Self-consistent compactification at finite temperature on $R\times S^5\times S^3$

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The self-consistency equations resulting from the Einstein equations for a space-time of the form $R\times S^5\times S^3$, with the vacuum-averaged energy-momentum tensor of a minimally coupled scalar field as the source, are solved using a one-loop finite-temperature calculation of this tensor. Solutions for low temperature are found to exist for large and small values of the radius ratio and also for the ratio close to $1/\sqrt{2}$. For the ratio equal to $1/\sqrt{2}$ a zero-temperature solution is found. There is a maximum temperature for the ratio larger than this.

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I. INTRODUCTION

This calculation is a further example of the one carried out in [1] where solutions of the Einstein equations for static space-times of the form $R\times S^3\times S^3$ were found, using the vacuum-averaged energy-momentum tensor of a minimally coupled scalar field at finite temperature as the source. The self-consistency equations were solved exactly up to numerical procedures and it was found that aside from a symmetric solution that existed at all temperatures, solutions did not exist for ratios of the sphere radii between certain values. The calculation in this paper uses a space-time of the form $R\times S^5\times S^3$ to investigate how the behavior changes when the symmetry between the spatial sections is removed. The method used is technically similar to that in the previous paper.

A Kaluza-Klein solution would have the radius of S^5 being small while that of S^3 was large. We investigate all possibilities. It is important to carry out a full analysis and, since these calculations are somewhat involved, the system considered has to be an idealized one for simplicity. The choice of dimensions for the spheres was intended to produce a tractable case that nevertheless was not too simple. The space-time dimension has been kept odd to avoid the problem of divergences.

One of the earliest papers on the effect of finite temperature in compactification was Ref. [2], using a space-time of the form $M^4\times S^1$. Subsequent calculations include that of Randjbar-Daemi *et al.* [3] who studied self-consistency at finite temperature with time dependence for space-times of the form $R\times S^3\times S^d$, calculating the effective action in the static approximation. They used approximate forms for both the zero- and finite-temperature parts. Yoshimura [4] and Acceta and Kolb [5] discuss the same case with similar approximations.

Calculations of the stability of various classical backgrounds against deformation produced by quantum fluctuations have been done by Shiraishi [6] and Okada [7] at low and zero temperatures, respectively; Szydłowski and Szczęsny study the stability of solutions with static microspaces for a space-time of the form Friedmann-Robertson-Walker$\times S^3\times S^3$ in a low-temperature approximation [8]. Szczęsny, Szydłowski, and Biesada investigate mixmaster models $R\times M^3\times B$ (where M^3 is Bianchi type IX) at high temperature [9]. Compactification using a nonlinear sigma model as the scalar field has been studied by Chakraborty and Parthasarathy [10].

Other work in this area includes the use of Epstein zeta functions by Shi and Li to calculate the vacuum energies of p-brane models [11], and Odintsov's use of zeta-function techniques to calculate Casimir energies of p-branes [12].

II. FIELD THEORY ON $S^5\times S^3$

The system considered in this paper is that of a gravitational field and a minimally coupled massless scalar field. The effective action for these fields is expanded to zeroth order and one loop, respectively; thus the effective scalar field will obey the classical equations of motion and will be taken to be zero. The effective gravitational field obeys the Einstein equations but with a source term consisting of a classical energy-momentum tensor evaluated at the effective field (and so zero here) and a one-loop quantum correction which will depend on the effective metric of the space-time.

In this paper the space-time will be restricted to the form $R\times S^5\times S^3$, with maximally symmetric spheres, so that the only parameters in the metric are the radii of the spheres. Since the calculations are at finite temperature we must work with a finite-temperature effective action or alternatively the free energy F for the scalar field, which for an ultrastatic space-time are the same thing up to a sign [13]. The effective action thus depends on the sphere radii and the temperature, and these may be varied to produce the effective field equations. This is done in Sec. III.

In this section we calculate the zeta function on the

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spatial section of the chosen space-time. From this the zero temperature vacuum energy can be found as [13,14]

$$E_0=\tfrac{1}{2}\zeta(-\tfrac{1}{2}) \ . \tag{1}$$

The zero-temperature free energy F_0 is equal to E_0 and the finite-temperature quantities can both be expressed as the sum of the zero-temperature values and a finite-temperature correction, and so it remains to calculate this. This is discussed at the end of this section.

The zeta function on the spatial section can be defined formally in terms of the Green's function of the relevant operator, but it is easier here to go to the resulting eigenvalue form. The zeta function of an operator (in this case $-\nabla_5^2/a^2-\nabla_3^2/b^2$ where a and b are the radii of the 5-sphere and the 3-sphere, respectively, and ∇_A^2 is the Laplacian on a unit A sphere) is given by

$$\zeta(s)=\sum_\alpha{}' \frac{1}{\lambda(\alpha)^s} \tag{2}$$

where $\lambda(\alpha)$ are the eigenvalues ($\alpha\in A$ where A is some index set) and the prime means omit from the sum $\{\alpha|\lambda(\alpha)=0\}$. The eigenvalues and degeneracies of the Laplacians in this case are

operator	eigenvalues	degeneracies
$-\nabla_5^2$	$m(m+4)$	$\frac{1}{12}(m+1)(m+2)^2(m+3)$
$-\nabla_3^2$	$n(n+2)$	$(n+1)^2$

where $m,n=0,1,\ldots$. So $\zeta(s)$ can be expressed as

$$\zeta(s)=\sum_{m,n=0}^{\infty}{}' \tfrac{1}{12}(m+1)(m+2)^2(m+3)(n+1)^2\left[\frac{m(m+4)}{a^2}+\frac{n(n+1)}{b^2}\right]^{-s} . \tag{3}$$

A redefinition of m and n is helpful:

$$M=m+2,\quad N=n+1 \tag{4}$$

in terms of which the zeta function becomes

$$\zeta(s)=\sum_{\substack{M=0\\N=0}}^{\infty}{}'' \tfrac{1}{12}(M^2-1)M^2N^2\left[\frac{M^2}{a^2}+\frac{N^2}{b^2}+c^2\right]^{-s} , \tag{5}$$

where the double prime means that the zero mode term $(M,N)=(2,1)$ is omitted, and $c^2=-4/a^2-1/b^2$.

This can be related to the *modified* Epstein zeta function [15]

$$Z(s)=\sum_{\substack{M=-\infty\\N=-\infty}}^{\infty}{}''' (\omega_{MN}^2)^{-s} , \tag{6}$$

where $\omega_{MN}=(M^2/a^2+N^2/b^2+c^2)^{1/2}$ and the triple prime means omitting the terms $(|M|<2,N)$, $(M,|N|<1)$, and $(|M|,|N|)=(2,1)$. $\zeta(s)$ can now be found in terms of $Z(s)$ and its derivatives:

$$\begin{aligned}\zeta(s)=\frac{1}{48}\Bigg[12Z(s)&-\left[7\frac{\partial}{\partial(1/a^2)}+12\frac{\partial}{\partial(1/b^2)}\right]\frac{Z(s-1)}{(s-1)}\\&+\left[\frac{\partial}{\partial(1/a^2)}\frac{\partial}{\partial(1/a^2)}+7\frac{\partial}{\partial(1/a^2)}\frac{\partial}{\partial(1/b^2)}\right]\frac{Z(s-2)}{(s-1)(s-2)}\\&-\frac{\partial}{\partial(1/a^2)}\frac{\partial}{\partial(1/a^2)}\frac{\partial}{\partial(1/b^2)}\frac{Z(s-3)}{(s-1)(s-2)(s-3)}\Bigg] .\end{aligned} \tag{7}$$

