Effect of topology on the thermodynamic limit for a string gas

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We discuss the thermodynamic limit for a gas of strings at high energy densities. This is defined by studying the statistical properties of the gas in a compact space and taking the size of the space to infinity keeping the energy density finite. We obtain a behavior that is different from earlier treatments where the gas is considered at the same energy density but living in a noncompact space. In particular we show that the gas is not dominated by a single energetic string above the Hagedorn energy density, but instead the number of energetic strings is $\ln R/\sqrt{\alpha'}$ where R is the radius of the universe and α' the slope parameter. The reason for the thermodynamic behavior being sensitive to topology is the existence of winding modes that can sense the large-scale structure of the space.

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I. INTRODUCTION

In the case of a gas of point particles one does not expect the thermal properties of the gas to depend upon the topology of the box in which it lives, provided the size of the box is sufficiently large. Effects sensitive to the large-scale structure of the box (or finite-size effects) are typically surface effects subleading in magnitude to the bulk or volume effects in the thermodynamic limit. In this limit quantities like the entropy and specific heat, etc., are extensive quantities.

One does not expect this to continue being the case for a string gas because strings have extension and individual strings could be sensitive to the large-scale structure of the space in which the gas lives. In fact, it has been known for a long time that, for an ideal gas of strings at energy densities high compared to the typical energy density in string theory (called the Hagedorn energy density and constructed out of the single dimensional parameter α'), a single energetic string can capture most of the energy of the system [1,2] (see also [3,4]). This result holds if the space in which the gas lives has at least three noncompact spatial dimensions ($d \geq 3$; d denotes the number of noncompact spatial dimensions). Introducing a volume V of the noncompact space as a bookkeeping device (in the thermodynamic limit the volume is taken to infinity) one finds that the system is not extensive, e.g., the specific heat is not proportional to the volume (and is, in fact, negative).

The single energetic string that dominates the system also occupies a large size. In fact, the extension in space of the wave function of a string of large energy ϵ is proportional to $\sqrt{\epsilon}$ [3], and for a string that captures a significant fraction of the total energy (i.e., $\epsilon \propto V$), this extension is of the order of $L^{d/2}$, where L denotes the linear size of the system. Thus, in the thermodynamic limit a single string acquires a spread infinitely greater than the size of the system itself. This means that mere bookkeeping of the size of the system is not enough; we must consider *ab initio* the effect of the size of the system on statistical properties of the system.

This and other reasons have motivated the study of an ideal string gas in compact spaces [5,6,4,7–10]. This space has no noncompact dimensions ($d=0$); all dimensions have a finite size and one starts with the full density of states explicitly including the states sensitive to the size of the space, for instance, the winding modes in the case where the space has the toroidal topology. In the noncompact studies mentioned earlier, such states are excluded in the bookkeeping. The statistical properties of this system turn out to be quite different from the noncompact case. E.g., the energy of the system is distributed uniformly among strings of all energies [4] and the specific heat can be positive.

However, up to now the studies of the compact case are valid only when the size of the space is itself small, of the order of $\sqrt{\alpha'}$ (and the total energy is much larger than $1/\sqrt{\alpha'}$ so that the energy density is above the Hagedorn energy density). In this situation the string modes sensitive to the size of the system (e.g., winding modes) are, of course, excited because, at this radius, these modes cost just as much energy as the ordinary modes. Thus, one does not know whether the difference between the results in this case and the noncompact space is a reflection of a characteristic pattern of excitation of size-sensitive modes in a string gas, or is a feature specific to the small-radius case.

In order to better understand the interplay between the size of the system and the modes excited in it above Hagedorn energy densities, and also to faithfully compare with the noncompact case it is necessary to consider the gas in a compact space, but allow the size of the space to be large ($R \gg \sqrt{\alpha'}$), keeping the energy density above the Hagedorn energy density. This defines the proper thermodynamic limit for the system. Compactness of the space means that we are forced to include modes sensitive to the large-scale structure of the space in the density of states, and the large radius means that we are letting such modes cost a lot of energy. The question is whether in this case the extended string modes are still excited at Hagedorn energy densities, and if so, how they affect the statistical properties of the gas. Is the picture obtained in the noncompact studies which excluded some of these modes from the density of states but ended up with a very energetic string still valid? This is the question we address in this paper. We study explicitly the example of a toroidal compactification.

Our conclusion is that, in the thermodynamic limit, the system is inherently stringy and different from a point-particle system in that it *is* sensitive to the large-scale structure of the space. In this limit it differs from both the compact, small-radius case and the noncompact case, but the behavior of modes at the upper end of the energy spectrum is more like the compact, small-radius case than the noncompact case. In particular, winding modes continue to be excited above Hagedorn energy densities when the number of expanding radii (denoted $\bar{d}$) is greater than or equal to three, there are a large number of energetic strings present in the thermodynamic limit (not just one) and the thermal properties such as the specific heat of the gas are different from the noncompact case at the same energy density.

Another motivation for this work comes from cosmology. According to the standard model of cosmology, the energy density ρ in the universe has evolved in time as $\rho \sim t^{-2}$ and the radius of the universe R as $R \sim t^{1/2}$ in the radiation-dominated era and as $R \sim t^{2/3}$ in the matter-dominated era. Extrapolating these in time from their present estimated values (e.g., assuming that the size of the universe today is 3000 Mps) backwards up to Planck time 10^{-19} GeV^{-1}, one finds that, at that time, the energy density of the universe was the Planck energy density while the radius of the universe was of the order of 10^{26} times the Planck length. Thus, we have a large (and possibly compact) universe at very high energy density. (Roughly speaking, since the present energy in the universe is much greater than the Planck mass, if all of it was compressed into a volume of Planck size, the energy density would be much greater than the Planck energy density. Conversely, therefore, if all this energy was compressed into a space with Planck energy density, that space must have a size much larger than the Planck radius. Assuming an adiabatic expansion gives the factor of 10^{26}.) This estimate is, of course, based on the original standard model without inflation. In fact, the appearance of such a large radius at early times is another way of stating the horizon problem of the standard model which inflation solves by restoring the size of the universe at Planck time to be about Planck size. Nevertheless, it is interesting that the energy densities and sizes we consider in this paper are quite conceivable in the early universe.

In the context of strings, a cosmological scenario for the very early universe based on $R \leftrightarrow 1/R$ duality and the existence of winding modes was proposed in [5]. Our results for the energy and winding-number distribution are relevant for this scenario.

In the next section we review known results about the string gas in a space that is compact and small, or noncompact. In Sec. III we consider the gas in a compact space with large radius. We discuss the density of states and the distribution of energy and winding number among individual strings and compare with results in the noncompact space. The latter discussion pertains to the case when $\bar{d} \geq 3$ (actually, for any $\bar{d} > 2$). The cases $\bar{d} = 1,2$ are described in Sec. IV. Section V summarizes the main conclusions.

