# General analytic solution of $R^2$ gravity with dynamical torsion in two dimensions

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Using light-cone variables, we show that $R^2$ gravity with dynamical torsion in two dimensions is one of the rare field theories whose *complete* classical solution in closed form can be obtained. It fulfils an invariant relation between the cosmological constant, the curvature scalar, and the scalar formed by the torsion tensor. We conjecture that this relation, interpreted as a local conservation law, is closely connected to the integrability of the theory. The solutions may possess a rich spectrum of singularities in curvature and torsion. Special cases, including one with nonvanishing torsion, can be used to elucidate some physical properties of the solution where by "physical" we imply the validity of concepts from general relativity such as measurements of distances and times and of extremal trajectories of a scalar test particle.

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## I. INTRODUCTION

The main incentive for nearly all novel field-theoretical ideas of the last two decades has been the desire to accommodate gravity as well in an otherwise increasingly coherent picture of the basic laws of Nature, cast into a renormalizable and perhaps even finite unified quantum field theory. Unfortunately, efforts turned out to be in vain in the case of supersymmetry and supergravity [1], and the great expectations in a "theory of everything" to emerge from superstrings [2], at least so far have not found convincing physical justification.

On the other hand, the rich new insights into the two-dimensional string world have motivated theoretical studies of gravity problems in lower dimensions. It is well known that a direct transfer of four-dimensional Einsteinian gravity with the Einstein-Hilbert (EH) action and vanishing torsion to $d=2$ leads to the appearance of Weyl symmetry which, together with two-dimensional diffeomorphisms, makes pure Einstein gravity topological. Nontrivial actions only result from the interaction with further fields [3]. Of course, a natural question to ask is why a diffeomorphism-invariant theory in $d=2$ as such needs to be of the Einsteinian type at all.

Precisely the lack of renormalizability of gravity in four dimensions has provoked numerous attempts to supplement more complicated invariants than just the curvature scalar $R$ to the EH action [4]. However, e.g., adding terms quadratic in the curvature, although mitigating the renormalizability problem, induces "ghost" particles with unwanted spins and, even worse, represented by higher-order poles in the propagator.

The idea to take the vielbeins $e^a$ and the spin connection $\omega^a{}_b$ as the basic variables of gravity is almost as old as Einstein's theory of gravity itself [5]. It somewhat miraculously also leads from the EH action to vanishing torsion and to the Einstein equations, expressed in terms of the metric field $g_{\mu\nu}$. Therefore, one way to attack the problem of renormalizability has been to use a supplemented EH action, with $e^a$ and $\omega^a{}_b$ playing a fundamental (instead of incidental) role. Maintaining still a vanishing torsion, it has been shown some time ago that even in four dimensions such a theory may become renormalizable, although ghosts survive [6]. This is still true for nonvanishing torsion [7]. However, by taking only the part with at most two derivatives, a nine-parameter action can be shown to contain no ghosts and no tachyons [8]; besides the graviton, only massive particles of spin 1 and 2 appear. Unfortunately, power-counting renormalizability being lost in this case is in agreement with the well-known fact that unitarity and power-counting renormalization exclude each other in $d=4$ gravity. The main technical problem in the analysis of such theories is the proliferation of possible terms to be added to the EH action.

Surprisingly enough, to the best of our knowledge, the field theoretic "laboratory" of $d=2$, which has played such an important role in understanding quantum field theory, has not been used in precisely this context until very recently [9]. Also for theories with $R^2$ terms and torsion the restriction to $d=2$ results in a substantial simplification of the problem. The most general action leading to field equations for $e_\mu{}^a$ and $\omega_\mu{}^a{}_b$ with at most two derivatives is simply

$$L_{\rm inv} = -\frac{1}{4M^2}\int d^2x\, e(\, R_{\mu\nu}{}^{ab}\, R^{\mu\nu}{}_{ab} + M^2\beta T_{\mu\nu}{}^a T^{\mu\nu}{}_a + 4M^2\lambda), \quad (1.1)$$

where instead of the two-forms of curvature $R^a{}_b = d\omega^a{}_b + \omega^a{}_c \wedge \omega^c{}_b$ and of torsion $T^a = de^a + \omega^a{}_b \wedge e^b$ [$a,b \in \{0,1\}$ are indices related to the tangential Lorentz space with Minkowski metric $\mathrm{diag}(\eta_{ab}) = (1,-1)$] already explicit "world" components ($\mu,\nu \in \{0,1\}$)

$$\begin{aligned} R_{\mu\nu}{}^a{}_b &= (\partial_\mu\omega_\nu - \partial_\nu\omega_\mu)\,\varepsilon^a{}_b =: F_{\mu\nu}\varepsilon^a{}_b, \\ T_{\mu\nu}{}^a &= \partial_\mu e_\nu{}^a + \omega_\mu \varepsilon^a{}_b e_\nu{}^b - (\mu \leftrightarrow \nu) \end{aligned} \quad (1.2)$$

with

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$$\omega_\mu{}^a{}_b =: \omega_\mu \varepsilon^a{}_b, \quad (\varepsilon^{ab}) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \tag{1.3}$$

have been introduced. $M$ has mass dimension one, $\beta M^2$ and the cosmological constant are free parameters with dimension two. The integration measure $e$ in (1.1) is expressed in terms of the determinant of the zweibein $e_\mu{}^a$. The EH term $eR$, a total divergence, is omitted. The simplification in $d = 2$ stems essentially from the fact that the curvature tensor $R_{\mu\nu}{}^{ab}$ only possesses one independent component so that

$$R_{\mu\nu}{}^{ab} R^{\mu\nu}{}_{ab} = R^2 \tag{1.4}$$

with

$$(\varepsilon^{\mu\nu}) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},$$