$Z(s)$ can be analytically continued using the incomplete gamma and Bessel functions $\Gamma(s,x)$, $\gamma(s,x)$, and $K_s(x,y)$ [1]: an expression for $\zeta(s)$ can thus be obtained:

$$\begin{aligned}\zeta(s)=\frac{\pi^{s-3}}{48\Gamma(s)}\Bigg\{&\sum_{\substack{M=-\infty\\N=-\infty}}^{\infty}{}''' \pi^3(M^2-1)M^2N^2\Gamma(s,\pi\omega_{MN}^2)\\&+\sum_{\substack{M=-\infty\\N=-\infty}}^{*}\Bigg[-\frac{a^3b^3\pi}{4}K_{3-s}+\left[\frac{3}{8}a^5b^3+\frac{\pi^2}{2}(N^2a^3b^5+M^2a^5b^3)\right]K_{4-s}\\&\qquad-\left[\frac{3}{4}\pi N^2a^5b^5+\frac{3}{2}\pi M^2a^7b^3+\pi^3M^2N^2a^5b^5\right]K_{5-s}\\&\qquad+\left[\frac{\pi^2}{2}M^4a^9b^3+3\pi^2M^2N^2a^7b^5\right]K_{6-s}-\pi^3M^4N^2a^9b^5K_{7-s}\Bigg]\\&+\frac{3}{8}a^5b^3\gamma(s-4,\pi c)-\frac{\pi a^3b^3}{4}\gamma(s-3,\pi c)-\frac{48\pi^3}{s}\Bigg\} .\end{aligned} \tag{8}$$

The arguments of the incomplete Bessel functions are always $[\pi(M^2a^2+N^2b^2),\pi c]$, and an asterisk means omitting the $(M,N)=(0,0)$ term.

The expression for E_0 is thus

$$E_0(a,b)=-\frac{1}{192\pi^4}\Bigg\{\sum_{\substack{M=-\infty\\N=-\infty}}^{\infty}{}''' \pi^3(M^2-1)M^2N^2\Gamma(-\tfrac{1}{2},\pi\omega_{MN}^2)$$
$$+\sum_{\substack{M=-\infty\\N=-\infty}}^{\infty}{}^{*}\Bigg[-\frac{a^3b^3\pi}{4}K_{7/2}+\left[\frac{3}{8}a^5b^3+\frac{\pi^2}{2}(N^2a^3b^5+M^2a^5b^3)\right]K_{9/2}$$
$$-\left[\frac{3}{4}\pi N^2a^5b^5+\frac{3}{2}\pi M^2a^7b^3+\pi^3M^2N^2a^5b^5\right]K_{11/2}$$
$$+\left[\frac{\pi^2}{2}M^4a^9b^3+3\pi^2M^2N^2a^7b^5\right]K_{13/2}-\pi^3M^4N^2a^9b^5K_{15/2}\Bigg]$$
$$+\frac{3}{8}a^5b^3\gamma(-9/2,\pi c)-\frac{\pi a^3b^3}{4}\gamma(-7/2,\pi c)+96\pi^3\Bigg\}\ . \tag{9}$$

As before an expression for $\partial E_0/\partial a$ can be found but will not be written out. The apparently incorrect scaling dimension of the last term in Eq. (9) is due to the introduction of a length scale in the continuation of $Z(s)$. The other terms in this equation also possess incorrect scaling but the overall expression has the correct dimensions.

The algorithms used to calculate these quantities are those of [16] for the Γ, γ, and K_s functions, which were the ones used in [1].

The remaining finite temperature parts of the free and vacuum energies can be calculated from the following mode sums [1,13]:

$$E'=\sum_{\substack{M=2\\N=1}}^{\infty}{}''\frac{1}{12}\frac{(M^2-1)M^2N^2\omega_{MN}}{(e^{\beta\omega_{MN}}-1)}+\frac{1}{\beta}\ ,$$
$$\frac{\partial F'}{\partial a}=-\sum_{\substack{M=2\\N=1}}^{\infty}{}''\frac{1}{12}\frac{(M^2-1)M^2N^2(M^2-4)}{a^3\omega_{MN}(e^{\beta\omega_{MN}}-1)}\ . \tag{10}$$

A typographical error in [1] should be pointed out here. In Eq. (9) of [1] the parentheses in the denominator of the fraction are incorrectly placed around $\beta\omega_{mn}-1$. They should enclose $\exp\beta\omega_{mn}-1$.

The trace condition can be derived from these two expressions and the equivalent one for $\partial F'/\partial b$. Because of the dependence of the free energy on the scale length in the presence of a zero mode there is an anomalous contribution to the trace [1,17]:

$$a\frac{\partial F}{\partial a}+b\frac{\partial F}{\partial b}+E=\frac{1}{\beta}\ . \tag{11}$$

III. SELF-CONSISTENCY FOR $S^5\times S^3$.

The Lagrangian for the system under consideration is

$$L=\frac{1}{16\pi\overline{G}}\int(R-2\overline{\Lambda})g_{00}^{1/2}d(\mathrm{vol})-F=L_G-F\ . \tag{12}$$

R is the Ricci scalar on $S^5\times S^3$, $\overline{\Lambda}$ is the cosmological constant, $d(\mathrm{vol})$ the spatial volume element, and F the free energy of the scalar field. Using the total free energy, $F-L_G$, a finite-temperature action can be defined,

$$\ln Z=-(F-L_G)\beta \tag{13}$$

and a, b, and β varied in (13) to produce the self-consistency equations. This gives

$$E=\frac{1}{16\pi\overline{G}}\Omega_5a^5\Omega_3b^3(R-2\overline{\Lambda})\ , \tag{14}$$

$$a\frac{\partial F}{\partial a}=\frac{\Omega_5\Omega_3}{16\pi\overline{G}}a^5b^3\left[\frac{60}{a^2}+\frac{30}{b^2}-10\overline{\Lambda}\right] \tag{15}$$

(where Ω_A is the area of a unit radius A sphere). There is a corresponding expression for $b\partial F/\partial b$ which will not be written out here. $\overline{\Lambda}$ can be found from the trace condition (11).

$$\overline{\Lambda}=\frac{1}{18}\left[7R-\frac{8\overline{G}}{\pi^4a^5b^3\beta}\right]\ , \tag{16}$$

where the values of π^3 and Ω_5 and $2\pi^2$ for Ω_3 have been used. The self-consistency equations now become

$$E(a,b,\beta)=\frac{\pi^4a^5b^3}{4\overline{G}}\frac{R}{9}+\frac{1}{9\beta}\ , \tag{17}$$

$$\frac{E(a,b,\beta)-\dfrac{1}{9\beta}}{\dfrac{\partial F}{\partial a}(a,b,\beta)-\dfrac{5}{9a\beta}}=a\left[\frac{3a^2+10b^2}{15a^2-40b^2}\right] . \tag{18}$$

Alternatively, (18) can be expressed in terms of $\partial F/\partial b$, but again this will not be written out here.

