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Cosmological constraints on cosmic-string gravitational radiation

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The primordial nucleosynthesis and pulsar timing noise constraints on cosmic-string gravitational radiation are computed. The computation consists of a numerical integration of the Friedmann-Robertson-Walker Einstein equations which describe a universe containing radiation, dust, and a "one-scale"-model cosmic-string component. The procedure takes into account the effects of the annihilations of massive particle species on the equation of state of the cosmological fluid. An expression for the power emitted per mode of oscillation by a cosmic-string loop, suggested by both analytic calculations and recent numerical simulations, is used. The results of the computation are spectra of the cosmic-string gravitational radiation at nucleosynthesis and at present. Comparison of these spectra with the observed bounds on pulsar timing noise, and the observed bound on the effective number of light neutrino species permitted by the model of nucleosynthesis, allows one to exclude a range of values of μ, the cosmic-string linear mass density, for certain values of α, the size of a newly formed loop as a fraction of the particle horizon radius. We find constraints to μ which are more restrictive than any previous limit.

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I. INTRODUCTION

Cosmic strings are tubes of the quanta of a grand unified field, which may have formed during a phase transition in the early Universe. The properties and behavior of cosmic strings are well detailed in the published literature [1–4]. During the past ten years, a great amount of analytical and numerical work has been undertaken to answer the question "can cosmic strings explain the formation and growth of galaxies and clusters?" (for example, see [3,4]). Parallel to this effort, periodic checks have been made to assure that successful cosmic-string scenarios are consistent with observation. These checks require that the phenomena associated with the presence of a cosmic-string network lie within the tolerances of three of the most important observations of the standard cosmology: (1) the isotropy of the microwave background radiation; (2) primordial nucleosynthesis; and (3) limits on a stochastic gravitational-radiation background observed through pulsar timing residuals.

The primary mechanism for energy loss by a cosmic-string network is through the formation of loops, which in turn emit gravitational radiation. A network of cosmic strings, through this process, will contribute both to a stochastic spectrum of background gravitational radiation, leaving a (rather indistinct) signature of noise in the measurement of pulsar periods, and to the radiation-driven expansion of the Universe. These effects have been translated into the two tests of cosmic strings which will be considered in this paper. These two tests will now be briefly described.

The presence of gravitational radiation in the Universe will cause a slight distortion of the signal from pulsars, which will be observed on Earth as timing residual noise; an upper limit to the amount of gravitational radiation consistent with observed timing noise has been formulated by Stinebring *et al.* [5]. They conclude that the gravitational radiation in the logarithmic frequency interval $f \in [f_{\rm obs}, ef_{\rm obs}]$, where $f_{\rm obs} = (7.1 \text{ yr})^{-1} = 4.5 \times 10^{-9}$ Hz, is constrained at the present time to contribute a fraction smaller than

$$\left.\frac{f}{\rho_{\rm crit}} \frac{d\rho_{\rm gr}}{df}\right|_{f_{\rm obs}} \leq 4 \times 10^{-7} h^{-2} \tag{1.1}$$

of the critical energy density. Here, the critical energy density is $\rho_{\rm crit} = 3c^2 H_0^2 / 8\pi G$, where the Hubble constant is $H_0 = 100h$ km/sec Mpc. The frequency $f_{\rm obs}$ is the lowest-frequency wave observable in the 7.1 years of observation.

The second constraint on cosmic strings arises from the highly successful "standard model" of primordial nucleosynthesis, based on the standard model of particle physics. In this framework, a certain number of light neutrino species, N_ν, are allowed by a successful model of primordial nucleosynthesis [6–9]. The presence of excess amounts of gravitational radiation at the time of nucleosynthesis will produce the same effects as an excess number of light neutrino species: an increase in the rate of expansion, leading to an overabundance of helium, in disagreement with observation. Thus, recent calculations [7] require $N_\nu \leq 3.4$, which may be written as a constraint on the cosmic-string-produced gravitational-radiation energy density at the time of nucleosynthesis:

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$$\Omega_{\rm gr} \le \frac{\frac{7}{8}(N_\nu - 3)}{1+3\times\frac{7}{8}+2\times\frac{7}{8}}\Omega_{\rm rad}$$
$$= 0.163\times(N_\nu-3)\Omega_{\rm rad}\ . \qquad (1.2)$$

(A derivation of these numerical factors is found in [10,11].) Here, $\Omega_{\rm gr}=\rho_{\rm gr}/\rho_{\rm crit}$ and $\Omega_{\rm rad}=\rho_{\rm rad}/\rho_{\rm crit}$ give the energy density in gravitational radiation and relativistic matter, respectively, as a fraction of the critical energy density at the time of nucleosynthesis. This equation assumes the three neutrino species in the standard particle-physics model to be light, so that N_ν gives the maximum number of light neutrino species consistent with nucleosynthesis.

In this paper, these constraints on the spectrum of gravitational radiation will be used to obtain constraints on values of μ, the linear mass density of the cosmic strings, for certain values of α, the size of a newly formed cosmic-string loop as a fraction of the particle horizon radius. We point out that the existing constraints that may be found in the literature generally refer only to μ, because the value of α is not a "free parameter"; in principle it can be determined from numerical simulations of the string network. There is currently no consensus regarding the value of this parameter, so we give results for a wide range of reasonable values of α. For this wide range of values of α, we find a maximum allowed value of μ consistent with cosmological constraints.

The constraints due to pulsar timing noise and primordial nucleosynthesis have been previously considered by a number of authors [12–22]. Successive authors have found a tighter bound on $G\mu/c^2$. This bound currently is of the order 10^{-6}, which is generally regarded as the minimum acceptable value of $G\mu/c^2$ for a cosmic-string network to make a useful contribution to galaxy formation. The tightening of the bound has been due to (1) improvement of pulsar timing, (2) a decreasing limit on the number of light neutrino species consistent with nucleosynthesis, and (3) the more accurate models of cosmic-string networks provided by detailed numerical simulations. A short summary of the constraints obtained by previous authors follows in Sec. II.

The bounds obtained in this work have several advantages over past work. First, they benefit from the most recent results of numerical simulations and astronomical observations. Second, an improved model of the emission of gravitational waves by an oscillating loop, obtained from recent numerical simulations, is used [14,23]. Another advantage is that our work attempts to take into account all processes which could affect the spectrum of cosmic-string gravitational radiation. These include the effects of the annihilations of massive particles on the equation of state of the cosmological fluid, and the deviation from pure radiation- or matter-dominated expansion caused by the presence of the cosmic-string network. The end result is that it is possible to generate an entire, realistic power spectrum of the cosmic-string gravitational radiation.

A description of the effects included and the methods used to carry out the computation are given in Sec. III. In Sec. IV, the results are presented and interpreted.

Note that Newton's constant G and the speed of light c have been retained in all equations which involve μ. In other equations, $k=c=G=1$, where k is Boltzmann's constant.

II. SUMMARY OF PREVIOUS CONSTRAINTS

The observed limits on a power spectrum of stochastic gravitational radiation at the present time, and at the time of nucleosynthesis, have been used by previous authors [12–22] to bound the cosmic-string linear mass density μ. This section will not duplicate work by other authors, but will summarize their key assumptions and results. A series of tables are included for comparison. The goal will be to indicate the weaknesses of past calculations, and motivate the more detailed computation which is the main thrust of this paper.

Basis for comparison of calculations

The one-scale model of cosmic strings, the model of gravitational-radiation emission by cosmic strings, and the cosmological model comprise the common elements of the past work. In this section, the key elements of these three models, and the pulsar and nucleosynthesis bounds, will be described with tables. The next five paragraphs describe the five different sections of these tables.

All calculations to date have used the "one-scale" model which describes the basic features and evolution of a cosmic-string network. This model assumes a homogeneous Friedmann-Robertson-Walker (FRW) metric $ds^2=-dt^2+a^2(t)d\mathbf{x}^2$. Its one characteristic length scale is the horizon radius $l(t)=a(t)\int_0^t dt'/a(t')$. The cosmic strings are described by three dimensionless parameters: A, B, and α. There will be on the average $A=\rho_\infty l^2(t)/c^2\mu$ cosmic strings of length greater than $l(t)$ per horizon volume. The energy density of these "long" strings is given by ρ_∞. The number of loops formed per horizon time per horizon volume will be $B=l^4(t)V^{-1}(t)dN_{\rm loops}/dl(t)$, where $N_{\rm loops}$ is the total number of loops present within $V(t)$, the comoving volume of the Universe. The size of a newly formed loop will be the fraction α of the horizon radius. However, the peculiar velocity of the loop rapidly redshifts to zero, reducing the length of the loop to a fraction f_r of its formation length. After this redshifting, the loop size may be written as $L_{\rm loop}=f_r\alpha l(t_{\rm loop})$. Note that, given values of the parameters A and α, the string equations of motion uniquely determine the value of B. [See Eq. (3.8).] Early calculations often set $f_r=1$, did not use A, and set B and α to ~ 1 for order-of-magnitude estimates. More recent work has taken advantage of the results of numerical simulations to obtain values for these parameters. The values of these parameters used in the calculations are listed in the following tables.

