Breaking Weyl invariance in the interior of a bubble

W. R. Wood and G. Papini
Department of Physics, University of Regina, Regina, Saskatchewan, Canada S4S 0A2
(Received 8 August 1991)

The basis on which Weyl's unified theory of gravitation and electromagnetism was rejected is reconsidered from a new perspective. It is argued that while Weyl's theory, as indeed any classical theory, is incapable of explaining atomic phenomena, this does not nullify the geometric interpretation of the exterior electromagnetic field; it simply reflects the fact that some form of quantization is needed to account for atomic standards of length. In support of this argument the Gauss-Mainardi-Codazzi formalism is employed to demonstrate that it is possible to construct a bubble in Weyl space where the exterior geometry is conformally invariant and the electromagnetic field can be given a geometric interpretation, while at the same time a standard of length can be introduced into the theory by breaking the conformal invariance in the interior of the bubble.

PACS number(s): 04.50.+h, 04.20.Cv

I. INTRODUCTION

Weyl's unified theory of gravitation and electromagnetism [1] extends the notion of parallel transport in general relativity to include the possibility that lengths, and not only directions, change when vectors are transported along a path. This nonintegrability of length provides a geometrical interpretation for the electromagnetic field and an elegant way to unify the two known long-range forces of nature. While null vectors remain null in the theory so that light signals behave as expected, the absence of what has historically become known as "absolute standards of length" [2] made Weyl's theory unpalatable to generations of physicists well aware of the precise standards of length that can be established, for instance, by means of atomic clocks. Lethal in this respect was Einstein's comment [3] that in Weyl's theory the frequency of spectral lines would depend on the history of the atom, in complete contradiction to known experimental facts.

In retrospect, the criticisms that brought about the downfall of Weyl's theory seem too harsh in two regards. First, it seems unfair to categorically state that a parallel transported vector in Weyl space changes length irrespective of the transport path selected when the corresponding Riemannian change in direction is known not to occur along a special class of lines (geodesics). Indeed, London [4] showed in 1927 that length changes in Weyl space can be made to vanish along closed paths for the proper choice of an arbitrary parameter (see also [5–7]), which leads us to our second point. It is now realized that perfect symmetries must frequently be broken to accommodate observed phenomena which are manifestly asymmetric, lessening the role of symmetries to one of guidance or simply to particular regions of spacetime. In the light of present trends it would therefore seem that all that is needed in the case of Weyl's theory is a plausible breaking mechanism for conformal symmetry that accounts for atomic standards of length. In fact, the problem is not so simple.

Alternatively, some attempts to rehabilitate Weyl's theory do not require any symmetry breaking. Most noticeable in this respect is that given by Dirac [2] who, following Milne, proposed the existence of an unmeasureable metric ds_E, affected by transformations in the standards of length, and of a measureable one, the conformally invariant atomic metric ds_A. Such a conceptual device would rescue the theory without recourse to symmetry breaking. In fact, any function $f(x)$ that tranformed as $f(x)/\sigma(x)$ under the transformation $g_{\mu\nu} \rightarrow \sigma^2 g_{\mu\nu}$ would provide the appropriate relationship between the two metrics:

$$f(x)ds_E = ds_A. \tag{1.1}$$

In effect (1.1) is equivalent to assuming that atomic and gravitational clocks are in principle affected by transformations over and above those provided by the special and general theories of relativity. A way of restating this is to assume that the relative strength of the gravitational and electromagnetic interactions changes in spacetime, which is, in fact, the suggestion advanced by Dirac. Rather than taking this view, the validity of which is severely restricted by the present [8] experimental limit $\dot{G}/G \leq (0.2 \pm 0.4) \times 10^{-11}\ \mathrm{yr}^{-1}$, we wish to revert to the notion of a single metric and address the issue of a unified theory of gravitation and electromagnetism without presupposing a variable G.

Standards of length can, in principle, be established by means of ideal standard rods which, by definition, dilate when transported in Weyl space. Were one to strictly adhere to the introduction of standards of length in this manner, then only the two metric approach mentioned above could conceivably resolve the difficulties inherent in Weyl's theory. Physical rods, in contrast, are made of atoms, and while this certainly adds to the complexity of the problem, it also suggests the possibility, already entertained by London, that "absolute standards of length" ultimately find their origin in the atomic structure of matter. In our opinion the real question that should be asked is why atoms and their constituents as sources of electromagnetic fields exhibit quantization at all, this being responsible for the existence of standards of length. Obviously the failure here is not only of Weyl's geometric interpretation of electromagnetism, but of all classical theories including Maxwell's theory which predicts that a bound electron will spiral into the nucleus of an atom. The solution of this problem calls for a theory that integrates quantum phenomena alongside gravitation and electromagnetism, a theory that, ironically, may be more readily formulated in Weyl's space [6, 7, 9, 10]. The magnitude of the problem can be gleaned from purely classical attempts to reproduce even a first level of quantization. The description of charged particles as regions of spacetime in which one field, gravitation, is uniquely defined, but not the other [6], leads to electromagnetic potentials that are not a pure gauge and hence to flux quantization. This is sufficient to restore some measure of length integrability to the theory [11]. Further investigations in this direction will be presented at a later date.

In this paper the more limited problem of a classical model of extended particles represented by bubbles in Weyl space is considered. Here, it is simply assumed that the conformal invariance is broken in the interior space without justifying this by means of a quantization mechanism. Weyl's geometric argument for the electromagnetic field in the conformally invariant exterior space applies without objection and without recourse to Dirac's assumption of two metrics.

The model of a particle that is presented here is based on the concept of an infinitesimally thin shell of matter. In 1962, Dirac [12] suggested that a similar (nongravitational) membrane model may be used to explain the similarities between the electron and muon. Dirac also considered a gravitational model [13] of a neutral particle based on an action principle. More recently, the Gauss-Mainardi-Codazzi (GMC) formalism has been used to facilitate the study of thin shells of matter in general relativity. The formalism was first put in a coordinate-independent form by Israel [14] in 1966 and applied to an uncharged spherical shell of dust. The motion of a charged shell was studied by Kuchař [15] a few years later. Interest in cosmological applications has since led to a number of comprehensive articles (see, e.g., [16, 17]) and just recently Barrabés and Israel [18] have extended the formalism to include the lightlike case. The GMC formalism is useful in the present context because it allows the possibility of sewing together two spacetime regions with different conformal properties. It is hoped that the consideration of boundary-value problems in Weyl space may lead to a more complete understanding of Weyl geometry and its physical implications.

In order to develop the particle model in Weyl space, it is necessary to first generalize the GMC formalism. Following a brief review of Weyl geometry and its associated gauge-covariant calculus in Sec. II, the equations of Gauss, Mainardi, and Codazzi in Weyl space are determined in Sec. III. The following section deals with the junction conditions for a theory that is linear in the scalar curvature. The formalism is then applied in Sec. V to Dirac's 1973 conformally invariant action [2] where a real scalar field is used to achieve conformal invariance. A static, spherically symmetric solution in the exterior conformally invariant space is found, which, together with the known interior solution and scalar-field-induced surface stress-energy tensor, comprise the model of the particle. The study of the particle model concludes with an analysis of the properties of the spherically symmetric thin shell. Section VI contains a discussion of the proposed symmetry-breaking approach to reconcile Weyl's geometric interpretation of the electromagnetic field with atomic standards of length.

II. WEYL SPACE

Weyl [1] generalized the Riemannian geometry of general relativity by supposing that a vector parallel transported around a closed circuit would not only undergo a change of direction, but would also experience a change in length. In order to describe this generalization mathematically, Weyl introduced a vector field κ^μ that, together with the metric $g_{\mu\nu}$, comprised the fundamental fields of the new geometry. It is a remarkable feature of this generalization that the properties of κ^μ coincide precisely with those of the electromagnetic potentials, suggesting that the long-range forces of electromagnetism and gravity have a common geometric origin.

If a vector of length ℓ is carried by parallel transport along an infinitesimal displacement δx^μ in Weyl space, the change in its length $\delta\ell$ is given by

$$\delta\ell = \ell\kappa_\mu\delta x^\mu. \tag{2.1}$$

For parallel transport around a small closed loop of area $\delta s^{\mu\nu}$ the change in the vector's length is

$$\delta\ell = \ell f_{\mu\nu}\delta s^{\mu\nu}, \tag{2.2}$$

where

$$f_{\mu\nu} = \kappa_{\nu,\mu} - \kappa_{\mu,\nu}. \tag{2.3}$$

If $\delta\ell \neq 0$ around a closed loop, it follows that the geometry will not support an "absolute standard of length." However, under the local scaling of lengths

$$\tilde{\ell} = \sigma(x)\ell, \tag{2.4}$$

the field κ_μ experiences the gauge transformation

$$\tilde{\kappa}_\mu = \kappa_\mu + (\ln\sigma)_{,\mu} \tag{2.5}$$

that is indicative of the independence of the physically significant $f_{\mu\nu}$ on the standard of length chosen.

Of course, the local scaling of lengths also affects the metric $g_{\mu\nu}$. As a consequence of (2.4), the metric tensor and its inverse undergo the conformal transformations

$$\tilde{g}_{\mu\nu} = \sigma^2 g_{\mu\nu}, \qquad \tilde{g}^{\mu\nu} = \sigma^{-2} g^{\mu\nu}. \tag{2.6}$$

One says that $g_{\mu\nu}$ and $g^{\mu\nu}$ have conformal weights 2 and -2, respectively, and writes $w(g_{\mu\nu}) = 2$ and $w(g^{\mu\nu}) =$

-2. It follows that $w(\sqrt{-g}) = 4$.