II. REVIEW OF KNOWN RESULTS FOR A STRING GAS

The quantities we will be interested in are the single-string density of states $f(\epsilon) \equiv \sum_a \delta(\epsilon - \epsilon_a)$, where ϵ_a is the energy of the single-string state a, and the microcanonical distribution function or the total density of states $\Omega(E) \equiv \sum_\alpha \delta(E - E_\alpha)$, where E_α is the energy of the state α of the whole system. $\Omega(E)$ gives the entropy, temperature, pressure, and specific heat of the gas. In addition, we are interested in knowing how the energy of the gas is distributed among individual strings. This is described by the function $\mathcal{D}(\epsilon;E)$, defined such that $\mathcal{D}(\epsilon;E)d\epsilon$ equals the average number of strings in the gas with individual energies between ϵ and $\epsilon + d\epsilon$ when the total energy in the gas is E, and given by the simple formula [4],

$$\mathcal{D}(\epsilon;E) = \frac{1}{\Omega(E)} f(\epsilon)\Omega(E-\epsilon) . \tag{2.1}$$

In principle, the single-string density of states if known for all ϵ determines the total density of states and hence the distribution of energy among individual strings. For example for an ideal gas of pointlike objects $f(\epsilon)$ varies as a positive power of ϵ, and hence $\mathcal{D}(\epsilon;E)$ as a function of ϵ at fixed E rises like a power of ϵ, peaks at an energy $\epsilon = \bar{\epsilon} \sim E/\bar{N}$ (where $\bar{N}$ is the average number of particles in the gas), and decays exponentially (for explicit formulas see, e.g., [7]). This is shown in Fig. 1 where $\epsilon\mathcal{D}(\epsilon;E)$ is plotted against ϵ [$\epsilon\mathcal{D}(\epsilon;E)d\epsilon$ equals the average energy carried by particles in the energy range ϵ to $\epsilon + d\epsilon$ in a gas of total energy E].

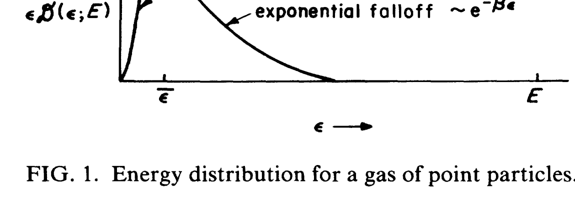

FIG. 1. Energy distribution for a gas of point particles.

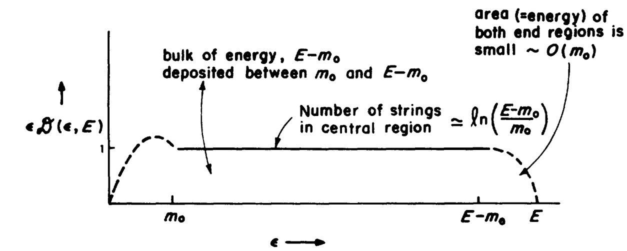

FIG. 2. Energy distribution for a string gas in compact space ($d=0$) of radius $\sim\sqrt{\alpha'}$.

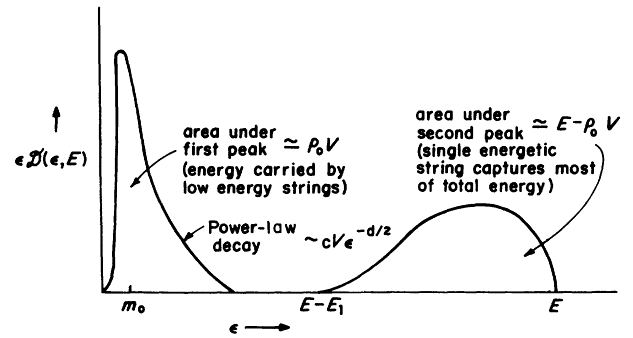

FIG. 3. Energy distribution for a string gas in noncompact space ($d\geq 3$).

A. String gas in a compact and small space

For a string gas in a space all of whose dimensions are compact ($d=0$) and small ($R\sim\sqrt{\alpha'}$), the corresponding energy distribution is flat [4] (see Fig. 2) implying that the total energy of the gas is shared equally among strings of all energies. This follows from the leading-order expressions for $f(\epsilon)$ [8,5] and $\Omega(E)$ [5,6]:

$$f(\epsilon)=e^{\beta_0\epsilon}/\epsilon\ , \tag{2.2}$$

$$\Omega(E)=\beta_0 e^{\beta_0 E+\lambda_0}\ , \tag{2.3}$$

valid for ϵ, $E>m_0\sim 1/\sqrt{\alpha'}$. Here β_0 is the inverse Hagedorn temperature, $\beta_0=(2\pi^2\alpha')^{1/2}(\sqrt{\omega_l}+\sqrt{\omega_r})$, where (ω_l,ω_r) is (2,2), (2,1), and (1,1), respectively, for the closed bosonic, heterotic, and type-II strings; and λ_0 is a constant. Consequently,

$$\mathcal{D}(\epsilon;E)=1/\epsilon \tag{2.4}$$

for $m_0<\epsilon<E-m_0$, and hence $\epsilon\mathcal{D}(\epsilon;E)$ is constant and unity as shown in Fig. 2.

B. String gas in noncompact space

On the other hand, for a string gas in a space with at least three dimensions noncompact ($d\geq 3$), at high energy densities a single energetic string captures most of the energy of the gas [1–4]. This conclusion follows if the expression

$$f(\epsilon)=cV\frac{e^{\beta_0\epsilon}}{\epsilon^{d/2+1}} \tag{2.5}$$

for the single-string density of states is assumed to hold for all $\epsilon>m_0\sim\sqrt{\alpha'}$. Here c is a constant $\sim\alpha'^{-3d/4}$ and V is the volume (to be taken to infinity in the thermodynamic limit) of the d noncompact dimensions. Equation (2.5) implies that the leading behavior of $\Omega(E)$ is given by

$$\Omega(E)=cV\frac{e^{\beta_0 E+\gamma_0 V}}{E^{d/2+1}} \tag{2.6}$$

for energy densities greater than the Hagedorn energy density $\rho_0\sim\alpha'^{-(d+1)/2}$, i.e., for $E>\rho_0 V$. ($\gamma_0\sim\alpha'^{-d/2}$ is a constant with dimensions of a number density.) From these expressions the energy distribution function can be analyzed. The result, depicted in Fig. 3 is that, at an energy density ρ ($\equiv E/V$) greater than the Hagedorn energy density ρ_0 [$(\rho-\rho_0)/\rho_0>1$, to be precise], a single very energetic string captures most of the energy (about $E-\rho_0 V$) of the system (the peak between $E-E_1$ and E in Fig. 3 is due to this string); the remainder of the energy, $\rho_0 V$, is shared by mostly low-energy strings which constitute a gas whose pressure, temperature (close to $1/\beta_0$), and number density (close to γ_0) are almost independent of the total energy. (For details see [4].) From (2.6) it also follows that the system has negative specific heat.