thus

$$eR = F_{\mu\nu}\varepsilon^{\mu\nu} , \tag{1.5}$$

and the torsion has only two independent components

$$T^a = -\frac{1}{2} e^{-1} \varepsilon^{\mu\nu} T_{\mu\nu}{}^a \tag{1.6}$$

with the torsion scalar $T^2$:

$$T_{\mu\nu}{}^a T^{\mu\nu}{}_a = -2T^a T_a = T^2. \tag{1.7}$$

The action (1.1) with (1.4) and (1.7) closely resembles an Abelian Yang-Mills theory in $\omega_\mu$ coupled to a non-Abelian (noncompact) gauge theory of $e_\mu{}^a$. As ordinary pure Yang-Mills theory in $d = 2$, (1.1) just misses the two physical transverse degrees of freedom, present in $d = 4$. This is reflected here in the absence of asymptotic states and of an $S$-matrix [10]. Still, the action (1.1) has a highly nontrivial content, corresponding to a "Coulomb"-like sector and is certainly relevant also if further interactions with other fields are introduced. Although by diffeomorphism and local Lorentz transformations

$$\delta x^\mu = -\xi^\mu(x), \quad \delta e_\mu{}^a = \gamma(x)\varepsilon^a{}_b e_\mu{}^b \tag{1.8}$$

from the original six degrees of freedom ($e_\mu{}^a$, $\omega_\mu$) three correspond to gauge transformations, the remaining three give rise to nontrivial classical solutions. Choosing in the conformal gauge

$$e_\mu{}^a = e^\varphi \delta_\mu^a \tag{1.9}$$

those variables to be $\omega_\mu$ and $\varphi$, the integral of the field equations has been given first in Ref. [9], up to the solution of a transcendental first-order differential equation.

Intuitively more appropriate for the separation of "physical" degrees of freedom, however, in theories with Yang-Mills structure seems the use of a light-cone (LC) gauge.

As shown in Sec. II, in that gauge a complete analytic solution without residual differential equation is possible. It depends on three arbitrary functions and implies a covariant relation between the curvature and torsion scalars (1.5) and (1.7). One sector of the solution is characterized by constant curvature and vanishing torsion [9] ("de Sitter" solution). It exists for $\lambda \geq 0$ only.

After a preliminary discussion of the role of singularities from the requirement of a "physical" time and length, in Sec. III special stationary and reflection-symmetric solutions (with respect to LC coordinates) are given.

The transformation to conformal coordinates (Sec. IV) makes the appearance of up to three singularities in these coordinates possible, if the arbitrary functions are fixed by appropriate boundary conditions. Two singularities of curvature (and also in torsion) may lie in finite "physical" distances. In the simple case of a de Sitter (deS) solution the transformation to conformal coordinates is obtained by elementary integrals and shows the salient points.

Section V is devoted to the study of selected extremals as defined by the passage of a test particle with action

$$L_{\text{matter}} = -\int d\kappa \sqrt{g_{\alpha\beta}\dot{x}^\alpha \dot{x}^\beta} \tag{1.10}$$

which "feels" only the Christoffel part of the affine connection.

## II. SOLUTION IN THE LIGHT-CONE GAUGE

In the following, $\omega_\mu$ and $e_\mu{}^a$ are expressed in LC components ($x^\pm = x^0 \pm x^1 = t \pm x$)

$$\omega_\pm = \tfrac{1}{2}(\omega_0 \pm \omega_1), \quad e_\pm{}^a = \tfrac{1}{2}(e_0{}^a \pm e_1{}^a), \tag{2.1}$$

where now also $a \in \{+,-\}$, corresponding to a metric

$$(\eta_{ab}) = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Although our solution is nonperturbative we anyhow introduce for convenience ($m > 0$)

$$e_\mu{}^a = \delta_\mu^a + \frac{\varphi_\mu{}^a}{m}. \tag{2.2}$$

The LC gauge is characterized by

$$\omega_+ = \varphi_+{}^\pm = 0 , \tag{2.3}$$

and we save indices below by introducing the abbreviations

$$(\omega_+, \omega_-, \varphi_+{}^+, \varphi_-{}^+, \varphi_+{}^-, \varphi_-{}^-) = (\bar{\omega}, \omega, \bar{\psi}, \psi, \bar{\varphi}, \varphi) . \tag{2.4}$$

The Lagrangian of (1.1) with (1.4) and (1.7) now reads

$$\mathcal{L}_{\text{inv}} = -\frac{4}{M^2 e}(F_{+-})^2 + \frac{2\beta}{e} T_{+-}{}^+ T_{+-}{}^- - e\lambda \tag{2.5}$$

written in terms of

$$F_{+-} = \partial_+\omega - \partial_-\bar{\omega}, \tag{2.6}$$

$$T_{+-}{}^{+} = \partial_+ \frac{\psi}{m} - \partial_- \frac{\bar\psi}{m} - \omega - \omega \frac{\bar\psi}{m} + \bar\omega \frac{\psi}{m}, \quad (2.7)$$

$$T_{+-}{}^{-} = \partial_+ \frac{\varphi}{m} - \partial_- \frac{\bar\varphi}{m} + \omega \frac{\bar\varphi}{m} - \bar\omega - \bar\omega \frac{\varphi}{m}. \quad (2.8)$$