The development of a conformal covariant calculus is facilitated by introducing the semimetric connection

$$\bar{\Gamma}^{\alpha}{}_{\mu\nu} = \Gamma^{\alpha}{}_{\mu\nu} + g_{\mu\nu}\kappa^{\alpha} - g^{\alpha}{}_{\mu}\kappa_{\nu} - g^{\alpha}{}_{\nu}\kappa_{\mu} \tag{2.7}$$

(which is here assumed to be symmetric) and a gauge-covariant (or cocovariant) derivative. In what follows, the conventions of Ref. [19] are adopted and an overbar is used to distinguish an object defined in terms of the gauge-covariant calculus from the corresponding object associated with the covariant calculus of Riemannian geometry. For example, the spacetime gauge-covariant derivative $\bar{\Box}$ gives rise to the Weyl spacetime curvature tensor $\bar{R}^{\mu}{}_{\nu\alpha\beta}$, whereas the Riemannian curvature tensor $R^{\mu}{}_{\nu\alpha\beta}$ is defined in terms of the spacetime covariant derivative $\Box$ and $\Gamma^{\alpha}{}_{\mu\nu}$ is the Christoffel symbol of the second kind. A property of the gauge-covariant derivative is that it does not change the conformal weight of a tensor on which it acts.

The Weyl curvature tensor can be written as $\bar{R}_{\mu\nu\alpha\beta} = \bar{R}_{(\mu\nu)\alpha\beta} + \bar{R}_{[\mu\nu]\alpha\beta}$, where

$$\bar{R}_{(\mu\nu)\alpha\beta} = \tfrac{1}{2}(\bar{R}_{\mu\nu\alpha\beta} + \bar{R}_{\nu\mu\alpha\beta}) = -g_{\mu\nu} f_{\alpha\beta} \tag{2.8}$$

and square brackets denote antisymmetrization. In what follows, all results are given in terms of $\bar{R}_{[\mu\nu]\alpha\beta}$ and the square brackets are dropped henceforth. One can show that

$$\bar{R}^{\mu}{}_{\nu} \equiv \bar{R}^{\alpha\mu}{}_{\alpha\nu} = R^{\mu}{}_{\nu} + 2(\Box_{\nu}\kappa^{\mu} + \kappa^{\mu}\kappa_{\nu}) + g^{\mu}{}_{\nu}(\Box_{\alpha}\kappa^{\alpha} - 2\kappa^{\alpha}\kappa_{\alpha}), \tag{2.9}$$

and

$$\bar{R} \equiv \bar{R}^{\mu}{}_{\mu} = R + 6(\Box_{\mu}\kappa^{\mu} - \kappa^{\mu}\kappa_{\mu}). \tag{2.10}$$

Also, for a dual vector field ω_{α} and a vector field v^{α} both of conformal weight λ,

$$\bar{\Box}_{\mu}\bar{\Box}_{\nu}\,\omega_{\alpha} - \bar{\Box}_{\nu}\bar{\Box}_{\mu}\,\omega_{\alpha} = \bar{R}_{\alpha}{}^{\beta}{}_{\mu\nu}\omega_{\beta} - (\lambda - 1)\omega_{\alpha} f_{\mu\nu}, \tag{2.11}$$

$$\bar{\Box}_{\mu}\bar{\Box}_{\nu}\,v^{\alpha} - \bar{\Box}_{\nu}\bar{\Box}_{\mu}\,v^{\alpha} = \bar{R}^{\alpha}{}_{\beta\mu\nu}v^{\beta} - (\lambda + 1)v^{\alpha} f_{\mu\nu}. \tag{2.12}$$

For a more complete review of the gauge-covariant calculus, see [6].

In Weyl space, the conformal weight of the scalar curvature is $w(\bar{R}) = -2$. However, for a theory to be conformally invariant, the Lagrangian must be of conformal weight -4. One possible way of achieving a conformally invariant action is to adopt, as Weyl originally did, an $\bar{R}^2$ theory of gravitation. Another approach, that is due to Dirac [2], is to retain an action that is linear in $\bar{R}$ by introducing a new field of conformal weight -2 that couples with $\bar{R}$. Dirac used a real scalar field β with $w(\beta) = -1$. This was later generalized to the case of a complex scalar field [6] and more recently, the coupling of the isodoublet Higgs field with $\bar{R}$ has been considered [20].

III. THE GMC EQUATIONS

In the GMC formalism, four-dimensional spacetime is viewed as being sliced up into three-dimensional hypersurfaces. In the initial-value problem, one considers spacelike hypersurfaces that define a foliation of spacetime. In the study of two-dimensional distributional sources, one considers a timelike hypersurface Σ that divides spacetime into two four-dimensional regions (V^I, V^E) both of which have Σ as their boundary, where Σ represents the history of an infinitesimally thin shell of matter. The GMC formalism in Weyl space is presented here for the latter problem, with the application to the initial-value problem following in a straightforward manner.

The intrinsic (2+1)-dimensional Weyl space on Σ is defined by the condition that the spacetime metrics in V^I and V^E induce the same intrinsic metric $h_{\mu\nu}$ on Σ according to the formula

$$h_{\mu\nu} = g_{\mu\nu} - n_{\mu}n_{\nu}, \tag{3.1}$$

where $n^{\mu} = g^{\mu\nu}n_{\nu}$ is the unit spacelike $(n^{\mu}n_{\mu} = 1)$ vector field normal to Σ that is taken to be directed from V^I to V^E. From (3.1) it follows that, since $w(g_{\mu\nu}) = 2$, $w(h_{\mu\nu}) = 2$ and $w(n_{\mu}) = 1$. Also, $w(n^{\mu}) = w(g^{\mu\nu}n_{\nu}) = -1$, so $n^{\mu}n_{\mu}$ is conformally invariant and n^{μ} is well defined as a unit vector field in Weyl space.

In order to describe the bending of Σ in V^I and V^E one defines the (three-dimensional) extrinsic curvature tensor by

$$\bar{K}_{\mu\nu} = -h_{\mu}{}^{\alpha}\,\bar{\Box}_{\alpha}\,n_{\nu}. \tag{3.2}$$

Since the gauge-covariant derivative acts on n_{μ} without changing its conformal weight, $w(\bar{K}_{\mu\nu}) = 1$. Then, by applying the rules of the gauge-covariant calculus, one finds that

$$\bar{K}_{\mu\nu} = K_{\mu\nu} + h_{\mu\nu}n^{\alpha}\kappa_{\alpha} = \bar{K}_{\nu\mu}, \tag{3.3}$$

where $K_{\mu\nu} = -h_{\mu}{}^{\alpha}\Box_{\alpha}n_{\nu}$ and $\kappa^{\mu} = g^{\mu\nu}\kappa_{\nu}$ is the vector field introduced by Weyl.

The intrinsic gauge-covariant derivative $\bar{D}$ associated with $h_{\mu\nu}$, which gives rise to the intrinsic curvature tensor ${}^3\bar{R}^{\mu}{}_{\nu\alpha\beta}$, is related to $\bar{\Box}$ according to the formula

$$\bar{D}_{\mu}T^{\alpha_1\ldots\alpha_j}{}_{\beta_1\ldots\beta_k} = h^{\alpha_1}_{\sigma_1}\cdots h_{\beta_k}^{\lambda_k}h_{\mu}{}^{\nu}\,\bar{\Box}_{\nu}\,T^{\sigma_1\ldots\sigma_j}{}_{\lambda_1\ldots\lambda_k}. \tag{3.4}$$

The relation (3.4) can be used to facilitate the decomposition of the spacetime curvature tensor into its normal and intrinsic components as follows. Begin by noting that for an intrinsic dual vector field ω_α of conformal weight λ,

$$\bar{D}_\mu \bar{D}_\nu \omega_\alpha - \bar{D}_\nu \bar{D}_\mu \omega_\alpha = {}^3\bar{R}_\alpha{}^\beta{}_{\mu\nu}\omega_\beta - (\lambda - 1) h_\alpha{}^\beta h_\mu{}^\gamma h_\nu{}^\delta \omega_\beta f_{\gamma\delta}. \quad (3.5)$$

Using (3.4) together with the results

$$h_\mu{}^\alpha h_\nu{}^\beta \,\bar{\Box}_\alpha\, h_\beta^\delta = \bar{K}_{\mu\nu} n^\delta, \quad (3.6)$$

$$h_\mu{}^\alpha n^\beta \,\bar{\Box}_\alpha\, \omega_\beta = \bar{K}_\mu{}^\nu \omega_\nu, \quad (3.7)$$

and (2.11), one finds

$$^3\bar{R}^\beta{}_{\alpha\nu\mu}\omega_\beta + (\lambda - 1) h_\alpha{}^\beta h_\mu{}^\gamma h_\nu{}^\delta \omega_\beta f_{\delta\gamma} = h_\mu{}^\lambda h_\nu{}^\delta h_\alpha{}^\gamma [\bar{R}^\sigma{}_{\gamma\delta\lambda} h_\sigma{}^\beta \omega_\beta + (\lambda - 1)\omega_\gamma f_{\delta\lambda}] + \bar{K}_{\mu\alpha}\bar{K}_\nu{}^\beta \omega_\beta - \bar{K}_{\nu\alpha}\bar{K}_\mu{}^\beta \omega_\beta. \quad (3.8)$$