III. THE STRING GAS IN COMPACT BUT LARGE SPACE

We would like to compute the large-radius corrections to the density of states when the space is compact. To do this we will employ the singularity structure of the thermal free energy $\Gamma(\beta)$ [6], in particular, the fact that $\Gamma(\beta)$ has radius-dependent singularities that move towards the Hagedorn singularity β_0 as the radius becomes much larger than $\sqrt{\alpha'}$.

A. Radius-dependent singularities of the thermal free energy

Both $f(\epsilon)$ and $\Omega(E)$ can be obtained as inverse Laplace transforms from $\Gamma(\beta)$:

$$f(\epsilon)=\int_C\frac{d\beta}{2\pi i}e^{\beta\epsilon}\Gamma(\beta)\ , \tag{3.1}$$

$$\Omega(E)=\int_C\frac{d\beta}{2\pi i}e^{\beta\epsilon}Z(\beta)\ , \tag{3.2}$$

where $Z(\beta)\equiv e^{\Gamma(\beta)}$ is the thermal partition function and the contour C is chosen to the right of all singularities of $\Gamma(\beta)$. This contour can be distorted to C_1 (see Fig. 4) since the rightmost singularities all lie on the real axis. The dominant contribution to the density of states at high energy comes from the horizontal part of C_1, the part that sees the singularities.

The Hagedorn singularity β_0 is independent of the radius but β_1, β_2, etc., depend upon the radius of the space in which the gas lives in a manner described below. Each singularity contributes a term proportional to $e^{\beta_i\epsilon}$ in $f(\epsilon)$; hence, for fixed radius at sufficiently large ϵ, successive terms are exponentially suppressed, and the leading contribution (2.2) comes by considering the effect of only the rightmost singularity β_0. As the radius becomes

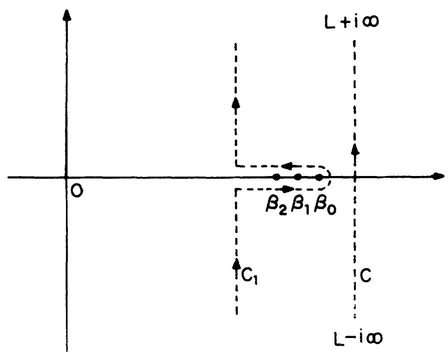

FIG. 4. Deformation of contour in the complex β plane for determining the density of state from the inverse Laplace transform. The new contour, C_1, "sees" the effect of the Hagedorn singularity β_0 as well as the subleading radius-dependent singularities β_1, β_2, etc.

large, β_1, β_2, etc., move up closer to β_0 and for a fixed ϵ, at sufficiently large radius the effect of all of them has to be included. This will give us the desired correction. Similar considerations hold for $\Omega(E)$.

Our considerations will be restricted to the toroidal geometry for space: $\bar{d}$ out of D spatial dimensions expand to circles of radii R_i, $i=1,\ldots,\bar{d}$ and the remaining $D-\bar{d}$ small dimensions are assumed to be circles of radius $\sim\sqrt{\alpha'}$. We denote $\bar{R}_i \equiv R_i/\sqrt{\alpha'}$. To be specific we consider the heterotic string, the conclusions hold for closed bosonic and type-II strings as well. The free energy has a logarithmic singularity for every $\mathbf{n}=(n_1,n_2,\ldots,n_{\bar{d}})$, n_i being integers subject to the constraint $\frac{1}{2}\sum_{i=1}^{\bar{d}}(n_i/\bar{R}_i)^s<1$. The singularity associated with $\mathbf{n}$ is denoted $\beta_{\mathbf{n}}$ and its location is

$$\beta_{\mathbf{n}}=(2\pi^2\alpha')^{1/2}\left[\left[1-\frac{1}{2}\sum_{i=1}^{\bar{d}}(n_i/\bar{R}_i)^2\right]^{1/2}+\left[2-\frac{1}{2}\sum_{i=1}^{\bar{d}}(n_i/\bar{R}_i)^2\right]^{1/2}\right]. \tag{3.3}$$

For $\mathbf{n}=0$ one recovers β_0 and for nonzero $\mathbf{n}$ one has $\beta_{\mathbf{n}}<\beta_0$. Thus, the free energy is given by

$$\Gamma(\beta)=-\sum_{\mathbf{n}}\ln\left[\frac{\beta-\beta_{\mathbf{n}}}{\beta_0}\right]+\lambda(\beta)\ , \tag{3.4}$$

where the sum goes over all the singularities mentioned above and $\lambda(\beta)$ is a regular function of β in the vicinity of these singularities. $\lambda(\beta)$ depends upon the other singularities of $\Gamma(\beta)$ further to the left of the $\beta_{\mathbf{n}}$ described above. For a derivation of this form of the free energy see [6,7].

Some of the singularities $\beta_{\mathbf{n}}$ can coalesce, e.g., if all the expanding radii are equal ($R_i=R$ for all i), $\beta_{\mathbf{n}}$ depends only on the magnitude n of $\mathbf{n}$ and (3.4) reduces to

$$\Gamma(\beta)=-\sum_{n=0}^{\sqrt{2}\bar{R}}g_n\ln\left[\frac{\beta-\beta_n}{\beta_0}\right]+\lambda(\beta)\ , \tag{3.5}$$

where g_n denotes the number of points in the lattice $Z^{\bar{d}}$ with the same magnitude n and

$$\beta_n=(2\pi^2\alpha')^{1/2}\{[1-\tfrac{1}{2}(n/\bar{R})^2]^{1/2}+[2-\tfrac{1}{2}(n/\bar{R})^2]^{1/2}\}\ . \tag{3.6}$$

For simplicity we will consider the case of all R_i equal, the generalization to unequal R_i is straightforward.

B. The single-string density of states

Every β_n is a logarithmic branch point of $\Gamma(\beta)$ where a branch cut originates and extends towards the negative real axis. In the computation of $f(\epsilon)$ from the inverse Laplace transform of $\Gamma(\beta)$ the contour C_1 picks up the discontinuity of the integrand across every cut. The contribution of a single term in (3.5) to the discontinuity of the integrand factor $e^{\beta\epsilon}\Gamma(\beta)$ is $2\pi i e^{\beta\epsilon}g_n\theta(\beta_n-\beta)$, where $\theta(x)$ is the step function, unity for x positive and vanishing for x negative. Hence,

$$f(\epsilon)=\sum_n g_n\frac{e^{\beta_n\epsilon}}{\epsilon} \tag{3.7}$$

up to a correction of the order of $e^{2\pi\sqrt{\alpha'}\epsilon}$. The $n=0$ term is precisely (2.2); the others are radius-dependent corrections.

