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Quantum fluctuations on domain walls, strings, and vacuum bubbles

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We develop a covariant quantum theory of fluctuations on vacuum domain walls and strings. The fluctuations are described by a scalar field defined on the classical world sheet of the defects. We consider the following cases: straight strings and planar walls in flat space, true vacuum bubbles nucleating in false vacuum, and strings and walls nucleating during inflation. The quantum state for the perturbations is constructed so that it respects the original symmetries of the classical solution. In particular, for the case of vacuum bubbles and nucleating strings and walls, the geometry of the world sheet is that of a lower-dimensional de Sitter space, and the problem reduces to the quantization of a scalar field of tachyonic mass in de Sitter space. In all cases, the root-mean-squared fluctuation is evaluated in detail, and the physical implications are briefly discussed.

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I. INTRODUCTION

Phase transitions in the early Universe may have generated nontrivial topologically stable field configurations (topological defects) such as domain walls and cosmic strings [1]. Of particular interest is the process of false vacuum decay through bubble nucleation, first studied by Voloshin *et al.* [2] and later described by Coleman in the language of instantons [3]. In this process, domain walls appear at the boundaries of the true vacuum bubbles and, similarly, strings may appear at the boundaries of circular holes nucleating in metastable domain walls. The instanton techniques have also been used recently to show that spherical domain walls and circular loops of string can nucleate spontaneously during an inflationary period in the early Universe [4]. The physical properties of topological defects have been extensively studied during the last decade and are now reasonably well understood.

In most of the research on strings and walls they have been treated as classical objects. Although this treatment is usually justified, quantum fluctuations on strings and walls can be important in a number of cosmological applications. For example, closed loops of string formed during inflation would all eventually collapse to black holes if they remained exactly circular. Quantum fluctuations cause some deviations from circular shape and thus determine the probability of black-hole formation. For a string network in de Sitter space, quantum fluctuations can lead to a rapid growth of string energy, suggesting the possibility of a self-consistent solution in which inflation is driven by the network [5]. Another interesting problem is the evolution of expanding vacuum bubbles. The question is how nonspherical the bubble walls can become due to quantum fluctuations.

The purpose of this paper is to develop a quantum theory of fluctuations on strings and walls. The corresponding classical theory of small perturbations has been presented in a previous paper [6]. There, we showed that perturbations on strings and walls can be described by a scalar field ϕ defined on the unperturbed world sheet. We shall see that the quantization of this field is not always straightforward since, in some interesting cases, the two-point function for ϕ is ill defined in the quantum state that respects the original symmetries of the classical world sheet. This happens, for instance, in the case of an infinite straight string in flat space. We shall see that the two-point function for ϕ is infrared divergent in the quantum state that respects the longitudinal Poincaré invariance of the string and that, as a result of long-distance correlations in the quantum fluctuations, the string will bend on very large scales.

Also, it is well known [3] that the classical solution describing a vacuum bubble after nucleation has the property of O(3,1) invariance, and therefore a vacuum bubble should look the same to all inertial observers. The classical world sheet describing the wall of the bubble has the internal geometry of (2+1)-dimensional de Sitter space, and fluctuations are represented by a scalar field of tachyonic mass $m^2=-3H^2$ living on the world sheet [6] (here H^{-1} is the radius of the bubble at nucleation). We shall see that it is possible to find an O(3,1)-invariant quantum state for this field, although there will be infrared divergences in the two-point function for ϕ due to the translational "zero" modes of the bubble. For the case of strings and walls nucleating during inflation, the internal geometry of the world sheet is also that of de Sitter space, and the quantization follows the same pattern as in the case of the vacuum bubble.

The rest of this paper is organized as follows. The basic formalism is described in Sec. II where we review the classical theory of perturbations on strings and walls and outline the quantization procedure. In Sec. III this formalism is applied to planar walls and straight strings, and in Sec. IV to expanding vacuum bubbles. Until Sec. V it is assumed that the thickness of strings and walls is small compared to all other relevant dimensions, so that they can be treated as infinitely thin lines and sheets. Generalization to thick walls and strings is discussed in Sec. V, using the formalism developed by Vachaspati and Vilenkin [7]. Quantum fluctuations on strings and walls in de Sitter space are studied in Sec. VI (again in the thin-string approximation) and our conclusions are summarized in Sec. VII. Some technical issues are dealt with in the Appendixes.

II. BASIC FORMALISM

Let us briefly recall, in this section, the classical theory of perturbations on domain walls and strings which was developed in Ref. [6], with the idea of setting up the notation for the rest of the paper. Here we shall also outline the procedure for quantizing the perturbations.

Following Ref. [6], we start by considering an infinitely thin domain wall evolving in $(N+1)$-dimensional Minkowski spacetime. During its time evolution, the wall sweeps out an N-dimensional timelike hypersurface (world sheet) Σ, which is parametrized by coordinates ξ^a ($a=0,\ldots,N-1$). The position of the world sheet is specified by the functions $x^\mu = x^\mu(\xi^a)$ and the metric on the world sheet is

$$g_{ab} = \partial_a x^\mu \partial_b x_\mu \ . \tag{1}$$

In general, a domain wall may separate two regions of space with different values of the vacuum energy density, say $\rho_v^{(1)}$ on one side of the wall and $\rho_v^{(2)}$ on the other side. Its dynamics will be given by the action

$$S = -\sigma \int_\Sigma \sqrt{-g}\, d^N\xi - \epsilon \int dt \int_{\text{Vol}} d^N x \ , \tag{2}$$

where $\epsilon = \rho_v^{(2)} - \rho_v^{(1)}$. The first term in the Nambu action, proportional to the area of the world sheet, σ is the surface tension of the wall, and g is the determinant of the metric induced on the world sheet. The second term accounts for the contribution of the vacuum Lagrangian, which is just a constant potential energy density. The $d^N x$ integral extends over the region occupied by the vacuum with $\rho_v = \rho_v^{(2)}$. The equations of motion resulting from (2) are [6]

$$g^{ab} K_{ab} = \frac{-\epsilon}{\sigma} \ , \tag{3}$$

where $K_{ab} = -\partial_a n_\mu \partial_b x^\mu$ is the extrinsic curvature of Σ and n^μ is the unit vector normal to Σ, pointing towards the region with $\rho_v = \rho_v^{(2)}$.

The classical theory of perturbations on a given solution x^μ of Eq. (3) can be summarized as follows. Noting that only motion perpendicular to the surface of the wall is physically observable, the perturbed solution $\tilde{x}^\mu$ can be written as

$$\tilde{x}^\mu = x^\mu + \phi n^\mu \ , \tag{4}$$

where $\phi = \phi(\xi^a)$ represents the proper magnitude of the perturbation (i.e., the perturbation as measured by a comoving observer "sitting" on the unperturbed wall). In Ref. [6] we showed that the linearized perturbations satisfy the "field equation"

$$-\Box\phi + \left[\mathcal{R} - \frac{\epsilon^2}{\sigma^2}\right]\phi = 0 \ , \tag{5}$$

where $\Box = g^{ab}\nabla_a\nabla_b$ stands for the covariant d'Alembertian in the curved geometry of the worldsheet (∇_a are covariant derivatives associated with the metric g_{ab}), and $\mathcal{R}$ is the (intrinsic) Ricci scalar on Σ. The theory of perturbations is thus formally equivalent to the theory of a scalar field living in the unperturbed world sheet. This field has a tachyonic mass $m^2 = -\epsilon^2/\sigma^2$ (or zero mass if $\epsilon = 0$) and a direct coupling to the curvature of the standard form $\xi\mathcal{R}\phi$, with $\xi = 1$ (see, e.g., Ref. [8]).

Equation (5) was derived in Ref. [6] by imposing that the perturbed solution (4) should satisfy the equations of motion (3) and working to linear order in ϕ. Also, one can proceed at the Lagrangian level, substituting (4) into the action (2) and then expanding to quadratic order in ϕ. Upon so doing one can show that the action takes the form $S = S_0 + S_\phi$, where S_0 is the action for the unperturbed solution and

$$S_\phi = -\frac{\sigma}{2}\int \sqrt{-g}\left[\phi_{,a}\phi^{,a} + \left[\mathcal{R} - \frac{\epsilon^2}{\sigma^2}\right]\phi^2\right] d^N\xi \ . \tag{6}$$

In deriving (6), boundary terms have been dropped, along with terms of order higher than quadratic in ϕ. (Of course linear terms do not appear because the unperturbed solution is an extremum of the action). One can recognize (6) as the action for a scalar field in a curved background, from which Eq. (5) follows.

Recall that the above formalism applies also to strings moving in $(2+1)$-dimensional Minkowski space (and, in general, to topological defects of codimension 1 in a flat space of arbitrary dimension). Strings in four-dimensional spacetime can be treated along similar lines. In this case there are two normal vectors orthogonal to the world sheet n_A^μ ($A=1,2$), with $n_A^\mu n_{B\mu} = \delta_{AB}$, and the perturbed world sheet can be written as

$$\tilde{x}^\mu = x^\mu + \phi^A n_A^\mu \tag{7}$$

[compare with (4)]. The picture is, however, more complicated than in the case of domain walls because, in general, the fields ϕ^A may be coupled to each other. In Secs. III and VI we will consider some physically interesting examples of strings in flat space as well as in curved space in which the fields ϕ^A are decoupled from each other. In these particular cases we are back to a situation similar to that of domain walls, where we had only one field ϕ.

The quantum theory of perturbations on strings and walls can now be developed using the standard methods of quantum field theory in curved spacetime [8]. The field ϕ [or ϕ^A, for strings in $(3+1)$ dimensions] will be treated as an operator which can be expressed in terms of creation and annihilation operators,

$$\phi(\xi) = \sigma^{-1/2}\sum_n [a_n\phi_n(\xi) + a_n^\dagger\phi_n^*(\xi)] \ , \tag{8}$$

satisfying the usual commutation relations $[a_n, a_{n'}^\dagger] = \delta_{nn'}$. The factor $\sigma^{-1/2}$ can be understood from Eq. (6): the perturbation ϕ has dimensions of length and the object with the usual dimensions of a scalar field is $\sigma^{1/2}\phi$. The mode functions $\phi_n(\xi)$ are normalized so that

$$(\phi_n, \phi_{n'}) = \delta_{nn'} \ ,$$

where

$$(\phi_n, \phi_{n'}) = -i\int_S \phi_n \overleftrightarrow{\partial}_a \phi_{n'}^* dS^a$$

is the Klein-Gordon scalar product and the integration is over a spacelike hypersurface on the world sheet. The vacuum state $|0\rangle$ of the field ϕ is now defined by

$$a_n|0\rangle = 0 ,$$

and quantum fluctuations of the world sheet in this state will be characterized by the two-point function

$$G(\xi',\xi'') = \langle 0|\phi(\xi')\phi(\xi'')|0\rangle = \sigma^{-1}\sum_n \phi_n(\xi')\phi_n^*(\xi'') . \tag{9}$$

It should be mentioned that the definition of the vacuum state in curved spacetime is notoriously ambiguous: it depends on the choice of "positive frequency" modes ϕ_n. No "natural" prescription for the choice can be given in the general case and, as we shall see, the quantum state of ϕ will have to be determined in each case by the physics of the problem.

III. PLANAR WALLS AND STRAIGHT STRINGS IN FLAT SPACE

In this section we apply our formalism to the simple cases of a planar domain wall and an infinite straight string in flat space. We shall see that, for the case of a straight string, a nontrivial effect arises due to long-distance correlations in the quantum fluctuations.

Let us start with the case of the planar wall. The worldsheet metric in this case is the flat metric. From (6) with $\mathcal{R}=\epsilon=0$, we have

$$S_\phi = \frac{-\sigma}{2}(\partial_a\phi\partial^a\phi)d^3\xi , \tag{10}$$

which is the action for a massless scalar field in (2+1)-dimensional Minkowski space. The natural vacuum state is thus the usual one associated with the set of positive-frequency modes

$$\phi_k = (2\pi)^{-1}(2\omega)^{-1/2}e^{i\mathbf{k}\cdot\mathbf{x}-i\omega t} , \tag{11}$$

where $\omega=|\mathbf{k}|,\mathbf{k}=(k_x,k_y)$, and we have parametrized the world sheet with coordinates $\xi^a=(t,x,y)$. This is in fact the only state that respects the tangential Poincaré invariance of the planar domain wall.