From (3.8) the purely intrinsic components of the spacetime curvature tensor are given by

$$h_\beta{}^\sigma h_\alpha{}^\rho h_\nu{}^\delta h^\mu{}_\lambda \bar{R}^\lambda{}_{\delta\rho\sigma} = {}^3\bar{R}^\mu{}_{\nu\alpha\beta} + \bar{K}_{\alpha\nu}\bar{K}_\beta{}^\mu - \bar{K}_{\beta\nu}\bar{K}_\alpha{}^\mu. \quad (3.9)$$

Next, using (3.1), (3.2), and (3.4) with $\bar{K}^\mu{}_\mu \equiv \bar{K}$, one has

$$\bar{D}_\mu \bar{K}^\mu{}_\nu - \bar{D}_\nu \bar{K} = h_\nu{}^\alpha h_\gamma{}^\beta (\bar{\Box}_\alpha \bar{\Box}_\beta\, n^\gamma - \bar{\Box}_\beta \bar{\Box}_\alpha\, n^\gamma) = h_\nu{}^\alpha h_\gamma{}^\beta \bar{R}^\gamma{}_{\lambda\alpha\beta} n^\lambda,$$

from which follows the result

$$n^\mu h_\alpha{}^\nu \bar{R}_{\mu\nu} = \bar{D}_\alpha \bar{K} - \bar{D}_\mu \bar{K}^\mu{}_\alpha. \quad (3.10)$$

Now define the intrinsic Lie derivative of the extrinsic curvature in the normal direction by

$$\mathcal{L}_n \bar{K}^\mu{}_\nu = n^\alpha \,\bar{\Box}_\alpha\, \bar{K}^\mu{}_\nu + \bar{K}^\mu{}_\alpha \,\bar{\Box}_\nu\, n^\alpha - \bar{K}^\alpha{}_\nu \,\bar{\Box}^\mu\, n_\alpha. \quad (3.11)$$

Imposing the condition that $n^\mu \,\bar{\Box}_\mu\, n_\nu \equiv \bar{\Box}_n\, n_\nu = 0$ (which requires n_μ to be extended off the surface), it follows from (3.11) together with (3.1) and (3.2) that

$$\mathcal{L}_n \bar{K}^\mu{}_\nu = \bar{\Box}_n\, \bar{K}^\mu{}_\nu = -n^\alpha h_\nu{}^\beta \,\bar{\Box}_\alpha \bar{\Box}_\beta\, n^\mu. \quad (3.12)$$

Combing (3.12) with the result

$$\bar{K}^\alpha{}_\mu \bar{K}^\nu{}_\alpha = -h_\mu{}^\alpha n^\beta \,\bar{\Box}_\alpha \bar{\Box}_\beta\, n^\nu, \quad (3.13)$$

the final nontrivial components of the spacetime curvature tensor are found to be

$$n^\alpha n^\beta h_\nu{}^\gamma h^\mu{}_\delta \bar{R}^\delta{}_{\beta\gamma\alpha} = \mathcal{L}_n \bar{K}^\mu{}_\nu - \bar{K}^\alpha{}_\nu \bar{K}^\mu{}_\alpha. \quad (3.14)$$

Equations (3.9), (3.10), and (3.14) are the Gauss-Mainardi-Codazzi equations in Weyl space. By contracting these equations in the appropriate manner and using the relation

$$\bar{K}_{\mu\nu} = -\tfrac{1}{2}\mathcal{L}_n h_{\mu\nu}, \quad (3.15)$$

the following expressions, given in terms of the generalized Einstein tensor

$$\bar{G}_{\mu\nu} = \tfrac{1}{2}(\bar{R}_{\mu\nu} + \bar{R}_{\nu\mu} - g_{\mu\nu}\bar{R}), \quad (3.16)$$

where

$$\bar{R}_{\mu\nu} - \bar{R}_{\nu\mu} = -2f_{\mu\nu}, \quad (3.17)$$

are determined:

$$n_\mu n^\nu \bar{G}^\mu{}_\nu = -\tfrac{1}{2}({}^3\bar{R} + \bar{K}_{\mu\nu}\bar{K}^{\mu\nu} - \bar{K}^2), \quad (3.18)$$

$$n_\mu h_\alpha{}^\nu \bar{G}^\mu{}_\nu = \bar{D}_\alpha \bar{K} - \bar{D}_\mu \bar{K}^\mu{}_\alpha - n_\mu h_\alpha{}^\nu f_\nu{}^\mu, \quad (3.19)$$

$$h^\alpha{}_\mu h_\beta{}^\nu \bar{G}^\mu{}_\nu = {}^3\bar{G}^\alpha{}_\beta + \mathcal{L}_n(\bar{K}^\alpha{}_\beta - h^\alpha{}_\beta \bar{K}) - \bar{K}\bar{K}^\alpha{}_\beta + \tfrac{1}{2}h^\alpha{}_\beta(\bar{K}_{\mu\nu}\bar{K}^{\mu\nu} + \bar{K}^2). \quad (3.20)$$

In applications to problems involving thin shells of matter (with stress energy), it is convenient to recast the above equations into a form that explicitly shows the Riemannian Einstein tensor. This is accomplished by substituting (2.9), (2.10), and (3.3) into (3.18)–(3.20). Following some rather lengthy manipulations (see Appendix A for details), one finds

$$n_\mu n^\nu G^\mu{}_\nu = -\tfrac{1}{2}({}^3R + K_{\mu\nu}K^{\mu\nu} - K^2) - D_\mu \kappa^\mu + 2h_\mu{}^\nu \kappa^\mu \kappa_\nu + 2K n^\mu \kappa_\mu, \quad (3.21)$$

$$n_\mu h_\alpha{}^\nu G^\mu{}_\nu = D_\alpha K - D_\mu K^\mu{}_\alpha, \quad (3.22)$$

$$h^\alpha{}_\mu h_\beta{}^\nu G^\mu{}_\nu = {}^3G^\alpha{}_\beta + (K^\alpha{}_\beta - h^\alpha{}_\beta K)_{,n} - KK^\alpha{}_\beta + \tfrac{1}{2}h^\alpha{}_\beta(K_{\mu\nu}K^{\mu\nu} + K^2) - 2(K^\alpha{}_\beta - h^\alpha{}_\beta K)n^\lambda \kappa_\lambda + 2h^\alpha{}_\beta h_\mu{}^\nu \kappa^\mu \kappa_\nu. \quad (3.23)$$

Equations (3.21)–(3.23) generalize the corresponding equations in Riemannian geometry, which are recovered from the above equations when κ_μ vanishes (cf. Ref. [19]). For a theory that is linear in $\bar{R}$, Eqs. (3.21)–(3.23) are to be equated to the corresponding components of the stress-energy tensor T^μ_ν, which necessarily contains a contribution from a dynamical field that is introduced to achieve conformal invariance.

IV. JUNCTION CONDITIONS

While the form of the GMC equations depends only on the nature of the geometry itself, the analysis of the junction conditions across a thin shell depends on the field equations and hence on the nature of the conformally invariant theory under consideration. In the present work, only theories that are linear in the scalar curvature are considered. In fact, for a thin shell of matter, it may be essential to work with a theory that is linear in the scalar curvature (and consequently also linear in the second derivatives of the metric), in order for the problem to be well defined in a distributional sense. The analysis of a thin shell of matter in Weyl space requires that the junction conditions for the electromagnetic field as well as the gravitational field be satisfied.

It is assumed here that the metric tensor field $g_{\mu\nu}$, the vector field κ^μ and the field that is conformally coupled to $\bar{R}$ are all smooth in the four-dimensional regions V^I and V^E (including their mutual boundary Σ), that they are all continuous across Σ, and their derivatives, which are taken to be discontinuous across Σ, are sufficiently continuous in V^I and V^E for the usual equations to apply. These assumptions are adequate to ensure that Σ exists and has a well-defined intrinsic Weyl geometry. The intrinsic stress-energy tensor on Σ, which is defined by

$$S^\mu{}_\nu \equiv \lim_{\varepsilon\to 0}\int_{-\varepsilon}^{\varepsilon} T^\mu{}_\nu dn, \tag{4.1}$$

depends on the properties of the field that is used to achieve a conformally invariant action that is linear in $\bar{R}$. The stress-energy tensor $S^\mu{}_\nu$ is particularly sensitive to the boundary conditions that are assumed for the derivatives of this field across Σ.

The junction conditions for the gravitational field are obtained by integrating the Einstein equations in the normal direction from $-\varepsilon$ to $+\varepsilon$ across Σ, and then taking the limit as $\varepsilon \to 0$. Using the above boundary conditions for $g_{\mu\nu}$ and κ^μ, integration of (3.21)–(3.23) when set equal to the corresponding components of (4.1) yields

$$\lim_{\varepsilon\to 0}\int_{-\varepsilon}^{\varepsilon} n_\mu n^\nu G^\mu{}_\nu dn = 0 = n_\mu n^\nu S^\mu{}_\nu, \tag{4.2}$$

$$\lim_{\varepsilon\to 0}\int_{-\varepsilon}^{\varepsilon} n_\mu h_\alpha{}^\nu G^\mu{}_\nu dn = 0 = n_\mu h_\alpha{}^\nu S^\mu{}_\nu, \tag{4.3}$$

$$\lim_{\varepsilon\to 0}\int_{-\varepsilon}^{\varepsilon} h^\alpha{}_\mu h_\beta{}^\nu G^\mu{}_\nu dn = \gamma^\alpha{}_\beta - h^\alpha{}_\beta\gamma = h^\alpha{}_\mu h_\beta{}^\nu S^\mu{}_\nu, \tag{4.4}$$

where

$$\gamma^\alpha{}_\beta \equiv [K^\alpha{}_\beta] \equiv \lim_{\varepsilon\to 0}[K^\alpha{}_\beta(n=+\varepsilon) - K^\alpha{}_\beta(n=-\varepsilon)] \tag{4.5}$$

represents the jump in the Riemannian extrinsic curvature and $\gamma \equiv \gamma^\mu{}_\mu$. The junction conditions (4.2)–(4.4) in Weyl space are formally the same as the junction conditions in Riemannian space. However, (4.4) differs fundamentally in that the existence of the stress-energy tensor (4.1) may be related to the conformal invariance of the geometry [see (5.23) below], rather than being introduced in an *ad hoc* manner.