Now consider (3.7) for large radius $R\gg\sqrt{\alpha'}$. Since the difference $\beta_0-\beta_1$ is of order $\alpha'^{3/2}/R^2$ for large enough energies ($\epsilon\gg R^2\alpha'^{-3/2}$) the correction terms are still exponentially suppressed compared to the leading term and $f(\epsilon)$ is still given by (2.2). However, for $\epsilon\ll R^2\alpha'^{-3/2}$, the corrections are comparable to the leading term and must be taken into account. For large R one can replace the sum over n by an integral, expand β_n in powers of $\bar{R}^{-1}$, and use the fact that $g_n\sim n^{\bar{d}-1}$, the area of a $\bar{d}-1$ dimensional sphere of radius n. The resultant Gaussian integration gives (2.5) with $V=(2\pi R)^{\bar{d}}$ and $\bar{d}$ replacing d. Thus,

$$f(\epsilon)=c'R^{\bar{d}}\frac{e^{\beta_0\epsilon}}{e^{\bar{d}/2+1}}\ ,\quad m_0<\epsilon\ll R^2\alpha'^{-3/2}\ , \tag{3.8a}$$

$$=\frac{e^{\beta_0\epsilon}}{\epsilon}\ ,\quad \epsilon\gg R^2\alpha'^{-3/2}\ . \tag{3.8b}$$

This shows that, when the radius of the compact space expands, the single-string density of states as a function of energy does *not* go over into the form (2.5) it has for a noncompact space. It does so only for sufficiently low energies $\epsilon\ll R^2\alpha'^{-3/2}$. For high energies $\epsilon\gg R^2\alpha'^{-3/2}$ it retains the form (2.2) that it has for a compact and small space. This has an explanation in terms of the winding modes which are frozen out at low energies when the radius is large and can be excited at sufficiently high energies (see [8] and below).

C. The total density of states $\Omega(E)$

We now turn to the question of how the total density of states is modified by the expansion. Since the single-string density of states changes, one expects $\Omega(E)$ to be modified as well. Naively one may expect that, as for the single-string density of states, for high energies one will remain with the compact, small-radius result (2.3) and recover the noncompact result (2.6) for low energies, the crossover energy being given by $R^2\alpha'^{-3/2}$. That turns out not to be the case. We find that large-radius corrections substantially modify $\Omega(E)$ even at high energies. However, for $\bar{d}\geq 3$ and at sufficiently high energies ($E \gg R^{\bar{d}}\alpha'^{-(\bar{d}+1)/2}$, not just $E\gg R^2\alpha'^{-3/2}$), the leading behavior of $\Omega(E)$ is proportional to (2.3), and the proportionality constant is a large radius-dependent quantity that is independent of the energy. There is no domain of energy and radius in which the form (2.6), obtained for a noncompact space, appears.

From (3.5) it follows that β_n is a pole of order g_n of the partition function $Z(\beta)$:

$$Z(\beta)=e^{\lambda(\beta)}\prod_n\left[\frac{\beta_0}{\beta-\beta_n}\right]^{g_n} . \tag{3.9}$$

C_1 can then be replaced by a sum of contours each encircling one pole (see Fig. 5). Therefore, $\Omega(E)$ can be thought of as the sum of terms each coming from a contour encircling a single singularity. The contribution Ω_0 from the contour encircling β_0 is

$$\Omega_0(E,R)=\beta_0 e^{\beta_0 E+\lambda(\beta_0)}\prod_n\left[\frac{\beta_0}{\beta_0-\beta_n}\right]^{g_n} , \tag{3.10}$$

where the product now excludes $n=0$. The crucial point is the R dependence of the product. One finds

$$\prod_n\left[\frac{\beta_0}{\beta_0-\beta_n}\right]^{g_n}=\exp[a\bar{R}^{\bar{d}}+O(\bar{R}^{\bar{d}-1}, \ln\bar{R})] , \tag{3.11}$$

where a is the positive number

$$a=-2^{\bar{d}/2}s_{\bar{d}}\int_0^1 dx\, x^{\bar{d}-1}\ln h(x) ,$$

$$h(x)=1-\frac{1}{1+\sqrt{2}}[(1-x^2)^{1/2}+(2-x^2)^{1/2}] ,$$

and $s_{\bar{d}}$ is the volume of the unit $(\bar{d}-1)$-dimensional sphere. Equation (3.11) follows simply by exponentiating the product; replacing the sum by an integral gives the leading term

$$\begin{aligned}-\sum_{n=1}^{\sqrt{2}\bar{R}}g_n\ln\left[\frac{\beta_0-\beta_n}{\beta_0}\right]&=-s_{\bar{d}}\int_1^{\sqrt{2}\bar{R}}dn\, n^{\bar{d}-1}\\&\quad\times\ln\left[\frac{\beta_0-\beta_n}{\beta_0}\right]+O(\bar{R}^{\bar{d}-1})\\&=a\bar{R}^{\bar{d}}+O(\bar{R}^{\bar{d}-1}, \ln\bar{R}) .\end{aligned}$$

Hence,

$$\Omega_0(E,R)=\beta_0 e^{\beta_0 E+\lambda(\beta_0)+a\bar{R}^{\bar{d}}+O(\bar{R}^{\bar{d}-1}, \ln\bar{R})} . \tag{3.12}$$

Thus, Ω_0 has the same form as (2.3), the total density of states in a compact and small space, as far as its energy dependence is concerned. However, by including the effect of β_n, it picks up a large multiplicative factor, the exponential of the expanded volume. Note that, while this factor arises because of the presence of subleading

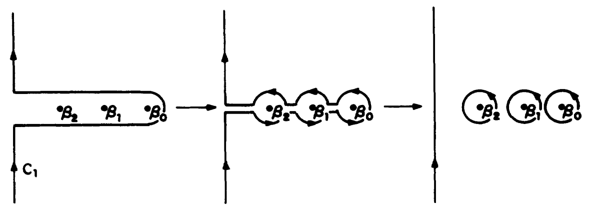

FIG. 5. Contour deformation for the computation of $\Omega(E)$.

radius-dependent singularities close to β_0, Ω_0 is simply the first term in the sum over contours encircling individual singularities. We still need to consider the additive contributions to $\Omega(E)$ coming from the contours encircling β_1, etc. In this sense the computation of the total density of states is different from the single-string density of states where subleading singularities produce purely additive corrections.

The function $\lambda(\beta)$ encodes information about other singularities of $Z(\beta)$, which lie in the complex β plane to the left of the β_n discussed here. There are infinitely many radius-independent singularities [6] and each one of them sees an accumulation of radius-dependent singularities (just like β_0) when the radius expands. Consequently, for the same reason as discussed above, the function $\lambda(\beta)$ is also proportional to $\bar{R}^{\bar{d}}$, the expanded volume. Further, since these singularities are poles and all of them are to the left of β_0, $e^{\lambda(\beta)}$ [and hence $\lambda(\beta)$] is a decreasing function of β at β_0. Consequently, close to β_0 we may write

$$\lambda(\beta)=R^{\bar{d}}[\lambda_0-(\beta-\beta_0)\sigma_0+O((\beta-\beta_0)^2)] , \tag{3.13}$$

where σ_0 is positive and has dimensions of an energy density.