The Poincaré invariance is manifest in the two-point function

$$\langle\phi(x'')\phi(x')\rangle = \frac{\sigma^{-1}}{(2\pi)^2}\int\frac{d^2k}{2\omega}\exp[ik(x''-x') - i\omega(t''-t')] = \frac{\sigma^{-1}}{4\pi\delta^{1/2}} , \tag{12}$$

which only depends on the interval $\delta=(\mathbf{x}''-\mathbf{x}')^2-(t''-t')^2$. Note that correlations in the perturbations at two different points decay with the distance.

Also, it is clear that $\langle\phi^2(x)\rangle=G(x,x)$ does not exist because (12) becomes infinite in the coincidence limit. As is well known [9], $\phi(x)$ is not a well behaved operator, but it is only an operator-valued distribution. Mathematically, this means that given a "smearing function" $f(x)$ one can construct an operator

$$\phi_f = \int f(y)\phi(\mathbf{y},t)d^2\mathbf{y} ,$$

which *is* well behaved. Physically, it means that we cannot measure the field at one point, but we can measure its value smeared over a certain region. In particular, taking f to be constant over a region of radius s centered at some point $\mathbf{x}$ and zero outside, the smeared field operator is

$$\phi_s(\mathbf{x},t) = \frac{1}{\pi s^2}\int_{|\mathbf{y}-\mathbf{x}|<s} d^2\mathbf{y}\,\phi(\mathbf{y},t) . \tag{13}$$

Usually, in quantum field theory, the problem of infinities is handled by renormalization, since one is seldom interested in the value of the field itself, but here ϕ represents the amplitude of the perturbations (the physical quantities that we are interested in) and we shall use the smeared field operator ϕ_s whenever it is needed.

We can calculate, for instance, the expectation value of the smeared field squared

$$\langle\phi_s^2(x)\rangle = \frac{1}{\pi^2 s^4}\int_{|\mathbf{y}-\mathbf{x}|<s} d^2\mathbf{y}\int_{|\mathbf{z}-\mathbf{x}|<s} d^2\mathbf{z}\langle\phi(y)\phi(z)\rangle . \tag{14}$$

Inserting in (14) the equal time correlation function (12) we find, after some algebra,

$$\langle\phi_s^2\rangle = (\tfrac{1}{2}+\mathcal{G})\frac{\sigma^{-1}}{s} , \tag{15}$$

where $\mathcal{G}=0.916\ldots$ is Catalan's constant [10]. Of course $\langle\phi_s^2\rangle$ depends on s and it goes to infinity as $s\to 0$. However, on physical grounds, we should not make the smearing distance arbitrarily small. Just as a point particle cannot be localized within a distance smaller than its Compton wavelength m^{-1}, there is also a natural limit on how small s can be. Demanding that s be larger than or equal to the Compton wavelength of a fragment of wall contained within a region of radius s, we have

$$s \gtrsim \sigma^{-1/3} . \tag{16}$$

A physical quantity of some interest to us is the "distortion" of the wall, $D(x,y)$, defined by

$$D^2(x,y) \equiv \langle(\phi_s(x)-\phi_s(y))^2\rangle = 2\langle\phi_s^2\rangle - 2\langle\phi_s(x)\phi_s(y)\rangle , \tag{17}$$

where the vacuum expectation values are taken at equal times. This characterizes the mean squared difference in transverse position at points x and y. The first term in the r.h.s. of (17) can be interpreted as the quantum fluctuations at the individual points x and y, while the second term represents the correlations between them. If the distance $|\mathbf{x}-\mathbf{y}|$ is much larger than s, then $\langle\phi_s(x)\phi_s(y)\rangle\approx\langle\phi(x)\phi(y)\rangle$. Since, from (12), the correlations decay with distance, we have that in the limit $|\mathbf{x}-\mathbf{y}|\gg s$

$$D^2(x,y) \approx 2\langle\phi_s^2\rangle = (1+2\mathcal{G})\frac{\sigma^{-1}}{s} = \text{const} . \tag{18}$$

The overall picture is, in summary, that we have

Poincaré-invariant quantum fluctuations on the domain wall, each small region undergoing quantum fluctuations of its own (as demanded by the uncertainty principle), but with perturbations being uncorrelated over large distances.

Let us now consider perturbations on a straight string. The situation in this case will be quite different (because of the dimensionality of the world sheet). Denoting by ϕ^A ($A=1,2$) the normal displacements in the two directions orthogonal to the string, the action for the perturbations is [6]

$$S_\phi = -\frac{\mu}{2}\int (\partial_a \phi^A \partial^a \phi^A) dx\, dt\ , \tag{19}$$

where we are using t and x as parameters on the world sheet, so now we have two uncoupled massless scalar fields in $(1+1)$-dimensional flat space.

As is well known (see, e.g., [11,12]), for a massless scalar field in $(1+1)$-dimensional Minkowski space, the Lorentz-invariant vacuum state based on the choice of modes

$$\phi_k = (2\pi)^{-1/2}(2\omega)^{-1/2}\exp(ikx - i|k|t)\ , \tag{20}$$

has an infrared-divergent two-point function,

$$\langle \phi(x'',t'')\phi(x',t')\rangle = \frac{1}{2\pi\mu}\int \frac{dk}{2|k|}\exp[ik(x''-x') - i|k|(t''-t')]\ . \tag{21}$$

(We have dropped the index A in ϕ^A). In order to obtain a finite two-point function one would have to use a different set of mode functions, but then the quantum state would not respect the Lorentz invariance of the world sheet.

However, using the same positive frequency modes as in (21), one can easily compute the quantity

$$\langle \phi(x'',t'')\phi(x',t') - \phi(y'',\tau'')\phi(y',\tau')\rangle = \frac{1}{4\pi\mu}\ln\frac{(y''-y')^2-(\tau''-\tau')^2}{(x''-x')^2-(t''-t')^2}\ , \tag{22}$$

which not only has a nice Lorentz invariant structure, but is also free from infrared divergences. Moreover, Eq. (22) is all that is needed to compute the "distortion" $D(x,y)$ introduced in Eq. (17). After some algebra we find that, in the limit $|x-y| \gg s$ (equal time),

$$D^2(x,y) \approx \frac{1}{2\pi\mu}\left[3 + 2\ln\left|\frac{x-y}{2s}\right|\right]\ . \tag{23}$$

Note that, unlike (18), the distortion (23) grows without bound as $|x-y| \to \infty$. Unlike the wall, the straight string develops unbounded distortions on very large scales, caused by long-distance correlations of the quantum perturbations. One can take the point of view that the state defined by the mode functions (20) is an acceptable physical state, and that the divergence of the two-point function simply reflects the fact that at very large distances the string deviates arbitrarily far from its unperturbed position.

An alternative approach is to insist that the vacuum state should be such that the two-point function $\langle 0|\phi(x)\phi(y)0\rangle$ is free from infrared divergences. In constructing such state one must necessarily break Lorentz invariance explicitly. For instance, in Ref. [12], a vacuum was constructed essentially by taking the usual modes $\phi_k \propto \exp(ikx - i\omega t)$ for $|k| > L^{-1}$, but changing the modes with $k < L^{-1}$ so that the integral (21) would be convergent. Here L is an arbitrarily introduced length scale. Alternatively, one can impose periodic boundary conditions on the string, again introducing a periodicity scale L which breaks Lorentz invariance, or one can consider a finite segment of string of length L with fixed end points. Using either of these three methods one reaches the same conclusion: if the "cutoff" scale L is much larger than the separation $|x-y|$ (and if, in the case of a finite segment, x and y are far from the end points of the string), then the results agree with Eqs. (22) and (23).

Finally, we should mention that the unboundedness of (23) has no cosmological implications, because the dependence of the distortion on the distance is only logarithmic. Even if $|x-y| \sim 10^{28}$ cm (the radius of our observable universe), we have $D(x,y) < 10\mu^{-1/2}$, which is certainly negligible for cosmological purposes.

IV. TRUE VACUUM BUBBLES

False vacuum decay may proceed through quantum nucleation of spherical bubbles of true vacuum in a "sea" of false vacuum [2,3]. After nucleation, the trajectory of a spherical domain wall separating the true vacuum phase from the false vacuum is

$$R^2 - t^2 = H^{-2}\ , \tag{24}$$

where R is the radius of the bubble and $H = \epsilon/N\sigma$ [we are considering bubbles nucleating in $(N+1)$-dimensional Minkowski space, with ϵ and σ as defined in Sec. II]. The bubble nucleates at $t=0$ with radius H^{-1} and then it starts expanding.

The solution (24) satisfies the energy conservation equation

$$E = \frac{R^{N-1}S^{(N-1)}\sigma}{(1-\dot{R}^2)^{1/2}} - R^N V^{(N-1)}\epsilon = 0\ , \tag{25}$$

where $S^{(N-1)}$ and $V^{(N-1)}$ are the surface and the volume of the unit $(N-1)$-sphere, respectively. The first term is the energy of the domain wall, while the second is the energy removed from the false vacuum by cutting out of it a spherical piece of radius R. Using $S^{(N-1)}/V^{(N-1)} = N$, Eq. (25) reads $R(1-\dot{R}^2)^{1/2} = 0$, which is readily satisfied by (24). The spherical symmetry of the bubble also guarantees that its three-momentum is zero, so the total four-momentum of the bubble vanishes

$$p^\mu = 0\ .$$

This is a consequence of the Lorentz invariance of (24), according to which the bubble looks the same in all inertial frames.

Equation (24) represents a hyperboloid embedded in $(N+1)$-dimensional Minkowski space, and therefore the metric induced on the world sheet is that of N-

dimensional de Sitter space, which can be written in the form

$$ds_{\Sigma}^2 = \frac{1}{H^2\cos^2\tilde{t}}(-d\tilde{t}^2 + d\Omega_{(N-1)}^2) . \quad (26)$$

Here $d\Omega_{(N-1)}^2$ is the line element on the $(N-1)$-sphere, and the timelike coordinate $\tilde{t}(-\pi/2<\tilde{t}<\pi/2)$, can be defined through the relation

$$R(t) = \frac{1}{H\cos\tilde{t}} , \quad (27)$$

where $R(t)$ is given by (24).

As we discussed in Sec. II, the perturbations to the solution (24) are described by a scalar field ϕ which has the meaning of a normal displacement of the world sheet. The scalar field ϕ satisfies the Klein-Gordon equation in N-dimensional de Sitter space [6]

$$-\Box\phi + m^2\phi = 0 , \quad (28)$$

where $\Box$ is the covariant d'Alembertian and

$$m^2 = -NH^2 . \quad (29)$$

It will be also convenient, for later discussion, to introduce a noncovariant parametrization in which the trajectory of the perturbed bubble is given by [13]

$$r(\Omega,t) = R(t) + \Delta(\Omega,t) . \quad (30)$$

Here r is the distance to the origin of coordinates, Ω is a set of angles on the sphere, $R(t)$ is the unperturbed radius (24) and $\Delta(\Omega,t)$ is the perturbation at time t as measured by an "external" observer at rest. The perturbation Δ is related to the proper perturbation ϕ by a Lorentz contraction factor [6]

$$\Delta = \sqrt{1-\dot{R}^2}\phi . \quad (31)$$

From (24), we have

$$\Delta = \cos\tilde{t}\phi . \quad (32)$$

In Ref. [6] we studied the classical solutions to Eq. (28) and their interpretation. Let us now address the quantum theory of perturbations.