The junction conditions for the electromagnetic field are similarly obtained by taking the limit as $\varepsilon \to 0$ of the integral of the Maxwell equations

$$\bar{\Box}_\nu f^{\mu\nu} = \Box_\nu f^{\mu\nu} = j^\mu, \tag{4.6}$$

$$\bar{\Box}_{[\alpha} f_{\mu\nu]} = f_{[\mu\nu,\alpha]} = 0, \tag{4.7}$$

where j^μ is the intrinsic current ($n_\mu j^\mu = 0$), defined only on Σ. Applying the boundary conditions for κ^μ in the integrals of the normal and intrinsic components of (4.6) and (4.7) yields the usual electromagnetic junction conditions (cf. Ref. [15])

$$[n_\mu h_\alpha{}^\nu f_\nu{}^\mu] = h_\alpha{}^\mu j_\mu, \qquad [h_\mu{}^\alpha h^\nu{}_\beta f^\mu{}_\nu] = 0. \tag{4.8}$$

Two further equations that are useful in determining the properties of the thin shell arise by taking the jump in the Einstein field equations. Introducing the notation

$$\{\Phi\} \equiv \lim_{\varepsilon\to 0}\tfrac{1}{2}[\Phi(n=+\varepsilon) + \Phi(n=-\varepsilon)] \tag{4.9}$$

it follows from (3.21) that

$$\{K^\mu{}_\nu\}S_\mu{}^\nu + [n_\mu n^\nu T^\mu{}_\nu] = 0 \tag{4.10}$$

which is identical to the Riemannian result [17]. Taking the jump in (3.22) and using (4.4), one recovers the Riemannian equation

$$D_\mu(h^\mu{}_\alpha h_\nu{}^\beta S^\alpha{}_\beta) + [n_\alpha h_\nu{}^\beta T^\alpha{}_\beta] = 0, \tag{4.11}$$

while the jump in (3.23) has been used to determine (4.4). Once a specific example has been chosen, Eqs. (4.10) and (4.11) can be used to describe the balance of stress-energy-momentum in the shell.

V. THE PARTICLE MODEL

A. Dirac's conformally invariant action

The simplest way in which a theory that is linear in $\bar{R}$ can be made conformally invariant is to introduce a real scalar field $\beta(x)$. This approach was adopted by Dirac [2], who took $w(\beta) = -1$ and considered the action

$$I_D = \int\Big(-\tfrac{1}{4}f_{\mu\nu}f^{\mu\nu} + \beta^2\bar{R} + k\,\bar{\Box}_\mu\,\beta\,\bar{\Box}^\mu\,\beta + \lambda\beta^4\Big)\sqrt{-g}\,d^4x. \tag{5.1}$$

Dirac chose to set $k = 6$, and by discarding a total divergence in the action, expressed (5.1) in the equivalent, although not manifestly invariant form

$$I_D = \int\Big(-\tfrac{1}{4}f_{\mu\nu}f^{\mu\nu} + \beta^2 R + 6\beta_{,\mu}\beta^{,\mu} + \lambda\beta^4\Big)\sqrt{-g}\,d^4x. \tag{5.2}$$

The field equations for the vacuum that follow from (5.2) are the Maxwell and Einstein equations

$$\Box_\nu f^{\mu\nu} = 0 \tag{5.3}$$

and

$$G_{\mu\nu} = \frac{1}{2\beta^2}E_{\mu\nu} + I_{\mu\nu} + \frac{1}{2}\lambda g_{\mu\nu}\beta^2 \equiv T_{\mu\nu}, \tag{5.4}$$

where

$$E_{\mu\nu} = f_{\mu\alpha} f_\nu{}^\alpha - \frac{1}{4} g_{\mu\nu} f_{\alpha\beta} f^{\alpha\beta}, \tag{5.5}$$

$$I_{\mu\nu} = \frac{2}{\beta}(\Box_\nu \Box_\mu \beta - g_{\mu\nu} \Box_\alpha \Box^\alpha \beta) - \frac{1}{\beta^2}(4\beta_{,\mu}\beta_{,\nu} - g_{\mu\nu}\beta_{,\alpha}\beta^{,\alpha}), \tag{5.6}$$

and the field equation for β which is identically the trace of (5.4). If the conformal invariance were to be broken by setting $\beta = 1$, the geometric interpretation of the electromagnetic field would be lost and one would be left with the Einstein-Maxwell theory (with cosmological constant) in units where $c^3(16\pi G)^{-1} = 1$.

The interest in Dirac's conformally invariant action here lies in the application of the GMC formalism to a thin shell of matter in Weyl space, where the field equations (5.3) and (5.4) hold in V^I and V^E. For simplicity, the formalism is applied to a spherically symmetric shell that is assumed to carry an electric charge q. The GMC formalism requires that the interior and exterior line elements be specified, as well as the intrinsic stress-energy tensor $S_{\mu\nu}$. For a spherically symmetric shell in Riemannian space, Birkoff's theorem may be used to specify the form of the line element and one arbitrarily chooses some particular form for $S_{\mu\nu}$. In the present case, one must determine the form of the line element from first principles, and then use (4.1) to determine $S_{\mu\nu}$ from the theory itself.

B. The static spherically symmetric solution

The general static spherically symmetric line element is written

$$ds^2 = -e^{\nu(r)}dt^2 + e^{\mu(r)}dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\phi^2). \tag{5.7}$$

The exterior and interior geometries are distinguished by writing $t_{E,I}$, $\nu_{E,I}$ and $\mu_{E,I}$ in $V^{E,I}$, respectively. The two equations that are required to completely solve for $\mu(r)$ and $\nu(r)$ are [21]

$$\frac{d}{dr}(re^{-\mu}) = 1 + r^2 T^0_{\ 0} \tag{5.8}$$

and

$$\mu' + \nu' = re^{\mu}(T^1_{\ 1} - T^0_{\ 0}), \tag{5.9}$$

where $T_{\mu\nu}$ is the tensor that is equated to the Riemannian Einstein tensor, and a prime denotes a derivative with respect to r.

In the interior space V^I it is assumed that (i) $\kappa_\mu = 0$ which establishes length integrability in the region of spacetime occupied by the extended particle and (ii) the scalar field acquires a constant value $\beta = \beta_0$ which breaks the interior conformal invariance and fixes the scale associated with the particle. Because of their continuity properties, $\kappa_\mu = 0$ and $\beta = \beta_0$ on Σ as well. The proposal that microscopic particles are responsible for the establishment of a standard of length suggests choosing β_0 to be very large since $\beta \sim (\text{length})^{-1}$. The idea that the $\lambda\beta_0^2$ term may play the role of a fundamental atomic constant was put forward by Lanczos [22] some time ago in his analysis of an action that is quadratic in the scalar curvature. In the present theory, λ remains an arbitrary constant. With the Maxwell tensor $E_{\mu\nu}$ and $I_{\mu\nu}$ vanishing in V^I, the interior stress-energy tensor reduces to

$$T^I_{\mu\nu} = \tfrac{1}{2}\lambda g_{\mu\nu}\beta_0^2, \tag{5.10}$$

which corresponds to a pressure $\lambda\beta_0^2/2$ for λ positive. The tensor (5.10) also represents a de Sitter ($\lambda < 0$), Minkowski ($\lambda = 0$), or anti–de Sitter ($\lambda > 0$) space with constant curvature $R = -2\lambda\beta_0^2$. Thus, the interior solution to (5.8) and (5.9) is given by the well-known result

$$e_I^{-\mu} = 1 + \tfrac{1}{6}\lambda\beta_0^2 r^2 = e_I^{\nu}. \tag{5.11}$$

The Coulomb potential of the charged shell gives rise to an exterior electromagnetic field that can be interpreted, when $\beta(r)$ is present to ensure conformal invariance, as a manifestation of the exterior Weyl space with a stress-energy tensor

$$T^E_{\mu\nu} = \frac{1}{2\beta^2}E_{\mu\nu} + I_{\mu\nu} + \frac{1}{2}\lambda g_{\mu\nu}\beta^2. \tag{5.12}$$

The required components of (5.12) expressed in terms of the metric (5.7) are given by (Appendix B)

$$(T^0_{\ 0})^E = -\frac{1}{4\beta^2}\frac{q^2}{r^4} - e_E^{-\mu}\left(2\frac{\beta''}{\beta} - \frac{\beta'^2}{\beta^2} - \mu'_E\frac{\beta'}{\beta} + \frac{4}{r}\frac{\beta'}{\beta}\right) + \frac{1}{2}\lambda\beta^2 \tag{5.13}$$

and

$$(T^1_{\ 1})^E = -\frac{1}{4\beta^2}\frac{q^2}{r^4} - e_E^{-\mu}\left(3\frac{\beta'^2}{\beta^2} + \nu'_E\frac{\beta'}{\beta} + \frac{4}{r}\frac{\beta'}{\beta}\right) + \frac{1}{2}\lambda\beta^2. \tag{5.14}$$

Substituting (5.13) and (5.14) into (5.8) and (5.9), the exterior solution is found to be (Appendix C)

$$e_E^{-\mu} = \left(1 + r\frac{\beta'}{\beta}\right)^{-2}\left(1 - \frac{2m}{\beta r} + \frac{q^2}{4\beta^2 r^2} + \frac{1}{6}\lambda\beta^2 r^2\right), \tag{5.15}$$

$$e_E^{\nu} = (\ell_0\beta)^{-2}\left(1 - \frac{2m}{\beta r} + \frac{q^2}{4\beta^2 r^2} + \frac{1}{6}\lambda\beta^2 r^2\right). \tag{5.16}$$

The integration constant ℓ_0, which arises in the radial integration of (5.9) from the boundary of the spherical shell to the field point r, has a dimension of length. Since $\ell_0 = \beta_0^{-1}(1 + r\beta'/\beta)|_{r=R}$ (see Appendix C), the exterior metric contains, in addition to the usual information regarding the mass and charge of the particle, information about the size and structure of the particle as well.