We now determine Ω_1, the contribution to $\Omega(E)$ from the contour encircling β_1. Its form is needed in order to determine under what conditions the leading term Ω_0 is dominant and also to determine the specific heat. Integrating (3.9) around β_1, it is clear that Ω_1 will contain derivatives of the integrand at β_1, since β_1 is a $(g_1=2\bar{d})$-order pole. Explicitly,

$$\begin{aligned}\Omega_1(E,R)=&\frac{\beta_0^{g_1}}{(g_1-1)!}e^{\beta_1 E+\lambda(\beta_1)}\left[\prod_{n\neq 1}\left[\frac{\beta_0}{\beta_1-\beta_n}\right]^{g_n}\right]\\&\times[(E+\lambda'(\beta_1))^{g_1-1}][1+O(\bar{R}^{2-\bar{d}})] .\end{aligned} \tag{3.14}$$

The product can be analyzed as before and we find that the ratio of the product in (3.14) and that in (3.10) is

$$\begin{aligned}\prod_{n\neq 1}\left[\frac{\beta_0}{\beta_1-\beta_n}\right]^{g_n}\Big/\prod_{n\neq 0}\left[\frac{\beta_0}{\beta_0-\beta_n}\right]^{g_n}\\=-\exp[b\bar{R}^{\bar{d}-2}+O(\bar{R}^{\bar{d}-3}, \ln\bar{R})] ,\end{aligned} \tag{3.15}$$

where $b=2^{(\bar{d}-5)/2}s_{\bar{d}}\int_0^1 dx\,[x^{\bar{d}-1}/h(x)]$ is a positive number. Substituting (3.15) and (3.11) in (3.14) gives $\Omega_1(E,R)$.

One can now compare Ω_0 and Ω_1:

$$|\Omega_1/\Omega_0| = \exp[-(\beta_0-\beta_1)(E-\rho_0 R^{\bar{d}}) + O(\bar{R}^{\bar{d}-3}, \ln E, \ln \bar{R})] , \quad (3.16)$$

where $\rho_0=\sigma_0+2^{5/2}b/\beta_0\alpha'^{d/2}$ is positive and defines a Hagedorn energy density. Since $\beta_0-\beta_1 \propto \alpha'^{3/2}/R^2$, the first term in the exponential dominates over the corrections only when $\bar{d}>2$. In that case it follows from (3.16) that, for Ω_1 to be much smaller than Ω_0, we must have $E>\rho_0 R^{\bar{d}}$, i.e., that the energy density in the expanded universe be greater than the Hagedorn energy density. We shall assume that

$$E-\rho_0 R^{\bar{d}} > \rho_0 R^{\bar{d}} , \quad \bar{d}>2 , \quad (3.17)$$

so that the total density of states for a string gas in a large but compact space is given by (3.12) plus a small correction (3.14):

$$\begin{aligned}\Omega(E,R) &\simeq \Omega_0(E,R)+\Omega_1(E,R) \\ &\simeq \beta_0 e^{\beta_0 E + a_0 \bar{R}^{\bar{d}} + O(\bar{R}^{\bar{d}-1})} \\ &\quad\times \left[1-\frac{(\beta_0 E)^{2\bar{d}-1}}{(2\bar{d}-1)!} e^{-(\beta_0-\beta_1)(E-\rho_0 R^{\bar{d}})}\right] , \quad (3.18)\end{aligned}$$

where $a_0=a+\lambda_0\alpha'^{\bar{d}/2}$.

Notice that this conclusion depends crucially on the fact that the ratio (3.15) of the products is *much smaller* than each individual product. The ratio has only $e^{b\bar{R}^{\bar{d}-2}}$ whereas each individual product contains $e^{a\bar{R}^{\bar{d}}+O(\bar{R}^{\bar{d}-1})}$. This is because β_0 and β_1 are very close, their difference being of order $\alpha'^{3/2}/R^2$. If the ratio had turned out to be larger than it is, our conclusion that Ω_0 dominates over Ω_1 at finite energy densities would have broken down, and (3.18) would be valid only at infinite energy densities.

D. Comparison with noncompact case, energy distribution among individual strings

The result (3.18) is quite different from the total density of states above the Hagedorn energy density obtained in noncompact space, Eq. (2.6). In particular, (3.18) implies that the temperature of the system is always less than the Hagedorn temperature $1/\beta_0$ and approaches it from below as the energy density goes to infinity whereas (2.6) implies that, at Hagedorn energy density, the temperature of the gas is higher than the Hagedorn temperature and approaches it from above in the infinite-energy-density limit. Consequently, the specific heat of the gas obtained from (2.6) is negative but from (3.18) is positive. Note that in both cases the specific heat is not extensive.

In addition, the distribution of total energy of the gas among individual strings is also different in the two cases. Since we know both $f(\epsilon)$ and $\Omega(E)$ for the compact, large-radius case, $\mathcal{D}(\epsilon;E)$ is immediately deduced from (2.1). The result is depicted in Fig. 6. The total energy satisfies (3.17), hence $\Omega(E)$ is given by essentially (3.12). Thus, $\mathcal{D}(\epsilon;E)=f(\epsilon)e^{-\beta_0\epsilon}$, where $f(\epsilon)$ is given by (3.8a) and (3.8b). For $\epsilon \ll R^2\alpha'^{-3/2}$, $f(\epsilon)$ is given by (3.8a), hence,

$$\mathcal{D}(\epsilon;E)=c'R^{\bar{d}}\epsilon^{-(\bar{d}/2+1)} . \quad (3.19)$$

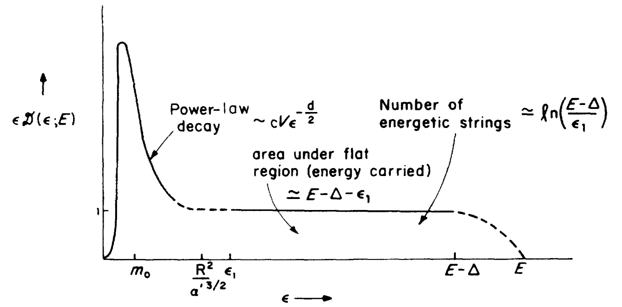

FIG. 6. Energy distribution for a string gas in a compact space of which three or more of the radii are much greater than $\sqrt{\alpha'}$ ($\bar{d}\geq 3$).

This accounts for the first peak in Fig. 6 and the distribution in this energy range is like the noncompact case in the same energy range. This is reasonable: low-energy strings have little extension, they do not see the structure of the space in the large and for them it does not matter whether space is noncompact or large and compact.