The first step is to choose a vacuum state for the perturbation field ϕ, based on the symmetries of the classical world sheet. We note that the N-dimensional de Sitter group $SO(N,1)$ coincides with the group of $(N+1)$-dimensional Lorentz transformations, and therefore the de Sitter-invariant state is the only state that does not break the Lorentz invariance of the bubble trajectory (24). There is also another reason for choosing a de Sitter invariant vacuum, based on the dynamics of bubble nucleation. The quantum state of a scalar field interacting with a nucleating vacuum bubble has been studied by Vachaspati and Vilenkin [7]. They consider a four-dimensional scalar field and in this respect their problem is different from ours. However, the two problems are closely related. A scalar field in four dimensions can be represented as an infinite set of scalar fields in a three-dimensional de Sitter space (that is, on the unperturbed world sheet of the bubble wall). By solving the Schrödinger equation for the system of bubble plus the scalar field, it can be shown [7] that the bubble nucleates with all these fields in de Sitter-invariant quantum states. This suggests that the field ϕ describing the perturbations of the nucleating bubble should also be in a de Sitter-invariant state. We shall see, however, that for the particular value of the mass that we have found, Eq. (29), the construction of a de Sitter-invariant state encounters an unexpected difficulty.

Let us briefly review the standard quantization of a scalar field of mass m in de Sitter space. The mode functions can be separated as

$$\phi_{LM} = \varphi_L(\tilde{t})Y_{LM}(\Omega) , \quad (33)$$

where Y_{LM} are harmonics on the N-sphere. They satisfy $\tilde{\Delta}Y_{LM} = -JY_{LM}$, where $\tilde{\Delta}$ is the Laplacian on the unit $(N-1)$-sphere and the eigenvalues are given by $J = L(L+N-2)$, with $L = 0,1,\ldots,\infty$. The field equation reduces to

$$\ddot{\varphi}_L + (N-2)\tan\tilde{t}\dot{\varphi}_L + \left[J + \frac{m^2}{H\cos^2\tilde{t}}\right]\varphi_L = 0 , \quad (34)$$

where here, and for the rest of this section, a dot indicates derivative with respect to $\tilde{t}$. Of course, this equation has two independent solutions.

The requirement of de Sitter invariance specifies the quantum state uniquely [14,15]. The corresponding mode solutions are given by [16]

$$\varphi_L = A_L(\cos\tilde{t})^{(N-1)/2}R_\nu^\lambda(\sin\tilde{t}) , \quad (35)$$

where we have introduced the combination of Legendre functions $R_\nu^\lambda = P_\nu^\lambda - (2i/\pi)Q_\nu^\lambda$, with

$$\lambda = \left[\frac{(N-1)^2}{4} - \frac{m^2}{H^2}\right]^{1/2} , \quad \nu = L + \frac{N-3}{2} . \quad (36)$$

The normalization constant is given by

$$A_L = \frac{\sqrt{\pi}}{2}H^{(N-2)/2}e^{i\lambda\pi/2}\left[\frac{\Gamma\left[L+\frac{N-1}{2}-\lambda\right]}{\Gamma\left[L+\frac{N-1}{2}+\lambda\right]}\right]^{1/2} . \quad (37)$$

The two-point function for this state is [17,18]

$$G_{m^2}(\xi,\xi') = \frac{H^{N-2}}{(4\pi)^{N/2}\sigma}\frac{\Gamma\left[\frac{N-1}{2}-\lambda\right]\Gamma\left[\frac{N-1}{2}+\lambda\right]}{\Gamma(N/2)} \times F\left[\frac{N-1}{2}+\lambda, \frac{N-1}{2}-\lambda; \frac{N}{2}; \frac{1+Z}{2}\right] , \quad (38)$$

where F is the hypergeometric function and Z is the scalar product of the position vectors at points ξ and ξ' (see, e.g., Ref. [17]),

$$Z(\xi,\xi') \equiv H^2 x^\mu(\xi) x_\mu(\xi') = \frac{-\sin\tilde{t}\sin\tilde{t}' + \cos\gamma}{\cos\tilde{t}\cos\tilde{t}'} . \quad (39)$$

Here γ is the angle between the two points on the bubble. Notice that the two-point function only depends on Z, which is a Lorentz invariant quantity in the embedding space.

We note that this two-point function is well defined both for $m^2 > 0$ and $m^2 < 0$. However it is divergent for a discrete set of masses

$$m^2 = -s(s+N-1)H^2, \quad s = 0, 1, \ldots \quad (40)$$

which make the argument of $\Gamma((N-1)/2 - \lambda)$ a negative integer or zero. For $s=1$ this gives $m^2 = -NH^2$, which is the mass squared of our field ϕ. Mathematically, the special status of the masses (40) is due to the fact that they are eigenvalues of the Laplacian on the N-sphere (the Euclideanized N-dimensional de Sitter space is an N-sphere of radius H^{-1}). The origin of the divergence in the two-point function becomes particularly clear when it is presented as an expansion in harmonics on the N-sphere (see, for example, Ref. [20]). In Appendix A we derive this representation using the Euclidean path integral approach. Let us emphasize that the special value (29) that we have found for the mass is not an artifact of the thin wall approximation. In Sec. V we shall see that in the general case, one of the three-dimensional fields representing perturbations of the bubble has exactly the mass (29). One could try to obtain a finite two-point function by taking the limit of G_{m^2} as $m^2 \to NH^2$, as it was done in Refs. [21,22] for the case of quantum perturbations around a soliton. However, in our case this gives an anomalous two-point function which is not a solution of the scalar field equation. This point is also explained in detail in Appendix A.

The origin of the difficulties can be understood in terms of the mode functions (35). Notice that the coefficients A_L with $L=0,1$ diverge as $m^2 \to -NH^2$. The reason is that for $m^2 = -NH^2$ and $L=0,1$ the mode functions R_ν^λ become real and their Klein-Gordon norm goes to zero. More specifically, these functions are given by

$$R_0 = B_0 \frac{\sin\tilde{t}}{(\cos\tilde{t})^{(N+1)/2}} ,$$
$$R_1 = B_1 \frac{1}{(\cos\tilde{t})^{(N+1)/2}} , \quad (41)$$

where R_0 and R_1 denote the functions R_ν^λ corresponding to the critical value of the mass [i.e., $\lambda = (N+1)/2$] for $L=0$ and $L=1$ respectively, and B_0, B_1 are constants, with $|B_0| = |B_1| = \pi^{-1} 2^{(N+1)/2} \Gamma((N+1)/2)$.

The resolution of the difficulties comes from the observation, made in Ref. [6], that the modes with $L=0,1$ correspond to space and time translations of the bubble. Indeed, from (32)–(35) and (41), we have

$$\Delta_{00} \equiv \cos\tilde{t}\,\phi_{00} \propto \sin\tilde{t} ,$$
$$\Delta_{1M} \equiv \cos\tilde{t}\,\phi_{1M} \propto Y_{1M}(\Omega) . \quad (42)$$

Note that Δ_{1M} is proportional to the spherical harmonics with $L=1$, which correspond to small spatial translations of the bubble. That Δ_{00} corresponds to a time translation can be seen by considering a small temporal shift, q^0, to the solution (24). We have

$$r = [(t+q^0) + H^2]^{1/2} \approx R(t) + q^0 \frac{t}{R(t)}$$
$$= R + q^0 \sin\tilde{t} , \quad (43)$$

so $\Delta = q^0 \sin\tilde{t}$. Therefore these modes do not represent excitations on the bubble, but rather correspond to going from one unperturbed bubble to another. The infinities in the normalization constants arise because the expansion of ϕ in terms of creation and annihilation operators becomes inadequate for the degrees of freedom corresponding to the zero modes.

In Appendix B we argue that a more suitable expansion is given by

$$\phi = \hat{\phi} + \sum_\mu [q^\mu \mathcal{Z}_\mu(\tilde{t}) + p^\mu \mathcal{N}_\mu(\tilde{t})] \mathcal{Y}_\mu(\Omega) , \quad (44)$$

where $\hat{\phi}$ is the usual expansion in terms of creation and annihilation operators for $L > 1$

$$\hat{\phi} = \sigma^{-1/2} \sum_{L>1} \sum_M (a_{LM} \phi_{LM} + a_{LM}^\dagger \phi_{LM}^*) . \quad (45)$$

In Eq. (44) the index μ ($\mu = 0, 1, \ldots, N$) runs over all translational zero modes. The functions $\mathcal{Z}_\mu$ are given by

$$\mathcal{Z}_0 = \tan\tilde{t} ,$$
$$\mathcal{Z}_i = \sec\tilde{t}, \quad (i = 1, \ldots, N) \quad (46)$$

and they correspond to small temporal and spatial shifts of the center of the hyperboloid, respectively. The "conjugate" modes $\mathcal{N}_\mu$ are independent solutions of Eq. (34), which can be chosen to satisfy the Wronskian condition

$$\mathcal{Z}_\mu \dot{\mathcal{N}}_\mu - \dot{\mathcal{Z}}_\mu \mathcal{N}_\mu = (H\cos\tilde{t})^{N-2} / \sigma S^{(N-1)} ,$$

(no summation over μ) where a dot indicates derivative with respect to $\tilde{t}$. The functions $\mathcal{Y}_\mu(\Omega)$ are real and properly normalized combinations of the spherical harmonics on the $(N-1)$-sphere. We take $\mathcal{Y}_0(\Omega) = Y_{00}$, and $\mathcal{Y}_i(\Omega)$ can be chosen as the Cartesian combinations of the spherical harmonics with $L=1$. The justification for an expansion such as (44) within the canonical Hamiltonian formalism can be found in Appendix B. We show also that the operators q^μ and p^μ have the interpretation of the position of the center of the hyperboloid and total momentum of the bubble, respectively, and they satisfy the usual commutation relation

$$[q^\mu, p^\nu] = i\delta^{\mu\nu} .$$

Now we define a vacuum state $|0\rangle$ by requiring

$$p^\mu |0\rangle = 0 ,$$
$$a_{LM} |0\rangle = 0 \quad (L > 1) . \quad (47)$$

The first equation implies that the vacuum has vanishing total momentum. In the "q" representation it means that the wave functional Ψ does not depend on the position of the center of the bubble,

$$p^\mu \Psi = -i\frac{\partial}{\partial q^\mu}\Psi = 0 . \tag{48}$$

Also, we have to show that $|0\rangle$ is a de Sitter-invariant state. This is not a trivial check since the two-point function $\langle 0|\phi(\xi)\phi(\xi')|0\rangle$ is ill defined, as expected from the fact that the solution of (48) is not normalizable. To probe de Sitter invariance we need to construct observables that do not depend on the translational zero modes. It is then natural to look for operators that characterize the geometry of the perturbed bubble. In Appendix C we show that, for the particular case at hand, all the information on the extrinsic and intrinsic geometry of the perturbed worldsheet is contained in the traceless part of the extrinsic curvature K_{ab} [see after Eq. (3)],

$$K^{\mathrm{T}}_{ab} \equiv K_{ab} - \frac{1}{N} K^c_c g_{ab} . \tag{49}$$

Also, we show that the two-point function

$$G_{aba'b'}(\xi,\xi') \equiv \langle 0|K^{\mathrm{T}}_{ab}(\xi) K^{\mathrm{T}}_{a'b'}(\xi')|0\rangle \tag{50}$$

is a well-defined and de Sitter-invariant bitensor. This establishes the de Sitter invariance of the state $|0\rangle$ defined by (47).

We can compute distortions to the spherical shape of the bubble by studying fluctuations in the truncated field $\hat{\phi}$ defined in (45), which excludes the monopolar and dipolar components of ϕ (associated with the zero modes). Note that the operation of truncating the field is not a de Sitter-invariant one, it involves a particular temporal slicing of the hyperboloid. The fluctuations can be computed using the truncated two-point function

$$\hat{G}(\xi,\xi') \equiv \langle 0|\hat{\phi}(\xi)\hat{\phi}(\xi')|0\rangle , \tag{51}$$

which is given as a sum over modes similar to (9) but without the $L=0$ and $L=1$ modes.

