. B **215**, 352 (1988); to the Gross-Neveu model in the previous reference and by H. F. Jones and M. Moshe, Phys. Lett. B **234**, 492 (1990); S. K. Gandhi, H. F. Jones, and M. B. Pinto, Nucl. Phys. **B359**, 429 (1991); to nonlinear differential equations and one-dimensional integrals by C. M. Bender, K. A. Milton, S. S. Pinsky, and L. M. Simmons, Jr., J. Math. Phys. **30**, 1447 (1989); C. M. Bender, F. Cooper, and K. A. Milton, Phys. Rev. D **39**, 3684 (1989); H. F. Jones and M. Monoyios, Int. J. Mod. Phys. A **4**, 1735 (1989); to quantum mechanics in the previous reference and by F. Cooper, H. F. Jones, and L. M. Simmons, Jr., Phys. Rev. D **43**, 3396 (1991); F. Cooper and M. Moshe, Phys. Lett. B **259**, 101 (1991).