For $\epsilon \gg R^2\alpha'^{-3/2}$, $\Omega(E)$ is still of the same form (3.12), but $f(\epsilon)$ is now given by (3.8b). Hence, one recovers $\mathcal{D}(\epsilon;E)=1/\epsilon$, the same result as for the compact space of radius $\sim\sqrt{\alpha'}$, Eq. (2.4). The only difference is that, for $\Omega(E-\epsilon)$, since the validity of the formula (3.18) requires the argument to be greater than $\rho_0 R^{\bar{d}}$, the flat region does not go all the way up to $E-m_0$ as in the small-radius case but only up to $E-\Delta$, where Δ is of the order of (and greater than) $\rho_0 R^{\bar{d}}$. This explains the flat part of the curve in Fig. 6. The fact that the behavior at the upper end of the spectrum agrees with the small-radius case means that the energetic strings contain excitations of the winding modes. This type of behavior is also reasonable because sufficiently energetic strings would stretch across the space easily, irrespective of its radius. The result, in particular, establishes the energy scale which separates the two types of strings in the gas, namely, at $\epsilon\sim R^2\alpha'^{-3/2}$.

The average number of energetic strings $N_{\rm energetic}$ is given by the integral of $\mathcal{D}(\epsilon;E)$ from an energy ϵ_1 which is $\propto R^2$ but much larger than $R^2\alpha'^{-3/2}$, to E. Thus,

$$N_{\rm energetic} \simeq \ln\frac{E-\Delta}{\epsilon_1} \sim \ln\frac{R}{\sqrt{\alpha'}} . \quad (3.20)$$

This result for the number of energetic strings in the gas is quite different from the noncompact case, where essentially a single energetic string dominates. In the thermodynamic limit the number of energetic strings becomes infinite. This means that, even at large sizes, the string gas is inherently sensitive to the large-scale structure of space.

E. Distribution of winding number in the gas

It is of interest to ask how winding modes are distributed in the gas, how many strings there are with a given winding number, and what are their energies. To do that it is instructive to understand the difference between the noncompact-space single-string density of states (2.5) and the compact-space expressions (3.8a) and (3.8b) in terms of the winding modes. For $\epsilon \gg R^2\alpha'^{-3/2}$, the two differ and the ratio between them, $(R^2\alpha'^{-3/2}/\epsilon)^{d/2}$, is much smaller than unity. There are many more states available to an energetic string in a compact space than in a noncompact space. It is not difficult to see that the extra states in compact space are due to the winding modes.

The energy ϵ of a string with winding number w_i, momentum m_i/R_i, and left and right oscillator level numbers N and $\bar{N}$ is

$$\epsilon^2=\frac{2}{\alpha'}(N+\bar{N}-2)+\tfrac{1}{2}\sum_{i=1}^{D}(p_i^2+\bar{p}_i^2)\ ,$$

where $p_i=m_i/R_i-w_iR_i/\alpha'$, $\bar{p}_i=m_i/R_i+w_iR_i/\alpha'$, and the quantum numbers obey the constraint $N=\bar{N}+\sum_i m_iw_i$. Therefore, for a state with a given energy, momentum, and winding, we have

$$N(\epsilon,m,w)=\frac{\alpha'}{4}(\epsilon^2-p^2)+1$$

and

$$\bar{N}(\epsilon,m,w)=\frac{\alpha'}{4}(\epsilon^2-\bar{p}^2)+1\ .$$

The degeneracy of oscillator states d_N at a fixed oscillator level number N is given by the asymptotic formula for large N, $d_N\simeq(1/\sqrt{2})N^{-27/4}e^{4\pi\sqrt{N}}$ (for closed bosonic strings).

Defining $f(\epsilon,m,w)$ as the number density of states at fixed momentum m and winding w [$f(\epsilon,m,w)d\epsilon$ = number of single-string states with fixed momentum m and winding w and energy in the range ϵ to $\epsilon+d\epsilon$], we have

$$\begin{aligned} f(\epsilon,m,w)&=d_{N(\epsilon,m,w)}d_{\bar{N}(\epsilon,m,w)}\frac{\partial}{\partial\epsilon}N(\epsilon,m,w)\\ &=\tfrac{1}{4}\alpha'\epsilon(N\bar{N})^{-27/4}e^{4\pi(\sqrt{N}+\sqrt{\bar{N}})}\ .\end{aligned}$$

Since $N>0$, we have $p<\epsilon$, and we can expand $\sqrt{N}$ in powers of p^2/ϵ^2. Then, to leading order at large ϵ we get

$$f(\epsilon,m,w)=\frac{1}{\epsilon}(2/\sqrt{\alpha'}\epsilon)^{25}e^{\beta_0\epsilon}e^{-a_im_i^2-b_iw_i^2}\ , \qquad (3.21)$$

where $a_i=2\pi\sqrt{\alpha'}/\epsilon R_i^2$ and $b_i=2\pi R_i^2/\epsilon\alpha^{3/2}$.

By definition $f(\epsilon)=\sum_{m,w}f(\epsilon,m,w)$. To determine $f(\epsilon)$ in various energy ranges we follow Ref. [8]. If $a_i,b_i\ll 1$, we can replace the corresponding sum by an integral. For the dimensions that stay small ($R_i\sim\sqrt{\alpha'}$), this holds for all $\epsilon\gg 1/\sqrt{\alpha'}$. For the $\bar{d}$ dimensions that become large, it is true for the corresponding a_i and we replace the momentum sum by an integral. Performing these integrations gives

$$f(\epsilon)=c'R^{\bar{d}}\frac{e^{\beta_0\epsilon}}{\epsilon^{\bar{d}/2+1}}\sum_{w}e^{-bw_i^2}\ , \qquad (3.22)$$

where the sum is now only over winding numbers in the large directions and $b=2\pi R^2/\epsilon\alpha^{3/2}$. $\bar{R}^{\bar{d}}$ is the familiar volume factor coming from the sum over momentum modes in the large directions. In the energy domain $\alpha'^{-1/2}\ll\epsilon\ll R^2\alpha'^{-3/2}$ we have $b\gg 1$, hence only the term with $w=0$ contributes to the sum, and we get (3.8a). In the energy domain $\epsilon\gg R^2\alpha'^{-3/2}$ we can integrate over w to reproduce (3.8b). Thus, it is clear that in this domain the increase in the single-string density of states is caused by the contribution of winding states.

Equation (3.21) is peaked at $m=w=0$. The reason is physically clear: the number of states is maximized by putting all of the energy ϵ into oscillator modes whose number grows exponentially and expending as little as possible into momentum and winding modes which grow only like a power law. This explains the Gaussian suppression and the dependence of a and b on energy and radius [7]. This also means that the "spread" in the winding number is of the order of $b^{-1/2}$ or that the mean-square winding number $\langle w^2\rangle$ of a string of energy ϵ is of the order of $\epsilon\alpha^{3/2}/R^2$.