The six equations of motion are obtained by varying the fields (2.4). In the gauge $\bar\omega = \bar\psi = \bar\varphi = 0$ they may be expressed in terms of the gauge-fixed quantities

$$R = \left.\frac{4F_{+-}}{e}\right|_{\mathrm{gf}}, \quad T^{\pm} = \left.T_{+-}{}^{\pm}\right|_{\mathrm{gf}}, \quad T^2 = -\frac{8}{e^2}T^+T^-, \quad e_0 = e|_{gf} = 1 + \frac{\varphi}{m} \quad (2.9)$$

and read

$$\partial_+ R - M^2\beta e_0^{-1}T^- = 0, \quad (2.10)$$

$$\partial_+ \left(e_0^{-1}T^-\right) = 0, \quad (2.11)$$

$$(R^2 - 4M^2\lambda) - 8M^2\beta e_0^{-1}\partial_+ T^+ = 0, \quad (2.12)$$

$$\partial_- R + M^2\beta\left(T^+ + e_0^{-1}T^-\frac{\psi}{m}\right) = 0, \quad (2.13)$$

$$(R^2 - 4M^2\lambda) - 8M^2\beta e_0^{-2}T^-\partial_+\frac{\psi}{m} + 8M^2\beta e_0^{-1}\partial_-\left(e_0^{-1}T^-\right) = 0, \quad (2.14)$$

$$(R^2 - 4M^2\lambda)\psi - 8M^2\beta[e_0^{-2}T^+T^-\psi + m\partial_-(e_0^{-1}T^+) + me_0^{-1}\omega T^+] = 0. \quad (2.15)$$

Starting with (2.11), Eqs. (2.10) – (2.12) may be trivially integrated for $x^+$. We consider first the case $\partial_+ e_0 \neq 0$ $[\zeta = mf(x^-)x^+,\ \Lambda = 4\lambda/M^2\beta^2]$:

$$\ln e_0 = \ln(1 + \varphi/m) = \zeta + g, \quad (2.16)$$

$$\omega = \frac{M^2\beta e_0}{4fm}(\zeta + h - 1) + ml, \quad (2.17)$$

$$\psi = \frac{M^2\beta e_0}{8mf^2}\left[(\zeta + h - 1)^2 + 1 - \Lambda\right] + \frac{rm\zeta}{f} + sm. \quad (2.18)$$

The arbitrary functions of $x^-$ $(f, g, h, l, r, s)$ in (2.16) – (2.18) are restricted by (2.13) – (2.15),

$$f' = mfr, \quad (2.19)$$

$$h' = m(fs + l - r), \quad (2.20)$$

$$(l' - r') = (l - r)\left[g' - m(fs + l)\right] \quad (2.21)$$

so that only three functions (e.g., $f \neq 0, h$ and $F$) and one constant $C_0$ remain free:

$$r = \frac{f'}{mf}, \quad (2.22)$$

$$s = f^{-1}\left(\frac{h'}{m} - l + r\right), \quad (2.23)$$

$$F := \frac{8(l - r)}{C_0} = f^{-1}\exp(g - h). \quad (2.24)$$

It is convenient to introduce dimensionless scalars for the curvature and for the torsion:

$$\hat R = R/M^2\beta = \zeta + h, \quad (2.25)$$

$$\hat T^2 = T^2/M^2\beta. \quad (2.26)$$

The expressions (2.25) and (2.26) depend precisely on those arbitrary functions which are still undetermined. The solution (2.16)–(2.18) may be expressed in terms of (2.25):

$$e_0 = Ffe^{\hat R}, \quad (2.27)$$

$$\omega = \frac{M^2\beta}{4m}Fe^{\hat R}\left(\hat R - 1\right) + \frac{mC_0F}{8} + \frac{f'}{f}, \quad (2.28)$$

$$\psi = \frac{M^2\beta F}{8mf}e^{\hat R}\left[(\hat R - 1)^2 + 1 - \Lambda\right] - \frac{mC_0F}{8f} + \frac{\partial_-\hat R}{f}. \quad (2.29)$$

Thus the dimensionless torsion scalar (2.26) turns out to be a covariant function of $\hat R$ and $\Lambda$

$$Q := e^{\hat R}\left[\hat T^2 + \left(\hat R - 1\right)^2 - \Lambda + 1\right] = \frac{C_0m^2}{M^2\beta} \quad (2.30)$$

with a nontrivial dependence on the constant $C_0$ alone. In the special case $l = r$, $C_0 = 0$ (then $F = e^{g-h}$), Eq. (2.30) implies a simple parabolic relation between $\hat R$ and $\hat T^2$. It is remarkable that *all* solutions of the field equation with nonvanishing torsion fulfil the covariant equation (2.30), which may be written as a conservation equation $\partial_\mu Q = 0$ for the local quantity $Q$. This strongly suggests a relation to the infinite number of conservation laws, typical for integrable theories. However an investigation of this question is outside the scope of our present work. We note in parentheses that a first-order differential equation, closely resembling (2.30) with $\hat R \to \Theta$ and $\hat T^2$ essentially $e^{-\Theta}|\Theta'|$, remains the only one not to be solved analytically when the solution is sought starting in the conformal gauge (1.9) [9].

The solutions for $\partial_+ e_0 = 0$ must be treated separately:

$$e_0 = 1 + \frac{\varphi}{m} = q, \quad (2.31)$$

$$\omega = \frac{R_0}{4}qx^+ + ml, \quad (2.32)$$

$$\psi = \frac{R_0 m}{8}q(x^+)^2 + m^2lx^+ + ms. \quad (2.33)$$

They correspond to vanishing torsion and to constant curvature

$$R_0 = \pm M^2 \beta \sqrt{\Lambda} \tag{2.34}$$

and thus exist for $\lambda \geq 0$ only. The arbitrary functions of $x^-$ are here $q$, $l$ and $s$. For this de Sitter (deS) solution the relation to the corresponding one in Ref. [9] is more easily established than in the general case (2.27)–(2.29). This point will be clarified below in connection with a transformation to conformal coordinates.