C. Properties of the thin shell

The intrinsic stress-energy tensor is determined by substituting $T_{\mu\nu}$ into the definition (4.1). Assuming that the normal derivative of $\ln\beta$ across Σ is discontinuous, i.e.,

$[n^\mu(\ln\beta)_{,\mu}] \equiv \alpha \neq 0$, one finds after decomposing $E_{\mu\nu}$ and $I_{\mu\nu}$ into their normal and intrinsic components (Appendix D),

$$n_\mu n^\nu E^\mu{}_\nu = \frac{1}{2} n_\mu n^\nu h_\alpha{}^\beta f^\alpha{}_\nu f_\beta{}^\mu - \frac{1}{4} h_\mu{}^\nu h_\alpha{}^\beta f^\mu{}_\beta f_\nu{}^\alpha, \quad (5.17)$$

$$n_\mu h_\alpha{}^\nu E^\mu{}_\nu = n_\mu h_\alpha{}^\nu h_\beta{}^\gamma f^\mu{}_\gamma f_\nu{}^\beta, \quad (5.18)$$

$$h^\alpha{}_\mu h_\beta{}^\nu E^\mu{}_\nu = h^\alpha{}_\mu h_\beta{}^\nu g_\gamma{}^\lambda f^\mu{}_\lambda f_\nu{}^\gamma - \frac{1}{4} h^\alpha{}_\beta (h_\mu{}^\lambda h_\nu{}^\gamma + 2 n_\mu n^\lambda h_\nu{}^\gamma) f^\mu{}_\gamma f_\lambda{}^\nu, \quad (5.19)$$

$$n_\mu n^\nu I^\mu{}_\nu = \frac{2}{\beta}(K\beta_{,n} - D_\mu D^\mu \beta) - \frac{1}{\beta^2}(3 n_\mu n^\nu - h_\mu{}^\nu)\beta^{,\mu}\beta_{,\nu}, \quad (5.20)$$

$$n_\mu h_\alpha{}^\nu I^\mu{}_\nu = \frac{2}{\beta}(h_\alpha{}^\mu \beta_{,\mu n} + K_{\alpha\mu}\beta^{,\mu}) - \frac{4}{\beta^2} n_\mu h_\alpha{}^\nu \beta^{,\mu}\beta_{,\nu}, \quad (5.21)$$

$$h^\alpha{}_\mu h_\beta{}^\nu I^\mu{}_\nu = {}^3 I^\alpha{}_\beta - \frac{2}{\beta}(K^\alpha{}_\beta - h^\alpha{}_\beta K)\beta_{,n} - n^\mu n^\nu h^\alpha{}_\beta \left(\frac{2}{\beta}\beta_{,\mu\nu} - \frac{1}{\beta^2}\beta_{,\mu}\beta_{,\nu}\right), \quad (5.22)$$

where $h_\mu{}^\nu \beta_{,\nu} = 0$ for $\beta = \beta(r)$, that the only nonvanishing components of $S_{\mu\nu}$ from (4.1) are

$$h^\alpha{}_\mu h_\beta{}^\nu S^\mu{}_\nu = -2\alpha h^\alpha{}_\beta. \quad (5.23)$$

This result, which is consistent with (4.2)–(4.4), indicates that for $\alpha > 0$ the thin shell is under a surface tension that opposes the Coulomb repulsion. The stress-energy tensor (5.23) is characteristic of a domain wall with surface energy density 2α. From (4.11) it follows that $\alpha =$ constant. This can be shown by substituting (5.23) into (4.11) and by making use of the jump condition

$$[n_\mu h_\alpha{}^\nu T^\mu_\nu] = 0 \quad (5.24)$$

that follows from (5.18) and (5.21).

To determine the physical content of (4.10) we follow the procedure used by Kuchař [15] and define the spacetime tensor associated with $S_{\mu\nu}$ by

$$\Im^\mu{}_\nu \equiv h^\mu{}_\alpha h_\nu{}^\beta S^\alpha{}_\beta \delta(n). \quad (5.25)$$

Kuchař considers the normal component of the divergence of the combined tensor $\Im^\mu{}_\nu + T^\mu_\nu$. For $\Im^\mu{}_\nu$ one has

$$n^\mu \Box_\nu \Im_\mu{}^\nu = S^\mu{}_\nu K_\mu{}^\nu. \quad (5.26)$$

The divergence of T^μ_ν for the present theory reduces to the simple form

$$n^\mu \Box_\nu T_\mu{}^\nu = -\frac{1}{2\beta_0^2} n^\mu f_\mu{}^\nu j_\nu \quad (5.27)$$

when the field equation (5.4) is used. Using (5.17) and (5.20), the jump condition required in (4.10) is determined to be

$$[n_\mu n^\nu T^\mu_\nu] = \frac{1}{4\beta_0^2} h_\mu{}^\nu j^\mu j_\nu + 2\alpha\{K\}. \quad (5.28)$$

Taking the average $\{(5.26)+(5.27)\}$, Eq. (4.10) becomes

$$\{n^\mu \Box_\nu (\Im_\mu{}^\nu + T_\mu{}^\nu)\} = -2\alpha\{K\}, \quad (5.29)$$

which demonstrates that, due to the presence of $I_{\mu\nu}$ in V^E, the normal forces acting on the respective sides of Σ are not equal. Taking the jump $[(5.26)+(5.27)]$, we recover Kuchař's result

$$[n^\mu \Box_\nu (\Im_\mu{}^\nu + T_\mu{}^\nu)] = \frac{1}{2\beta_0^2} j_\mu j^\mu + S^\mu{}_\nu S_\mu{}^\nu - \frac{1}{2} S^2. \quad (5.30)$$

For a domain-wall tensor, the right-hand side of (5.30) is negative definite.

With the geometry in V^E and V^I fixed [Eqs. (5.11), (5.15), and (5.16)] as well as the form of $S_{\mu\nu}$ (5.23), it remains to consider the dynamical properties of the spherical shell. One usually begins by writing the intrinsic metric on Σ in the form

$$ds^2 = -d\tau^2 + R^2(\tau)(d\theta^2 + \sin^2\theta\, d\phi^2), \quad (5.31)$$

where the equation $r = R(\tau)$ describes the motion of the spherical shell in terms of the proper time τ. The four-velocity of a point on the shell as seen by an observer in $V^{E,I}$ is then given by

$$u^\alpha_{E,I} = (X_{E,I}, \dot{R}, 0, 0), \quad (5.32)$$

where $X_{E,I} \equiv dt_{E,I}/d\tau \equiv \dot{t}_{E,I}$ and

$$X_{E,I} = \pm(e^{\mu-\nu}_{E,I}\dot{R}^2 + e^{-\nu}_{E,I})^{1/2}. \quad (5.33)$$

The spacelike unit normals that are orthogonal to $u^\alpha_{E,I}$ are

$$n^\alpha_{E,I} = (e^{-\nu}_{E,I}\dot{R}, e^{-\mu}_{E,I} X_{E,I}, 0, 0). \quad (5.34)$$

By applying the rules of the gauge-covariant calculus to (3.15) one can show that

$$K_{\mu\nu} = -\tfrac{1}{2} n^\alpha h_{\mu\nu,\alpha}. \quad (5.35)$$

As discussed in Ref. [16], the components of the extrinsic curvature tensor that are particularly useful are

$$(K^\theta{}_\theta)^{E,I} = -\frac{1}{R} e^{-\mu}_{E,I} X_{E,I}. \quad (5.36)$$

Now, from (4.4) and (5.23), it follows that

$$\gamma^\theta{}_\theta = S^\theta{}_\theta - \tfrac{1}{2} h^\theta{}_\theta S = \alpha, \quad (5.37)$$

so that at $r = R$, the $\theta\theta$ component of (4.5) becomes

$$\pm e_I^{-\mu}(e_I^{\mu-\nu}\dot{R}^2 + e_I^{-\nu})^{1/2} \mp e_E^{-\mu}(e_E^{\mu-\nu}\dot{R}^2 + e_E^{-\nu})^{1/2} = \alpha R. \quad (5.38)$$