From the field theory expressions for E_0 and the finite temperature corrections in terms of the eigenvalues some scaling relations can be derived:

$$bE(a,b,\beta)=E\left[\frac{a}{b},1,\frac{\beta}{b}\right] ,$$

$$b^2\frac{\partial F}{\partial a}(a,b,\beta)=\frac{\partial F}{\partial a'}(a',1,\lambda_b)\Bigg|_{\substack{a'=a/b\\ \lambda_b=\beta/b}} , \tag{19}$$

and employing these in (18) we find

$$\frac{E(a',1,\lambda_b)-\dfrac{1}{9\lambda_b}}{\dfrac{\partial F}{\partial a'}(a',1,\lambda_b)-\dfrac{5}{9a'\lambda_b}}=a'\left[\frac{3a'^2+10}{15a'^2-40}\right] , \tag{20}$$

where $a'=a/b$ and $\lambda_b=\beta/b$. Solutions of this equation in the form of values of λ_b and a' (or of the corresponding equation involving $\partial F/\partial b'$ in the form of values of λ_a and b' where $b'=b/a$ and $\lambda_a=\beta/a$) can now be found numerically, evaluating the left-hand side from the field theory expressions. These can then be used in the scaled version of (17):

$$E(a',1,\lambda_b)=\frac{\pi^4a'^5b^7}{18\bar{G}}\left[\frac{10}{a'^2}+3\right]+\frac{1}{9\lambda_b} \tag{21}$$

to find b and hence to rescale a' and λ_b to a and β [or by using the b' version of (21) to find a, to rescale b' and λ_a to b and β]. The results of this somewhat lengthy procedure will be described later but we will first look at the specific cases of $R\times S^5\times R^3$ and $R\times R^5\times S^3$ as these require treatment *ab initio* and also provide a check on the numerical solutions by "linking up" with them as the radius ratio goes to infinity.

IV. LIMITING CASES

A. Replacing S^3 by R^3

The free and vacuum energies must now be expressed as densities, defined by

$$\eta_a(a,\beta)=\lim_{b\to\infty}\frac{E(a,b,\beta)}{2\pi^2b^3},\quad \phi_a(a,b,\beta)=\lim_{b\to\infty}\frac{F(a,b,\beta)}{2\pi^2b^3} \tag{22}$$

and with scaling relations (from those for E and F)

$$\eta_a(1,\lambda_a)=a^4\eta_a(a,\beta),\quad \phi_a(1,\lambda_a)=a^4\phi_a(a,\beta) . \tag{23}$$

The Lagrangian density becomes

$$\mathcal{L}=\frac{\pi^2}{16\bar{G}}a^5\left[\frac{20}{a^2}-\bar{\Lambda}\right]-\phi_a(a,\beta) \tag{24}$$

and the trace condition is

$$a\frac{\partial\phi_a}{\partial a}+3\phi_a+\eta_a=0 . \tag{25}$$

After variation of $\mathcal{L}$ these produce

$$a^7=\frac{6\bar{G}}{5\pi^2}[2\eta_a(1,\lambda_a)+\phi_a(1,\lambda_a)] \tag{26}$$

and

$$\bar{\Lambda}_a=\frac{10}{3a^2}\left[\frac{4\eta_a(1,\lambda_a)+3\phi_a(1,\lambda_a)}{2\eta_a(1,\lambda_a)+\phi_a(1,\lambda_a)}\right] . \tag{27}$$

For zero temperature this gives

$$a_0^7=\frac{18\bar{G}}{5\pi^2}\eta_a(1,\infty),\quad \eta_a(1,\infty)\simeq 4.28\times10^{-4} . \tag{28}$$

This equation agrees with that found by Candelas and Weinberg [18]. The value for $\eta_a(1,\infty)$ is taken from [19], and agrees with the value found using the method of calculating zeta functions adopted here.

B. Replacing S^5 by R^5

The calculation for $R\times R^5\times S^3$ proceeds similarly and gives

$$b^7=\frac{2\bar{G}}{3\pi}[4\eta_b(1,\lambda_b)+5\phi_b(1,\lambda_b)] ,$$
$$\phi_b(1,\lambda_b)=b^6\phi_b(b,\beta) , \tag{29}$$

$$\bar{\Lambda}_b=\frac{3}{b^2}\left[\frac{2\eta_b(1,\lambda_b)+5\phi_b(1,\lambda_b)}{4\eta_b(1,\lambda_b)+5\phi_b(1,\lambda_b)}\right] ,$$
$$\eta_b(1,\lambda_b)=b^6\eta_b(b,\beta) , \tag{30}$$

and for zero temperature

$$b_0^7=\frac{6\bar{G}}{\pi}\eta_b(1,\infty),\quad \eta_b(1,\infty)\simeq 7.09\times10^{-6} . \tag{31}$$

The expression for the free energy of the field when one of the spheres is replaced by R_n is given in Appendix A 1 below: this expression is used to calculate the variation of a and b with β for each of the cases and the results are plotted in Figs. 1 and 2. The behavior here is qualitative-

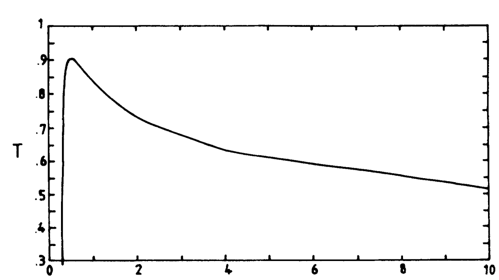

FIG. 1. The variation of temperature with radius for $R\times S^5\times R^3$.

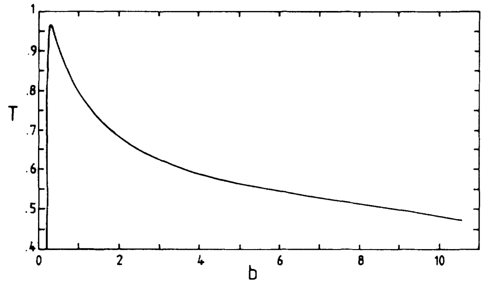

FIG. 2. The variation of temperature with radius for $R \times R^5 \times S^3$.

ly the same as in the case $R \times S^3 \times R^3$, with a maximum temperature found in both the above geometries for particular values of the radius of the remaining sphere. In the $R \times S^5 \times R^3$ case the maximum temperature is $T \simeq 0.904$ and occurs at $a \simeq 0.503 \bar{G}^{1/7}$. In the $R \times R^5 \times S^3$ case the relevant quantities are $T \simeq 0.964$ and $b \simeq 0.412 \bar{G}^{1/7}$.

The values for a_0 and b_0 found in this section agree with those found by calculating a or b from (21) using the numerical solutions as b' or a' becomes large. For a' and $b' \approx 30$ the values agree to three decimal places, and at the same time the temperature is going to zero, thus "linking" with the limiting solutions.

V. RESULTS

The results obtained from solving the self-consistency equations (20) and (21) are plotted in Fig. 3. The plot is of temperature $(=\beta^{-1})$ against $r=(b'-1)/(b'+1)$. There are four limiting regions: $r \to -1$, $r \to 1$, $r \to -0.17$ $(a' \to \sqrt{2})$, and $r \to -0.27$ $(a' \to 1.737)$. As the solutions approach these limits the values of (a,b) tend respectively to (∞, b_0), (a_0, ∞), (∞, ∞), and $(0,0)$, where a_0 and b_0 are the self-consistent radii at zero temperature calculated in Secs. IV A and IV B. Figure 4 is a plot against r of λ_b in the left half and λ_a in the right half of the figure. Two different variables are used so as to afford a better comparison between the two halves of the graph.

To give some idea of where Fig. 4 comes from Figs. 5–10 reproduce plots of the left- and right-hand sides of (20) against a' for various values of λ_b. The behavior of the $b'>1$ solutions is fairly straight forward and this is reflected in the simple behavior of the right-hand side of

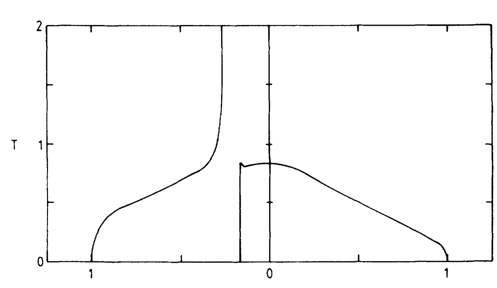

FIG. 3. Solutions of the self-consistency equations plotted as temperature against r.