Thus, low-energy strings (those under the first peak of Fig. 6) have zero winding number, those with energy $\sim R^2\alpha'^{-3/2}$ (near the beginning of the flat part of the curve) have winding number of the order of unity, and the most energetic strings at the upper end of the flat region have a mean-square winding number $\sim\bar{R}^{\bar{d}-2}$. In the flat region of Fig. 6, since the number of strings in energy range ϵ to $\epsilon+d\epsilon$ is given by $\mathcal{D}(\epsilon;E)d\epsilon$, the mean-square winding number carried by them is of the order of $(\alpha^{3/2}/R^2)d\epsilon$, which, being independent of ϵ, also gives a flat distribution. That is, the mean-square winding number is also distributed uniformly among strings of all energies above ϵ_1.

If we take into account the conservation of winding number and keep the total winding number of the gas fixed, the form of $\Omega(E)$ is modified. Since we have seen that the typical extent of winding number for strings of high energy is proportional to the square root of their energy, and since the distribution of strings at high energies scales with the total energy of the system, each winding number constraint will result in a factor of $E^{-1/2}$, as also happens when all radii are small [4,9,10]. This would mean that the overall specific heat of the system would be negative, and some common properties (like the transitivity of thermal equilibrium between systems) would not be valid in this case. However, as observed in [4] for the cases studied there, the distribution of strings at various energies remained the same even though $\Omega(E)$ changed. We expect the same to hold here, i.e., the distribution to be given by Fig. 6 even when the winding-number conservation is imposed, especially for large R, where there is a large number $\sim\ln\bar{R}$ of strings among which the winding number is effectively shared.

IV. THE STRING GAS IN A SPACE WITH ONE OR TWO LARGE DIMENSIONS

When space has only one or two large spatial dimensions, $\bar{d}=1$ or 2, the situation is quite different. For $\bar{d}=1$ at finite energy densities the total energy is proportional to R and that is not enough to excite string modes which see the structure of the space. For a string to span the whole space either it should have a nonzero winding number, which, as seen earlier, requires its energy should be at least $\sim R^2$, or, the spread of its oscillator wave function should be of the order R which again requires its energy to be $\sim R^2$. When the total energy is of order R, no individual string mode can have such energies. The modes that are excited are much smaller than the size of the space, which to them looks very large (effectively noncompact) and hence one expects the compact, large-radius results to be the same as the noncompact case. For $\bar{d}=2$ one might expect that at finite but large energy densities since $E \propto R^2$ modes that see the structure of the space are excited. However, it turns out that $\bar{d}=2$ is a marginal case in which the system requires energies of the order $\sim R^2 \ln R$ (i.e., infinite energy densities in the thermodynamic limit) to see such modes; at finite energy densities such modes are, in fact, not excited. This is because, in the latter case while the *total* energy is $\sim R^2$, this is distributed among many strings and no *individual* string energy is able to cross the $O(R^2)$ threshold required for that string to stretch across the large dimensions of space. Thus, in this case also at finite energy densities the gas behaves as if space were noncompact.

In this section we first present the results for the noncompact cases $d=1,2$ for finite but large energy densities and then discuss why the conclusion is the same if we consider the compact case with large radius ($\bar{d}=1,2$).

For $d=1,2$ we may use the canonical ensemble to discuss the statistical properties of the system at finite energy densities. This is because (i) there exists the prefactor V in (2.5) which suppresses fluctuations at the saddle point in the Laplace transform connecting $\Omega(E)$ and $Z(\beta)$, and (ii) the mean energy density $\langle\rho\rangle$ in the canonical ensemble diverges as $\beta\to\beta_0$, which allows one to assign a canonical temperature to the system for arbitrarily large energy densities. [In the cases $d=\bar{d}=0$ and $d\geq 3$ at large energy densities, condition (i) and condition (ii), respectively, are violated.]

Within the canonical ensemble, quantities scale with V in the standard manner; specific heat is extensive, the temperature depends on energy density, and $\mathcal{D}(\epsilon;E)$ is $f(\epsilon)e^{-\beta\epsilon}$. β differs from β_0 by a function of ρ, hence, energetic strings are exponentially suppressed. Explicit expressions for $\Omega(E)$ and $\mathcal{D}(\epsilon;E)$ are given in [4]. To illustrate for $d=2$, using (2.5) in $\Gamma(\beta)=\int_0^\infty d\epsilon\, f(\epsilon)e^{-\beta\epsilon}$, we have

$$\Gamma(\beta)\simeq V\int_{m_0}^{\infty}\frac{d\epsilon}{\epsilon^2}e^{(\beta_0-\beta)\epsilon} .$$

For $\beta\to\beta_0$,

$$\Gamma(\beta)\simeq R^2[c(\beta-\beta_0)\ln(\beta-\beta_0)m_0 + c_1 - c_2(\beta-\beta_0)+\cdots] , \tag{4.1}$$

where the c's are dimensional constants involving α'. Since

$$\langle E\rangle = -\frac{\partial\Gamma(\beta)}{\partial\beta}\simeq V[-c\ln(\beta-\beta_0)m_0+\rho_0+O(\beta-\beta_0)] ,$$

we have

$$\frac{\partial}{\partial E}\ln\Omega(E)=\beta=\beta_0+\frac{1}{m_0}e^{-\bar{\rho}/c}+O(e^{-2\bar{\rho}/c}) ,$$

whence

$$\Omega(E)\simeq c_0\exp V[\beta_0\rho - e^{-\bar{\rho}/c}+O(e^{-2\bar{\rho}/c})] \tag{4.2}$$

and

$$\mathcal{D}(\epsilon;E)\simeq\frac{cV}{\epsilon^2}e^{-a\epsilon} , \quad a=c_3e^{-\bar{\rho}/c} , \tag{4.3}$$

where $\bar{\rho}\equiv\rho-\rho_0$. Thus, for any finite energy density we have a positive extensive specific heat, and energetic strings (strings carrying an appreciable fraction of the total energy) are exponentially suppressed. The situation is similar in fact to the point-particle gas described by Fig. 1.

We now ask whether this picture is modified if we use, instead of (2.5), the expressions (3.8a) and (3.8b) for $f(\epsilon)$, which takes into account winding-number modes. The two expressions differ when $\epsilon\gg R^2\alpha'^{-3/2}$. Since, for $d=1,2$, strings with such large energies are not present in the gas at finite energy densities, the system effectively does not get a chance to sense the difference between (2.5) and (3.8b). Thus, in the thermodynamic limit, we expect the noncompact calculation to remain valid for arbitrarily high (but finite) energy densities. This is unlike for $d\geq 3$, where the naive noncompact calculation itself led to a single-string capturing energy $\sim R^d$, which is well into the range where $f(\epsilon)$ crosses over to (3.8b). There the noncompact calculation itself told us to expect a difference in the proper thermodynamic limit because modes that sensed the difference between Eqs. (3.8a) and (3.8b) and Eq. (2.5) were excited.