From the explicit form (2.16)–(2.18) of the solution and from (2.30), by a suitable choice of three boundary conditions on a "plane" $x^+ =$ const (e.g., by $R$, $\partial_+ R$, $T^2$ the functions $f$, $h$, and $F$ are determined) it is always possible to produce singular ones in $x^-$. In the LC gauge only the singularity in $x^+$ is fixed by the dynamics of (2.5). It is evident that a rich structure of singularities may be produced in this manner.

Just as the deS solution exists only for positive or vanishing cosmological constant, Eq. (2.30) obviously implies certain restrictions on $\hat{T}^2$ for given curvature so that the fields remain real. The range where this holds is surrounded by a "curve of singularity" which need not be immediately related to coordinate singularities determined by boundaries where measurements of "length" and "time" become singular. In general, $x^+ \to +\infty$ or $x^+ \to -\infty$ will lie at the boundary of the region mentioned above. At these points $\hat{R}$ and $\hat{T}^2$ diverge.

A Hamiltonian may be constructed from the action (2.5) with canonical momenta $\Pi^\omega$, $\Pi^\psi$, and $\Pi^\varphi$ for $\omega, \psi$, and $\varphi$, respectively. At $\bar{\omega} = \bar{\psi} = \bar{\varphi} = 0$ it is a conserved quantity in the "time" $x^+$. The resulting expression

$$\begin{aligned} H &= \int_A^B dx^- \, e_0 \left( \lambda - \frac{4}{M^2 {e_0}^2} (\partial_+ \omega)^2 + \frac{2\beta}{m^2 {e_0}^2} (\partial_+ \varphi)(\partial_+ \psi) \right) \\ &= \int_A^B dx^- \, m f' \end{aligned} \tag{2.35}$$

for an interval $A \leq x^- \leq B$ depends on $f$ alone (and vanishes for the deS solution). An appropriate $f$, e.g., with $f'$ behaving like $(x-A)^{-\alpha}$, $0 < \alpha < 1$, at the point $A$ and similarly at $B$, may produce any value of $H$.

Quantizing around one of our classical solutions, the value of the action with (2.27) - (2.29) or (2.31) - (2.33) will eventually be needed. With (2.30) it is readily obtained as ($\partial_+ e_0 \neq 0$)

$$\tilde{L}_{\text{inv}} = -\frac{M^2 \beta}{4} \int_V dx^+ dx^- \, e_0 \left( \hat{R} + \Lambda - 1 - \frac{2 C_0 m^2}{M^2 \beta} e^{\hat{R}} \right) \tag{2.36}$$

and ($\partial_+ e_0 = 0$)

$$\tilde{L}_{\text{inv}} = -\lambda \int_V dx^+ dx^- \, q, \tag{2.37}$$

respectively. For infinite space-time volume $V$ both expressions in general diverge.

## III. PROPERTIES OF SOLUTIONS IN LIGHT-CONE COORDINATES

### A. "Time" and "length"

The motivation of our study of $R^2$ gravity in $d = 2$ has been its possible relevance for an "extended" theory of gravity in $d = 4$. Thus if classical "observables" are searched in this non-Einsteinian theory, it is tempting to proceed as one is used to for ordinary gravity [11]. In $d = 2$, for $g_{00} > 0$ coordinate time $x^0 = t$ is simply defined as that variable in the line element $(ds)^2 = g_{\alpha\beta} \, dx^\alpha dx^\beta$ which at a fixed "space" point $dx^1 = dx = 0$ leads to a real proper time $\tau$ with

$$d\tau = \sqrt{g_{00}} \, dt \, , \tag{3.1}$$

whereas a measurement of a space distance by means of a light ray is determined by the induced metric for a fixed time

$$d\ell = \sqrt{\frac{-g}{g_{00}}} \, dx \, . \tag{3.2}$$

For $g_{00} < 0$ the role of $t$ and $x$ has to be interchanged. Inserting our solution (2.16) - (2.18) into $g_{\alpha\beta} = e_\alpha{}^a e_\beta{}^b \eta_{ab}$ yields

$$d\tau^2 = e_0 (1 + \psi/m) \, dt^2, \tag{3.3}$$

$$d\ell^2 = e_0 (1 + \psi/m)^{-1} \, dx^2 \, . \tag{3.4}$$

$e_0$ and/or $1 + \psi/m$ may exhibit zeros or poles. It is evident from the explicit solution that those zeros and poles may occur at finite and infinite values of $x^\pm$ with only the $x^+$ behavior somewhat limited. The corresponding surfaces (curves) determine the coordinate range of $x^\pm = t \pm x$ to be considered in possibly disconnected parts of the "universes." For brevity we will call those connected regions "regular" regions.

### B. Stationary and space-symmetric light-cone solutions

The very nature of LC variables $x^\pm$ precludes an immediate interpretation of a special solution (2.16) - (2.18) in terms of more physical coordinates [i.e., closer to $\tau$ and $\ell$ in (3.1) and (3.2)]. In this respect, a transition to conformal coordinates ($\bar{g}_{++} = \bar{g}_{--} = 0$) has advantages (cf. Sec. IV). Still, the peculiar asymmetry induced by the gauge choice (2.3) is absent for "stationary" and "space-symmetric" solutions. The deS solution (2.31) - (2.33) trivially fulfils both conditions $\partial_0 R = (\partial_+ + \partial_-) R = 0$

(stationary) and $R(t,x) = R(t,-x)$ (space symmetric). For the general solution with $\partial_+ e_0 \neq 0$, Eq. (2.30) automatically couples the respective behavior of $\hat{R}$ to the one of $\hat{T}^2$.