The model of gravitational-radiation emission by loops is based on the expression for the power radiated by a loop: $P=\sum_{n=1}^{\infty}P_n G\mu^2 c=\gamma G\mu^2 c$. The dimensionless parameter γ describes the rate at which a loop converts its energy into gravitational radiation. A looped formed with length $L_{\rm loop}$ at time $t_{\rm loop}$ will radiate at the time-dependent frequencies $f_n(t)=2n/[L_{\rm loop}-\gamma G\mu(t-t_{\rm loop})]$ until it has radiated all its energy, and disappears. The constants P_n characterize the power $P_n G\mu^2 c$ radiated at frequency f_n. Early calculations made order-of-magnitude estimates of γ, and assumed that all emission occurs in the $n=1$ mode at fixed frequency. This has been noted in the tables.

While all past work has examined the cosmic-string network in an FRW background, the further details of the cosmological model have varied greatly. These include the transition from the radiation- to matter-dominated expansion eras, the massive particle annihilations which serve to reheat the cosmological fluid, and the effects of the cosmic strings and cosmic-string gravitational radiation on the expansion rate of the Universe. The details of the different cosmological models are listed in the following tables.

The upper limit on the energy density in a stochastic gravitational-wave background [5] has lowered considerably since the observation of pulsar timing noise began eight years ago. This is due to the fact that during this time, the upper limit has decreased as

$$\frac{f}{\rho_{\rm crit}}\frac{d\rho_{\rm gr}}{df}\propto T_{\rm yrs}^{-4} ,$$

where $T_{\rm yrs}$ is the observation time in years. (It is unlikely that the limit will continue to decrease in proportion to $T_{\rm yrs}^{-4}$; see Sec. IV.) Thus, the later papers have established constraints on cosmic strings which are tighter than those imposed by the early papers. In the following tables, we list the limit to

$$\Omega_{\rm gr}(f_{\rm obs})\equiv\frac{f}{\rho_{\rm crit}}\frac{d\rho_{\rm gr}}{df} ,$$

the frequency $f_{\rm obs}$, and the resultant limit on $G\mu/c^2$. The reader should note that $\Omega_{\rm gr}(f_{\rm obs})$ differs from $\Omega_{\rm gr}$. The former is the fraction of critical energy density in the logarithmic frequency interval $f\in[f_{\rm obs},ef_{\rm obs}]$. The latter is the total fraction of critical energy density. We retain this notation throughout the paper.

The upper limit on the number of light neutrino species consistent with the standard model of nucleosynthesis has also dropped in the past decade, tightening the limit on cosmic-string models. In the following tables, we list the limit to N_ν, and the resultant limit on $G\mu/c^2$.

We now examine past calculations of the nucleosynthesis and pulsar timing limits on the cosmic-string-produced gravity-wave spectrum. Note that the parameters listed in the following tables may differ from those used in the original papers by powers of two. This is because our parameters are defined in terms of the horizon radius $l(t)$ rather than the time t. In the radiation-dominated era, $l(t)/ct=2$.

Vilenkin (1981)

The earliest estimate of the energy density and power spectrum of the gravitational radiation produced by the loops of a cosmic-string network was made by Vilenkin [12]. In this purely analytic calculation, he made an order-of-magnitude comparison of the energy density in radiation emitted by the string loops with the bounds on energy density consistent with the observed cosmology. One assumption made (see Table I) was that a loop formed at time $t_{\rm loop}$ was assumed to radiate only in the fundamental mode, at the constant frequency $\sim t_{\rm loop}^{-1}$ until it evaporated at time $\sim(G\mu/c^2)^{-1}t_{\rm loop}$. Another assumption was that B and α were of order unity; the parameter A did not enter directly into this calculation. In 1981 the most stringent limit on cosmological sources of gravitational radiation was $\Omega_{\rm gr}\leq\Omega_{\rm rad}$ at the time of nucleosynthesis [24,25]. (It appears to have been incorrectly reported that the first nucleosynthesis constraints on cosmic strings were calculated much later.) While this limit was originally expressed in terms of a fraction of the radiation energy density, it may also be expressed as an

TABLE I. Summary of the evaluation of constraints on μ by Vilenkin [12] and Hogan and Rees [13].

Parameter	Vilenkin [12]	Hogan and Rees [13]
$\rho_\infty l^2(t)/c^2\mu=A$		
$l^4(t)V^{-1}(t)dN_{\rm loops}/dl=B$	~ 4	
$L_{\rm loop}/l(t)=\alpha$	~ 0.5	
$-\dot E/(Gc\mu^2)=\gamma$	~ 1	~ 1
Power per mode P_n	$P_1=\gamma\sim 1$	$P_1=\gamma\sim 1$
Emission $f_n(t)$	constant	constant
Particle annihilation		
Strings affect expansion		
Observed limit on $\Omega_{\rm gr}$		$\Omega_{\rm gr}\leq 1.6\times 10^{-4}h^{-2}$
Sensitive frequency		$f_{\rm obs}\sim 1\ {\rm yr}^{-1}$
Pulsar timing limit on μ		$G\mu/c^2\sim 10^{-6}$ ok
Observed limit on N_ν	$N_\nu\lesssim 6$	
Nucleosynthesis limit on μ	$G\mu/c^2\lesssim 10^{-3}$	

equivalent number of light neutrino species, yielding $N_\nu \lesssim 6$. The primary conclusion of Vilenkin's work was that values of μ as large as $G\mu/c^2 \sim 10^{-3}$ were consistent with the limits of observational cosmology and satisfied the requirements of galaxy formation.

Hogan and Rees (1984)

The first check of the cosmic-string loop gravitational-wave spectrum against the timing noise of the millisecond pulsar PSR1937+21 was made by Hogan and Rees [13]. Their approach did not explicitly use the one-scale model, although the assumptions regarding the behavior of the cosmic-string network were not different from those of the one-scale model. They made order-of-magnitude estimates based on assumptions similar to those of Vilenkin (see Table I) in order to calculate the energy density in gravitational waves caused by the radiating cosmic-string loops. These assumptions included the simplification that the cosmic-string loops radiate at a single, constant frequency. They determined that the millisecond pulsar timing noise was sensitive to the gravitational waves emitted by cosmic-string loops which formed shortly after nucleosynthesis. Because the pulsar timing measurements were only ~ 1 year old, in the logarithmic frequency interval at $f_{\rm obs} \sim 1\ {\rm yr}^{-1}$ the pulsar constraint was a loose $\Omega_{\rm gr}(f_{\rm obs}) \le 1.6\times 10^{-4} h^{-2}$. Consequently, they found that μ was not severely restricted at the present time. The essential shape of the spectrum $\Omega_{\rm gr}(f) = (f/\rho_{\rm crit})(d\rho_{\rm gr}/df)$ of stochastic gravitational radiation produced by a network of cosmic-string loops was first sketched in their paper. (For the spectra obtained in the present paper, see Fig. 3.) Ultimately, their main conclusion was to show that $G\mu/c^2 \sim 10^{-6}$, a desirable value for seeding galaxy formation, may contradict the bounds set by refined pulsar timing measurements within the near future.

Vachaspati and Vilenkin (1985)

In 1985, Vachaspati and Vilenkin [14] reexamined the work of Vilenkin, in order to improve the earlier, crude treatment of cosmic-string gravitational radiation. In their work, the gravitational radiation emitted by an oscillating cosmic-string loop was examined in detail. They found $\gamma \sim 100$, and that the power emitted per mode of oscillation of a particular class of loops behaved as the power law $P_n \propto n^{-4/3}$ for the higher modes of emission. Empirical values of the P_n's were found for the lower emission modes, for the particular class of loops studied. They exploited this work to derive an expression for the total energy density in gravitational waves, emitted by a cosmic-string network in a radiation-dominated universe (see Table II). They made an improvement over past calculations by accurately expressing the frequency of emission per mode of oscillation as $f_n(t) = 2n/L_{\rm loop}(t,t')$, which changes with time as the loop shrinks. In this equation, $L_{\rm loop}(t,t') = L_{\rm loop}(t',t') - \gamma G\mu(t-t')/c$ gives the size at time t of a loop formed with size $L_{\rm loop}(t',t')$ at time t'. They approximated the effects of the first 100 modes of gravitational-wave emission, ultimately arriving at the expression

$$\Omega_{\rm gr}(f_{\rm obs}) \approx \frac{32\pi}{9}\alpha^{3/2} B (G\mu/\gamma c^2)^{1/2} \Omega_\gamma \qquad (2.1)$$

for the fraction of critical energy density in gravitational waves within the logarithmic frequency interval bounded below by $f_{\rm obs} \sim 1\ {\rm yr}^{-1}$. This equation, taken from [14], does not properly show the dependence of $\Omega_{\rm gr}(f_{\rm obs})$ on α. Rather, $\Omega_{\rm gr}(f_{\rm obs}) \propto (\alpha G\mu/c^2)^{1/2}$. This is due to the fact that B is not an independent parameter, but can be expressed in terms of A and α; the parameter A, however, was not used directly in this calculation. [See Appendix B for a derivation of the correct expression; see Eq. (3.8) for the relation between A, B, and α.] They did find, as did Hogan and Rees, that the millisecond pulsar timing

TABLE II. Summary of the evaluation of constraints on μ by Vachaspati and Vilenkin [14] and Davis [15].