For simplicity, let us consider the case $N=2$ first. In this case, the $t=$const sections of the hyperboloid (24) are just circles S^1 which can be parametrized by an angle $\theta, 0\le\theta<2\pi$, and the spherical harmonics in (33) reduce to $Y_L(\theta)=(2\pi)^{-1/2}\exp(i\theta L)$. The equal-time-truncated two-point function is then

$$\hat{G}(\xi,\xi') = \frac{1}{4\mu}\sum_{L=2}^{\infty} \cos L(\theta-\theta')\frac{\Gamma(L-1)}{\Gamma(L+2)} \times \cos\tilde{t}\,|R^{3/2}_{L-1/2}(\sin\tilde{t})|^2 , \tag{52}$$

where we have replaced σ by μ to remind that for $N=2$ the wall is a stringlike object. The Legendre functions in (52) are given by

$$R^{3/2}_{L-1/2}(\sin\tilde{t}) = -\left[\frac{2}{\pi}\right]^{1/2}\frac{e^{iL(\tilde{t}-\pi/2)}}{(\cos\tilde{t})^{3/2}} \times(\sin\tilde{t}+iL\cos\tilde{t}) .$$

Substituting in (52) we have

$$\hat{G}(\xi,\xi') = \frac{1}{2\pi\mu}\sum_{L=2}^{\infty}\frac{\cos L(\theta-\theta')}{L} \times\left[\frac{1}{(L^2-1)\cos^2\tilde{t}}+1\right] . \tag{53}$$

After some straightforward algebra, this sum can be given in closed form as

$$\hat{G}(\theta-\theta',\tilde{t}) = \frac{-1}{4\pi\mu}\left[\left[1+\frac{\delta H^2}{2}\right]\left[\ln\left|\frac{\delta H^2}{4}\right| + \ln(4\cos^2\tilde{t})\right] + (\cos^2\tilde{t}-\tfrac{3}{4})\delta H^2 - \frac{1}{2\cos^2\tilde{t}} + 2\right] , \tag{54}$$

where we have used the de Sitter-invariant interval

$$\begin{aligned}\delta &\equiv -(x^\mu - x'^\mu)(x_\mu - x'_\mu)\\ &= 2H^{-2}(Z-1)\\ &= \frac{\cos(\theta-\theta')-\cos(\tilde{t}-\tilde{t}')}{H^2\cos\tilde{t}\cos\tilde{t}'}\end{aligned}$$

evaluated at equal times. Note that $\delta\to 0$ in the coincidence limit.

Deformations from sphericity are characterized by the mean squared fluctuation in the radial coordinate. From (32),

$$\langle \Delta^2(\tilde{t})\rangle = \cos^2\tilde{t}\langle\hat{\phi}^2(\tilde{t})\rangle = \cos^2\tilde{t}\lim_{\theta\to\theta'}\hat{G}(\theta-\theta',\tilde{t}) . \tag{55}$$

Again, this quantity is formally divergent because of the $\ln\delta H^2$ term in Eq. (54). To deal with the divergence one has to introduce a smearing distance s (as explained in Sec. III) and work with the smeared field operator $\hat{\phi}_s$. It is easy to see from (54) that the smeared two-point function $\hat{G}_s \equiv \langle \hat{\phi}^2_s\rangle$ at large times (i.e., $\tilde{t}\to\pi/2$) is dominated by the s independent term

$$\hat{G}_s(0,\tilde{t}) \approx \frac{1}{8\pi\mu}\frac{1}{\cos^2\tilde{t}} = \frac{H^2R^2(t)}{8\pi\mu} . \tag{56}$$

More precisely (for $s<H^{-1}$) the above approximation is valid when the radius of the bubble $R(t) \gg H^{-1}|\ln Hs|$. Therefore for large times the mean square fluctuation in the radius "freezes out" at a constant value

$$\bar{\Delta}^2 \equiv \langle \Delta^2_s\rangle = \cos^2\tilde{t}\hat{G}_s(0,\tilde{t}) \approx \frac{1}{8\pi\mu} \quad (N=2) . \tag{57}$$

Similarly one can consider the case $N=3$, that is, bubbles nucleating in ordinary four-dimensional Minkowski space. Although for $N=3$ it is not possible to give $\hat{G}$ in closed form, one can calculate the asymptotic behavior of

$\bar{\Delta}$ for large times. The details are reported in Appendix D. We find

$$\bar{\Delta}^2 \approx \frac{H}{3\pi^2\sigma} \quad (N=3) . \tag{58}$$

Both for $N=2$ and $N=3$, the relative fluctuation $\bar{\Delta}/R$ goes to zero as $R\to\infty$, so the bubble actually becomes more and more spherical as it expands. This behavior is completely analogous to that of classical perturbations, studied in Refs. [13,6]. In the classical analysis, the amplitude at which each individual mode "freezes out" is a constant related to the initial conditions, which were left as free parameters. Here, given the quantum state, we have been able to compute the particular amplitude at which the perturbations freeze out.

V. DEFECTS OF ARBITRARY THICKNESS

In this section we summarize and extend the results of Ref. [7] regarding the quantum state of a nucleating bubble. We show in particular that fluctuations of an arbitrary thick wall bubble always include a branch described by a tachyonic field of mass $m^2=-NH^2$. We will also consider thick planar walls and straight strings.

Consider the Lagrangian

$$\mathcal{L} = -\tfrac{1}{2}[\partial_\mu\sigma\partial^\mu\sigma + V(\sigma)] - \tfrac{1}{2}[\partial_\mu\Phi\partial^\mu\Phi + M^2(\sigma)\Phi^2] . \tag{59}$$

We assume that the potential $V(\sigma)$ has the shape of a double well with nondegenerate minima (the true and the false vacua), so that the scalar field σ can undergo a first-order phase transition from the false to the true vacuum through bubble nucleation (see Fig. 1). The field Φ is a "test" scalar field, which is interacting with the bubble through the term $M^2(\sigma)\Phi^2$, where $M^2(\sigma)\geq 0$.

In Ref. [7] the wave functional describing the nucleating bubble plus the scalar field Φ was studied in the semiclassical approximation. It was assumed that the wave functional for σ is strongly peaked around a family of field configurations that can be obtained as $t=$const slices of the O(3,1) symmetric solution

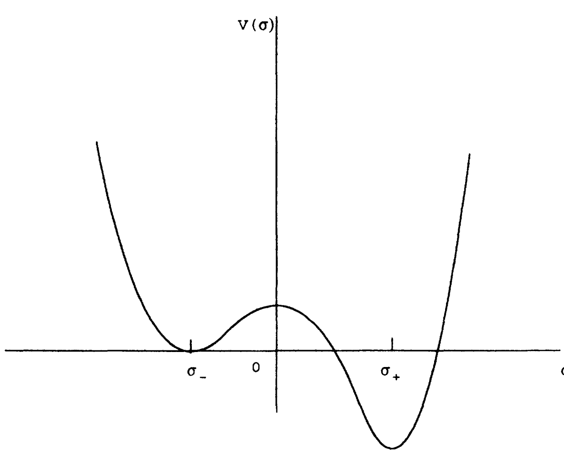

FIG. 1. The typical shape of the potential for the field σ, with a false vacuum at σ_- and a true vacuum at σ_+.

$$\sigma=\sigma_0(\zeta) . \tag{60}$$

Here $\zeta\equiv H^{-1}\ln[H(r^2-t^2)^{1/2}]$, where H^{-1} is an arbitrary length scale, r is the radial coordinate and σ_0 is the analytic continuation to real time of the so-called Euclidean bounce solution [3]. Solving the functional Schrödinger equation for the system of the bubble plus the scalar field Φ (with suitable boundary conditions) it was shown that, after nucleation, the resulting quantum state for the field Φ is Lorentz invariant.

To understand how these results may be related with the problem of small perturbations on the bubble, it is convenient to review in some detail how the field Φ was described in Ref. [7] (see also Ref. [23]). From (59), the classical field equation for Φ is

$$-\Box\Phi + M^2(\zeta)\Phi = 0 , \tag{61}$$

where we have used (60) to express σ as a function of ζ. $\Box$ denotes the d'Alembertian in $(N+1)$-dimensional Minkowski space. Note that the $\zeta=$const hypersurface are hyperboloids embedded in flat space, and therefore they have the internal geometry of de Sitter space. Denoting by $\xi^a (a=1,\ldots,N)$ a set of coordinates on the hyperboloid, one can use the coordinates (ζ,ξ^a) to write the Minkowski metric as [23]

$$ds^2 = e^{2H\zeta}[d^2\zeta + ds_H^2] . \tag{62}$$

Here ds_H^2 is the line element of N-dimensional de Sitter space of "radius" H^{-1}, given in Eq. (26). The d'Alembertian in flat space can then be split as

$$\Box = e^{-2H\zeta}\left[\frac{d^2}{d\zeta^2} + (N-1)H\frac{d}{d\zeta} + \Box_H\right] , \tag{63}$$

where $\Box_H$ is the covariant d'Alembertian in N-dimensional de Sitter space. Expanding the field Φ as

$$\Phi(\zeta,\xi^a) = \sum_{q,\beta} e^{(1-N)H\zeta/2} F_q^\beta(\zeta)\phi_q(\xi^a) , \tag{64}$$

and using (63), one can easily see that the field equation (61) separates into

$$-\frac{d^2F_q^\beta}{d\zeta^2} + e^{2H\zeta}M^2(\zeta)F_q^\beta = \left[m_q^2 - \frac{(N-1)^2}{4}H^2\right]F_q^\beta , \tag{65}$$

and

$$-\Box_H\phi_q(\xi^a) + m_q^2\phi_q(\xi^a) = 0 . \tag{66}$$

Here m_q^2 are separation constants, and the index β labels the possible degeneracy for a given m_q^2. In summary, the field Φ is described in terms of an infinite set of scalar fields ϕ_q living on the hypersurface $\zeta=0$. The result of Ref. [7] is that the bubble nucleates with all these fields in de Sitter-invariant quantum states. The Lorentz invariance of the corresponding quantum state for Φ follows immediately.

The spectrum of masses m_q^2 is determined by the fact that $\Phi(\zeta,\xi^a)$ must have a well defined Klein-Gordon norm. If the fields ϕ_q have the usual normalization in de

Sitter space, then one must demand that the functions F_q^β be normalized as

$$\int_{-\infty}^{\infty} d\zeta F_q^\beta(\zeta) F_{q'}^{\beta'}(\zeta) = \delta_{qq'}\delta_{\beta\beta'} . \tag{67}$$

Equation (65) can be seen as a Schrödinger equation with potential $e^{2H\zeta}M^2(\zeta) \geq 0$, and therefore all of its "energy" eigenvalues must be positive, i.e.,

$$m_q^2 \geq \frac{(N-1)^2}{4} H^2 .$$

Let us now try to apply the above formalism to the perturbations on the bubble. For this we write

$$\sigma = \sigma_0(\zeta) + \delta\sigma(\zeta, \xi^a) , \tag{68}$$

where σ_0 is the unperturbed solution (60) and $\delta\sigma$ is a small perturbation. The field equation for σ is

$$-\Box\sigma + V'(\sigma) = 0 , \tag{69}$$

where a prime indicates derivative with respect to the argument. Inserting (68) in this equation, we have

$$-\Box\delta\sigma + V''(\sigma_0)\delta\sigma = 0 . \tag{70}$$

This is analogous to (61), with the replacements

$$\delta\sigma \to \Phi , \quad V''(\sigma_0) \to M^2(\zeta) . \tag{71}$$

There is, however, an interesting difference. Since $V(\sigma)$ has the shape of a (nondegenerate) double well, it is clear that V'' will not be positive-definite. In fact, it will be negative on the "hump" that separates the two minima of the potential (that is, at the core of the domain wall separating the true from the false vacuum). In Fig. 2 we illustrate the typical shape of the effective potential of the Schrödinger equation $V_{\text{eff}} = e^{2H\zeta}M^2(\zeta)$, for $M^2(\zeta) = V''(\sigma_0(\zeta))$, where $V(\sigma)$ is a double well such as the one depicted in Fig. 1. Notice that there is a region in which $V_{\text{eff}} < 0$, so the Schrödinger equation (65) may have some bound states with negative energy, and we may even have $m_q^2 < 0$.