Equation (5.38) may be applied in general to any spherical domain-wall problem. For example, the analogous equations given in Refs. [16] and [23] are recovered from (5.38) by taking the interior (exterior) solutions to be de Sitter (Schwarzschild) and Minkowski (Reissner-Nordström), respectively. In the analyses given in [16, 23], the surface tension is an arbitrary input parameter. In the present model α is not arbitrary since it arises by assuming that the normal derivative of $\ln\beta$ is discontinuous across Σ. In fact, using (5.33) and (5.34) in the definition of α, it follows that, at $r = R$,

$$\alpha = \pm e_E^{-(\mu+\nu)}(e_E^{\mu-\nu}\dot{R}^2 + e_E^{-\nu})^{-1/2}\omega, \tag{5.39}$$

where $\omega \equiv (\beta'/\beta)|_{r=R}$. The requirement that α in (5.38) take the form given in (5.39) leads to the relationship

$$e_I^{-\mu}(e_I^{\mu-\nu}\dot{R}^2 + e_I^{-\nu})^{1/2} = (1 + \omega R + e_E^{\mu}\dot{R}^2)e_E^{-(\mu+\nu)} \times(e_E^{\mu-\nu}\dot{R}^2 + e_E^{-\nu})^{-1/2}, \tag{5.40}$$

wherein the sign ambiguity is lost. Substituting the expressions for $e^{\mu}_{E,I}(R)$ and $e^{\nu}_{E,I}(R)$ for the static case from Eqs. (5.11), (5.15), and (5.16) into (5.40) with $\dot{R} = 0$ and squaring, yields the equilibrium solution

$$R_{\rm eq} = \frac{q^2}{8\beta_0 m}, \tag{5.41}$$

the existence of which is required if the bubble is to be interpreted as representing a particle. Indeed, a particle interpretation further suggests that the static solution (5.41) should be reasonably stable under small perturbations about $R = R_{\rm eq}$. To determine if the spherical shell is stable under small radial oscillations one would like to solve for the motion of the bubble in general. Attempting to solve the Einstein equations in V^E with the stress-energy tensor (5.12) for $\mu_E(r,t)$ and $\nu_E(r,t)$ is a rather formidable task. However, an approximate solution, valid to order $\partial\beta/\partial t$, retains the same functional forms for $e_E^{-\mu}$ and e_E^{ν} given by (5.15) and (5.16), where $\beta(r,t)$ is now a slowly varying function of time. By solving (5.40) for $\dot{R}^2$ and then taking the proper time derivative, one obtains the equation

$$\ddot{R} = -\xi^2 R + \eta, \tag{5.42}$$

where ξ and η are expressed entirely in terms of the parameters $R_{\rm eq}$, ω, λ, β_0, and m. The fact that (5.42) yields an oscillating solution indicates that the equilibrium solution is stable under small radial oscillations. The equilibrium solution (5.41), together with the one that follows from (5.42) when $\ddot{R} = 0$, provide a relationship between the parameters in the model that is general enough so as not to place any restrictions on the value that λ, for example, may have.

In Einstein-Maxwell theory, the exterior metric is guaranteed by Birkhoff's theorem to retain the static form, even if the bubble is pulsating [24]. The generalization of Birkhoff's theorem fails in the present model because the scalar field β establishes a link between the thin shell and the exterior metric; a pulsating bubble with a scalar-field-induced surface stress-energy tensor will, in general, affect the exterior conformally invariant geometry. Hence, it appears that a time-dependent solution to the Einstein equations is required in order to analyze the dynamics of spherical shells in the present model beyond the small oscillations approximation. While the existence of a static bubble is sufficient to determine if Weyl's geometric interpretation of the electromagnetic field can be reconciled with atomic standards of length based on the proposed symmetry-breaking approach, a dynamical solution would be of interest in its own right.

VI. DISCUSSION

It is our contention that Weyl's theory has been unduly penalized for not accounting for the existence of atomic standards of length. The introduction of such standards requires, in our opinion, that "atoms" be incorporated into a classical theory not only as matter in the energy-momentum tensor, but also as entities that somehow produce standards of length. Far from claiming that we have achieved the desired unification of a rigorous description of quantum phenomena with a classical geometric theory, the "atoms" that we have introduced must simply be construed as regions of spacetime in which Weyl invariance is violated. The "atoms" are entirely classical in nature and though their dimensions are not dictated by any atomic constant such as Planck's constant, their dimensions can be truly microscopic. Even at such a scale, however, the breaking of Weyl invariance can affect the structure of the resulting spacetime in a truly macroscopic way.

The particular "atom" or "particle" considered consists of a static, charged, spherically symmetric bubble in Weyl's space. On the surface and in the interior $\kappa_\mu = 0$. As discussed in Sec. V., the interior space can be Minkowski, de Sitter, or anti–de Sitter. These spaces are frequently considered in the literature as possible realizations of confinement at the elementary particle level and naturally appear as consequences of Dirac's Lagrangian (5.2) for β = const and λ arbitrary. It is also interesting to notice that if one wants to maintain linearity in $\bar{R}$, which appears necessary from the point of view of the GMC formalism, this is most easily accomplished by means of the constraining field β introduced by Dirac. For Dirac's conformally invariant action, β then (i) ensures conformal invariance in V^E, (ii) fixes the scale in V^I, and (iii) induces the surface tension needed for stability, a tension which is frequently introduced by hand in the literature.

In the conformally invariant exterior space V^E, a transport path linking two arbitrary points A and D for which $\delta\ell = 0$ can always be constructed by employing a curve that has a segment BC lying entirely on the particle's world tube with end points B and C that are connected to A and D via the radial segments AB and CD. In this way, for the simple particle model considered, the broken scale invariance in V^I can be used to establish a uniform standard of length in V^E. Physical objects cannot, of course, follow the path indicated. In this case, one would have to argue that the mechanism which binds the "atoms" together is also responsible for maintaining the standard of length of the macroscopic object. Such an argument lies beyond the scope of the present work which is limited to a single particle model. An observer would, however, be able to measure lengths by carrying an "atom" along the world line. An alternative way would be to look for certain paths along which no dilation occurs. Steps in this direction have been considered in the past [4–6]. More recently, Wheeler [7] has considered such paths in a theory of measurement in Weyl geometry where test particles "do not exist" and found that "weightful bodies follow the preferred classical trajectories and therefore experience no dilation."

Although the "atom" or "particle" model introduced is very primitive, it supports the main point of the paper that atomic standards of length can coexist with Weyl's geometric interpretation of the electromagnetic field. Possible refinements could be achieved by introducing a second scalar field. This leads, for some models, still of the Weyl-Dirac type, to a wave equation [25] and therefore to a real first-quantization mechanism.

ACKNOWLEDGMENTS

This work was supported in part by the Natural Sciences and Engineering Research Council of Canada.

APPENDIX A: THE GMC EQUATIONS IN TERMS OF $G_{\mu\nu}$

The GMC equations, as expressed in the manifestly gauge-invariant form in (3.18)–(3.20), are here rewritten in terms of the Riemannian Einstein tensor $G_{\mu\nu}$. Using (2.9) and (2.10) in (3.16) one has

$$\bar{G}^{\mu}{}_{\nu} = G^{\mu}{}_{\nu} + \Box_{\nu}\kappa^{\mu} + \Box^{\mu}\kappa_{\nu} + 2\kappa^{\mu}\kappa_{\nu} - g^{\mu}{}_{\nu}(2\Box_{\alpha}\kappa^{\alpha} - \kappa_{\alpha}\kappa^{\alpha}), \tag{A1}$$

from which it follows that

$$n_{\mu}n^{\nu}\bar{G}^{\mu}{}_{\nu} = n_{\mu}n^{\nu}G^{\mu}{}_{\nu} - 2D_{\mu}\kappa^{\mu} + 3n_{\mu}n_{\nu}\kappa^{\mu}\kappa^{\nu} + h_{\mu}{}^{\nu}\kappa^{\mu}\kappa_{\nu}, \tag{A2}$$

$$n_{\mu}h_{\alpha}{}^{\nu}\bar{G}^{\mu}{}_{\nu} = n_{\mu}h_{\alpha}{}^{\nu}G^{\mu}{}_{\nu} + n_{\mu}h_{\alpha}{}^{\nu}(\Box_{\nu}\kappa^{\mu} + \Box^{\mu}\kappa_{\nu} + 2\kappa^{\mu}\kappa_{\nu}), \tag{A3}$$

$$h^{\alpha}{}_{\mu}h_{\beta}{}^{\nu}\bar{G}^{\mu}{}_{\nu} = h^{\alpha}{}_{\mu}h_{\beta}{}^{\nu}G^{\mu}{}_{\nu} + h^{\alpha}{}_{\mu}h_{\beta}{}^{\nu}(\Box_{\nu}\kappa^{\mu} + \Box^{\mu}\kappa_{\nu} + 2\kappa^{\mu}\kappa_{\nu}) - h^{\alpha}{}_{\beta}(2\Box_{\mu}\kappa^{\mu} - \kappa_{\mu}\kappa^{\mu}), \tag{A4}$$

and

$$^{3}\bar{G}^{\alpha}{}_{\beta} = {}^{3}G^{\alpha}{}_{\beta} + h^{\alpha}{}_{\mu}h_{\beta}{}^{\nu}(\Box_{\nu}\kappa^{\mu} + \Box^{\mu}\kappa_{\nu} + 2\kappa^{\mu}\kappa_{\nu}) - h^{\alpha}{}_{\beta}(2D_{\mu}\kappa^{\mu} - h_{\mu\nu}\kappa^{\mu}\kappa^{\nu}). \tag{A5}$$

By rewriting now the right-hand sides of (3.18)–(3.20) in terms of Riemannian operators and fields and then equating these results to (A2)–(A4), the GMC equations in terms of the components of $G_{\mu\nu}$ may be obtained.