Parameter	Vachaspati and Vilenkin [14]	Davis [15]
$\rho_\infty l^2(t)/c^2\mu = A$		
$l^4(t)/V(t) dN_{\rm loops}/dl = B$	4	~ 4
$L_{\rm loop}/l(t) = \alpha$	~ 0.5	~ 0.5
$-\dot{E}/(Gc\mu^2) = \gamma$	100	100
Power per mode P_n	$P_n \propto n^{-4/3}$ for $n \gg 1$	$P_1 = \gamma$
Emission $f_n(t)$	$2n/L(t)$	constant
Particle annihilation		
Strings affect expansion		
Observed limit on $\Omega_{\rm gr}$	$\Omega_{\rm gr} \le 10^{-5}$	
Sensitive frequency	$f_{\rm obs} \sim 1\ {\rm yr}^{-1}$	
Pulsar timing limit on μ	$G\mu/c^2 \sim 10^{-6}$ ok	
Observed limit on N_ν		$N_\nu \le 4.0$
Nucleosynthesis limit on μ		$G\mu/c^2 \sim 10^{-6}$ not ok

noise is sensitive primarily to the gravitational waves emitted by loops formed shortly after nucleosynthesis. (This is not necessarily so, as will be shown in Sec. IV.) Consequently, the value $G\mu/c^2 \sim 10^{-6}$ satisfied the observed constraint. This result agrees with earlier, order-of-magnitude estimates of the gravitational-radiation energy density.

Davis (1985)

The restrictions posed by primordial nucleosynthesis on cosmic strings were considered by Davis [15]. Using the same, simple framework as Vachaspati and Vilenkin (see Table II), the accumulated gravitational-radiation energy density was compared to the energy density of the cosmological fluid at the time of nucleosynthesis. Davis found an expression of the form

$$\rho_{gr} = \frac{16\pi}{9} B\alpha^{3/2}(G\mu/c^2\gamma)^{1/2}\ln(t_{nuc}/t_f)\rho_{rad} \qquad (2.2)$$

which describes the total energy density in gravitational waves produced by loops which have evaporated prior to the onset of nucleosynthesis. In this expression, t_f is the time at which the cosmic-string network begins to move freely, and the first loops are formed. The time t_{nuc} refers to the time at which nucleosynthesis begins. (A derivation of the exact expression will be given in Appendix B.) Davis found that, for $G\mu/c^2 \sim 10^{-6}$, the cosmic-string gravitational-radiation energy density might exceed the observed limits on the energy density of approximately one additional species of light neutrino. At the time, it was commonly estimated that all the cosmic-string parameters were of the order of 1. However, using Eq. (3.8) to determine B from A and α, this order-of-magnitude calculation would have found cosmic strings in agreement with observation. Thus, it was the limitations of the early, simple model, not an overly restrictive cosmological constraint, which caused the cosmic strings to disagree with observations in this calculation.

Brandenberger, Albrecht, and Turok (1986)

A comprehensive effect to focus solely on the constraints on cosmic strings through various cosmological phenomena was made by Brandenberger, Albrecht, and Turok [16] (see Table III). They correctly used the equations of motion to determine the relationship between B, α, and A [Eq. (3.8) in this paper]. However, they assumed that a loop radiates in its fundamental mode only, at a constant frequency, $f = 2/L_{loop}$. In order to model the change in the number of effective degrees of freedom after the QCD phase transition, the energy density was reduced by a numerical factor. They showed that the energy density in gravitational waves up to the time of nucleosynthesis was dominated by an expression of the form similar to that found by Davis. Thus, they find that the nucleosynthesis constraint for the 1984 limit of $N_\nu = 4.0$ [6] is satisfied for $G\mu/c^2 \sim 10^{-6}$. (As in many other papers, the conversion from a limit on N_ν to a limit on ρ_{gr} was incorrectly reported as $\rho_{gr}\rho_\gamma^{-1} \le 0.18$; the correct limit is $\rho_{gr}\rho_\gamma^{-1} \le 0.163$. However, the overall result was not affected.) With regards to the pulsar timing constraint, an expression similar to that of Vachaspati and Vilenkin is derived. Comparing the energy density in gravitational

TABLE III. Summary of the evaluation of constraints on μ by Brandenberger, Albrecht, and Turok [16] and Bennett [17].

Parameter	Brandenberger, Albrecht, and Turok [16]	Bennett [17]
$\rho_\infty l^2(t)/c^2\mu = A$	~ 4	10
$l^4(t)/V(t)dN_{loops}/dl = B$	equations of motion	equations of motion
$L_{loop}/l(t) = \alpha$	~ 0.5	0.05
$-\dot{E}/(Gc\mu^2) = \gamma$	50	20π
Power per mode P_n	$P_1 = \gamma$	$P_1 = \gamma$
Emission $f_n(t)$	constant	constant
Particle annihilation	correction for annihilations at QCD transition	corrections for annihilations due to minimal and maximal GUT
Strings affect expansion		back reaction on expansion by strings and radiation
Observed limit on Ω_{gr}	$\Omega_{gr} \le 4.3\times 10^{-6} h^{-2}$	
Sensitive frequency	$f_{obs} \sim 0.3\ \mathrm{yr}^{-1}$	
Pulsar timing limit on μ	$G\mu/c^2 \le 1.5\times 10^{-3}$	
Observed limit on N_ν	$N_\nu \le 4.0$	$N_\nu \le 4.0$
Nucleosynthesis limit on μ	$G\mu/c^2 \sim 10^{-6}$ ok	$G\mu/c^2 \le 4\times 10^{-6}$

waves in the logarithmic frequency interval at the 1986 frequency $f_{\rm obs} \sim 0.3\ {\rm yr}^{-1}$, the limit $G\mu/c^2 \leq 1.5\times10^{-3}$ is found. This is the first calculation to not simply test a favored value of μ, but to find the upper limit on acceptable values of μ. As was indicated by Hogan and Rees, Brandenberger *et al.* stress that unless the timing noise increased, within 10 years cosmic-string scenarios which required $G\mu/c^2 \sim 10^{-6}$ would be ruled out.

Bennett (1986)

An extremely thorough treatment of the evolution of a system of cosmic strings, which included a calculation of the nucleosynthesis constraint, was carried out by Bennett in a pair of papers [17]. The first of these papers predates the Brandenberger, Albrecht, and Turok work (Table III refers to Bennett's second paper). Treating the cosmic strings as a dilute gas of long strings and loops in a FRW background, he obtained evolution equations for the gravitational-radiation energy density. The values used for A and α were suggested by the early numerical simulations of Albrecht and Turok [26]. This calculation included the effects due to redshifting of a loop's peculiar velocity (this is discussed further in Sec. III); at the time, the numerical simulations [26] suggested that only the fraction $f_r = 0.97$ of the loop energy is converted into gravitational radiation. The analytic calculation models the effect of the cosmic strings on the FRW cosmological fluid, which increased the radiation-dominated expansion rate. The calculation also includes numerical corrections to the ratio of the gravitational radiation to cosmological fluid energy density, in order to compensate for the massive particle annihilations in a minimally extended [no particles with masses between the Weinberg-Salam symmetry-breaking energy scale and the grand-unified-theory (GUT) transition energy scale] GUT model. Bennett found that the nucleosynthesis constraint $(\rho_{\rm strings}+\rho_{\rm gr})\rho_{\rm rad}^{-1} \leq 0.17$ (again, the exact limit is 0.163) is marginally satisfied for $G\mu/c^2 \sim 10^{-6}$. The resultant upper bound on μ is $G\mu/c^2 \leq 4\times10^{-6}$. At this time, nucleosynthesis imposed the most restrictive constraint on cosmic strings.

Accetta and Krauss (1989)

Accetta and Krauss [18] numerically computed the power spectrum of the gravitational radiation produced by the loops in a cosmic-string network in order to test the validity of certain cosmic-string scenarios against the pulsar timing and nucleosynthesis bounds. The analytic expression, which they adapted to a numerical calculation, was similar to the work of Vachaspati and Vilenkin, and Davis. Many improvements were made to the calculation of the constraints. Numerical factors were used to model the effects of the adiabatic reheating of the cosmological fluid by massive particle annihilations for several species of particles in a minimal GUT model. A smooth transition between the radiation- and matter-dominated eras was manufactured by making phenomenological corrections to the expansion scale factor. The behavior of the power P_n as a function of the emission mode n found by Vachaspati and Vilenkin was adopted. Thus, they used the empirical values of P_n for small values of n, and $P_n \propto n^{-4/3}$ for large values of n. The first 200 modes of oscillation by the cosmic-string loops were included in the sum of the emitted power P_n. In their work, the sizes of newly formed loops are modeled by a size probability distribution. The values of B, γ, and a range for α were suggested by the numerical simulations of Bennett and Bouchet [27] (see Table IV). However, neither the equations of motion nor A were used in this calculation. As a results, because they numerically computed the power spectrum of gravitational radiation for different values of B and α, the dependent value A differed. Thus it was not possible to make a fair comparison of the various power spectra, nor conclude what values of the parameters were in agreement with observation. The main result of this work was that for a variety of values of α and B, $G\mu/c^2 \sim 10^{-6}$ was consistent with pulsar timing and nucleosynthesis bounds.