In the stationary case from

$$\partial_0 \hat{R} = (\partial_+ + \partial_-)\hat{R} = 0 \tag{3.5}$$

evidently (the index zero denotes an arbitrary constant)

$$f = f_0 \neq 0, \qquad h = h_0 - m f_0 x^-, \tag{3.6}$$

so that

$$\hat{R} = 2 m f_0 x + h_0, \tag{3.7}$$

and the other arbitrary functions by (2.19) - (2.21) are restricted to

$$l - r = l = -f_0(1+s) = \frac{C_0}{8 f_0} \exp(g + m f_0 x^- - h_0)\,. \tag{3.8}$$

The curvature $\hat{R}$ diverges here for $x = \pm\infty$. Whether or not these points lie inside a regular region can be seen from (3.3) and (3.4) with

$$e_0 = -\frac{8 f_0^2 (1+s)}{C_0} \exp(2 m f_0 x + h_0), \tag{3.9}$$

$$1 + \frac{\psi}{m} = (s+1) \times \left(1 - \frac{M^2 \beta}{m^2 C_0} e^{\hat{R}} \left[(\hat{R}-1)^2 + 1 - \Lambda\right]\right)\,. \tag{3.10}$$

The behavior in $x^- = t - x$ of (3.9) and (3.10) still depends on the free function $s(x^-)$. In any case, Eq. (3.10) may show up to three zeros in $x$, depending how the parabola in $x$ transects $\exp(-\hat{R} + h_0)$.

If the stronger condition $(\partial_+ + \partial_-)(\text{fields}) = 0$ is imposed, in Eq. (3.8) $s = s_0$, $g = g_0 - f_0 x^-$ are further specified.

Space symmetry with respect to $x$ implies $R(x^+, x^-) = R(x^-, x^+)$ so that

$$\hat{R} = m^2 \alpha_0 (t^2 - x^2) + 2 m \rho_0 t + \sigma_0 \tag{3.11}$$

and

$$\begin{aligned} f &= \rho_0 + m \alpha_0 x^-, \\ h &= \sigma_0 + m \rho_0 x^-\,. \end{aligned} \tag{3.12}$$

From (3.11) the curvature spreads symmetrically with the speed of light in both directions $x = \pm t$, increasing linearly with $t$. Using (2.30) the corresponding behavior of the torsion scalar can be deduced easily.

## IV. CONFORMAL COORDINATES

In conformal coordinates $\bar{g}_{++} = \bar{g}_{--} = 0$ the measurements of length and time (3.1) and (3.2) are determined by one single expression $\bar{g}_{+-}$

$$\frac{d\tau}{d\bar{t}} = \frac{d\ell}{d\bar{x}} = \sqrt{2\bar{g}_{+-}}\,. \tag{4.1}$$

The LC solution (2.16) - (2.18) yields in LC coordinates

$$\begin{aligned} g_{++} &= g^{--} = 0, \\ g_{+-} &= (g^{+-})^{-1} = \frac{w}{2}, \\ g_{--} &= -\frac{w^2}{4} g^{++} = z = vw, \\ -g &= e_0^2 = w^2, \end{aligned} \tag{4.2}$$

where new abbreviations $e_0 = w$, $v = \psi/m$, $z = vw$ have been introduced. In order to achieve $\bar{g}_{++} = \bar{g}_{--} = 0$ a diffeomorphism $x^\mu(\bar{x}^\alpha)$ must satisfy

$$\frac{\partial x^-}{\partial \bar{x}^+} = 0, \qquad \frac{\partial x^-}{\partial \bar{x}^-} = -\frac{1}{v}\frac{\partial x^+}{\partial \bar{x}^-} \tag{4.3}$$

or a similar relation with $\bar{x}^+ \leftrightarrow \bar{x}^-$. Equation (4.3) leaves $\partial x^+/\partial \bar{x}^+ \neq 0$ completely free. Furthermore $\partial x^-/\partial \bar{x}^-$ may be fixed conveniently, e.g., by $x^- = \bar{x}^-$. The remaining first-order differential equation

$$m\frac{\partial x^+}{\partial \bar{x}^-} = -\psi(x^+(\bar{x}^+, \bar{x}^-), \bar{x}^-) \tag{4.4}$$

with (2.18) again shows the structure of the residual differential equation when the solution is sought directly in conformal coordinates [9].

### A. Torsion and curvature independence of $x^-$

Most salient features of a solution with nonvanishing torsion and curvature can be deduced from a special case with the functions $f$, $h$ and $F$ replaced by constants $f_0$, $h_0$ and $F_0$ in (2.18), i.e. ($\xi = m f_0 x^+$)

$$f_0 v = \frac{M^2 \beta F_0}{8 m^2} e^{\xi + h_0} \left[(\xi + h_0 - 1)^2 + 1 - \Lambda\right] - \frac{C_0 F_0}{8}\,. \tag{4.5}$$

This equation may be simplified by a shift in $x^+$ to

$$f_0 v = A_0 + B_0 e^{\xi} (\xi^2 + D_0) \tag{4.6}$$

with constants $A_0$, $B_0$ and $D_0$. A qualitative discussion of the integral of (4.4)

$$\begin{aligned} G(x^+) := \int_{\xi_0}^{\xi} d\xi' \left[A_0 + B_0 e^{\xi'} (\xi'^2 + D_0)\right]^{-1} \\ = -m[\bar{x}^- - \Phi(\bar{x}^+)] \end{aligned} \tag{4.7}$$

is straightforward, choosing, e.g., the arbitrary differentiable function $\Phi(\bar{x}^+) = \bar{x}^+$. In that case $x^+$ becomes a function of the conformal space coordinate $\bar{x}$ alone. We have to distinguish $\bar{g}_{+-} > 0$ and $\bar{g}_{+-} < 0$. In the second case conformal time $\bar{t}$ may be simply exchanged with conformal space $\bar{x}$.