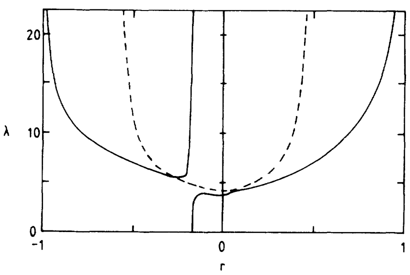

FIG. 4. Solutions of first self-consistency equation [(19) or its equivalent in terms of $\partial F/\partial b'$] plotted as λ_b against r (left), λ_a against r (right). The curve delimiting physical solutions (above the curve a and b are negative) is also shown.

Fig. 4. The solutions calculated from (20), however, have a more complex behavior. The story is roughly as follows.

There are two solutions for $\lambda_b \to \infty$, one of which is at infinity (Fig. 5), and these persist as λ_b decreases (Figs. 6 and 7) until the two branches converge in Fig. 8, at $a' \simeq 1.69$ and $\lambda_b \simeq 5.47$. There is then a range of λ values for which there is no solution for any a' for which the central branch of the curve of the left-hand side of (20) in Fig. 8 passes between the two branches of the right-hand side before a solution appears at $a' \simeq 1.19$ and $\lambda_b = 3.97$ in Fig. 9. This then splits and one of the solutions goes below $a'=1$ (Fig. 10); the other drops to $\lambda_b=0$ while $a' \to \sqrt{2}$, the limit described in Sec. V A 3.

As in [1] there is still a region of ratio values for which there is no solution, but now the symmetry present in the former case has (not surprisingly) disappeared. However there is still a zero-temperature solution for a finite value of the radius ratio, corresponding to the symmetric branch of the solution in the $R \times S^3 \times S^3$ case, and the left-hand branch of the solution here (when S^3 is the internal space) also qualitatively resembles the solutions in [1], although the functional dependence of T on r, even in the approximate case, is different.

The precise shape of the right-hand branch of the curve in Fig. 3 means that as T varies the number of solutions changes, from three at low values of T to four at a critical value, then to five, back to four again, then three and finally to just one for high enough values of the temperature. While no deep significance is to be attached to these facts, it does show that the solution sets can have a

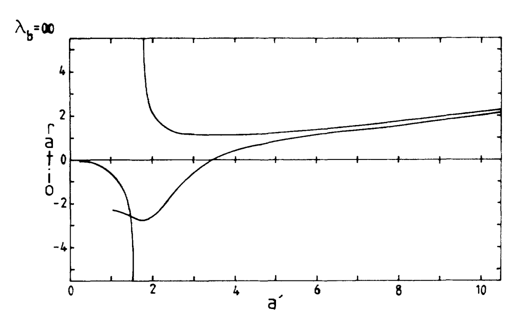

FIG. 5. Plot of the left- and right-hand sides of Eq. (19) against a' for $\lambda_b=\infty$ showing how the solutions change as λ_b varies.

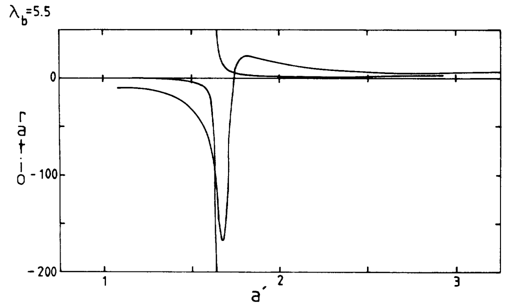

FIG. 6. Same as Fig. 5 for $\lambda_b = 5.5$.

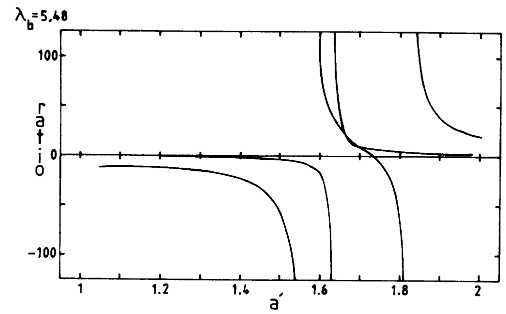

FIG. 8. Same as Fig. 5 for $\lambda_b = 5.48$.

more complex behavior that might at first be thought and hints that in a dynamic model unexpected behavior could show up. For $R \times S^3 \times S^3$ the number of solutions for a given temperature was three and did not change as T varied.

A. Approximate solutions

The expression in Appendix A 2 applies when rT for one of the spheres is large. The procedure adopted in this section is to use this expression to solve the self-consistency equations with the restrictions that rT and λ are large, the reasons for the second condition being (1) that it simplifies the solution greatly and (2) that the exact solutions show that when λ is large the radius ratio is too, which is the condition we are actually interested in from a physical point of view.

1. *The large-a′ case*

For large aT, E, and F can be expanded:

$$E(a',1,\lambda_b) = \mathcal{A}(\lambda_b)a'^5 + \mathcal{B}(\lambda_b)a'^3 , \tag{32}$$

$$F(a',1,\lambda_b) = \mathcal{C}(\lambda_b)a'^5 + \mathcal{D}(\lambda_b)a'^3 . \tag{33}$$

For large λ_b and large a' the self-consistency equation can be solved and gives

$$a'^2 = \frac{30\mathcal{A}(\infty) - 2\mathcal{B}(\infty)}{5[\mathcal{A}(\lambda_b) - \mathcal{C}(\lambda_b)]} . \tag{34}$$

From the expression in the Appendix with $d=3$, $n=5$,

$$\mathcal{A}(\lambda_b) - \mathcal{C}(\lambda_b) = 12\zeta_R(6)\lambda_b^{-6}, \text{ where } \zeta_R(6) \simeq 1.01734,$$

$$\mathcal{A}(\infty) = \frac{1}{768}\zeta_3'(-3) \simeq \pi^3 \times 7.09 \times 10^{-6} ,$$

$$\mathcal{B}(\infty) = -\frac{5}{384}\zeta_3'(-2) \simeq (5\pi^2/6) \times 7.57 \times 10^{-5} .$$

So

$$a'^2 \simeq 8.764 \times 10^{-5} \times \lambda_b^6 . \tag{35}$$

Since a' and λ_b are large, $b \approx b_0$ which is given by (31). Using this we have $b_0 = \bar{b}_0 \bar{G}^{1/7}$, where $\bar{b}_0 \simeq 0.2016$.