Bennett and Bouchet (1990,1)

Two recent studies by Bennett and Bouchet [19] have combined Bennett's earlier, analytic work with results from their high-resolution numerical simulations in order to generate constraints on the gravity-wave background produced by cosmic strings (see Table IV). The two bounds were calculated through rather different means, and so will be discussed individually.

The nucleosynthesis limit was essentially a reevaluation of the 1986 calculation, with a new set of parameters as suggested by the recent numerical simulations. Thus, Bennett and Bouchet found that $N_\nu \leq 3.4$ implies $G\mu/c^2 \leq 6\times10^{-6}$. Recent work has suggested the possibility of an inhomogeneous QCD phase transition [28,29]. This would allow for a weakening of nucleosynthesis constraints, by increasing the number of light neutrino species to $N_\nu = 3.7$ in the case of an inhomogeneous transition. In such a case, they found $G\mu/c^2 \leq 1.1\times10^{-5}$. The conversion from the limit on the number of neutriono species to the limit on the energy density in gravitational waves, however, was done incorrectly. The limits on μ listed above, and in [19] are too tight by $\sim 20\%$. The limit $G\mu/c^2 \leq 6\times10^{-6}$ corresponds to $N_\nu \leq 3.3$, and $G_\mu/c^2 \leq 1.1\times10^{-5}$ corresponds to $N_\nu \leq 3.6$.

The calculation of the pulsar timing limit was similar to the work of Vachaspati and Vilenkin, and Brandenberger *et al.* Assuming that cosmic-string loops radiate only in the fundamental mode of oscillation, they derived an expression for the energy density in gravitational waves within the frequency interval to which pulsar timing measurements are sensitive. They found that these gravitational waves were emitted in the radiation-dominated era. They included the effects due to redshifting of a loop's peculiar velocity; recent high-resolution numerical simulations [30] suggest that $f_r = 0.71$. They also considered the limit in which the length of newly formed loops is very small. The motivation for having α very small is that the recent, high-resolution numerical simulations found loops forming on the smallest length scales resolved. Thus, for the calculation of the pulsar

TABLE IV. Summary of the evaluation of constraints on μ by Accetta and Krauss [18] and Bennett and Bouchet [19].

Parameter	Accetta and Krauss [18]	Bennett and Bouchet [19]
$\rho_\infty l^2(t)/c^2\mu = A$		52
$l^4(t)/V(t)dN_{\text{loops}}/dl = B$	$.5 < B < 5$	equation of motion
$L_{\text{loop}}/l(t) = \alpha$	$5\times10^{-3} < \langle\alpha\rangle < 1$	limit $\alpha\to0$
$-\dot{E}/(Gc\mu^2) = \gamma$	$55 < \gamma < 114$	50
Power per mode P_n	$P_n \propto n^{-4/3}$ for $n \gg 1$	$P_1 = \gamma$
Emission $f_n(t)$	$2n/L(t)$	constant
Particle annihilation	minimal GUT	minimal and maximal GUT
Strings affect expansion		back reaction of ρ_∞ and ρ_{gr}
Observed limit on Ω_{gr}	$\Omega_{\text{gr}} \le 2.2\times10^{-6}h^{-2}$	$\Omega_{\text{gr}} \le 4\times10^{-7}h^{-2}$
Sensitive frequency	$f_{\text{obs}} \sim 0.16$ yr^{-1}	$f_{\text{obs}} \sim 0.14$ yr^{-1}
Pulsar timing limit on μ	$G\mu/c^2 \sim 10^{-6}$ ok	$G\mu/c^2 \le 4\times10^{-5}$
Observed limit on N_ν	$N_\nu \le 4.0$	$N_\nu \le 3.4$
Nucleosynthesis limit on μ	$G\mu/c^2 \sim 10^{-6}$ ok	$G\mu/c^2 \le 6\times10^{-6}$

timing limit, an expression for $\Omega_{\text{gr}}(f_{\text{obs}})$ similar to that of Vachaspati and Vilenkin is found. This quantity is then evaluated for $\alpha \sim (f_{\text{obs}}t_{\text{present}})^{-1} \sim 10^{-9}$. (In fact the same result holds for $10^{-9} < \alpha < \gamma G\mu/c^2$. For smaller values of α, no cosmic-string gravitational radiation is observed in f_{obs} at the present.) In this limit of small loop size, Bennett and Bouchet found the constraint $G\mu/c^2 \le 4\times10^{-5}$. They also note that this constraint depends sensitively on the shape of the gravitational-wave power spectrum. The gravitational-wave power spectrum may rise at frequencies near f_{obs}, in which case, for $\alpha > 10^{-9}$, the constraint on μ may tighten.

Other work

There have also been other papers which examine the nucleosynthesis and pulsar timing constraints on cosmic strings, using updated values for N_ν and $\Omega_{\text{gr}}(f_{\text{obs}})$, and for the dimensionless parameters found in the cosmic-string model. We have not included descriptive tables for these papers, whose discussions of cosmological constraints on cosmic strings are primarily updates of earlier calculations.

In 1989, Albrecht and Turok [20] carried out a rough analytic calculation similar to that of [16]. In this calculation, they correctly showed that the shape of the power spectrum is flat, regardless of the behavior of P_n, in a radiation-dominated cosmology. They found that the value $G\mu/c^2 = 10^{-6}$ disagreed with the observations of both nucleosynthesis and pulsar timing noise. They pointed out, however, that effects such as loop fragmentation and the dilution of the gravitational-radiation energy density caused by massive particle annihilations might bring cosmic strings into agreement with observation.

In 1990, Brandenberger and Kung [31] updated the earlier work in [16] for the pulsar timing constraint. Using the most current values of various parameters and limits, they found that cosmic-string scenarios for which $G\mu/c^2 = 10^{-6}$ were not ruled out.

Several papers by Sanchez and Signore [21] have considered the restrictions on cosmic-string models. These papers have carried out calculations similar to those of [16] in order to restrict the parameter μ. Additionally, they considered the restrictions on the emission of electromagnetic waves by superconducting cosmic strings, in analogy with the models for the emission of gravitational waves by cosmic strings.

A recent paper by Quiros [22] has examined the nucleosynthesis limit on cosmic-string-produced gravitational radiation. This paper uses the simple, analytic expression for the total energy density in gravitational waves present at the time of nucleosynthesis (as is derived in Appendix B). What distinguishes this work is its detailed analysis of the restriction placed on the energy density in gravitational waves by the observed values and uncertainties in the helium abundance, the baryon density, and the neutron lifetime. This analysis ultimately arrives at a limit equivalent to restricting $N_\nu \le 3.2$.

While the calculations of Bennett and Bouchet and Accetta and Krauss have been the most realistic to date, several aspects may be improved. The energy density in string loops was not included by Bennett and Bouchet in their calculation of the nucleosynthesis limit, although the authors estimate that a change in the limit on μ by more than a factor of 4 is ruled out by their string simulations. The effect of the cosmic-string and gravitational-radiation energy densities on the expansion rate was calculated, rather than the full effect on the evolution of the FRW equations. That is, the effects on the horizon radius, the rate of loop production, and the cosmic-string and gravitational-wave energy densities were neglected. Rather than including the smoothly varying number of

TABLE V. Summary of the evaluation of constraints on μ by Caldwell and Allen (this paper).

Parameter	Caldwell and Allen (this paper)
$\rho_\infty l^2(t)/c^2\mu = A$	$A_{\text{radiation}}=52$, and $A_{\text{matter}}=31$
$l^4(t)/V(t)dN_{\text{loops}}/dl = B$	derived from equations of motion
$L_{\text{loop}}/l(t)=\alpha$	$\gamma G\mu/c^2<\alpha<0.1$
$-\dot{E}/(Gc\mu^2)=\gamma$	50
Power per mode P_n	$P_n \propto n^{-4/3}$
Emission $f_n(t)$	$2n/L(t)$
Particle annihilation	minimal GUT
Strings affect expansion	ρ_∞, ρ_{loops}, and ρ_{gr} included in FRW evolution
Observed limit on Ω_{gr}	$\Omega_{\text{gr}} \le 4\times 10^{-7}h^{-2}$
Sensitive frequency	see Fig. 3
Pulsar timing limit on μ	see Fig. 1
Observed limit on N_ν	$N_\nu \le 3.4$
Nucleosynthesis limit on μ	see Fig. 1

particle degrees of freedom in the energy density of the cosmological fluid, in many calculations, numerical corrections for the overall change in the number of particle degrees of freedom were made. To date, no single calculation has attempted to include all the aspects of a realistic model for the production of gravitational radiation by a cosmic-string network. Therefore, it appears that there is room for improvement of the gravitational-radiation limits to cosmic-string models (see Table V).

III. METHOD AND DETAILS OF COMPUTATIONS

The methods used in this paper to calculate the nucleosynthesis and pulsar timing constraints on cosmic strings will now be presented. This section will begin with a detailed description of our calculation, which was outlined in Table V. We will present the one-scale model of cosmic strings, the model of emission of gravitational waves by the string loops, and the cosmological model, which are used to derive the time evolution equations for all physical quantities of interest. We will then present the initial conditions used in our computation, and finally describe two details regarding the cosmological fluid. The calculation outlined in this section is adapted to a numerical computation, as discussed in Appendix A.