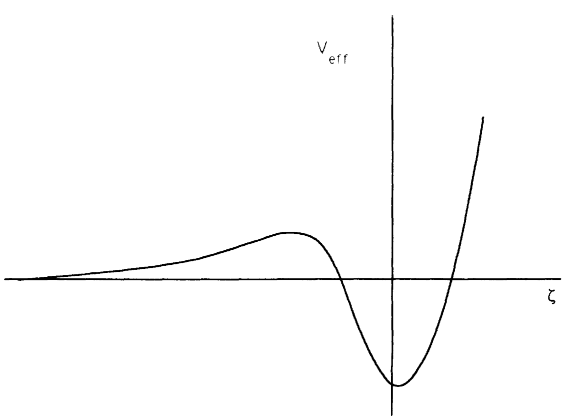

FIG. 2. The typical shape of $V_{\text{eff}}(\zeta)$ for a potential $V(\sigma)$ such as the one represented in Fig. 1. The width of the region in which $V_{\text{eff}} < 0$ is of the order of the thickness of the wall. The core of the wall is in the vicinity of $\zeta = 0$.

Indeed, this is what happens. Using (63) the field equation for $\sigma_0(\zeta)$ reads

$$e^{-2H\zeta}\left[\frac{d^2}{d\zeta^2}\sigma_0 + (N-1)H\frac{d}{d\zeta}\sigma_0\right] - V'(\sigma_0) = 0 .$$

Upon differentiation we find

$$\frac{d^2}{d\zeta^2}\sigma_0' + (N-3)H\frac{d}{d\zeta}\sigma_0' - 2H^2(N-1)\sigma_0' - V''(\sigma_0)e^{2H\zeta}\sigma_0' = 0 ,$$

which, in terms of the new variable

$$F \equiv e^{(N-3)H\zeta/2}\sigma_0' , \tag{72}$$

gives

$$-\frac{d^2}{d\zeta^2}F + V''(\sigma_0)e^{2H\zeta}F = \left[-N - \frac{(N-1)^2}{4}\right]H^2F .$$

Therefore F is a solution of Eq. (65) with $m_q^2 = -NH^2$. From the fact that σ_0 approaches a constant as $\zeta \to \infty$ and $d\sigma_0/dr = 0$ at $r = 0$ (that is $\zeta \to -\infty$), it is easy to see that F is normalizable according to (67).

As a result, we have that

$$m_q^2 = -NH^2 \tag{73}$$

is the mass of one of our fields ϕ_q describing the excitations of the bubble (even for walls of finite thickness). Let us denote this particular field by $\phi_0(\xi^a)$. To interpret ϕ_0 geometrically we write, from (71) and (64)

$$\delta\sigma = Ae^{(1-N)H\zeta/2}F(\zeta)\phi_0(\xi^a) ,$$

where A is a small parameter. It is easy to check that, to linear order in A, the perturbed solution $\sigma = \sigma_0 + \delta\sigma$ is equivalent to

$$\sigma(\zeta, \xi^a) = \sigma_0(\zeta + Ae^{-H\zeta}\phi_0(\xi^a)) , \tag{74}$$

from which we see that ϕ_0 corresponds to excitations on the bubble that do not change its profile σ_0, but only shift it along the ζ direction in a position- and time-dependent way. (This kind of excitations is the only one that one needs to consider in the zero thickness limit, as we saw in Sec. IV.)

Since the mass (72) corresponds to a bound state of the Schrödinger equation, it is a nondegenerate eigenvalue, and there are no other excitations (i.e., other F's) with the same mass. Also, it can be shown [3] that $\sigma_0' > 0$, so that $F > 0$. Since F has no nodes, it corresponds to the lowest energy eigenstate. All the other masses must satisfy

$$m_q^2 > -NH^2 \quad (q \neq 0) .$$

In principle, some of the m_q^2 may be in the interval $-NH^2 < m_q^2 < 0$. Perhaps it would be interesting to investigate this possibility in some particular model, since it would involve fields of tachyonic mass in de Sitter space for which a de Sitter-invariant two point function is well defined.

To close this section, we will show that the results of Sec. III, concerning the behavior of quantum fluctuations on planar walls and straight strings, also apply in the case when the defects have a finite thickness.

For the case of domain walls one can start with a Lagrangian given by the first term in Eq. (59) where, now, $V(\sigma)$ is a double-well potential with degenerate minima. For a planar domain wall parallel to the (x,y) plane, the unperturbed solution has the form

$$\sigma=\sigma_0(z) . \qquad (75)$$

Writing the perturbed solution as $\sigma=\sigma_0(z)+\delta\sigma$ the equation for the perturbations is just Eq. (70), with σ_0 given by (75). Writing

$$\delta\sigma=\sum_{q,\beta} F_q^\beta(z)\phi_q(x,y,t) , \qquad (76)$$

the equation for the perturbations separates into

$$-\frac{d^2F_q^\beta}{dz^2}+V''(\sigma_0)F_q^\beta=m_q^2F_q^\beta , \qquad (77)$$

and

$$-\Box\phi_q(x,y,t)+m_q^2\phi_q(x,y,t)=0 . \qquad (78)$$

Here, as before, m_q^2 are separation constants, and the functions F_q^β are normalized according to (67). It is then easy to see that $m^2=0$ is one of the eigenvalues of the Schrödinger equation (77), with eigenfunction $F=\sigma_0'(z)$. This corresponds to perturbed solutions of the form

$$\sigma=\sigma_0[z+A\phi_0(x,y,t)] ,$$

which represent transverse wiggles on the wall that do not change its profile σ_0. This establishes that the results that we obtained in Sec. III for infinitely thin planar walls also apply to the thick wall case. Using the same arguments that we used for the case of the vacuum bubble we can see that all the other excitations must have $m_q^2>0$.

For the case of strings of finite thickness the analysis is not as simple, since at least one complex scalar field plus gauge fields are involved. However, it has been shown by Vachaspati and Vachaspati [24] that transverse waves of arbitrary shape (and amplitude) traveling along a straight string at the speed of light are exact solutions of the coupled field equations for the scalar and gauge fields. This indicates that, in the linear approximation, transverse excitations on a straight string of finite thickness will also be described by a massless field ["living" in $(1+1)$-dimensional flat space].

VI. STRINGS AND WALLS IN DE SITTER SPACE

Here we are going to consider quantum perturbations on circular loops of string and spherical domain walls that can spontaneously nucleate during inflation [4]. Also, we shall comment on the case of a straight string in de Sitter space, which has received some attention in the context of string driven inflation [5]. As we shall see below, both cases are very closely related.

We shall use the standard representation of the de Sitter space, which describes the inflationary universe, as a hyperboloid embedded in a five-dimensional Minkowski space,

$$-(X^0)^2+(X^1)^2+(X^2)^2+(X^3)^2+(X^4)^2=H^2 . \qquad (79)$$

Here H is the expansion rate of the inflationary universe and $X^\alpha(\alpha=0,\ldots,4)$ is the position vector in the embedding Minkowski space. The metric on the hyperboloid can be written in the flat Friedmann-Robertson-Walker (FRW) form

$$ds^2=-dt^2+e^{2Ht}(d\mathbf{x})^2 , \qquad (80)$$

where the coordinates $(t,\mathbf{x})$ are defined by

$$\begin{aligned} t&=H^{-1}\ln[H(X^4+X^0)] ,\\ x^i&=H^{-1}X^i(X^4+X^0)^{-1} \quad (i=1,2,3) . \end{aligned} \qquad (81)$$

Conversely, using (81) and (79),

$$\begin{aligned} X^0&=H^{-1}\sinh(Ht)+\frac{H\mathbf{x}^2}{2}e^{Ht} ,\\ X^4&=H^{-1}\cosh(Ht)-\frac{H\mathbf{x}^2}{2}e^{Ht} ,\\ X^i&=x^ie^{Ht} . \end{aligned} \qquad (82)$$

The world sheet of a circular loop of string after nucleation is given by [4]

$$\begin{aligned} &-(X^0)^2+(X^1)^2+(X^2)^2=H^{-2} ,\\ &X^3=X^4=0 , \end{aligned} \qquad (83)$$

which is a $(1+1)$-dimensional hyperboloid of maximal "radius" embedded in the hyperboloid (79). The internal geometry of this world sheet is, of course, that of $(1+1)$-dimensional de Sitter space. To see what the string looks like in the FRW coordinates (t,x^i), one uses (82) and (83) to find

$$\begin{aligned} &x^2+y^2=H^{-2}(1+e^{-2Ht}) ,\\ &z=0 , \end{aligned} \qquad (84)$$

where we have adopted the notation $x^i=(x,y,z)$. This is a loop of physical radius

$$R(t)=H^{-1}\sqrt{e^{2Ht}+1} \qquad (85)$$

centered at $x=y=0$.

Different configurations for the string after nucleation can be obtained by applying de Sitter transformations to the solution (83). These transformations result in arbitrary spacetime translations and rotations of the loop [4]. In particular, it is instructive to consider the rotation

$$\begin{aligned} X^4&\to X^4\cos\theta+X^2\sin\theta ,\\ X^2&\to X^4\sin\theta+X^2\cos\theta , \end{aligned} \qquad (86)$$

which transforms the hyperboloid (83) into

$$\begin{aligned} &-(X^0)^2+(X^1)^2+(X^2\cos\theta-X^4\sin\theta)^2=H^{-2} ,\\ &X^3=0, \quad X^2=-X^4\cot\theta . \end{aligned} \qquad (87)$$

In the FRW coordinates this is seen as a circular loop in the $z=0$ plane, centered at $(x,y)=(0,H^{-1}\tan\theta)$, and with physical radius

$$R(t)=H^{-1}\sqrt{e^{2Ht}(1+\tan^2\theta)+1}\ .$$

Note that for $\theta=\pi/2$ we have $X^2=X^3=0$ and $R\to\infty$. Therefore a straight string lying on the x axis is obtained as the limiting case in which the radius of the loop goes to infinity.

Let us now turn to the problem of small perturbations on the solution (83). The perturbed solution can be parametrized as

$$\begin{aligned}&-(X^0)^2+(X^1)^2+(X^2)^2=H^{-2}-\phi_A\phi_A \quad (A=1,2)\ ,\\ &X^3=\phi_1,\quad X^4=\phi_2\ .\end{aligned} \tag{88}$$

The fields $\phi_A(\xi^a)$ correspond to perturbations in two directions orthogonal to the unperturbed world sheet [6]. To clarify their interpretation, one can use (82) to find (to linear order in ϕ_A)

$$\begin{aligned}&\phi_1(\xi^a)=e^{Ht}z\equiv z_{\rm phy}(\xi^a)\ ,\\ &\phi_2(\xi^a)=-(r-R)\frac{HR}{(H^2R^2-1)^{1/2}}\ .\end{aligned} \tag{89}$$

Here $r(\xi^a)=e^{Ht}(x^2+y^2)^{1/2}$ is the perturbed physical radius, and R is the unperturbed one, given by (85). Therefore, the field ϕ_1 represents proper perturbations transverse to the plane of the loop, while the field ϕ_2 represents, essentially, perturbations of the radius.

The effective action for the perturbations is given by [6]

$$S_\phi=\frac{\mu}{2}\int_\Sigma\sqrt{g}\,(g^{ab}\phi_{A,a}\phi_{A,b}-2H^2\phi_A\phi_A)\,d^2\xi\ , \tag{90}$$

where g_{ab} is the metric on the unperturbed world sheet. The perturbations behave as two uncoupled scalar fields of mass $m^2=-2H^2$ in $(1+1)$-dimensional de Sitter space. This is the same mass that we obtained for the case of a vacuum bubble, so here we can simply borrow from the results of Sec. IV (with $N=2$).

In the present case there will be three zero modes for each field. For ϕ_1, one of them $(L=0)$ corresponds to translations of the loop in the z direction and the other two $(L=1)$ to changes in the orientation of the loop. For ϕ_2, one of the zero modes corresponds to time translations $(L=0)$ and the others $(L=1)$ to spatial translations in the plane of the loop. The six zero modes correspond to the six independent de Sitter transformations that do not leave the world sheet (83) invariant. However, the de Sitter group, O(4,1) is a 10-parameter group. The remaining four independent transformations leave the unperturbed world sheet invariant. If we think of O(4,1) as the Lorentz group in the embedding five-dimensional Minkowski space, then these four transformations are the 3 Lorentz transformations in the (X^0,X^1,X^2) space plus rotations in the plane (X^3,X^4).