2. *The large-b′ case*

E and F can be expanded similarly to the above case:

$$E(1,b',\lambda_a) = \mathcal{A}'(\lambda_a)b'^3 + \mathcal{B}'(\lambda_b)b' , \tag{36}$$

$$F(1,b',\lambda_a) = \mathcal{C}'(\lambda_a)b'^3 + \mathcal{D}'(\lambda_a)b' . \tag{37}$$

Now the self-consistency equation gives

$$b'^2 = \frac{27\mathcal{A}'(\infty) - 20\mathcal{B}'(\infty)}{30[\mathcal{A}'(\lambda_a) - \mathcal{C}'(\lambda_a)]} , \tag{38}$$

and using the expression in the appendix but with $d=5$, $n=3$ we get $\mathcal{A}'(\lambda_a) - \mathcal{C}'(\lambda_a) = 8\zeta_R(4)\lambda_a^{-4}$, where $\zeta_R(4) \simeq 1.08232$,

$$\mathcal{A}'(\infty) = -\frac{1}{32}\zeta_5'(-2) \simeq 2\pi^2 \times 4.28 \times 10^{-4}$$

$$\mathcal{B}'(\infty) = \tfrac{1}{16}\zeta_5'(-1) \simeq -\pi \times 1.11 \times 10^{-2} .$$

Inserting these values gives

$$b'^2 \simeq 3.563 \times 10^{-3} \times \lambda_a^4 . \tag{39}$$

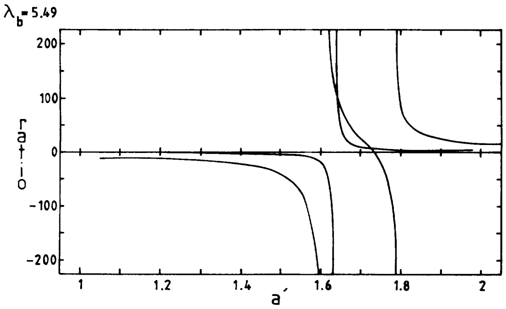

FIG. 7. Same as Fig. 5 for $\lambda_b = 5.49$.

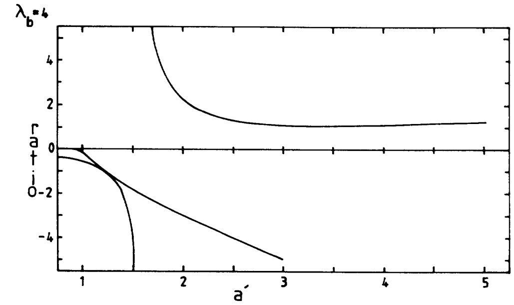

FIG. 9. Same as Fig. 5 for $\lambda_b = 4$.

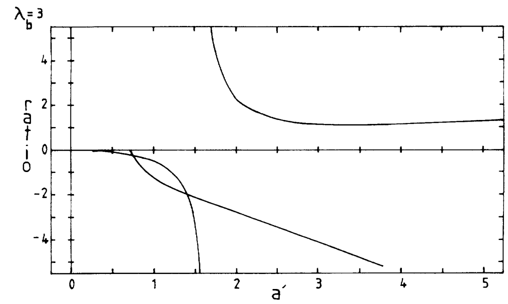

FIG. 10. Same as Fig. 5 for $\lambda_b=3$.

Now $a \approx a_0$ which is given by (28). Using this we have $a_0=\bar{a}_0\bar{G}^{1/7}$, where $\bar{a}_0 \simeq 0.2859$. The approximate forms calculated here agree with the numerical solutions in the regions where the approximations apply.

It is necessary to have a value for $\bar{G}$ in order to calculate explicit numbers for a_0 and b_0. If the Lagrangian in nine space-time dimensions is compared to the Einstein-Hilbert Lagrangian in four dimensions (here we are talking about the large b' case) then the coefficient of R_4, the Ricci scalar in four dimensions, will be $1/16\pi G$ where G is the "observed" four dimensional Newton's constant. (The renormalization of the gravitational coupling constant by one-loop effects in Kaluza-Klein space-times has been examined in Ref. [20].) From this comparison we get

$$\frac{a_0^2}{G}=\pi^3\bar{a}_0^7-\frac{4\mathcal{D}'(\lambda_a)}{3\pi} . \tag{40}$$

At zero temperature, which because λ_a is large is all that is necessary to the accuracy here, this gives $a_0^2/G \approx 0.02$. In order to give a numerical value to $\bar{G}$ we will take G to have its observed value which means that $a_0 \approx 0.14 L_P = 2.3\times 10^{-34}$ cm. Then $b \approx 4\times 10^{29}$ cm $\Longrightarrow T \approx 6$ K, and in units with $G=1$, we find $\bar{G}\approx 0.007$. The effective four-dimensional cosmological constant can also be found, and in this case has a value $\approx 10^{-56}$ m^{-2}, which is within observational limits.

3. *The other limits*

The other two limiting cases are different. The case $T\rightarrow\infty$ is caused by the same effect that produced the minimum value of r in [1]. Since

$$b^7=\frac{4\bar{G}}{\pi^4}\left[\frac{E(a',1,\lambda_b)-1/9\lambda_b}{10a'^3+3a'^5}\right] \tag{41}$$

which is a version of (21), then as $E(a',1,\lambda_b)-1/9\lambda_b \rightarrow 0$, so $b\rightarrow 0$ and $T(=1/b\lambda_b)\rightarrow\infty$. This effect can be see in Fig. 4, of solutions in the ratio vs λ plane, which shows the contour $E(a',1,\lambda_b)-1/9\lambda_b=0$. At this limit a and $b\rightarrow 0$ and $a'\rightarrow 1.737$.

The remaining limit can be derived analytically. It is in a region where λ_b is small and so a high temperature expression for $F(a',1,\lambda_b)$ can be used. From [19] we have

$$F(a',1,\lambda_b)=-2\sqrt{\pi}a'^5\zeta_R(9)\Gamma(\tfrac{9}{2})\lambda_b^{-9} , \tag{42}$$

$$E(a',1,\lambda_b)=-8F(a',1,\lambda_b) . \tag{43}$$

Substituting these into the self-consistency relation (20) gives a solution for λ_b in terms of a:

$$\lambda_b^8=27\sqrt{\pi}\zeta_R(9)\Gamma(\tfrac{9}{2})a'^5(2-a'^2) . \tag{44}$$

So as $a'\rightarrow\sqrt{2}$ from above, $\lambda_b\rightarrow 0$ as can be seen in Fig. 4. This value is confirmed to 7 places of decimals by the exact solutions. Using the approximate expression for $E(a',1,\lambda_b)$, (43), in (21) gives

$$b^7=\frac{2\bar{G}}{3\pi^4}\frac{1}{\lambda_b a'^3(2-a'^2)} \tag{45}$$

and so $b\rightarrow\infty$ as $a'\rightarrow\sqrt{2}$ and $a=ba'\rightarrow\infty$ as well. When (44) is used to substitute for λ_b, and using $T=1/b\lambda_b$ we obtain

$$T^7=\frac{3\pi^4}{2\bar{G}}\left[\frac{1}{27\sqrt{\pi}\zeta_R(9)\Gamma(\frac{9}{2})a'}\right]^{3/4}(2-a'^2)^{1/4} \tag{46}$$

which shows that $T\rightarrow 0$ also.

So there exists a low or zero-temperature solution for a finite value ($\sqrt{2}$) of the radius ratio, although the radii themselves become very large as the limit is approached.

VI. CONCLUSIONS

The same limitations apply to this calculation as to the one in [1], that is the lack of time dependence in the metric. However the behavior is qualitatively different, albeit in a region far removed from present reality. This behavior shows interesting features which do not appear in the approximations usually used in this type of calculation (for example those of high temperature and low curvature), and as such merits some interest. The toy model considered here can of course be generalized in many ways. There is clearly no difficulty in extending this calculation to $R\times S^m\times S^n$ although the numerical procedures become tedious. One or both of the spheres could be distorted by squashing; the spatial section could be made noncompact, replacing S^n by H^n, the n-dimensional hyperboloid, for example, as in [21,22]. The most important extension would be to see if the results found here in the static case had any effect on dynamical Kaluza-Klein cosmologies [3–5]. The simplest procedure would be to use the static approximation and apply the results here directly. Better would be an adiabatic expansion in powers of time derivatives (see for example [23]) which can have significant effects on the effective action and on questions of stability. The exact results should certainly produce some effect if the cosmology concerned were to stray away from the region for which the high-temperature and low curvature approximations are valid. The other obvious possibility is to use other fields: spin-$\frac{1}{2}$ [6,24], sigma models [10] or the gravitational field [25–27] for which it would be interesting and necessary to use the modified definition of the effective action given by Vilkovisky [28] and DeWitt to ensure the gauge independence of the results [29].