This conclusion can be verified by inverse Laplace transforming $Z(\beta)=e^{\Gamma(\beta)}$. It is not necessary to do this for $d=1$; there, for any energy density ρ, the total energy of the system is linear in R, so that the crossover in $f(\epsilon)$ at $\sim R^2$ is inconsequential because no string can ever possess this much energy. For $\bar{d}=2$, we have from (3.8a) and (3.8b),

$$\Gamma(\beta)\simeq c'R^2\int_{m_0}^{\lambda R^2}\frac{d\epsilon}{\epsilon^2}e^{(\beta_0-\beta)\epsilon}+\int_{\lambda R^2}^{\infty}\frac{d\epsilon}{\epsilon}e^{(\beta_0-\beta)\epsilon} ,$$

where λ sets the scale of the crossover point. This gives back (4.1) but now only in the domain (to be called domain 1) $m_0^{-1}\gg\beta-\beta_0\gg(\lambda R^2)^{-1}$. In the domain $\beta-\beta_0\ll(\lambda R^2)^{-1}$ (domain 2), one gets instead

$$\begin{aligned}\Gamma(\beta)\simeq& -\ln[(\beta-\beta_0)/\beta_0]\\ &+R^2[c_1+(\beta-\beta_0)(-c_2\ln\bar{R}+c_3)\\ &+c_4(\beta-\beta_0)^2+\cdots] .\end{aligned} \tag{4.4}$$

[This conclusion also follows from summing over singularities in (3.5).] Thus, very close to β_0 (domain 2) the free energy sees a pure logarithmic singularity at β_0, since other singularities are relatively far away, whereas not so close to β_0 (domain 1), the singularity is seen to be weaker [that given by (4.1)] which is the smeared out effect of the subleading singularities accumulating at β_0.

The question is whether the modification of the free energy in domain 2 changes $\Omega(E)$ at finite energy densities. It is not difficult to see that it does not; in fact, for $E<O$

($R^2 \ln R$) the same expression (4.2) holds since, for these energies, the saddle point is still in domain 1 where (4.1) is valid. Thus, we recover the noncompact picture for finite energy densities, namely, that energetic strings are exponentially suppressed. The effect of the modification (4.4) in domain 2 becomes important only for energies greater than O ($R^2 \ln R$) (infinite energy densities in the thermodynamic limit) where the behavior becomes similar to the $\bar{d} \geq 3$ case studied in the previous section.

V. CONCLUSIONS

To summarize, we have studied the ideal string gas in the thermodynamic limit (space is compact but large, energy density is finite) and compared its behavior to that obtained when space is treated to be noncompact to begin with. We find certain things in common between the two cases but they also differ in certain key respects at energy densities greater than the Hagedorn energy density. The common feature is that, in both cases the distribution of low-energy strings is the same (compare Figs. 6 and 3). Also, in both cases most of the energy of the gas is carried by energetic strings. However, the distribution of energy among energetic strings is different when the number of large dimensions is greater than 2. While in the noncompact case a single energetic string captures most of the energy, we find that, in the proper thermodynamic limit, the energy is uniformly distributed among $\sim \ln R/\sqrt{\alpha'}$ number of strings. This large number is due to the existence of winding modes which are left out if space is treated *a priori* as noncompact. We have shown that the least energetic of these energetic strings has an energy of the order $R^2\alpha'^{-3/2}$. This also explains why the energetic strings are not present when the number of large dimensions is less than or equal to 2, because then at finite energy densities the total system does not have enough energy to excite them.

The winding modes stretch across the space and are sensitive to the topology of the space. In addition, the spread of the oscillator wave function of strings with energies $\sim R^2\alpha'^{-3/2}$ is also of the order of the radius of the space. Hence, individual strings in the gas can see the global structure of the space and their presence destroys the extensivity of the gas.

The flat energy distribution among energetic strings in the thermodynamic limit is similar to that in a space which is compact and small with $R \sim \sqrt{\alpha'}$ (compare Figs. 6 and 2). Although we have only considered the case of toroidal compactification here, from the physical arguments about the string sizes that we have discussed, and from the fact that the distribution for a compact small space is universal and independent of the compactification [7], we expect a similar picture to be valid for the general case.

Our analysis of the string gas in a compact space resolves a question about the behavior of the system as energy is pumped into the string gas. At low energy densities ($\rho \ll \rho_0$) the gas behaves like a gas of point particles, since stringy modes are not excited. Its temperature grows essentially like a power of the energy density. It has been conjectured that as one pumps in more energy, after a certain stage when the system approaches the Hagedorn temperature, the energy goes into the creation of a long string rather than into the gas of low-energy strings [3,9,10]. However, this argument has been based on the noncompact expressions for the density of string states. In the high energy density domain ($\rho \gg \rho_0$) an analysis based on the naive thermodynamic limit (which treats space as noncompact from the outset) leads to the temperature approaching the Hagedorn temperature from *above* as $\rho \to \infty$. This is well known to follow from (2.6) [1,2] and means that, as energy is pumped into the system, at some intermediate energy densities the temperature rises *above* the Hagedorn temperature and then falls; clearly this behavior is quite unphysical. The calculation we have presented in this paper deals with the compact system rigorously, and shows that the scenario previously conjectured is qualitatively correct in that, after a certain point, the energy does go into making very energetic strings rather than low-energy strings, but with the difference that the number of energetic strings is not 1 but depends logarithmically on the size of the universe. In addition we find that the temperature also behaves reasonably: it increases monotonically with energy and approaches the Hagedorn temperature asymptotically from below [from (3.18) it follows that $T_H - T \sim (1/R^2)\exp(-cE/R^2)$].

A. Duality

Finally it is worth mentioning that, while we have discussed the case $R \gg \sqrt{\alpha'}$ in this paper, the same analysis goes through for $R \ll \sqrt{\alpha'}$. The singularities β_n in (3.3) are then quite far from the Hagedorn singularity, but their place is taken by other singularities given by (3.3) with $\bar{R}_i$ replaced by $1/\bar{R}_i$. In particular, the distribution of Fig. 6 remains the same when $R \ll \sqrt{\alpha'}$ provided we replace $\bar{R} \to 1/\bar{R}$ and interchange the roles of momentum and winding. If we start from a large radius, pass through the duality radius $\bar{R}=1$, and go on to smaller radii, the energy distribution among strings will start from Fig. 6, pass through Fig. 2 at $\bar{R} \sim 1$ where the peak disappears, and return to Fig. 6 at $\bar{R} \ll 1$. At small radii the low-energy peak is due to winding modes and the flat region contains momentum excitations.

Note that these distributions are only valid for sufficiently high energy densities or "dual densities," or, if energy E is held fixed, only for radii in the range $(c\sqrt{\alpha'}E)^{-1} < \bar{R}^d < c\sqrt{\alpha'}E$, where c is a dimensionless number of order unity. For radii outside this range the total energy starts becoming insufficient for supporting winding modes (if radius is too large) or momentum modes (if radius is too small). In view of the cosmological scenario of Ref. [5], it is of interest to investigate these transition regions.

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