One-scale model

The starting point for most analytic calculations of the properties of a cosmic-string network is the one-scale model. The ability of this simple model to accurately describe the evolution of a cosmic-string network has been supported by numerical simulations [23,30,32]. This model consists of the following elements.

(1) The Universe is described by a homogeneous, spatially flat, $\Omega=1$ FRW cosmology with the metric $ds^2=-dt^2+a^2(t)(dx^2+dy^2+dz^2)$. The horizon radius $l(t)=a(t)\int_0^t dt'/a(t')$ serves as the characteristics length scale. All physically interesting quantities describing the cosmic-string network will be expressed in terms of this length scale.

(2) For simplicity, the physical volume of the Universe in which these calculations are made is expressed as $V(t)=a^3(t)L^3$. The constant coordinate volume L^3 ultimately cancels from all calculations.

(3) The energy density in long cosmic strings is given by

$$\rho_\infty = A\mu c^2/l^2(t)\ . \tag{3.1}$$

The parameter A represents the number of long strings present per horizon sized volume. Numerical simulations [30] suggest the values $A=52$ in the radiation-dominated era, and $A=31$ in the matter-dominated era. We smoothly interpolate between these two values when the Universe undergoes the radiation-to-matter transition.

(4) The size of a newly formed loop, chopped off the long string network at time t_{loop}, is a constant fraction α of the horizon radius. Loops tend to be cut off with large (relativistic) peculiar velocities: the loop center of mass (c.m.) is moving with respect to the rest frame of the cosmological fluid. Thus the energy of a loop at the time it is formed is the sum of two terms: a "kinetic" term arising from its peculiar velocity, and a "potential" term arising from its c.m. energy. In an expanding universe, just as for any relativistic massive particle, the "kinetic" part of the energy rapidly redshifts to zero. Albrecht and Turok [20] have calculated this redshifting effect on the loop energy. For a single loop with initial energy $E(t_i)$ and velocity v_i at time t_i, they found that the loop energy behaves as $E(t)=E(t_i)\{[a(t_i)v_i/a(t)]^2+1-v_i^2\}^{1/2}$. As $a(t)$ increases, this mechanism quickly damps away the peculiar velocity, leaving the loop with only the fraction $f_r=(1-v_i^2)^{1/2}$ of its initial energy. This redshifting of peculiar velocities does not affect the loop production rate, but it does change the size of a loop immediately after its formation to $L_{\text{loop}}(t_{\text{loop}})=f_r\alpha l(t_{\text{loop}})$. [Numerical simulations have suggested that loops chop them-

selves into smaller loops or "fragment" rapidly after formation until they reach a stable size. We take this stable size, after the loops have lost their peculiar velocity, to be $f_r \alpha l(t_{\rm loop})$.] Hence, only a fraction f_r of the total loop energy is converted into gravitational radiation; the rest is lost to redshifting. From their numerical simulations, Bennett and Bouchet [30] estimate that this fraction is $f_r = 0.71$.

In order to calculate the effect of the cosmic-string network on the expansion of the Universe, it is necessary to determine the cosmic-string equation of state. The procedure for deriving the equation of state of the cosmic-string network has been presented in other work [20,33]. Nevertheless, we will quickly outline the necessary steps. The stress tensor for the network of infinite strings and string loops may be written as the integral

$$T^{\mu\nu}(t,\mathbf{y}) = \mu a^{-4}(t) \int d\sigma (\epsilon \dot{x}^{\mu}\dot{x}^{\nu} - \epsilon^{-1} x'^{\mu} x'^{\nu}) \times \delta^3(\mathbf{x}(\sigma,\eta) - \mathbf{y}) \ , \tag{3.2}$$

where an overdot and a prime are derivatives with respect to η and σ, the temporal and spatial parameters on the world sheet of the cosmic string [20]. The vector $\mathbf{x}(\sigma,\eta)$ gives the coordinate position on the cosmic-string world sheet. The integral over σ runs over the entire network of long strings and loops. Upon averaging over the comoving volume, and using the gauge condition which defines $\epsilon, x'^2 = \epsilon^2(1-\dot{x}^2)$, we find the equation of state for the long strings to be

$$p_\infty = \tfrac{1}{3}\rho_\infty [2\langle v^2 \rangle - 1] \ , \tag{3.3}$$

where $\langle v^2 \rangle$ is the mean-squared velocity of the long strings. In the case of ultrarelativistic strings ($\langle v^2 \rangle = 1$) the equation of state is $p_\infty = \rho_\infty / 3$; the infinite strings behave like radiation. On the other hand, slow moving or static strings ($\langle v^2 \rangle = 0$) have the equation of state $p_\infty = -\rho_\infty / 3$. In this case, the cosmological expansion pumps energy into the long strings, which rapidly dominate the Universe. The numerical simulations of Bennett and Bouchett find that the long strings have the values $\langle v^2 \rangle = 0.43$ in the radiation-dominated era, and $\langle v^2 \rangle = 0.37$ [30] in the matter-dominated era; we have used these values in our calculation, smoothly interpolating between the two when the Universe undergoes the transition from radiation- to matter-dominated expansion. In the case of loops, which are by definition smaller than the horizon radius and do not feel the effects of the curvature of spacetime, $\langle v^2 \rangle = 1/2$ in the c.m. frame, so that

$$p_{\rm loops} = 0 \ . \tag{3.4}$$

The loop equation of state is similar to that of dust.

The rate of loop formation may be calculated using the conservation of the cosmic-string stress-energy tensor. The cosmic-string network does not directly interact with any of the particles which comprise the cosmological fluid; the only effect the string network has on the fluid is indirect (through its effect on the expansion rate of the Universe). Therefore, the stress tensor of the cosmological fluid and that of the cosmic-string network are separately conserved. From the conservation of the string stress tensor (3.2), we find the equation of energy conservation:

$$\frac{d}{dt}[a^3(t)(\rho + p)] = \dot{p} a^3(t) \ . \tag{3.5}$$

Here, an overdot indicates a derivative with respect to time t. In this equation, $\rho = \rho_\infty + \rho_{\rm loops} + \rho_{\rm gr}$ and $p = p_\infty + \rho_{\rm gr}/3$. Now, the long strings form loops, which then radiate gravitational waves. The rate at which the long strings form loops is independent of whether the loops emit gravitational waves, save for the indirect effect of $\rho_{\rm gr}$ on the rate of cosmological expansion. Therefore, we may ignore the gravitational-radiation terms in Eq. (3.5) in solving for the rate of loop formation. Using the long string equation of state, we may solve for the loop energy which is related to the number of loops by the differential equation

$$\frac{dE_{\rm loops}}{dt} = \mu f_r \alpha l(t) c^2 \frac{dN_{\rm loops}}{dt} \ . \tag{3.6}$$

Thus, we find that the rate of loop formation is

$$\frac{dN_{\rm loops}}{dt} = -\frac{V(t)}{\mu \alpha l(t) c^2} \left[\dot{\rho}_\infty + 2\frac{\dot{a}}{a}\rho_\infty [1 + \langle v^2 \rangle] \right] \ . \tag{3.7}$$

Here, $N_{\rm loops}(t)$ gives the total number of loops produced up to the present time t within the volume $V(t)$, and $\langle v^2 \rangle$ refers to the mean squared velocity of the long cosmic strings.

We may now use the differential equation for the rate of loop formation to determine the relationship between the parameters α, B, and A. While past work has often made order-of-magnitude estimates of B, an exact expression for B may be given. The number of loops formed per horizon time is

$$B = \frac{l^4(t)}{V(t)} \frac{dN_{\rm loops}}{dl} = 2\frac{A}{\alpha}[1 - m(1 + \langle v^2 \rangle)] \ , \tag{3.8}$$

where $m = 1/2$ in the radiation-dominated era, and $m = 2/3$ in the matter-dominated era. This expression is given only for comparison with the earlier work presented in Sec. II. Equation (3.7), rather than expression (3.8) is used in all our calculations.

Emission of gravitational radiation by string loops

The model of the emission of gravitational radiation by cosmic-string loops, necessary to calculate the energy in cosmic-string loops and cosmic-string loop-produced gravitational waves, is composed of the following three elements.

(1) A loop is assumed to radiate energy at the constant rate $dE/dt = -\gamma G \mu^2 c$. Recent numerical simulations suggest that $\gamma \approx 50$ [23] is a reasonable value for the average loop produced by the cosmic-string network. It follows that the length at time t of a loop formed at time t' is $L(t,t') = f_r \alpha l(t') - \gamma G \mu (t - t')/c$. Such a loop will have lost all its energy and thus disappear, or "evaporate" at a later time $t'' = t'/\beta(t')$, where we define $\beta(t') = [1 + f_r \alpha l(t') c / \gamma G \mu t']^{-1}$. Because loops never

fragment in this model, a loop at time t is uniquely identified by its time of "birth" or by its time of "death." It will be easier to discuss the birth and death of loops if we define the following functions. First, $t_D(t)=t/\beta(t)$ gives the time of death of a loop which formed at time t. Second, $t_B(t)$ gives the time of birth of a loop which evaporated at time t. Here, the time t_B is found by solving for the root of the equation $t_D[t_B(t)]=t$.