The only quantum state for the perturbations ϕ_A that shares the four-parameter symmetry of the world sheet is, just as in the case of a vacuum bubble, the de Sitter-invariant state. [Here, de Sitter invariance is understood in the $(1+1)$-dimensional sense.] The construction of such state was discussed at length in Sec. IV. Here, as we did in the case of vacuum bubbles, we shall use $\hat{G}$ [see Eq. (52)] to estimate the deviations of the loop from circular shape. The average radial and transverse fluctuations are given, from (89), by

$$\begin{aligned}&\bar{\Delta}^2\equiv\langle(r-R)^2\rangle=\langle(\hat{\phi}_2(t))^2\rangle\frac{H^2R^2-1}{H^2R^2}\ ,\\ &\bar{z}^2_{\rm phy}\equiv\langle(\hat{\phi}_1(t))^2\rangle\ .\end{aligned} \tag{91}$$

Using (56) we have (for $R\gg H^{-1}$)

$$\bar{\Delta}\approx\bar{z}_{\rm phys}\approx\frac{HR}{\sqrt{8\pi}\mu^{1/2}}\ . \tag{92}$$

The ratio of the mean fluctuation amplitude to the loop radius is

$$\bar{\Delta}/R\sim H\mu^{-1/2}\ . \tag{93}$$

This result differs from that of Ref. [4], where this ratio (at the time of formation) was estimated to be of order $H^4\mu^{-2}$. The argument that was used there, however, involved only the $L=0$ mode, which does not change the shape of the loop and therefore should not be included.

After inflation, the nucleated circular loops will start collapsing and, in fact, they would all form black holes were it not for the perturbations Δ. For a given loop to form a black hole we need $\Delta/R\lesssim 4\pi G\mu$ [25,26]. Taking $H\mu^{-1/2}\sim 1$ (this seems to be the interesting range of parameters [4]) we see, from (93), that the probability of black hole formation will be very small (unless μ is close to the Planck scale). An estimate of this probability and a discussion of the cosmological implications of the nucleating strings scenario will be reported elsewhere.

Spherical domain walls nucleating during inflation can be treated in a similar way. The unperturbed world sheet is $(2+1)$-dimensional de Sitter space and the proper perturbations in the direction normal to the world sheet are represented by a scalar field ϕ of mass $m^2=-3H^2$ living on the unperturbed world sheet [6]. Therefore we can use the results that we have obtained in Appendix D for the case of a vacuum bubble $(N=3)$. From (D6) we have

$$\frac{\bar{\Delta}}{R}\approx\frac{H^{3/2}}{\sqrt{3\pi^2\sigma}}\ . \tag{94}$$

An important difference with the case of strings is that the Schwarzschild radius of a wall grows quadratically with R, instead of linearly. As a result, walls with radius

$$R>(8\pi G\sigma)^{-1}\left[\frac{H^3}{3\pi^2\sigma}\right]^{1/2}$$

will all collapse to black holes.

VII. CONCLUSIONS

We have developed a quantum theory of fluctuations on topological defects, such as vacuum domain walls and strings. Fluctuations are represented by a scalar field ϕ that "lives" on the unperturbed world sheet of the defect and has the meaning of a normal displacement of the world sheet. To avoid ultraviolet divergences, it is often necessary to smear the field ϕ over some distance s [see Eq. (13)]. The amplitude of the fluctuations can then be characterized by $\langle \phi_s^2 \rangle$, where ϕ_s is the smeared field. Another quantitative measure of the fluctuations is given by the "distortion,"

$$D^2(x,y)=\langle (\phi_s(x)-\phi_s(y))^2 \rangle \ , \tag{95}$$

where the expectation value is taken at equal times. We have applied our formalism to a number of different cases. Here we shall summarize our main results .

(1) For planar walls and straight strings, ϕ is a massless field in an N-dimensional Minkowski space, where $N=3$ and $N=2$ for walls and strings, respectively. In the case of walls, we find

$$\langle \phi_s^2 \rangle \approx 1.4(\sigma s)^{-1} \ , \tag{96}$$

where σ is the wall tension and s is the smearing distance. The wall can be treated semiclassically if the amplitude of the fluctuations is much smaller than the wall thickness δ for $s \gtrsim \delta$, that is, if

$$\delta \gg \sigma^{-1/3} \ . \tag{97}$$

For walls appearing in the scalar field theory with the potential

$$V(\varphi)=\lambda(\varphi^2-\eta^2)^2 \ , \tag{98}$$

$\sigma \sim \lambda^{1/2}\eta^3$, $\delta \sim \lambda^{-1/2}\eta^{-1}$, and the semiclassical approximation is justified in the weak coupling limit, $\lambda \ll 1$.

We have also calculated the distortion for a planar wall. For points separated by a distance much greater than the smearing length s, the distortion is independent of the separation, $D^2 \approx 2\langle \phi_s^2 \rangle$. The reason is that quantum fluctuations are uncorrelated at large distances.

(2) In the case of quantum fluctuations on straight strings, the picture is entirely different. With the usual choice of the positive-frequency mode functions (20), we find that $\langle \phi_s^2 \rangle \to \infty$. The distortion is finite, but grows as the separation of the two points is increased,

$$D^2(x,y) \approx \frac{1}{\pi\mu} \ln \left| \frac{x-y}{s} \right| , \quad |x-y| \gg s \ , \tag{99}$$

indicating that at large distances the string deviates arbitrarily far from its unperturbed position. The growth of D^2 with distance is only logarithmic, and the unbounded quantum fluctuations (99) are unlikely to have any cosmological effect. Even if $|x-y|$ is equal to the present Hubble length, the amplitude of the fluctuations is $D \lesssim 10\mu^{-1/2}$ and cannot much exceed the string thickness.

The infrared divergence in $\langle \phi_s^2 \rangle$ can be cut off if one chooses a different quantum state for ϕ, defined by mode functions that are different from (20) at wavelengths above some cutoff length L. If the cutoff is introduced, the behavior of quantum fluctuations on scales much smaller than L remains unaffected, and the distortion is still given by Eq. (99) for $L \gg |x-y| \gg s$.

(3) We have studied several cases in which the unperturbed world sheet is an N-dimensional de Sitter space, with $N=3$ for walls and $N=2$ for strings. The field ϕ then has a tachyonic mass,

$$m^2=-NH^2 \ , \tag{100}$$

where H is the expansion rate of the de Sitter space. For an expanding vacuum bubble H is determined by the wall tension and the false vacuum energy density, while for strings and walls nucleating during inflation the expansion rate on the world sheet is the same as in the background space.

In order to preserve the Lorentz invariance of the expanding bubble, the quantum state of the field ϕ should be de Sitter invariant. The construction of a de Sitter-invariant state for ϕ requires careful treatment of the lowest modes ($L=0$ and $L=1$) in the expansion of the field operator. These modes do not correspond to deformations of the bubble, but to infinitesimal translations of the unperturbed world sheet as a whole.

Using the de Sitter-invariant quantum state, we have calculated the rms amplitude of the fluctuations in the bubble radius, $\bar{\Delta}$ [Δ is related to ϕ through Eq. (32)]. At large times $\bar{\Delta}$ approaches a constant value,

$$\bar{\Delta}^2 \approx \frac{H}{3\pi^2\sigma} \ , \tag{101}$$

and the relative fluctuation $\bar{\Delta}/R$ goes to zero. Thus the bubble becomes more and more spherical as it expands, in agreement with the classical behavior (see Refs. [13,6]). Similar results are obtained for expanding holes in a planar domain wall.

(4) A different behavior of the fluctuations is found for a string loop or a spherical wall nucleating in de Sitter space. There, the fluctuations grow with the loop radius, and the relative distortion approaches a constant. We have

$$\frac{\bar{\Delta}}{R} \approx \frac{H}{\sqrt{8\pi\mu}} \tag{102}$$

for a string loop, and

$$\frac{\bar{\Delta}}{R} \approx \frac{H^{3/2}}{\sqrt{3\pi^2\sigma}} \tag{103}$$

for a spherical wall.

(5) The results summarized so far have been obtained neglecting the wall and string thickness. In the case of a thick domain wall, the fluctuations can be described by an infinite number of scalar fields with masses determined by eigenvalues of a certain operator. We have shown, however, that for an expanding vacuum bubble one of the fields has exactly the mass (100). This field describes distortions of the bubble shape, while all the other fields, which have larger values of m^2, describe internal excitations of the bubble wall. Since the mass of the field is still given by (100) the results obtained for a thin wall bubble [such as Eq. (101)] are applicable to thick wall bubbles as well. Also, the results obtained in Sec. III are applicable to thick planar walls and straight strings.

Cosmological implications of the results presented here will be discussed in a separate paper.

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APPENDIX A: EUCLIDEAN GREEN'S FUNCTION AND ANOMALY

In this appendix we use the path integral approach to derive the Euclidean Green's function for the perturbations on a vacuum bubble. For the case $N=2$ and working in units in which $H=1$ the Euclidean version of (24) is the unit 2-sphere. Rescaling the bubble tension into the definition of the fields ($\sigma^{1/2}\phi \to \phi$), the action for the perturbations is

$$S_E = \tfrac{1}{2}\int \sqrt{g}\,(g^{ab}\phi_{,a}\phi_{,b} + m^2\phi^2)\,d^2\xi = \tfrac{1}{2}\int \phi(-\Delta + m^2)\phi\,d\Omega\ , \tag{A1}$$

where $m^2=-2, g_{ab}$ is the metric on the two sphere, $d\Omega$ is the differential solid angle and Δ is the Laplacian on the sphere. The Green's function is given by

$$G^E_{m^2}(\Omega,\Omega') = \langle \phi(\Omega)\phi(\Omega')\rangle \equiv \frac{\int \mathcal{D}\phi\phi(\Omega)\phi(\Omega')e^{-S_E}}{\int \mathcal{D}\phi e^{-S_E}}\ . \tag{A2}$$

Expanding the field in spherical harmonics

$$\phi(\Omega) = \sum C_{LM} Y_{LM}(\Omega)\ , \tag{A3}$$

with $C_{LM} = C^*_{L-M}$, we have

$$S_E = \tfrac{1}{2}\sum_{LM} |C_{LM}|^2 [L(L+1) + m^2]\ .$$

(Notice that now we are expanding the field in terms of eigenmodes of the Laplacian, and not in terms of solutions of the field equation.) For $m^2=-2$ we see that the $L=0$ mode gives a negative contribution to the Euclidean action. We call this a negative mode. Also, we see that the three $L=1$ modes are zero modes. Note that $L=0,1$ here do not have the same meaning as in Sec. IV. Here we are dealing with spherical harmonics on the world sheet, as opposed to harmonics on the spatial sections of the world sheet. The negative and zero modes make the path integral ill defined for $m^2=-2$, but we can evaluate it for sufficiently large m^2 and then analytically continue the result back to $m^2=-2$. The measure in the path integral can be written as [3]

$$\mathcal{D}\phi = \prod_{LM} dC_{LM}\ .$$

Substituting the expansion (A3) into (A2), the path integral is reduced to Gaussian integrations. Using standard manipulations one finds [18]

$$G^E_{m^2} = \sum_{LM} \frac{Y_{LM}(\Omega)Y^*_{LM}(\Omega')}{L(L+1)+m^2}\ ,$$

which is a solution of the field equation with a delta function source

$$(-\Delta + m^2)G^E_{m^2}(\Omega,\Omega') = \sum_{LM} Y_{LM}(\Omega)Y^*_{LM}(\Omega') = \delta(\Omega-\Omega')\ .$$

Although the contribution of the negative mode to $G^E_{m^2}$ remains finite, the contribution of the $L=1$ modes

$$G^{\text{div}}_E = \sum_M \frac{Y_{1M}(\Omega)Y^*_{1M}(\Omega')}{2+m^2} = \frac{3}{4\pi}\frac{Z}{2+m^3}$$

becomes divergent as $m^2 \to -2$. Here $Z = \mathbf{x}(\Omega)\cdot\mathbf{x}(\Omega') = \cos\alpha$, where α is the angle between Ω and Ω'. This is the Euclidean analog of the variable Z used in Sec. IV. Since the spherical harmonics with $L=1$ do not correspond to excitations on the sphere, but to infinitesimal translations of the unperturbed solution as a whole, it seems natural to drop them from the expansion of the field and, hence, from the Green's function. Upon so doing we obtain the new object

$$G^A_E(Z) = \lim_{m^2\to -2} (G^E_{m^2} - G^{\text{div}}_E)\ .$$

However, it is clear that

$$(-\Delta - 2)G^A_E = \delta(\Omega-\Omega') - \frac{3}{4\pi}Z\ ,$$

and therefore G^A_E is not a Green's function. Using the identity

$$\sum_M Y_{LM}(\Omega)Y^*_{LM}(\Omega') = \frac{2L+1}{4\pi}P_L(Z)\ ,$$

where P_L are the Legendre polynomials, we can write

$$G^A_E = \frac{1}{4\pi}\sum_{L\neq 1}\frac{2L+1}{L(L+1)-2}P_L(Z)\ .$$

One can check (by using the orthogonality relations for P_L) that this is the Legendre expansion of the function

$$G^A_E = \frac{-1}{4\pi}\left[\tfrac{4}{3}Z + 1 + Z\ln\frac{1-Z}{2}\right]\ . \tag{A4}$$

The analytic continuation of G^A_E to real time simply amounts to using the definition (39) for Z.