ACKNOWLEDGMENTS

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APPENDIX A. APPROXIMATE FORMS WHEN ONE OF THE SPHERES IS LARGE

The heat kernel for one of the spheres can be expanded in a short-time asymptotic series which produces a high temperature expansion for the traced zeta function on that sphere. This in turn produces a "large-rT" expansion for the free energy (where r is the radius of the sphere) and this expansion can therefore be used to produce an approximation to the free energy when rT for one of the spheres is $\gg 1$.

An expression for the free energy of the field for a manifold of the form $R\times\mathcal{M}^t = R\times\mathcal{M}^n\times\mathcal{M}^d$ when $\mathcal{M}^n = R^n$ is given in [19]. We need a generalization of this to the case where the curvature of $\mathcal{M}^n$ is small but nonzero.

In general the free energy for a manifold with an ultrastatic metric can be expressed in the form

$$F=-L(\beta)=\frac{1}{2}i\lim_{s\to 1}\frac{L^{-2(s-1)}}{s-1}\mathrm{tr}_t\zeta_{t+1}(s-1,\beta)\ , \qquad \text{(A1)}$$

where L is an arbitrary scaling length , t is the total number of spatial dimensions (8 here) and $\mathrm{tr}_t\zeta_{t+1}(s,\beta)$ is the covariant integration over the spatial part of the manifold of the coincidence limit of the finite-temperature zeta function on the whole space-time, i.e.,

$$\mathrm{tr}_t\zeta_{t+1}(s,\beta)=\int d^tx\, g^{1/2}(x)\zeta_{t+1}(x,x,s,\beta)\ . \qquad \text{(A2)}$$

From [19] we have that

$$\begin{aligned}\mathrm{tr}_t\zeta_{t+1}(s,\beta)&=\frac{i}{\beta}\frac{e^{i\pi s/2}}{\Gamma(s)}\int_0^\infty d\tau\,\tau^{s-1}\theta_3(0|-4\pi\tau/\beta^2)[K_t(\tau)-d_0^{(t)}]+\frac{2i}{\beta}d_0^{(t)}\left[\frac{\beta}{2\pi}\right]^{2s}\zeta_R(2s)\\ &={}_tX_{t+1}(s,\beta)+{}_tY_{t+1}(s,\beta)\end{aligned} \qquad \text{(A3)}$$

where

$${}_tY_{t+1}(s,\beta)=\frac{2i}{\beta}d_0^{(t)}\left[\frac{\beta}{2\pi}\right]^{2s}\zeta_R(2s)$$

and also, by using the standard identity of the theta function, that

$$\mathrm{tr}_t\zeta_{t+1}(s,\beta)=i\frac{e^{i\pi s/2}}{\Gamma(s)}\int_0^\infty d\tau\frac{\tau^{s-1}}{(4\pi i\tau)^{1/2}}\sum_{m=-\infty}^{\infty}\exp(im^2\beta^2/4\tau)[K_t(\tau)-d_0^{(t)}]+\frac{2i}{\beta}d_0^{(t)}\left[\frac{\beta}{2\pi}\right]^{2s}\zeta_R(2s)\ . \qquad \text{(A4)}$$

Because we are working on a product manifold we can factorize the heat kernel and the zero-mode degeneracies as

$$K_t(\tau)=K_n(\tau)K_d(\tau),\quad d_0^{(t)}=d_0^{(n)}d_0^{(d)}\ . \qquad \text{(A5)}$$

${}_tX_{t+1}(s,\beta)$ can now be expressed as follows:

$${}_tX_{t+1}(s,\beta)=\frac{i}{\beta}\frac{e^{i\pi s/2}}{\Gamma(s)}\int_0^\infty d\tau\,\tau^{s-1}\theta_3(0|-4\pi\tau/\beta^2)[K_n(K_d-d_0^{(d)})+d_0^{(d)}(K_n-d_0^{(n)})]\ . \qquad \text{(A6)}$$

Now $K_n(\tau)$ can be expanded asymptotically for small τ in the following series (which is also a small curvature expansion):

$$K_n(\tau) \sim (4\pi i \tau)^{-n/2} \sum_{l=0,1/2,1,\ldots} C_l^{(n)}(i\tau)^l \ . \tag{A7}$$

When this is substituted into (A6) and use made of the identity

$$\int_0^\infty d\tau\, \tau^{s-1}[\theta_3(0|i\tau^2)-1] \equiv \pi^{-s/2}\Gamma(s/2)\zeta_R(s) \tag{A8}$$

then we obtain

$$\mathrm{tr}_t \zeta_{t+1}(s,\beta) = \sum_{l=0,1/2,1,\ldots} C_l^{(n)}(4\pi)^{-n/2}\frac{\Gamma(s-n/2+l)}{\Gamma(s)}\mathrm{tr}_d\zeta_{d+1}(s-n/2+l,\beta) + d_0^{(d)}\frac{i}{\beta}\zeta_n(s,\infty) \ . \tag{A9}$$

The free energy is now given by (A1). So we require

$$F = \frac{1}{2} i \lim_{s\to 1}\frac{1}{s-1}\left[\sum_{l=0,1/2,1,\ldots} C_l^{(n)}(4\pi)^{-n/2}\frac{\Gamma(s-1-n/2+l)}{\Gamma(s-1)}\mathrm{tr}_d\zeta_{d+1}(s-1-n/2+l,\beta) + d_0^{(d)}\frac{i}{\beta}\zeta_n(s-1,\infty)\right] \tag{A10}$$

where the length scale L has been set equal to 1 for simplicity. Obviously we need to know

$$\frac{1}{2} i \lim_{s\to 1}\frac{1}{s-1}(4\pi)^{-n/2}\frac{\Gamma(s-1-n/2+l)}{\Gamma(s-1)}\mathrm{tr}_d\zeta_{d+1}(s-1-n/2+l,\beta) \ . \tag{A11}$$

$\mathrm{tr}_d\zeta_{d+1}(s,\beta)$ [which is given by (A3) or (A4) with t replaced by d] can be divided into three parts:

(i) The $m=0$ term in the sum in (A4), which is the zero-temperature part. This can be expressed in terms of ζ_d ($\equiv \mathrm{tr}_d\zeta_d$), the zeta function on $\mathcal{M}^d$ as follows [13]:

$$\mathrm{tr}_d\zeta_{d+1}(s,\infty) = \mathrm{tr}_d\zeta_{d+1}(s) = \frac{i}{\sqrt{4\pi}}\frac{\Gamma(s-\frac{1}{2})}{\Gamma(s)}\zeta_d(s-\tfrac{1}{2}) \ . \tag{A12}$$

(ii) The $m\neq 0$ terms, which give part of the finite-temperature correction. Call these $\bar{\zeta}_{d+1}(s,\beta)$.

(iii) The "Planckian" term in β^{2s-1}.

Different values of l produce different poles in the various terms above. $\zeta_d(s)$ has poles at $s=(d-p)/2$ where $p=0,1,\ldots,d-1,d+1,d+3,\ldots$ [19] with residues

$$\rho_p = \frac{(4\pi)^{-d/2}}{\Gamma(\frac{1}{2}(d-p))}C_{p/2}^{(d)} \tag{A13}$$

and remainders $R_d(s)$. The highest pole of $\zeta_d(s-1-\frac{1}{2}(n+1-2l))$ as $s\to 1$ will thus be at $l=(d+n+1)/2=(t+1)/2$. For $l>(t+1)/2$ the limits are all finite and the series can be left as it is. For $l\le(t+1)/2$ we wish to separate the finite part and find the residues.