(2) The frequency of the gravity waves emitted at time t by a loop formed at time t' is given by $f_n(t,t')=2n/L(t,t')$, where $n=1,2,3,\ldots$ labels the oscillation mode.

(3) The power emitted in each mode of oscillation at frequency f_n is described by the mode coefficients P_n, where the total emitted power is $P=\sum_{n=1}^{\infty}P_n G\mu^2 c$. Recent studies by Allen and Shellard [23] indicate that $P_n=\gamma n^{-4/3}/\sum_{k=1}^{\infty}k^{-4/3}$ is a reasonable description of the distribution of power per mode of emission. Because this sum converges slowly, higher emission modes contribute significantly to the total power. Our calculation includes all modes of emission.

We may now construct the differential equation describing the rate of change of the energy in loops present in a volume $V(t)$ at time t. The energy in cosmic-string loops is μ times the total length present in loops. This may be written as the integral

$$E_{\text{loops}}(t)=\int_{\max[t_f,t_B(t)]}^{t}dt'\frac{dN_{\text{loops}}}{dt'}\mu L(t,t') \ . \tag{3.9}$$

The lower bound of this integral is the time of formation of the oldest loops still present in the network at time t. We are interested in finding a differential equation for the rate of change in loop energy. Taking the derivative of expression (3.9), we find

$$\dot{E}_{\text{loops}}(t)=\mu f_r \alpha l(t)c^2\frac{dN_{\text{loops}}}{dt}-\gamma G\mu^2 c C_{\text{loops}}(t) \ . \tag{3.10}$$

The first term on the right-hand side (RHS) represents the rate of change in energy by loops formed in the time step from t to $t+dt$. The second term represents the rate of change in energy due to the emission of gravitational waves by the loops present throughout the time step. We have defined

$$C_{\text{loops}}(t)=N_{\text{loops}}(t)-N_{\text{loops}}(\max[t_f,t_B(t)]) \tag{3.11}$$

to be the total number of loops still present at time t. [The difference $N_{\text{loops}}(t)-C_{\text{loops}}(t)$ is the total number of loops which have evaporated by time t.] The differential equation (3.10) may be integrated in order to find $\rho_{\text{loops}}=E_{\text{loops}}(t)/V(t)$, the loop energy density. A description of the numerical integration, and the special handling given to Eq. (3.11) is found in Appendix A.

The differential equation for the rate of change of the energy in gravitational waves may be obtained from the conservation of the string stress tensor. Using Eq. (3.5) and (3.10), with $\rho=\rho_\infty+\rho_{\text{loops}}+\rho_{\text{gr}}$ and $p=p_\infty+\rho_{\text{gr}}/3$, we find

$$\dot{E}_{\text{gr}}(t)=-\frac{\dot{a}}{a}E_{\text{gr}}(t)+\gamma G\mu^2 c C_{\text{loops}}(t) \ . \tag{3.12}$$

The first term on the RHS represents the rate of change in energy of gravitational waves due to the redshift. The second term represents the rate at which new radiation is emitted by the cosmic-string loops present. This differential equation may be integrated in order to find $\rho_{\text{gr}}=E_{\text{gr}}(t)/V(t)$, the gravitational-wave energy density. The numerical integration of Eq. (3.12), and the method by which we calculate the power spectrum of gravitational radiation, are discussed in Appendix A. An analytic approximation to ρ_{gr} is given in Appendix B.

Cosmological model

We will now construct a cosmological model which will serve as the framework for the evolution of the cosmic-string network. Expressions for the evolution of the energy of the cosmic strings and gravitational radiation were given above. In the following discussion, we will present a method for computing the energy density of the cosmological fluid which surrounds the cosmic strings.

The energy density used to determine the expansion of an FRW spacetime in the presence of the cosmic-string network is composed of (1) the energy density of the long strings ρ_∞, (2) the energy density of the loops ρ_{loops}, (3) the energy density of the loop-produced gravitational radiation ρ_{gr}, and (4) the energy density of the cosmological fluid ρ_{fluid}. Thus, the total energy density is

$$\rho_{\text{total}}=\rho_{\text{fluid}}+\rho_\infty+\rho_{\text{loops}}+\rho_{\text{gr}} \ . \tag{3.13}$$

The cosmological fluid, which is the focus of the following discussion, is composed of a relativistic component and a nonrelativistic, matter component

$$\rho_{\text{fluid}}=\rho_{\text{rel}}+\rho_{\text{matter}} \ . \tag{3.14}$$

The matter term is simply

$$\rho_{\text{matter}}(t)=\rho_{\text{matter}}(t_f)\left[\frac{a(t_f)}{a(t)}\right]^3 , \tag{3.15}$$

where $\rho_{\text{matter}}(t_f)$ is a constant, which will be discussed in the next part of this section.

The energy density of the relativistic part of the cosmological fluid, a gas of relativistic particles in thermal equilibrium at a temperature T is

$$\rho_{\text{rel}}(T)=T^4\sum_i\left[\frac{T_i}{T}\right]^4\frac{g_i}{2\pi^2}\int_{x_i}^{\infty}\frac{u^2\sqrt{u^2-x_i^2}du}{e^u\pm 1}$$
$$=\frac{\pi^2}{30}g(T)T^4 \ . \tag{3.16}$$

The sum is the over the i species of fermions and bosons present in the cosmological fluid, each of which may have a different temperature and number of degrees of freedom. (See [10] for a complete discussion.) The effective number of energy degrees of freedom of the relativistic gas is $g(T)$. Here, m_i, T_i, and g_i are the mass, temperature, and number of degrees of freedom of the ith particle species respectively, and $x_i=m_i/T$. The plus and minus signs in the denominator of the integrand are for Fermi-

Dirac and Bose-Einstein statistics, respectively. For the particles in a minimal GUT model (see Table VI), we may numerically integrate Eq. (3.16). It then follows that for a given temperature, the energy density of the relativistic gas may be computed.

In order to determine $\rho_{\rm rel}$ as a function of time rather than temperature, we will take advantage of the conservation of entropy by the relativistic component of the cosmological fluid. When the temperature of the relativistic cosmological fluid drops below the rest mass of any of the various particle species, the equilibrium abundance of that particle will rapidly drop to zero. As this occurs, the characteristic interaction times of the particles in question are always smaller than the characteristic expansion time $a/\dot{a}$. For this reason, the entropy

$$S(t,T)=V(t)T^3\sum_i \left[\frac{T_i}{T}\right]^3 \frac{g_i}{2\pi^2} \times \int_{x_i}^{\infty} \frac{u^2\sqrt{u^2-x_i^2}+\frac{1}{3}(u^2-x_i^2)^{3/2}}{e^u\pm 1}du = V(t)\frac{2\pi^2}{45}g_S(T)T^3 \tag{3.17}$$

of this fluid will remain constant throughout its evolution:

$$S|_{T_1,t_1}=S|_{T_2,t_2} . \tag{3.18}$$

The effective number of entropic degrees of freedom, $g_S(T)$, for the particles in a minimal GUT model (see Table VI; note that the massless particle g is the gluon) may be found by integrating Eq. (3.17). Then for a given temperature, the entropy of the relativistic gas may be computed. The sole purpose of $g_S(T)$ in this calculation will be, for a given T_1,t_1, and t_2, to invert Eq. (3.18) and solve for the new temperature T_2. Therefore, given T_1 and t_1, we may find $\rho_{\rm rel}(t_2)$ at any later time t_2.

The behavior of this cosmology is determined by the total energy density given in Eq. (3.13), through the differential equation

$$\dot{a}(t)=a(t)\left[\frac{8\pi G}{3}\rho_{\rm total}\right]^{1/2} , \tag{3.19}$$

for the cosmological scale factor. It is clear that, because of the presence of the cosmic strings, the expansion of the Universe will not be purely radiation, $a(t)\propto t^{1/2}$, or matter, $a(t)\propto t^{2/3}$, driven. For this reason the horizon radius will deviate from $l(t)=2ct$ in the radiation-dominated era, and $l(t)=3ct$ in the matter-dominated era.

Initial conditions

We will now describe the initial conditions for the evolution of this cosmology. We will show how to find the initial temperature T_f and energy density $\rho_{\rm matter}(t_f)$ when the cosmic-string network begins to evolve at time t_f.

We begin by determining the time t_f. The cosmic-string network will move freely of the friction of the cosmological fluid when the force of friction becomes weaker than the string tension. This will occur [34,35] at time $t_f=t_{\rm Planck}c^4/G^2\mu^2$. Thus, the choice of linear mass density μ determines the initial time t_f.

TABLE VI. Particles and masses in a standard, minimal, GUT model.

Particle	Mass
Higgs boson	100 GeV
Z^0	92.4 GeV
t	90 GeV
$W^\pm$	81 GeV
b	4.7 GeV
$\tau^\pm$	1.784 GeV
c	1.5 GeV
s	200 MeV
$\pi^{0,\pm}$	140 MeV
$\mu^\pm$	105.7 MeV
d	9 MeV
u	5 MeV
$e^\pm$	0.511 MeV
$\gamma,\nu_e,\nu_\mu,\nu_\tau,g$	Massless

The temperature T_f is determined by matching the total energy density before and after the time t_f. Prior to the free evolution of the cosmic-string network, the cosmological fluid was a radiation-dominated gas. The energy density was $\rho_{\rm total}=3c^2/32G\pi t_f{}^2$. The initial value of the horizon radius was $l(t)=2t_f$, which allows us to determine the initial value of ρ_∞. Matching the energy densities

$$\rho_{\rm total}(t_f)=\rho_\infty(t_f)+\rho_{\rm rel}(T_f) , \tag{3.20}$$

we may solve this equation for the temperature T_f at which the cosmic strings begin to evolve freely of the friction of the cosmological fluid.