Equation (A4) can be compared with the limit of G_{m^2} [Eq. (38)] as $m^2 \to 2H^2$. Following Ref. [27], we expand the hypergeometric function in (38) around $m^2=-2H^2$ (this corresponds to $\lambda = \frac{3}{2}$) [10],

$$G_{m^2}(Z) = G^{\text{div}}_{m^2}(Z) - \frac{1}{4\pi\mu}\left[Z + 1 + Z\ln\left|\frac{1-Z}{2}\right|\right] + O(\tfrac{3}{2}-\lambda)\ , \tag{A5}$$

where

$$G^{\rm div}_{m^2}(Z)=\frac{1}{4\pi\mu}\frac{Z}{(\frac{3}{2}-\lambda)} \tag{A6}$$

(we have replaced σ by μ to remind us that for $N=2$ the domain wall is a stringlike object). In terms of the variable Z (and under the assumption of de Sitter invariance) the Klein-Gordon operator can be cast into the simple form [17]

$$H^{-2}[-\Box+m^2]=(Z^2-1)\frac{d^2}{dZ^2}+NZ\frac{d}{dZ}+m^2H^{-2} . \tag{A7}$$

It is seen that the divergent part of G_{m^2} is a solution of the scalar field equation

$$(-\Box-2H^2)G^{\rm div}_{m^2}(Z)=0 ,$$

so one might naively expect that simply by dropping $G^{\rm div}_{m^2}$ from G_{m^2} one would get a well behaved de Sitter-invariant two-point function,

$$G_A\equiv\lim_{m^2\to-2H^2}[G_{m^2}-G^{\rm div}_{m^2}]$$

$$=-\frac{1}{4\pi\mu}\left[Z+1+Z\ln\left|\frac{1-Z}{2}\right|\right] . \tag{A8}$$

However, from (A7) and (A8), one can see that this is not the case, since G_A is not a solution of the field equation

$$(-\Box-2H^2)G_A=\frac{-3H^2}{4\pi\mu}Z\neq0 . \tag{A9}$$

Note that G_A is essentially the analytic continuation of G^A_E. The difference between both expressions is proportional to Z, and it stems from the fact that the divergent term that we have dropped from $G^E_{m^2}$ includes not only the pole in $(\frac{3}{2}-\lambda)$, but also a finite part proportional to Z.

The same "anomaly" (A9) can be obtained without an explicit calculation by considering the limit $\lambda\to\frac{3}{2}$ of the equation

$$(-\Box+m^2)G_{m^2}=0 , \tag{A10}$$

as it was done in Ref. [27] for the massless minimally coupled case. Noting, from (36), that

$$m^2=-2H^2+H^2(\tfrac{3}{2}+\lambda)(\tfrac{3}{2}-\lambda)$$

and using (A6) one finds

$$(-\Box-2H^2)\left[\lim_{m^2\to2H^2}G_{m^2}(Z)\right]=\frac{-3H^2}{3\pi\mu}Z , \tag{A11}$$

in agreement with (A9).

APPENDIX B: QUANTIZATION OF ZERO MODES

In this appendix we argue that the expansion of the field ϕ in terms of creation and annihilation operators is not appropriate for the degrees of freedom corresponding to the zero modes (see Refs. [19,28,29]). We present an alternative expansion which will allow for the definition of a de Sitter-invariant state for the quantum fluctuations on vacuum bubbles.

The situation is analogous to that of a quantum mechanical harmonic oscillator. The expansion of position and momentum (x and p) in terms of creation and annihilation operators becomes inappropriate in the limit when the frequency of the oscillator w goes to zero (the free particle case). In the Heisenberg picture we have

$$x(t)=\frac{1}{\sqrt{2Mw}}(ae^{-i\omega t}+a^\dagger e^{+i\omega t}) ,$$

$$p(t)=-i\left[\frac{Mw}{2}\right]^{1/2}(ae^{-i\omega t}-a^\dagger e^{+i\omega t}) ,$$

where M is the mass of the particle. Of course, such expansions are invalid in the limit $w\to0$, reflecting the physical fact that for a free particle the spectrum of the Hamiltonian is continuous and the number operator is not an adequate tool. Instead we can consider the expansions

$$x(t)=x_0+p_0t , \qquad p(t)=p_0 , \tag{B1}$$

where the new operators p_0 and x_0 satisfy the commutation relation $[x_0,p_0]=i$. At the classical level, x_0 and p_0 have the interpretation of the initial position and momentum (and therefore are constants), hence (B1) can be regarded as a Hamilton-Jacobi canonical transformation, in which the new canonical variables are constants of motion.

Similarly, we have to reconsider the expansion of the field ϕ. Instead of (8) we shall write

$$\phi(\tilde{t},\Omega)=\sum_{LM}\chi_{LM}(\tilde{t})\mathcal{Y}_{LM}(\Omega) , \tag{B2}$$

where $\mathcal{Y}_{LM}$ are real and properly normalized independent combinations of the usual spherical harmonics. For $L>1$ we take

$$\chi_{LM}(\tilde{t})=\sigma^{-1/2}[a_{LM}\varphi_L(\tilde{t})+a^\dagger_{LM}\varphi^*_L(\tilde{t})] , \tag{B3}$$

as usual, with φ_L given by (35). However, for the lowest modes we need an expansion analogous to (B1). Let us consider, for instance, the $L=0$ mode. We write

$$\chi_0(\tilde{t})=q^0\mathcal{Z}_0(\tilde{t})+p^0\mathcal{N}_0(\tilde{t}) , \tag{B4}$$

where $\mathcal{Z}_0$ is the zero mode solution that we can read off from (42),

$$\mathcal{Z}_0=\tan\tilde{t} .$$

$\mathcal{N}_0$ is a second solution of (34), which can be chosen to satisfy the Wronskian condition

$$\mathcal{Z}_L\dot{\mathcal{N}}_L-\dot{\mathcal{Z}}_L\mathcal{N}_L=(H\cos\tilde{t})^{N-2}/\sigma S^{(N-1)} , \tag{B5}$$

where $S^{(N-1)}$ is the surface of the unit $(N-1)$-sphere. The explicit form of $\mathcal{N}_L(\tilde{t})$ depends on the dimension N and is unimportant here. Explicit expressions can be found in Ref. [6] for $N=2$ and $N=3$. Here and throughout this appendix, a dot indicates derivative with respect to $\tilde{t}$.

From the action (6) one can find the canonical momentum associated with the coordinate χ_{LM}

$$\pi_{LM}=\frac{\partial L}{\partial\dot{\chi}_{LM}}=\sigma(H\cos\tilde{t})^{2-N}\dot{\chi}_{LM}\ ,$$

which, combined with (B4), suggests the expansion

$$\pi_0=\sigma(H\cos\tilde{t})^{2-N}[q^0\dot{\mathcal{Z}}_0+p^0\dot{\mathcal{N}}_0]\ . \tag{B6}$$

Equations (B4) and (B6) implement a canonical transformation between χ_0,π_0, and the new variables q^0,p^0. Indeed, the canonical commutation relation

$$[q^0,p^0]=i$$

follows from $[\chi_0,\pi_0]=i$ and the condition (B5). Also, one can check that the new canonical Hamiltonian for these variables vanishes identically, indicating that q^0 and p^0 are constant operators (as expected).

Classically, the constants q^0 and p^0 have a simple interpretation. We saw in Eq. (43) q^0 is the temporal coordinate of the center of the "shifted" hyperboloid representing the world sheet of the bubble. Also, using (B5) and (25) one can see that p^0 is equal to the energy of the bubble ($p^0=E$, to linear order).

In the same way we can treat the $L=1$ modes. We write

$$\chi_{1M}(\tilde{t})=q^M\mathcal{Z}_1(\tilde{t})+p^M\mathcal{N}_1(\tilde{t})\ , \tag{B7}$$

where the "zero mode" solution

$$\mathcal{Z}_1=\sec\tilde{t}$$

can be read off from (42), and $\mathcal{N}_1$ can be found from (B5). Classically, q^m are the spatial coordinates of the center of the bubble and p^M are the components of the three-momentum. Quantum mechanically, we will have the commutation relations

$$[q^M,p^N]=i\delta^{MN}\ .$$

To summarize, we have expanded the field as

$$\phi=\hat{\phi}+\sum_{\mu}[q^\mu\mathcal{Z}_\mu+p^\mu\mathcal{N}_\mu]\mathcal{Y}_\mu\ , \tag{B8}$$

where $\hat{\phi}$ is the usual expansion in terms of creation and annihilation operators for $L>1$ and we have used an obvious notation in which the index μ ($\mu=0,1,\ldots,N$) runs over all zero modes. Here, $\mathcal{Y}_0=Y_{00}(\Omega)$, and $\mathcal{Y}_i(i=1,\ldots,N)$ can be chosen as the Cartesian real combinations of the spherical harmonics with $L=1$, which are simply proportional to the components of the unit normal $\hat{n}^i$ expressed in spherical coordinates. The position and momentum operators satisfy the canonical commutation relation

$$[q^\mu,p^\nu]=i\delta^{\mu\nu}\ .$$

The expansion (B8) is used in Sec. IV to define a de Sitter-invariant state for the fluctuations on the vacuum bubble.

APPENDIX C: DE SITTER INVARIANCE OF QUANTUM FLUCTUATIONS ON A VACUUM BUBBLE

In this appendix we show that the quantum state defined by (47) is de Sitter invariant. As we mentioned in Sec. IV, the de Sitter invariance cannot be checked directly in the field two-point function $\langle 0|\phi(\xi)\phi(\xi')|0\rangle$ since this quantity is ill defined. Instead, we have to consider the two-point function for operators describing the geometry of the perturbed world sheet.