The integrand in (4.7) has at most three poles, in complete analogy to the short discussion of the "stationary" solution above.

*1. No pole*

$x^+$ lies in $[-\infty,+\infty]$ and the initial value $x_0^+$ lies in $(-\infty,+\infty]$, but due to the convergence of $G$ at $x^+ \to \infty$ (we take all constants positive, which also guarantees $\psi/m > 0$) we find that $\bar{x}$ is restricted to the interval $(\bar{x}_{-\infty} = -\infty, \bar{x}_{+\infty}]$ representing the regular region in the conformal coordinates. According to (2.25) the curvature diverges at $\bar{x}_{\pm\infty}$. From

$$\bar{g}_{+-}(\bar{x}) = -\frac{w}{2}\frac{\partial x^+}{\partial \bar{x}^-} \tag{4.8}$$

it is easy to see that $\bar{g}_{+-}|_{\bar{x}_{-\infty}} = 0$ with $\partial\ /\partial\bar{t}$ representing a Killing vector field. The total spatial "physical" size between the singularities $\bar{x}_{\pm\infty}$ from (4.1), after transforming back to $\xi = mf_0x^+$, is proportional to the infinite integral of $\exp(\xi/2)[G'(\xi)]^{1/2}$ which is finite. Thus this solution corresponds to two singularities in curvature [and by (2.30) also in torsion], situated in a finite distance.

*2. One pole*

When the parabola and the exponential in the denominator of the integral of (4.7) touch, a double pole at $x^+ = {}^{(1)}x^+$ is produced. This happens for $A_0 = -D_0B_0$. Choosing $B_0 > 0$ yields $\bar{g}_{+-} > 0$. For $x^+ \geq {}^{(1)}x^+$ (we take, e.g., $\xi_0$ in $G$ to be $+\infty$) due to the divergence of the integral at $x^+ = {}^{(1)}x^+$, $\bar{x}$ now resides in the interval $(-\infty, 0]$, with $\bar{g}_{+-}$ now diverging at $\bar{x} \to -\infty$ $(x^+ = {}^{(1)}x^+)$. Thus (in infinite "physical" distance) for finite $R$ a coordinate singularity exists, whereas the singularity in $R$ is placed at infinity. Turning to the range $x^+ \leq {}^{(1)}x^+$ (with $f_0 > 0$ as before and $\xi_0$ any negative real number) both coordinate singularities are infinitely far away, none of them related to a singularity of $R$ and $T$.

*3. Two or three poles*

The occurrence of two (single) poles ${}^{(1)}x^+ > {}^{(2)}x^+$ in (4.7) entails the separation into three regions of $x^+$. In the cases $x^+ \geq {}^{(1)}x^+$ and $x^+ \leq {}^{(2)}x^+$ we have essentially the situation of Sec. IV A 2. For ${}^{(2)}x^+ \leq x^+ \leq {}^{(1)}x^+$, on the other hand, both coordinate singularities lie at infinite distances. These considerations may be simply extended to the three-pole case.

As already mentioned above, for $\bar{g}_{+-} < 0$ with the same choice $\Phi(\bar{x}^+) = \bar{x}^+$ the reinterpretation $\bar{t} = \bar{x}', \bar{x} = \bar{t}'$ shows that all the singularities in conformal coordinates discussed so far turn into (finite and infinite) beginnings and ends of "time." In all cases the respective second internal variable ($\bar{t}$ for $\bar{g}_{+-} > 0$, $\bar{x}$ for $\bar{g}_{+-} < 0$) is unrestricted. Hence the space-time volume of the respective regular regions diverges for all special solutions discussed on the basis of (4.5). However, in a completely general case with an appropriate $x^-$ dependence, solutions with finite space-time volume can be expected.

**B. De Sitter solution with $x^-$ dependence**

The deS solution (2.31) - (2.33) in the special case ($\hat{l}_0$ and $\hat{s}_o$ are constants)

$$l = q(x^-)\hat{l}_0, \qquad s = q(x^-)\hat{s}_0 \tag{4.9}$$

permits a "separation of variables" by fixing in (4.3)

$$\frac{\partial x^-}{\partial \bar{x}^-} = q^{-1} \tag{4.10}$$

so that by (4.9) in (4.3)

$$-\frac{\partial x^+}{\partial \bar{x}^-} = \frac{\psi}{qm} = \frac{R}{8}{x^+}^2 + \hat{l}_0 m x^+ + \hat{s}_0 \tag{4.11}$$

can even be reduced to elementary integrals. For $R \neq 0$ without loss of generality we set $\hat{l}_0 = 0$ and $\hat{s}_0 =: \pm(R/8)a^2$ $(a > 0)$, depending upon whether $\hat{s}_0/R \gtrless 0$.

The nontrivial relation from (4.10)

$$\int_{x_0^-}^{x^-} d\xi\ q(\xi) = \bar{x}^- = \bar{t} - \bar{x} \tag{4.12}$$

with appropriately selected function $q(\xi)$ now allows situations with $\bar{x}^-$ limited to a finite regular interval.