When these limits are taken for the different ranges of l in (A11) and the expressions combined we find, after some work, the following result for the free energy:

$$
\begin{aligned}
F=&-\sum_{l=0}^{(n-1)/2} C_l^{(n)}\left[\frac{1}{2}(4\pi)^{-(t+1)/2}C^{(d)}_{(t+1-2l)/2}\left[\frac{1}{s-1}+\psi(\tfrac{1}{2}(n+3-2l))+\gamma\right]\right.\\
&\qquad+\frac{(4\pi)^{-(n+1)/2}}{2\Gamma(\frac{1}{2}(n+3-2l))}A_l-\frac{i}{2}\frac{(4\pi)^{-n/2}}{2\Gamma(\frac{1}{2}(n+2-2l))}B_l(\beta)\\
&\qquad\left.+(4\pi)^{-(n+1)/2}d_0^{(d)}\Gamma[\tfrac{1}{2}(n+1-2l)]\zeta_R(n+1-2l)\left[\frac{\beta}{2}\right]^{2l-n-1}\right]\\
&-C^{(n)}_{n/2}\left[\frac{1}{2}(4\pi)^{-(t+1)/2}C^{(d)}_{(d+1)/2}\left[\frac{1}{s-1}+2-2\ln 2\right]+\frac{(4\pi)^{-(n+1)/2}}{\sqrt{\pi}}A_{n/2}\right.\\
&\qquad\left.-\frac{i}{2}(4\pi)^{-n/2}B_{n/2}(\beta)-\frac{(4\pi)^{-n/2}}{\beta}d_0^{(d)}\ln\beta\right]\\
&-C^{(n)}_{(n+1)/2}\left\{\frac{1}{2}(4\pi)^{-(t+1)/2}C^{(d)}_{d/2}\left[\frac{1}{s-1}\right]+\frac{(4\pi)^{-(n+1)/2}}{2}A_{(n+1)/2}\right.\\
&\qquad\left.-\frac{i}{2\sqrt{\pi}}(4\pi)^{-n/2}B_{(n+1)/2}(\beta)+(4\pi)^{-(n+1)/2}d_0^{(d)}\left[\ln\left[\frac{\beta}{4\pi}\right]+\gamma\right]\right\}\\
&-\sum_{l=(n+2)/2}^{(t+1)/2} C_l^{(n)}\left[\frac{1}{2}(4\pi)^{-(t+1)/2}C^{(d)}_{(t+1-2l)/2}\left[\frac{1}{s-1}+\psi(\tfrac{1}{2}(2l-n-1))+\gamma\right]\right.\\
&\qquad+\frac{1}{2}(4\pi)^{-(n+1)/2}\Gamma(\tfrac{1}{2}(2l-n-1))R_d(\tfrac{1}{2}(2l-n-1))\\
&\qquad-\frac{i}{2}(4\pi)^{-n/2}\Gamma(\tfrac{1}{2}(2l-n))\overline{\zeta}_{d+1}(\tfrac{1}{2}(2l-n),\beta)\\
&\qquad\left.+(4\pi)^{-n/2}\frac{d_0^{(d)}}{2\pi}\Gamma(\tfrac{1}{2}(2l-n))\zeta_R(2l-n)\left[\frac{\beta}{2\pi}\right]^{2l-n-1}\right]\\
&-\sum_{l=(t+2)/2}^{\infty} C_l^{(n)}\left[\frac{1}{2}(4\pi)^{-n/2}\Gamma(\tfrac{1}{2}(2l-n-1))\zeta_d(\tfrac{1}{2}(2l-n-1))\right.\\
&\qquad-\frac{i}{2}(4\pi)^{-n/2}\Gamma(\tfrac{1}{2}(2l-n))\overline{\zeta}_{d+1}(\tfrac{1}{2}(2l-n),\beta)\\
&\qquad\left.+(4\pi)^{-n/2}\frac{d_0^{(d)}}{2\pi}\Gamma(\tfrac{1}{2}(2l-n))\zeta_R(2l-n)\left[\frac{\beta}{2\pi}\right]^{2l-n-1}\right]\\
&+\frac{d_0^{(t)}}{2\beta}\frac{1}{s-1}-\frac{d_0^{(d)}}{2\beta}\zeta_n'(0)\;. \qquad \text{(A14)}
\end{aligned}
$$

In this expression

$$
A_l=\begin{cases}(-1)^{(1/2)(n+2-2l)}\pi R_d(-\tfrac{1}{2}(n+1-2l)), & n-2l \text{ even },\\ (-1)^{(1/2)(n+1-2l)}\zeta_d'(-\tfrac{1}{2}(n+1-2l)), & n-2l \text{ odd },\end{cases} \qquad \text{(A15)}
$$

$$
B_l(\beta)=\begin{cases}(-1)^{(1/2)(n-2l)}\overline{\zeta}\,'_{d+1}(-\tfrac{1}{2}(n-2l),\beta) & n-2l \text{ even },\\ (-1)^{(1/2)(n+1-2l)}\pi\overline{\zeta}_{d+1}(-\tfrac{1}{2}(n-2l),\beta) & n-2l \text{ odd }.\end{cases} \qquad \text{(A16)}
$$

This is a generalization of Eq. (24) in [19].

The expressions for $B_l(\beta)$ remain to be evaluated. The finite-temperature quantity $\overline{\zeta}_{d+1}(s,\beta)$ is given by

$$
\overline{\zeta}_{d+1}(s,\beta)=i\frac{e^{i\pi s/2}}{\Gamma(s)}\int_0^\infty d\tau\frac{\tau^{s-1}}{(4\pi i\tau)^{1/2}}\sum_{m=-\infty}^{\infty}{}'\exp(im^2\beta^2/4\tau)[K_d(\tau)-d_0^{(d)}]\;, \qquad \text{(A17)}
$$

where the prime means omitting the $m=0$ term. By substituting an eigenvalue form for K_d the integral can be performed to give a series of MacDonald Bessel functions which can then be given an integral representation. For $l<(n+2)/2$ the sum over m can then be performed. This is the range we require in (A14). For $l \geq (n+2)/2$ the sum cannot easily be performed. We find the following two expressions.

$$B_l(\beta)=2i \sum_{\omega_i^{(d)}}{}' D_i^{(d)} \omega_i^{(d)n+1-2l} \int_1^\infty dx \frac{(x^2-1)^{(1/2)(n-2l)}}{e^{m\beta\omega_i^{(d)}x}-1}, \quad l<\frac{n+2}{2} . \tag{A18}$$

and

$$\zeta_{d+1}(\tfrac{1}{2}(2l-n),\beta)=\frac{2i}{\Gamma^2(\frac{1}{2}(2l-n))} \sum_{\omega_i^{(d)}}{}' D_i^{(d)} \left[\frac{\beta}{2}\right]^{2l-n-1} \times \int_1^\infty dx\, e^{-\beta\omega_i^{(d)}x} \Phi(e^{-\beta\omega_i^{(d)}x}, n+1-2l,1)(x^2-1)^{(1/2)(2l-n-2)} \quad l>n/2 , \tag{A19}$$

where $\Phi(\tau,s,v)=\sum_{k=0}^{\infty}(v+k)^{-s}\tau^k$ and $v>0$. Unfortunately $\Phi(\tau,s,v)$ is hard to represent simply for $s<0$, which is the range we are interested in.