Another initial value needed to evolve our system of equations is $\rho_{\rm matter}(t_f)$, which through Eq. (3.15) determines the matter component of the cosmological fluid energy density. The presence of this term brings about a smooth radiation-to-matter transition, and determines the critical energy density at the present time. When the cosmological fluid cools to $T=2.74$ K at the present time, all but 1 part in 10^4 of the total energy density is in nonrelativistic matter. Then, the equation

$$\rho_{\rm matter}(t_{\rm present})=\frac{3c^2}{8\pi G}H_0^2 \tag{3.21}$$

may be used to solve for the Hubble parameter h, where $H_0=100h$ km/sec Mpc. Thus, our choice of $\rho_{\rm matter}(t_f)$ determines the value of h. We have chosen the initial value $\rho_{\rm matter}(t_f)$ such that $h=1$ in all our calculations.

We have now specified the initial time t_f and temperature T_f, and the initial values for $\rho_{\rm fluid}(T_f),\rho_\infty(t_f)$, and $l(t_f)$. The remaining initial conditions, used in the integration of the differential equations (3.7), (3.10), and (3.12) are, $N_{\rm loops}(t_f)=E_{\rm loops}(t_f)=E_{\rm gr}(t_f)=0$ and $a(t_f)=1$.

Chiral transition and neutrino decoupling

The thermal history of the standard $SU(3)\times SU(2)\times U(1)$ particle-physics model, which determines the behavior of the energy density of the relativistic fluid, is not complete without a discussion of the chiral transition and the neutrino decoupling.

The composition of the relativistic fluid changes dramatically when the chiral symmetry of the SU(3) field is broken near the temperature $T_{\rm chiral}\sim 250$ MeV; the gluons and up and down quarks freeze into pions. Thus, the number of effective degrees of freedom changes rapidly in the short temperature span $\Gamma\sim 100$ MeV of this transition. It is important for both a realistic model of the chiral transition and the accuracy of the numerical integration that the number of effective degrees of freedom $g(T)$ is continuous at $T=T_{\rm chiral}$. Thus, for temperatures $T_{\rm chiral}+\frac{1}{2}\Gamma > T > T_{\rm chiral}-\frac{1}{2}\Gamma$, the contributions of g,u, and d to $g(T)$ are "Boltzmann suppressed" as their abundances diminish. Simultaneously, the contributions of $\pi^{0,\pm}$ are exponentially increased as their number density grows to an equilibrium value. We model this behavior with the expression

$$g_{[g,u,d,\pi]}(T)=g_{[g,u,d]}(T)\tfrac{1}{2}(1+\tanh\Delta)+g_{[\pi]}(T)\tfrac{1}{2}(1-\tanh\Delta)\ , \qquad (3.22)$$

where $\Delta=(T-T_{\rm chiral})/\Gamma$. Our model of the effective number of degrees of freedom is similar to the function $g(T)$ shown in Ref. [10], Fig. 3.5, p. 65.

At low temperatures, $T<m_\mu$, the light neutrinos decouple from the cosmological fluid. Since the thermal equilibrium can no longer be maintained, the temperature of the neutrinos, T_ν, becomes different from the temperature of the photons, T_γ. Still, the neutrinos continue to contribute to the relativistic energy density $\rho_{\rm rel}$ of the cosmological fluid. This phenomena, and the method we use to calculate the effect of the neutrino decoupling on $g(T)$ and $g_S(T)$, is described in [11] and [10].

The final synthesis of the cosmic-string one-scale model, the model of gravitational radiation, and the cosmological model presented in this section yields a method for computing many physical quantities of interest in the evolution of a cosmic-string network in a realistic universe. We now possess the "tools" to evaluate the nucleosynthesis and pulsar timing constraints on cosmic strings. We may numerically integrate Eq. (3.12) from the time t_f to the time when the temperature drops to $T=1$ MeV, to determine the gravitational-wave energy present at the onset of nucleosynthesis. The nucleosynthesis constraint, which depends on the *total* energy in gravitational waves, in all frequencies, is determined by Eq. (1.2). It is straightforward to integrate Eq. (3.12) from t_f to the present time, when the temperature drops to $T=2.74$ K. The constraint equation (1.1), however, is frequency dependent; we must compute the gravitational-wave power spectrum, not simply the total gravitational-radiation energy density, in order to determine the energy density present in the logarithmic frequency interval at $f_{\rm obs}$ for the pulsar timing limit. The methods by which we numerically integrate the differential equations, and compute the power spectrum are given in Appendix A.

IV. RESULTS AND INTERPRETATION

The final products of the numerical computation outlined in the preceding section are the equivalent number of neutrino species represented by the cosmic-string gravitational radiation at the onset of nucleosynthesis, and the spectrum of cosmic-string gravitational radiation at the present time. We have produced numerical data for a large range of values of α and μ. We shall present and discuss the data first with respect to the nucleosynthesis constraint. Next, we will describe the gravitational-radiation power spectra produced by the numerical calculation, and then present the pulsar timing constraint.

The nucleosynthesis and pulsar timing limits on cosmic strings are best displayed as contours of constant $\Omega_{\rm gr}$ in $\log_{10}(\alpha)-\log_{10}(\mu)$ parameter space. The general shape of such contours depends on the constraint considered; $\Omega_{\rm gr}$ (nucleosynthesis) and $\Omega_{\rm gr}(f_{\rm obs})$ (pulsar timing) depend differently on the parameters α and μ. To leading order, both $\Omega_{\rm gr}$ and $\Omega_{\rm gr}(f_{\rm obs})$ are increasing functions of μ. In the limit of small $\alpha,\Omega_{\rm gr}$, the fractional energy density in gravitational radiation at all frequencies, decreases to a minimum value. Thus, in the case of the nucleosynthesis limit, the contour lies at a constant value of μ for $\alpha<\gamma G\mu/c^2$, then slopes downward for $\gamma G\mu/c^2<\alpha$. However, $\Omega_{\rm gr}(f_{\rm obs})$ depends on the parameter α in a more complicated manner, discussed later in this section. We will see that $\Omega_{\rm gr}(f_{\rm obs})$ increases for α approaching $\gamma G\mu/c^2$ from either side. Thus, in the case of the pulsar timing limit, there is a maximum in the constraining contour near $\alpha=\gamma G\mu/c^2$. Also, as pointed out by Bennett and Bouchet [19], for $\alpha<(f_{\rm obs}t_{\rm present})^{-1}$, there is no limit on μ; the entire gravitational-wave power spectrum lies at frequencies above $f_{\rm obs}$.

We may thus express the restrictions on cosmic-string models in terms of a "minimum" constraint on μ. This minimum constraint occurs near the intersection of the line $\alpha=\gamma G\mu/c^2$ with the nucleosynthesis and pulsar timing limit contours. This nucleosynthesis minimum constraint on μ must be satisfied for all values of α. However, the pulsar timing minimum constraint on μ holds only if $(f_{\rm obs}t_{\rm present})^{-1}<\alpha$.

The requirement that the cosmic-string gravitational-radiation energy density lies below the equivalent energy density of $N_\nu-3.0=0.4$ excess neutrino species at the time of nucleosynthesis results in a simple constraint to the cosmic-string model. In Fig. 1, the points which lie below the contour satisfy this constraint. For small values of α, the contour becomes flat as the dependence on α diminishes. In the limit $\alpha\to 0$, the long cosmic strings lose their energy directly into gravitational radiation, rather than loops. The asymptotic limit of this contour is in excellent agreement with the limiting curve given by Bennett and Bouchet for the case $\alpha\ll\gamma G\mu/c^2$. At the minimum expected value $\alpha=\gamma G\mu/c^2$, the nucleosynthesis bound requires $G\mu/c^2\le 7\times 10^{-6}$. While the constraint on μ depends on α, a conservative upper limit using the bound on μ at $\alpha=\gamma G\mu/c^2$ may be set. Thus, for *all* reasonable values of α nucleosynthesis constrains $G\mu/c^2 \le 7\times10^{-6}$.

The value $G\mu/c^2=10^{-6}$ has historically been a favorite value for successful large-scale structure forming cosmic-string scenarios. At this value of μ, we find that nucleosynthesis constrains $\alpha \le 8\times10^{-3}$. Thus, the early cosmic-string models which found $\alpha \sim 0.1$ would have disagreed with observation for $G\mu/c^2=10^{-6}$. More recent simulations find $\alpha \le 0.01$.

Should the nucleosynthesis constraint on the number of excess light neutrino species change in the future, or should the present quoted limit of $N_\nu \le 3.4$ prove unreasonable, additional contours of constant N_ν, equivalent to constant Ω_{gr} through Eq. (1.2), have been plotted in Fig. 2. In this event, one may simply refer to Fig. 2 to determine the new limits on α and μ.