For Eq. (3.18), the right-hand side becomes

$$n_{\mu}n^{\nu}\bar{G}^{\mu}{}_{\nu} = -\tfrac{1}{2}({}^{3}R + K_{\mu\nu}K^{\mu\nu} - K^{2}) - 3D_{\mu}\kappa^{\mu} + 3g_{\mu}{}^{\nu}\kappa^{\mu}\kappa_{\nu} + 2Kn_{\mu}\kappa^{\mu}. \tag{A6}$$

For Eq. (3.19), one uses the relationships

$$\bar{D}_{\mu}\bar{K}^{\mu}{}_{\alpha} = D_{\mu}\bar{K}^{\mu}{}_{\alpha} + h_{\mu}{}^{\nu}h_{\alpha}{}^{\beta}(\bar{K}^{\mu}{}_{\nu}\kappa_{\beta} - 2\bar{K}^{\mu}{}_{\beta}\kappa_{\nu}) = D_{\mu}K^{\mu}{}_{\alpha} - 3K_{\mu\alpha}\kappa^{\mu} + Kh_{\alpha}{}^{\mu}\kappa_{\mu} + n_{\mu}h_{\alpha}{}^{\nu}(\Box_{\nu}\kappa^{\mu} + \kappa^{\mu}\kappa_{\nu})$$

and

$$\bar{D}_{\alpha}\bar{K} = D_{\alpha}K - 3K_{\mu\alpha}\kappa^{\mu} + Kh_{\alpha}{}^{\mu}\kappa_{\mu} + 3n_{\mu}h_{\alpha}{}^{\nu}(\Box_{\nu}\kappa^{\mu} + \kappa^{\mu}\kappa_{\nu})$$

to obtain

$$n_{\mu}h_{\alpha}{}^{\nu}\bar{G}^{\mu}{}_{\nu} = -D_{\mu}K^{\mu}{}_{\alpha} + D_{\alpha}K + n_{\mu}h_{\alpha}{}^{\nu}(2\Box_{\nu}\kappa^{\mu} + 2\kappa^{\mu}\kappa_{\nu} - f_{\nu}{}^{\mu}). \tag{A7}$$

For Eq. (3.20), the relation $h^{\mu}{}_{\nu,n} = 0$, that follows from the definition of the Lie derivative (3.11), is used to show that

$$h^{\alpha}{}_{\mu}h_{\beta}{}^{\nu}\bar{G}^{\mu}{}_{\nu} = {}^{3}\bar{G}^{\alpha}{}_{\beta} + (K^{\alpha}{}_{\beta} - h^{\alpha}{}_{\beta}K)_{,n} - KK^{\alpha}_{\beta} + \tfrac{1}{2}h^{\alpha}{}_{\beta}(K_{\mu\nu}K^{\mu\nu} + K^{2}) - 2h^{\alpha}{}_{\beta}(n_{\mu}\kappa^{\mu})_{,n} - 2(K^{\alpha}{}_{\beta} - h^{\alpha}{}_{\beta}K)n^{\lambda}\kappa_{\lambda} + h^{\alpha}{}_{\beta}n_{\mu}n_{\nu}\kappa^{\mu}\kappa^{\nu}. \tag{A8}$$

Finally, substituting (A5) into (A8) and equating (A2)–(A4) to (A6)–(A8), respectively, the resulting equations can be solved for the different components of $G_{\mu\nu}$ to yield

$$n_{\mu}n^{\nu}G^{\mu}{}_{\nu} = -\tfrac{1}{2}({}^{3}R + K_{\mu\nu}K^{\mu\nu} - K^{2}) - D_{\mu}\kappa^{\mu} + 2h_{\mu}{}^{\nu}\kappa^{\mu}\kappa_{\nu} + 2Kn^{\mu}\kappa_{\mu}, \tag{A9}$$

$$n_{\mu}h_{\alpha}{}^{\nu}G^{\mu}{}_{\nu} = D_{\alpha}K - D_{\mu}K^{\mu}{}_{\alpha}, \tag{A10}$$

$$h^{\alpha}{}_{\mu}h_{\beta}{}^{\nu}G^{\mu}{}_{\nu} = {}^{3}G^{\alpha}{}_{\beta} + (K^{\alpha}_{\beta} - h^{\alpha}{}_{\beta}K)_{,n} - KK^{\alpha}_{\beta} + \tfrac{1}{2}h^{\alpha}{}_{\beta}(K_{\mu\nu}K^{\mu\nu} + K^{2}) - 2(K^{\alpha}_{\beta} - h^{\alpha}{}_{\beta}K)n^{\lambda}\kappa_{\lambda} + 2h^{\alpha}{}_{\beta}h_{\mu}{}^{\nu}\kappa^{\mu}\kappa_{\nu}. \tag{A11}$$

APPENDIX B: $T^{E}_{\mu\nu}$ IN A STATIC, SPHERICALLY SYMMETRIC METRIC

The exterior metric components of the static, spherically symmetric metric (5.7) are

$$g_{00} = -e_{E}^{\nu(r)}, \quad g_{11} = e_{E}^{\mu(r)}, \quad g_{22} = r^{2}, \quad g_{33} = r^{2}\sin^{2}\theta. \tag{B1}$$

The required nonvanishing components of the Christoffel symbol of the second kind,

$$\Gamma^{\alpha}{}_{\mu\nu} = \tfrac{1}{2}g^{\alpha\beta}(g_{\mu\beta,\nu} + g_{\nu\beta,\mu} - g_{\mu\nu,\beta}), \tag{B2}$$

are determined to be

$$\Gamma^{1}{}_{00} = \tfrac{1}{2}\nu'_{E}e_{E}^{\nu-\mu}, \quad \Gamma^{1}{}_{11} = \tfrac{1}{2}\mu'_{E}, \tag{B3}$$

where a prime denotes differentiation with respect to r.

In the metric (B1) the tensor

$$I_{\mu\nu} = \frac{2}{\beta}(\Box_\nu \Box_\mu \beta - g_{\mu\nu}\Box_\alpha \Box^\alpha \beta) - \frac{1}{\beta^2}(4\beta_{,\mu}\beta_{,\nu} - g_{\mu\nu}\beta_{,\alpha}\beta^{,\alpha}) \tag{B4}$$

is determined as follows. The Laplace-Beltrami operator

$$\Box_\alpha \Box^\alpha = \frac{1}{\sqrt{-g}}[\partial_\mu(\sqrt{-g}g^{\mu\nu}\partial_\nu)], \tag{B5}$$

acting on $\beta = \beta(r)$ becomes

$$\Box_\alpha \Box^\alpha \beta(r) = \frac{1}{2}e_E^{-\mu}(\nu_E' - \mu_E')\beta' + \frac{2}{r}e_E^{-\mu}\beta' + e_E^{-\mu}\beta''. \tag{B6}$$

One also has

$$\beta_{,\alpha}\beta^{,\alpha} = e_E^{-\mu}\beta'^2. \tag{B7}$$

Using (B6) and (B7), along with the connection coefficients (B3) in the term

$$\Box_\mu \Box_\nu \beta = \beta_{,\mu\nu} - \Gamma^\alpha{}_{\mu\nu}\beta_{,\alpha}, \tag{B8}$$

the required components of $I_{\mu\nu}$ become

$$I^0{}_0 = -e_E^{-\mu}\left(2\frac{\beta''}{\beta} - \frac{\beta'^2}{\beta^2} - \mu_E'\frac{\beta'}{\beta} + \frac{4}{r}\frac{\beta'}{\beta}\right) \tag{B9}$$

and

$$I^1{}_1 = -e_E^{-\mu}\left(3\frac{\beta'^2}{\beta^2} + \nu_E'\frac{\beta'}{\beta} + \frac{4}{r}\frac{\beta'}{\beta}\right). \tag{B10}$$

To express the Maxwell tensor $E_{\mu\nu}$ in the static metric (B1), begin by writing

$$\kappa_0 = \varphi(r), \quad \kappa_1 = f(r), \quad \kappa_2 = \kappa_3 = 0. \tag{B11}$$

Then, by noting that the gauge transformation $\tilde{\kappa}_\mu = \kappa_\mu + (\ln\sigma)_{,\mu}$ by itself leaves the action (5.2) invariant, one can gauge away κ_1 (so that length integrability exists along radial directions) leaving

$$f_{01} = -f_{10} = -\varphi' \tag{B12}$$

with all other $f_{\mu\nu} = 0$. Integrating the Maxwell equation $(e_E^{(\mu+\nu)/2}r^2 f^{10})_{,1} = 0$ yields the result

$$e_E^{-(\mu+\nu)}\varphi'^2 = \frac{q^2}{r^4}, \tag{B13}$$

where q is an integration constant. Substituting (B12) and (B13) into the Maxwell tensor, one obtains

$$E^0{}_0 = -\frac{1}{2}\frac{q^2}{r^4} = E^1{}_1. \tag{B14}$$

Using the results (B9), (B10), and (B14) in the stress-energy tensor

$$T^E_{\mu\nu} = \frac{1}{2\beta^2}E_{\mu\nu} + I_{\mu\nu} + \frac{1}{2}\lambda g_{\mu\nu}\beta^2 \tag{B15}$$

one finally obtains the required terms

$$(T^0{}_0)^E = -\frac{1}{4\beta^2}\frac{q^2}{r^4} - e_E^{-\mu}\left(2\frac{\beta''}{\beta} - \frac{\beta'^2}{\beta^2} - \mu_E'\frac{\beta'}{\beta} + \frac{4}{r}\frac{\beta'}{\beta}\right) + \frac{1}{2}\lambda\beta^2 \tag{B16}$$

and

$$(T^1{}_1)^E = -\frac{1}{4\beta^2}\frac{q^2}{r^4} - e_E^{-\mu}\left(3\frac{\beta'^2}{\beta^2} + \nu_E'\frac{\beta'}{\beta} + \frac{4}{r}\frac{\beta'}{\beta}\right) + \frac{1}{2}\lambda\beta^2. \tag{B17}$$

APPENDIX C: THE EXTERIOR SOLUTION

A static, spherically symmetric solution to the Einstein field equations is obtained by solving the two equations

$$\frac{d}{dr}(re^{-\mu}) = 1 + r^2 T^0_0 \tag{C1}$$

and

$$\mu' + \nu' = re^{\mu}(T^1_1 - T^0_0). \tag{C2}$$

Substituting the expressions (B16) and (B17) into (C2) and rearranging terms, one finds that

$$\mu_E' + \nu_E' = \frac{d}{dr}\left[2\ln\left(\frac{1}{\beta} + r\frac{\beta'}{\beta^2}\right)\right], \tag{C3}$$

which can then be integrated immediately to yield

$$e_E^{\nu} = e_E^{-\mu}(\ell_0\beta)^{-2}\left(1 + r\frac{\beta'}{\beta}\right)^2. \tag{C4}$$

The integration constant $\ell_0 \equiv \beta_0^{-1}(1 + r\beta'/\beta)|_{r=R}$ has a dimension of length.