The expression (A14) is utilized at two points in the text: for the cases when one of the spheres (S^n) is replaced by R^n, and for the approximate forms when the ratio of the radii of the spheres becomes large.

1. The n manifold becomes flat

In this case the total free energy will diverge for the simple reason that the space-time volume is infinite and so it is necessary to look at densities on the flat space. To this end the trace over the flat space-time can be "divided out" of (A14) which will have the effect of replacing the integrated Minakshisundaram coefficients $C_l^{(n)}$ by the unintegrated versions $a_l^{(n)}$ and of dividing the final terms in (A14) by the volume of the n sphere. When the radius of the n sphere is taken to infinity these will vanish, and since R^n has no boundary the half-odd-integer coefficients will also disappear. For $l>0$ and integer the quantities $a_l^{(n)}$ involve positive powers of the curvature and will vanish when $\mathcal{M}^n=R^n$, and so the only term that will survive in the expansion (A14) is the $l=0$ term. If in addition the total space-time dimension $t+1$ is odd the quantities $C_{(t+1-2l)/2}^{(d)}$ will vanish for integral l and in particular for $l=0$; the term in $C_{(t+1)/2}^{(d)}$ in (A14) is therefore zero. This leaves for the free energy density

$$\phi=-\frac{1}{2}\frac{(4\pi)^{-(n+1)/2}}{\Gamma(\frac{1}{2}(n+3))}A_0+\frac{i}{2}\frac{(4\pi)^{-n/2}}{\Gamma(1+n/2)}B_0(\beta) -d_0^{(d)}\frac{\Gamma(\frac{1}{2}(n+1))}{\pi^{(n+1)/2}}\zeta_R(n+1)\beta^{-n-1} . \tag{A20}$$

The vacuum energy density can now be evaluated from $\eta=\partial(\beta\phi)/\partial\beta$ and the expression found is the same as that in [19]. Specializing to the case where $\mathcal{M}^d$ is a d sphere, and ϕ becomes a function of r_d, the radius of the d sphere, and β, enables the dependence of r_d on temperature to be calculated from (26) and (29) [here $B_0(\beta)$ is evaluated by numerical integration], and it is this which is plotted in Figs. 1 and 2. The zero-temperature values of n are expressed in (A20) in terms of the zeta function on the internal space, in this case a sphere. These values can be calculated using the Epstein zeta function method used in this paper or by an alternative method such as that used in [19]. The values found by these two methods agree.

2. The curvature of the n manifold is small

These are the cases considered in Sec. V A. Assuming that $\partial\mathcal{M}^n=\emptyset$, the $l=$ half-odd-integer terms will be zero as $C_l^{(n)}$ will vanish. The series in (A14) is taken to terms linear in the curvature of $\mathcal{M}^n$ which means discarding terms with $l>1$. The final term will have a logarithmic dependence on the scale of $\mathcal{M}^n$. In the case of $\mathcal{M}^n$ being a sphere the $l=0$ and $l=1$ terms in the series will depend on r_n as r_n^n and r_n^{n-2} which, for $n>2$ and r_n large, will swamp the logarithm. The penultimate term does not contribute to either the vacuum energy or $\partial F/\partial$(radius) which are the expressions appearing in the self-consistency conditions, and will thus be omitted. If β is also considered large (as in Sec. V A) then the terms in $B_l(\beta)$ can be dropped and the final expression becomes (for n odd)

$$F(r_n,r_d,\beta)=C_0^{(n)}\left[-\frac{1}{2}\frac{(4\pi)^{-(n+1)/2}}{\Gamma(\frac{1}{2}(n+3))}\zeta_d'(-\tfrac{1}{2}(n+1))-d_0^{(d)}\frac{\Gamma(\frac{1}{2}(n+1))}{\pi^{(n+1)/2}}\zeta_R(n+1)\beta^{-n-1}\right] +C_1^{(n)}\left[-\frac{1}{2}\frac{(4\pi)^{-(n+1)/2}}{\Gamma(\frac{1}{2}(n+1))}\zeta_d'(-\tfrac{1}{2}(n-1))-d_0^{(d)}\frac{\Gamma(\frac{1}{2}(n-1))}{4\pi^{(n+1)/2}}\zeta_R(n-1)\beta^{-n+1}\right] . \tag{A21}$$

With the appropriate values for d and n, the expressions in Sec. V A can be obtained.

[1] J. S. Dowker and I. H. Jermyn, Class. Quantum Grav. **7**, 965 (1990).
[2] M. A. Rubin and B. Barkan-Roth, Nucl. Phys. **B226**, 444 (1983).
[3] S. Randjbar-Daemi, A. Salam, and S. Strathdee, Phys. Lett. **135B**, 388 (1984).
[4] M. Yoshimura, Phys. Rev. D **30**, 344 (1984).
[5] F. S. Accetta and E. W. Kolb, Phys. Rev. D **34**, 1798 (1986).
[6] K. Shiraishi, Prog. Theor. Phys. **77**, 1253 (1986).
[7] J. Okada, Class. Quantum Grav. **3**, 221 (1986).
[8] M. Szydłowski and J. Szczęsny, Phys. Rev. D **38**, 3625 (1988).
[9] J. Szczęsny, M. Szydłowski, and M. Biesiada, Europhys. Lett. **4**, 761 (1987).
[10] B. Chakraborty and R. Parthasarathy, Class. Quantum Grav. **6**, 1455 (1989).
[11] X. Shi and X. Li, Class. Quantum Grav. **8**, 75 (1991).
[12] S. D. Odintsov, J. Nucl. Phys. **51**, 1787 (1990).
[13] J. S. Dowker and G. Kennedy, J. Phys. A **11**, 895 (1978).
[14] G. W. Gibbons, in *On Functional Integrals in Curved Space-Time,* edited by K. Bleuler, H. R. Petry, and A. Reetz, Lecture Notes in Mathematics No. 676 (Springer, Berlin, 1978), p. 513.
[15] P. Epstein, Math. Ann. **56**, 615 (1903).
[16] R. J. Terras, Comput. Phys. **31**, 146 (1979).
[17] J. S. Dowker, Phys. Rev. D **39**, 1196 (1988).
[18] P. Candelas and S. Weinberg, Nucl. Phys. **B237**, 397 (1984).
[19] J. S. Dowker, Class. Quantum Grav. **1**, 359 (1984).
[20] M. A. Awada and D. J. Toms, Phys. Lett. **135B**, 283 (1984).
[21] H. Y. Lee and N. Sakai, Phys. Rev. D **34**, 2956 (1986).
[22] P. Roy, Int. J. Mod. Phys. A **5**, 353 (1990).
[23] G. Gilbert, B. McLain, and M. A. Rubin, Phys. Lett. **142B**, 28 (1984).
[24] J. S. Dowker, Phys. Rev. D **29**, 2773 (1984).
[25] A. Chodos and E. Myers, Ann. Phys. (N.Y.) **156**, 412 (1984).
[26] S. Randjbar-Daemi and M. H. Sarmadi, Phys. Lett. **151B**, 343 (1985).
[27] M. H. Sarmadi, Nucl. Phys. **B263**, 187 (1986).
[28] G. A. Vilkovisky, in *Quantum Theory of Gravity. Essays in Honor of the 60th Birthday of Bryce DeWitt,* edited by S. M. Christensen (Hilger, Bristol, 1985), p. 169.
[29] E. S. Fradkin and A. A. Tseytlin, Nucl. Phys. **B234**, 509 (1984).