The geometry of the world sheet is characterized by the extrinsic and intrinsic curvature. The extrinsic curvature is defined by

$$K_{ab}\equiv-\partial_a n_\mu\partial_b x^\mu\ . \tag{C1}$$

For the unperturbed world sheet we have $n^\mu=Hx^\mu$ (this is easily seen in Euclidean space, where the world sheet is a sphere of radius H^{-1}), so

$$K_{ab}=-H\partial_a x^\mu\partial_b x_\mu=-Hg_{ab}\ . \tag{C2}$$

For the perturbed world sheet we have $\tilde{x}^\mu=x^\mu+n^\mu\phi$, and the perturbed extrinsic curvature is

$$\tilde{K}_{ab}=K_{ab}+\delta K_{ab}\ ,$$

where $\delta K_{ab}=\nabla_a\partial_b\phi-K_a^cK_{cb}\phi$ (see Ref. [6]). Combining with (C2) we have

$$\tilde{K}_{ab}=K_{ab}+[\nabla_a\partial_b\phi-H^2g_{ab}\phi]\ .$$

Using the field equation for ϕ one finds

$$\tilde{g}^{ab}\tilde{K}_{ab}=g^{ab}K_{ab}=-NH^2\ , \tag{C3}$$

where $\tilde{g}^{ab}$ is the inverse of the perturbed metric

$$\tilde{g}_{ab}=(1+2H\phi)g_{ab}\ .$$

Equation (C3) shows that the trace of the extrinsic curvature is the same for the perturbed and the unperturbed solution, so we need only consider the traceless part

$$K^{\mathrm{T}}_{ab}\equiv\tilde{K}_{ab}+H\tilde{g}_{ab}\ .$$

From the preceding equations we find

$$K^{\mathrm{T}}_{ab}=\nabla_a\partial_b\phi+H^2g_{ab}\phi\ . \tag{C4}$$

Let us now consider the internal geometry. For $N=2$ and $N=3$, the basic object is the Ricci curvature R_{ab}. We showed in Ref. [6] that the Ricci scalar $\mathcal{R}=R_c^c$, is not affected by the perturbations ϕ as long as the equations of motion are satisfied. So, again, we only need to consider the traceless part. From Ref. [6]

$$\tilde{R}_{ab}-\frac{1}{N}\mathcal{R}\tilde{g}_{ab}=-H(N-2)[\nabla_a\partial_b\phi+H^2g_{ab}\phi]$$

$$=-H(N-2)K^{\mathrm{T}}_{ab}\ ,$$

implying that all the information on the intrinsic and the extrinsic curvature is actually contained in K^{T}_{ab}.

Now we want to check the de Sitter invariance of the two-point function

$$G_{aba'b'}(\xi,\xi') \equiv \langle 0|K^{\mathrm{T}}_{ab}(\xi)K^{\mathrm{T}}_{a'b'}(\xi')|0\rangle$$
$$= \langle 0|Q_{ab}[\phi(\xi)]Q_{a'b'}[\phi(\xi')]|0\rangle ,$$

where we have introduced the operator notation

$$Q_{ab} \equiv \nabla_a \partial_b + H^2 g_{ab} .$$

We recall that from Sec. IV, the field operator can be expanded as

$$\phi = \hat{\phi} + \sum_{\mu} [q^{\mu} \mathcal{Z}_{\mu}(\tilde{t}) + p^{\mu} \mathcal{N}_{\mu}(\tilde{t})] \mathcal{Y}_{\mu}(\Omega) ,$$

where $\hat{\phi}$ is the truncated field (45) and we use the notation in which the index μ runs over all zero modes. It is easy to check that Q_{ab} annihilates the zero modes

$$Q_{ab}[\mathcal{Z}_{\mu}(\tilde{t})\mathcal{Y}_{\mu}(\Omega)] = 0 , \qquad \text{(C5)}$$

for all μ. This is expected, since for a translated hyperboloid the traceless part of the extrinsic curvature is still vanishing. Also, the vacuum satisfies $p^{\mu}|0\rangle = 0$. As a result,

$$Q_{ab}[\phi]|0\rangle = Q_{ab}[\hat{\phi}]|0\rangle ,$$

and we have

$$G_{aba'b'}(\xi,\xi') = Q_{ab} \circ Q_{a'b'}[\hat{G}(\xi,\xi')] ,$$

with $\hat{G}$ defined in (51).

The de Sitter invariance of $G_{aba'b'}$ is not manifest in the previous equation. However, $G_{aba'b'}$ will be (by construction) a de Sitter-invariant bitensor if we can prove the following equality:

$$Q_{ab} \circ Q_{a'b'}[\hat{G}(\xi,\xi')] = \lim_{m^2 \to -NH^2} Q_{ab} \circ Q_{a'b'}[G_{m^2}(Z)] , \qquad \text{(C6)}$$

where G_{m^2} is given by Eq. (38).

We recall that G_{m^2} is given as a sum over all modes, of the form

$$G_{m^2}(Z) = \sum_{LM} \phi_{LM}(\xi)\phi^*_{LM}(\xi') ,$$

whereas $\hat{G}$ is given by a similar sum but without the $L=0$ and $L=1$ terms. Therefore, all we have to show is that

$$Q_{ab} \circ Q_{a'b'} \left[\lim_{\epsilon \to 0} \sum_{L=0,1} \sum_{M} \phi_{LM}(\xi)\phi^*_{LM}(\xi') \right] = 0 , \qquad \text{(C7)}$$

where $\epsilon \equiv [\lambda - (N-1)/2]$, with λ defined in Eq. (36) (clearly, $\epsilon \to 0$ as $m^2 \to -NH^2$). From (37), the normalization constants A_L for the modes $L=0$ and $L=1$ behaves like $\epsilon^{-1/2}$, whereas the functions R^{λ}_{ν} are analytic in λ, and therefore in ϵ. As a result we can expand, for $L=0$ and $L=1$,

$$\phi_{LM} = \epsilon^{-1/2}[f_{LM}(\tilde{t}) + \epsilon g_{LM}(\tilde{t}) + \cdots] Y_{LM}(\Omega) . \qquad \text{(C8)}$$

As we have repeatedly emphasized, the term

$$f_{LM}(\tilde{t}) Y_{LM}(\Omega) \qquad \text{(C9)}$$

represents a zero mode. The other term, $g_{LM} Y_{LM}$ is not a zero mode (it is not even a solution of the field equation). To check Eq. (C7) we have to multiply (C8) by its complex conjugate and sum over $L=0,1$. The result is

$$T_{\epsilon} = \sum_{L=0,1} \sum_{M} [\tfrac{1}{\epsilon} f_{LM}(\tilde{t}) f^*_{LM}(\tilde{t}') + f_{LM}(\tilde{t}) g^*_{LM}(\tilde{t}') + f^*_{LM}(\tilde{t}') g_{LM}(\tilde{t})] Y_{LM}(\Omega) Y^*_{LM}(\Omega') + O(\epsilon) .$$

Since Q_{ab} acting on (C9) is zero, it is clear that

$$\lim_{\epsilon \to 0} Q_{ab} \circ Q_{a'b'}[T_{\epsilon}] = 0 ,$$

hence establishing the truth of (C7). This completes our proof.

APPENDIX D: DISTORTION OF VACUUM BUBBLES FOR $N=3$

In this appendix we derive Eq. (58), which gives the asymptotic behavior, for large times of the fluctuation in the radial coordinate of a vacuum bubble (for the case $N=3$).

From Eq. (55), we need the equal time two point function $\hat{G}$, which, from (9) and (33)–(37), is given as

$$\hat{G}(\Omega,\Omega';t') = \frac{H}{8\sigma} \cos^2\tilde{t} \sum_{L=2}^{\infty} C^{1/2}_L(\cos\gamma)(L+\tfrac{1}{2}) \frac{\Gamma(L+1-\lambda)}{\Gamma(L+1+\lambda)} |R^{\lambda}_L(\sin\tilde{t})|^2 . \qquad \text{(D1)}$$

Here γ is the angle between Ω and Ω', and we have used the relation

$$\frac{4\pi}{(2L+1)} \sum_{M=-L}^{L} Y^*_{LM}(\Omega) Y_{LM}(\Omega') = C^{1/2}_L(\cos\gamma) ,$$

where $C^{1/2}_L$ are Gegenbauer polynomials. The parameter λ is related to the mass through Eq. (36), and we have $\lambda \to 2$ as $m^2 \to -3H^2$.

As explained in Sec. IV, the terms with $L=0$ and $L=1$ have been dropped from the sum over modes because they do not correspond to deformations of the bubble, but to temporal and spatial translations of the unperturbed bubble as a whole. The sum (D1) can be cast into the form

$$\hat{G} = G_{m^2} - \frac{H}{8\sigma} \cos^2\tilde{t} \left[\frac{1}{2} \frac{\Gamma(1-\lambda)}{\Gamma(1+\lambda)} |R^{\lambda}_0(\sin\tilde{t})|^2 + \tfrac{3}{2}\cos\gamma \frac{\Gamma(2-\lambda)}{\Gamma(2+\lambda)} |R^{\lambda}_1(\sin\tilde{t})|^2 \right] , \qquad \text{(D2)}$$

where G_{m^2} is a sum over modes similar to (D1) but including the $L=0$ and $L=1$ terms. G_{m^2} is given in closed form by Eq. (38), with $N=3$. The Legendre functions in (D2) are

$$R_0^\lambda(\sin\tilde{t})=\frac{-i}{\pi}\frac{\Gamma(\lambda+1)}{\lambda}(S^{\lambda/2}-e^{i\pi\lambda}S^{-\lambda/2}) ,$$

$$R_1^\lambda(\sin\tilde{t})=\frac{i}{\pi}\frac{\Gamma(\lambda+1)}{\lambda-1}\left[\left[1+\frac{\sin\tilde{t}}{\lambda}\right]S^{\lambda/2}+e^{i\pi\lambda}\left[1-\frac{\sin\tilde{t}}{\lambda}\right]S^{-\lambda/2}\right] ,$$

with $S=(1-\sin\tilde{t})/(1+\sin\tilde{t})$.

Expanding (D2) in the neighborhood of $\lambda=2$ and then taking the limit as $\lambda\to 2$ we find, after some lengthy algebra,

$$\hat{G}=G_A(Z)-\frac{H}{2\pi^2\sigma}Z\left\{\frac{5}{6}+\sin\tilde{t}\left[\left[1+\frac{\cos^2\tilde{t}}{2}\right]\ln S+\frac{\cos^2\tilde{t}}{2}-1\right]\right\}-\frac{H}{2\pi^2\sigma}\tan^2\tilde{t}\left[\frac{1}{3}-\frac{\cos^2\tilde{t}}{2\sin\tilde{t}}\ln S-\frac{1+\sin^2\tilde{t}}{2}\sin\tilde{t}\right] , \tag{D3}$$

where the anomalous Green's function G_A is the analog of (A8) for $N=3$

$$G_A(Z)\equiv\lim_{m^2\to -3H^2}\left[G_{m^2}(Z)+\frac{H}{2\pi^2\sigma}\frac{Z}{(\lambda-2)}\right] . \tag{D4}$$

Expanding the hypergeometric function in (38) in powers of its argument, it is easy to check that the second term in the right-hand side of (D4) exactly cancels the pole of G_{m^2} as $m^2\to -3H^2$, so G_A is well defined.

The mean fluctuation in the radial coordinate is given by [see (55)]

$$\langle\Delta^2(\tilde{t})\rangle=\cos^2\tilde{t}\langle\phi^2(\tilde{t})\rangle=\cos^2\tilde{t}\lim_{\gamma\to 0}\hat{G}(\gamma,\tilde{t}) , \tag{D5}$$

[recall that $\hat{G}$ depends on γ through Z, defined in (39)]. As usual, $\hat{G}$ is formally divergent in the coincidence limit (because G_A is), so one has to smear it out over a region of radius s in order to find a finite answer. They key observation is that the two-point function $G_A(Z)$ is a de Sitter-invariant function, so at equal times it only depends on the geodesic distance between the two points. As a result, if we smear $\hat{G}(\gamma,\tilde{t})$ over a region of constant radius, the contribution from G_A remains time independent. It is then easy to see from (D3) that at large times ($\tilde{t}\to\pi/2$), the smeared two-point function $\hat{G}_s$ is dominated by the term

$$\hat{G}_s(0,\tilde{t})\approx\frac{H}{3\pi^2\sigma}\frac{1}{\cos^2\tilde{t}}=\frac{H^3R^2(t)}{3\pi^2\sigma} , \tag{D6}$$

and from (D5), Eq. (58) follows.

Note that Eq. (58) could also have been guessed on dimensional grounds. The factor σ^{-1} comes from the normalization of the modes [see Eq. (8)]. Once σ^{-1} has been factored out from the two-point function, we are left with a field theory in de Sitter space, in which the only dimensionful parameter is H. Since $\bar{\Delta}^2$ has dimensions of length squared, we must have $\bar{\Delta}^2\sim H\sigma^{-1}$.

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