Using (B16), Eq. (C1) becomes

$$\frac{d}{dr}(re_E^{-\mu}) = 1 + r^2\left[-\frac{q^2}{4\beta^2 r^4} - e_E^{-\mu}\left(2\frac{\beta''}{\beta} - \frac{\beta'^2}{\beta^2} - \mu_E'\frac{\beta'}{\beta} + \frac{4}{r}\frac{\beta'}{\beta}\right) + \frac{1}{2}\lambda\beta^2\right]. \tag{C5}$$

Multiplying (C5) by e_E^{μ} and introducing the fields $y \equiv re_E^{-\mu}$ and $z \equiv 1 + r\beta'/\beta$ leads to the equation

$$y' + fy = g, \tag{C6}$$

where

$$f \equiv 2\frac{z'}{z} + \frac{1}{r}(z-1) = \frac{d}{dr}\ln\left[\beta\left(1 + r\frac{\beta'}{\beta}\right)^2\right] \tag{C7}$$

and

$$g \equiv \frac{1}{z}\left(1 - \frac{q^2}{4\beta^2 r^2} + \frac{1}{2}\lambda\beta^2 r^2\right). \tag{C8}$$

Equation (C6) is a standard differential equation with solution

$$y = e^{-F}\left(c_1 + \int g(r)e^F dr\right), \tag{C9}$$

where

$$F = \int f(r)dr = \ln\left[c_2\beta\left(1 + r\frac{\beta'}{\beta}\right)^2\right]. \tag{C10}$$

The integral in (C9) also takes on a trivial form:

$$\int ge^F dr = c_2 \int \frac{d}{dr}\left[\beta r + \frac{q^2}{4\beta r} + \frac{1}{6}\lambda\beta^3 r^3\right] dr, \tag{C11}$$

which leads to yet another integration constant c_3. Defining $2m \equiv -(c_1 + c_3)/c_2$ and reexpressing the solution y in terms of $e_E^{-\mu}$ we obtain the final result

$$e_E^{-\mu} = \left(1 + r\frac{\beta'}{\beta}\right)^{-2}\left(1 - \frac{2m}{\beta r} + \frac{q^2}{4\beta^2 r^2} + \frac{1}{6}\lambda\beta^2 r^2\right), \tag{C12}$$

and from (C4),

$$e_E^{\nu} = (\ell_0\beta)^{-2}\left(1 - \frac{2m}{\beta r} + \frac{q^2}{4\beta^2 r^2} + \frac{1}{6}\lambda\beta^2 r^2\right). \tag{C13}$$

APPENDIX D: DECOMPOSITION OF $I_{\mu\nu}$

The decomposition of the Maxwell tensor $E_{\mu\nu}$ into its normal and intrinsic components follows in a straightforward manner by application of (3.1). To decompose $I_{\mu\nu}$, one must first determine the relationships

$$h^{\alpha}{}_{\mu}h_{\beta}{}^{\nu}\Box^{\mu}\Box_{\nu}\beta = D^{\alpha}D_{\beta}\beta - K^{\alpha}{}_{\beta}\beta_{,n}, \tag{D1}$$

$$n_{\mu}h_{\alpha}{}^{\nu}\Box^{\mu}\Box_{\nu}\beta = h_{\alpha}{}^{\mu}\beta_{,\mu n} + K_{\alpha\mu}\beta^{,\mu}, \tag{D2}$$

$$n_{\mu}n^{\nu}\Box^{\mu}\Box_{\nu}\beta = n_{\mu}n^{\nu}\beta^{,\mu}{}_{,\nu}, \tag{D3}$$

and

$$\Box_{\mu}\Box^{\mu}\beta = D_{\mu}D^{\mu}\beta + n_{\mu}n^{\nu}\beta^{,\mu}{}_{,\nu} - K\beta_{,n}, \tag{D4}$$

where the definition (3.4) has been used extensively. Employing (D1)–(D4) in the definition of $I_{\mu\nu}$, Eq. (5.6), one obtains

$$n_{\mu}n^{\nu}I^{\mu}{}_{\nu} = \frac{2}{\beta}(K\beta_{,n} - D_{\mu}D^{\mu}\beta) - \frac{1}{\beta^2}(3n_{\mu}n^{\nu} - h_{\mu}{}^{\nu})\beta^{,\mu}\beta_{,\nu}, \tag{D5}$$

$$n_{\mu}h_{\alpha}{}^{\nu}I^{\mu}{}_{\nu} = \frac{2}{\beta}(h_{\alpha}{}^{\mu}\beta_{,\mu n} + K_{\alpha\mu}\beta^{,\mu}) - \frac{4}{\beta^2}n_{\mu}h_{\alpha}{}^{\nu}\beta^{,\mu}\beta_{,\nu}, \tag{D6}$$

$$h^{\alpha}{}_{\mu}h_{\beta}{}^{\nu}I^{\mu}{}_{\nu} = {}^3I^{\alpha}{}_{\beta} - \frac{2}{\beta}(K^{\alpha}{}_{\beta} - h^{\alpha}{}_{\beta}K)\beta_{,n} - n^{\mu}n^{\nu}h^{\alpha}{}_{\beta}\left(\frac{2}{\beta}\beta_{,\mu\nu} - \frac{1}{\beta^2}\beta_{,\mu}\beta_{,\nu}\right). \tag{D7}$$

[1] H. Weyl, Sitzungsber. Preuss. Akad. Wiss. **465** (1918).
[2] P. A. M. Dirac, Proc. R. Soc. London **A333**, 403 (1973). Referring to quantum phenomena, Dirac introduced the expression "absolute standard of length" in this paper. However, the term "absolute" is somewhat of a misnomer since a measurement process is inherently a relative process.
[3] A. Einstein, Sitzungsberg. Preuss. Akad. Wiss. **478** (1918).
[4] F. London, Z. Phys. **42**, 375 (1927).
[5] E. R. Caianiello, Lett. Nuovo Cimento **25**, 225 (1979).
[6] D. Gregorash and G. Papini, Nuovo Cimento **B63**, 487 (1981).
[7] J. T. Wheeler, Phys. Rev. D **41**, 431 (1990).
[8] R. W. Hellings, P. J. Adams, J. D. Anderson, M. S. Keesey, E. L. Lau, E. M. Standish, V. M. Canuto, and I. Goldman, Phys. Rev. Lett. **51**, 1609 (1983).
[9] E. Santamato, Phys. Rev. D **29**, 216 (1984); C. Castro, University of Texas at Austin Report No. DOE-ER40200-111, 1977 (unpublished).
[10] G. Papini and W. R. Wood, in Proceedings of the Conference on Structure: from Physics to General Systems, 1991, edited by G. Scarpetta (unpublished).
[11] G. Papini, in *High-Energy Physics*, edited by S. L. Mintz and A. Perlmutter (Plenum, New York, 1985), pp. 179–198.
[12] P. A. M. Dirac, Proc. R. Soc. London **A268**, 57 (1962).
[13] P. A. M. Dirac, in *Conférence Internationale sur les Théories Relativistes de la Gravitation,* Warszawa and Jablonna, Poland, 1962, edited by L. Infeld (Gauthier-Villars, Paris, 1964).
[14] W. Israel, Nuovo Cimento **B44**, 1 (1966); **B48**, 463 (1967).
[15] K. Kuchař, Czech. J. Phys. **B18**, 435 (1968).
[16] S. K. Blau, E. I. Guendelman, and A. H. Guth, Phys. Rev. D **35**, 1747 (1987).
[17] V. A. Berezin, V. A. Kuzmin, and I. I. Tkachev, Phys. Rev. D **36**, 2919 (1987).
[18] C. Barrabès and W. Israel, Phys. Rev. D **43**, 1129 (1991).
[19] C. W. Misner, K. S. Thorne, and J. A. Wheeler, *Gravitation* (Freeman, San Francisco, 1973).
[20] H. Cheng, Phys. Rev. Lett. **61**, 2182 (1988).
[21] J. L. Synge, *Relativity: The General Theory* (North-Holland, Amsterdam, 1960).
[22] C. Lanczos, Rev. Mod. Phys. **29**, 337 (1957).
[23] C. A. López, Phys. Rev. D **38**, 3662 (1988).
[24] R. M. Wald, *General Relativity* (University of Chicago Press, Chicago, 1984).
[25] G. Papini, in *Fifth Marcel Grossman Meeting,* Proceedings, Perth, Australia, 1988, edited by D. G. Blair and M. J. Buckingham (World Scientific, Singapore, 1989), pp. 787–796.