The integral of (4.11) in the absence of poles [$\hat{s}_0 = (R/8)a^2$, $a$ real, the arbitrary function $\Phi(\bar{x}^+) = \bar{x}^+$ as in the preceding subsection] becomes

$$x^+ = a\tan\left(\frac{Ra\bar{x}}{4}\right), \tag{4.13}$$

$$\bar{g}_{+-} = \frac{Ra^2}{16}\cos^{-2}\left(\frac{Ra\bar{x}}{4}\right). \tag{4.14}$$

For $R > 0$ the positivity of $\bar{g}_{+-}$ is guaranteed. The spacelike-separated coordinate singularities in (4.13) and (4.14) are at $Ra\bar{x} = \pm 2\pi$.

If $R < 0$ as in Sec. IV A we identify $\bar{x} = \bar{t}'$ and obtain a solution for a finite interval of conformal time.

In both cases the respective "other" coordinate is determined by the solution of (4.10). It seems remarkable that, for that type of solution, $\bar{g}_{+-}$ of (4.13) turns out to be independent of $q$, the volume determinant $e_0$ in the measure, expressed in LC coordinates.

For $\hat{s}_0 = -(R/8)a^2$ a situation like the one in Sec. IV A 3 develops. If $(x^+, x_0^+)$ are inside the two poles at ${}^{(1,2)}x^+ = \pm a$, we obtain $[\Phi(\bar{x}^+) = \bar{x}^+]$

$$x^+ = -a\tanh\left(\frac{Ra\bar{x}}{4}\right), \tag{4.15}$$

$$\bar{g}_{+-} = -\frac{Ra^2}{16}\cosh^{-2}\left(\frac{Ra\bar{x}}{4}\right) \tag{4.16}$$

and formally $|\bar{x}| \leq \infty$. In this case $\bar{g}_{+-} > 0$ only for $R < 0$. The physical distance between the coordinate singularities now becomes finite:

$$S = \int_{-\infty}^{+\infty} d\bar{x}\ \sqrt{2\bar{g}_{+-}} = \sqrt{\frac{2}{|R|}}\pi. \tag{4.17}$$

Positive curvature implies $\bar{g}_{+-} = \bar{g}_{+-}(\bar{t}')$ and a finite "life-time" (4.17) with infinite "space."

If the region outside the two poles in $x^+$, say $x^+ \geq a$, is considered, (4.15) and (4.16) are replaced by

$$x^+ = -a \coth\left(\frac{Ra\bar{x}}{4}\right), \tag{4.18}$$

$$\bar{g}_{+-} = -\frac{Ra^2}{16}\sinh^{-2}\left(\frac{Ra\bar{x}}{4}\right) . \tag{4.19}$$

For $R < 0$ the range of $\bar{x}$ is $\bar{x} \geq 0$. Again for $R > 0$ the replacement $\bar{x} = t'$ is necessary.

## V. EXTREMAL TRAJECTORIES

Motion of a test particle according to the action (1.10) in the background provided by a certain solution (2.16) – (2.18) is determined by only part of the affine connections (without the contorsion part also at $T^2 \neq 0$)

$$\tilde{\Gamma}_{\alpha\beta}{}^{\gamma} = \frac{1}{2} g^{\gamma\rho}(\partial_\alpha g_{\beta\rho} + \partial_\beta g_{\rho\alpha} - \partial_\rho g_{\alpha\beta}) \tag{5.1}$$

with (4.2) inserted. The trajectories $x^\alpha(\kappa)$ given by the extremum of (1.10) obey

$$\ddot{x}^+ + \frac{\partial_+ w}{w}(\dot{x}^+)^2 + 2\frac{\partial_+ z}{w}\dot{x}^+\dot{x}^- - \frac{2z}{w^2}(\partial_- w - \partial_+ z)(\dot{x}^-)^2 = 0, \tag{5.2}$$

$$\ddot{x}^- + \frac{1}{w}(\partial_- w - \partial_+ z)(\dot{x}^-)^2 = 0 . \tag{5.3}$$

Besides the invariant "energy" $\sigma_0$ ($\sigma_0 = 0$ for lightlike trajectories)

$$2g_{+-}\dot{x}^+\dot{x}^- + g_{--}(\dot{x}^-)^2 = w\dot{x}^-(\dot{x}^+ + v\dot{x}^-) = \sigma_0 \tag{5.4}$$

in $d = 2$, only one further integral of motion suffices for the integration. If, e.g., as in the examples of Sec. IV A, one Killing vector is $\partial/\partial x^-$ (or $\partial/\partial\bar{t}$ in conformal coordinates) the corresponding second constant of motion becomes

$$\frac{w}{2}\dot{x}^+ + z\dot{x}^- = \rho_0 . \tag{5.5}$$

On the other hand, working in conformal coordinates, in the example in Sec. IV B for $\bar{g}_{+-}(\bar{x})$ the Killing vector $\partial/\partial\bar{t}$ yields

$$g_{00}\frac{d\bar{t}}{d\kappa} = 2\bar{g}_{+-}(\bar{x})\dot{\bar{t}} = \rho_0 . \tag{5.6}$$

### A. General lightlike extremals

For $\sigma_0 = 0$ in (5.4) we first consider the solution $x^- = x_0^- =$ const. Equation (5.3) is fulfilled identically, and from (5.2) we conclude

$$x^+(\kappa) = \frac{1}{mf_0}\ln\mu_0(\kappa - \kappa_0), \tag{5.7}$$

where $f_0 = f(x_0^-)$, $\kappa_0$ and $\mu_0$ are constants. Equation (5.7) just means that $-\infty \leq x^+ \leq +\infty$. Another set of lightlike extremals follows for continuous $\dot{x}^- \neq 0$ and $\sigma_0 = 0$ in (5.4):

$$\frac{\dot{x}^+}{\dot{x}^-} = \frac{dx^+(x^-)}{dx^-} = -v = -\frac{1}{m}\psi(x^+(x^-), x^-). \tag{5.8}$$

The trajectory $x^+(x^-)$ thus fulfils essentially the same differential equation as the one to be solved for transformation to conformal coordinates. In the examples considered in Sec. IV we may simply replace $2\bar{x} = \Phi(\bar{x}^+) - \bar{x}^-$ by $-x^-$, to obtain the solution of (5.8). For example, from the solution (4.7), the trajectory solving (5.8) is just $G(x^+) = -mx^-$. Transformed into conformal coordinates according to (4.7) this yields $\Phi(\bar{x}^+) = 0$, i.e., $\bar{x} = -\bar{t} + \text{const}$, the expected trivial lightlike trajectories in conformal coordinates.

### B. Extremals for particles

To illustrate some "physical" properties of the solutions discussed in Sec. IV the extremals are an excellent tool. For the separable deS solution (4.9) in Sec. IV B all integrals are elementary also here. Extremals here coincide with usual geodesics.

From (5.4) (we choose $\sigma_0 = 1$) and (5.6) for $R > 0$ with (4.14) and initial conditions $\bar{x}_0 = \kappa_0 = 0$,

$$\tan\left(\frac{Ra\bar{x}}{4}\right) = \beta\sin\left(\sqrt{\frac{R}{2}}\kappa\right) \tag{5.9}$$

describes a periodic oscillation of $\bar{x}$ or, equivalently, of the physical distance

$$\sqrt{\frac{R}{2}}\ell = \ln\tan\left(\frac{Ra\bar{x}}{8} + \frac{\pi}{4}\right) \tag{5.10}$$

with frequency $\sqrt{R/2}$ and an amplitude determined by $\beta^2 = 8\rho_0^2/a^2R - 1$, provided $\beta^2 > 0$. It should be noted that the family of trajectories (5.9) for finite $\beta$ always stays inside the regular region $Ra\bar{x} \leq 2\pi$.

The infinite interval for $R < 0$ in conformal $\bar{x}$ of (4.15) and (4.16) maps onto a finite distance, say $\sqrt{|R|/2}\ell \leq \pi/2$, as long as $x^+$ remains between the poles at $\pm a$. Choosing the integration constant $\rho_0$, again so that $8\rho_0^2/a^2|R| - 1 = \beta^2 > 0$, the relation which is the analogue of (5.9) becomes

$$\tanh\left(\frac{|R|a\bar{x}}{4}\right) = \beta\sinh\left(\sqrt{\frac{|R|}{2}}\kappa\right) . \tag{5.11}$$

Because $d\tau/d\kappa > 0$ and finite for finite $\bar{x}$, the singularity $\bar{x} = \infty$, corresponding to $\ell\sqrt{|R|/2} = \pi/2$, is reached after a finite interval of proper time. A similar situation is obtained for $1 \geq 1 - 8\rho_0^2/a^2|R| > 0$ with cosh replacing sinh in (5.11). Here no geodesics are found to exist with $\tanh(|R|a\bar{x}/4) < \sqrt{-\beta^2}$.

The behavior after crossing the coordinate singularity at $\ell\sqrt{|R|/2} = \pi/2$ can be analyzed (again for $R < 0$) with (4.19). Clearly a wide range of further solutions may be investigated.

## VI. SUMMARY AND OUTLOOK

Few models of field theories with complete analytic solutions are known. As we show in our present work, non-Einsteinian gravity in $d = 2$ represents one of them, with the full solution to be obtained in the LC gauge in terms of elementary functions. The interpretation of some salient features of the solution is transparent by transforming to conformal coordinates. As in previous works [9], a sector of de Sitter–type solutions with $R =$ const and another one with nontrivial curvature and torsion scalar are found. In order to obtain the latter without having to solve any residual differential equation, the use of LC variables is essential. The general structure of our solution shows that singularities in curvature and torsion at infinite LC coordinate $x^+$ and arbitrary $x^-$ may occur. Coordinate singularities for these solutions are preferably discussed in conformal coordinates. We find e.g., that, a singularity in $x^+$ corresponding to diverging curvature and torsion may appear in finite "physical" distances. Extremals, as defined in analogy to general relativity, are used to elucidate the geometry of special cases of the general solution.

There are numerous ways in which our present sketchy discussion of this model can be extended, and even more directions of generalization of the model itself exist, the most obvious ones being the introduction of scalar and fermion "coordinate" fields by analogy to string theory, and supersymmetry. Also test particles with spin should be sensitive to the torsion part of our solutions.

The original motivation for the present work was the quantization of the action (1.1) in a suitable gauge. With quantum fluctuations around a flat background at vanishing cosmological constant $\lambda$ this quantum theory was found to be renormalizable [10]. Insight into the quantization of nontrivial manifolds may be obtained by starting from a suitable nontrivial classical solution, including especially solutions periodic in the space coordinate, which were excluded from our present work.

*Note added in proof.* Working in conformal coordinates, recently M. O. Katanaev [J. Math. Phys. **32**, 2483 (1991)] has given a complete discussion of the extremals and geodesics. As suggested by his work, the residual gauge invariance left over after fixing the gauge may be used also in our case. It allows one to normalize the auxiliary functions to $f = 1$, $F = 1$, and $h = 0$. The special example of Sec. IV A, therefore, really covers the most general situation. This point, but especially a novel symmetry on "phase space" responsible for the integrability of (1.1), is treated elsewhere [M. Grosse, W. Kummer, P. Prešnajder, and D. J. Schwarz (in preparation)].

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