An improved calculation of the nucleosynthesis constraint by [9] yields $N_\nu \le 3.3$. This translates to the constraint $G\mu/c^2 \le 6\times10^{-6}$.

Recent interest in the possibility of an inhomogeneous QCD phase transition has led to estimates that the number of excess light neutrino species may be as high as $N_\nu - 3.0 = 0.7$ without contradicting observation [28,29]. In this case, we find the minimum constraint imposed is $G\mu/c^2 \le 1.2\times10^{-5}$.

Uncertainties in the nuclear reaction rates suggest that

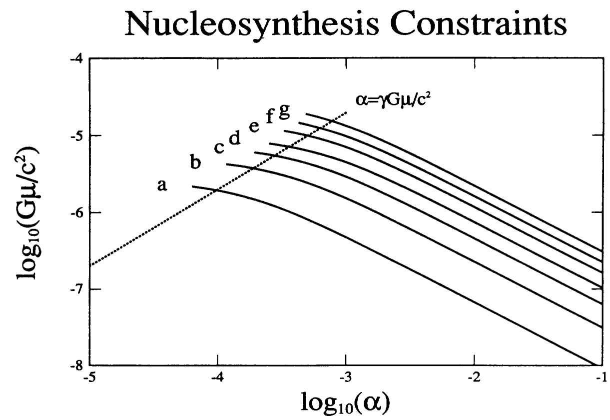

FIG. 2. Nucleosynthesis constraints to cosmic strings are shown. Curves a, b, c, d, e, f, g represent contours of constant N_ν for 3.1, 3.2, 3.3, 3.4, 3.6, 3.8, 4.0, which intersect the line $\alpha=\gamma G\mu/c^2$ at $G\mu/c^2 = 2,4,6,7,10,13,16\times10^{-6}$, respectively. The present limit requires $N_\nu \le 3.4$. Because of the uncertainties in the standard model of nucleosynthesis, speculation regarding an inhomogeneous QCD transition, and the possibility of a spectrum of relic gravitons, the limit on N_ν may change. Thus, an array contours of constant N_ν are shown.

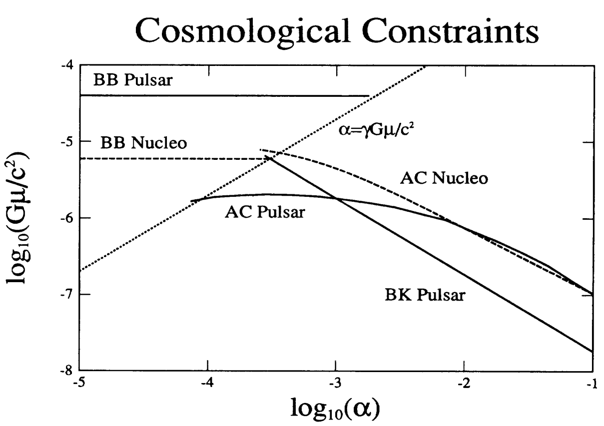

FIG. 1. The restrictions on cosmic-string parameters α and μ due to the nucleosynthesis and pulsar timing constraints are shown. The contour of constant Ω_{gr} at the time of nucleosynthesis, for $N_\nu=3.4$, is given by the AC Nucleo line. The contour of constant $(f/\rho_{crit})(d\rho_{gr}/df)$ at the present time, in the logarithmic frequency interval at $f_{obs}=$ (7.1 yr)$^{-1}$ is given by the AC Pulsar line. This paper finds that points which lie below these two contours, and the line $\alpha=\gamma G\mu/c^2$, represent cosmic-string scenarios which are in accord with observation. The line $\alpha=\gamma G\mu/c^2$ intersects the AC Nucleo line at $G\mu/c^2=7\times10^{-6}$, and intersects the AC Pulsar line at 2×10^{-6}. Constraints to the cosmic-string parameters, based on the nucleosynthesis and pulsar timing limits found by previous authors, are shown for comparison. The Brandenberger-Kung pulsar constraint is valid for $\alpha \gg \gamma G\mu/c^2$. The Bennett-Bouchet pulsar and nucleosynthesis lines are valid in the limit $\alpha \le \gamma G\mu/c^2$. (A more stringent pulsar timing constraint has been obtained using recent, unpublished observational results [36]—see Figs. 7 and 8.)

the limit $N_\nu \le 3.4$ may be overly optimistic. Rather, $N_\nu \le 3.6-3.8$ may be a more realistic, or at least a more conservative bound [37]. In this case, the "standard" nucleosynthesis constraint is, similar to the inhomogeneous nucleosynthesis constraint, $N_\nu \le 3.7$. This requires that $G\mu/c^2 \le 1.2\times10^{-5}$ at $\alpha=\gamma G\mu/c^2$.

It is also possible that there exists a thermal background of relic gravitons which decoupled from the cosmological fluid at very early times (see, for example, [10]). In this case the relic gravitons are described by a blackbody fluid with temperature $T_{relic}=[3.91/106.75]^{1/3}T_\gamma$ contributing 0.1 light neutrino species to the cosmological fluid at the time of nucleosynthesis. (At the present time, this temperature would be $T_{relic}=0.91$ K.) To adjust the limits on α and μ in the event that this relic gravitational radiation is present, one may simply refer to the next contour of constant N_ν. Thus, the limit on cosmic strings would change from $N_\nu \le 3.4$ to $N_\nu \le 3.3$. At $\alpha=\gamma G\mu/c^2$, it is required that $G\mu/c^2 \le 6\times10^{-6}$ in order that cosmic-strings agree with observation.

In contrast with the constraints from nucleosynthesis, the constraints arising from pulsar timing noise depend very strongly on the way in which loops emit gravitational waves. One of the features distinguishing our work from others is that the spectrum of P_n's that we consider contains a larger fraction of the gravitational-wave energy at higher frequencies. Our model of the emission of gravitational waves by loops used in this paper has allowed us to produce the power spectrum of the gravitational radiation emitted by the loops of a cosmic-string network for any set of parameters (as described in Appendix A). Before discussing the pulsar timing constraint on cosmic strings, we will describe the characteristic shape of the power spectrum, and discuss the behavior of the power spectrum under changes in α and μ.

A sample gravitational-radiation power spectrum is shown in Fig. 3 for the parameters $\alpha=10^{-3}$ and $G\mu/c^2=10^{-6}$. The energy in gravitational waves produced in the radiation-dominated era is shown separately from that produced in the matter-dominated era. The frequency $f_{\rm peak}$ marks the peak in the power spectrum, whose primary contribution is due to the oldest currently radiating loops at time $t_{\rm present}$. The frequency $f_{\rm match}$ marks the frequency, as observed today, at which the energy densities of gravitational radiation produced in the radiation and matter eras are equal. (Note that this frequency is unrelated to the time of equal matter and radiation.) At frequencies $f_{\rm peak}<f<f_{\rm match}$, the spectrum behaves roughly as

$$\frac{f}{\rho_{\rm crit}}\frac{d\rho_{\rm gr}}{df}\propto f^{-1/3}\ ,$$

a result of the $P_n \propto n^{-4/3}$ behavior of the power emitted per mode of oscillation. An effect of the smoothly changing number of particle degrees of freedom $g(T)$, and the gradual transition from radiation to matter domination, is to smooth the power spectrum, which changes from slope $-1/3$ to 0 (see Fig. 3), in a wide range of frequencies across $f_{\rm match}$. Now, the frequency at which pulsar timing noise observations constrain the energy density in gravitational waves lies in the interval $f_{\rm obs}\in[f_{\rm peak},f_{\rm match}]$. In most earlier calculations, it was assumed that loops radiated primarily in the $n=1$ mode, from which it follows that gravitational waves with frequency $f_{\rm obs}$ today were emitted in the radiation era. However, because we use a model in which the radiated loop energy is distributed over a wider range of modes, the majority of the energy with frequency $f_{\rm obs}$ is found to be emitted in the matter era. Thus, the key aspects of our calculation, (1) the gravitational-wave emission model, (2) the thermodynamic evolution of the relativistic component of the cosmological fluid, (3) the smooth radiation-to-matter transition, all affect the pulsar timing constraint on cosmic strings.

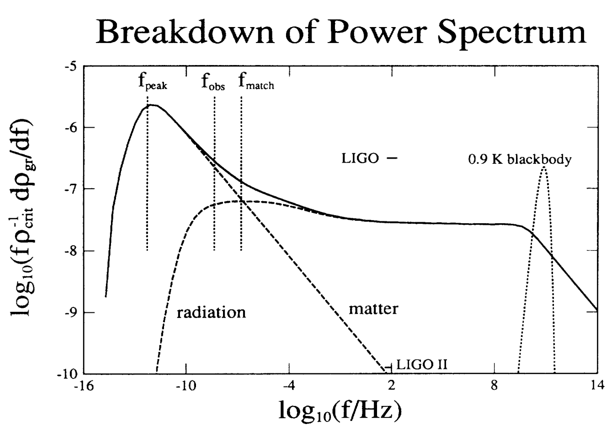

FIG. 3. A sample gravitational-wave power spectrum produced in this computation is shown. The cosmic-string parameters used are $\alpha=10^{-3}$ and $G\mu/c^2=10^{-6}$. These parameters are allowed by both the nucleosynthesis and the pulsar timing constraint. The contributions to the power spectrum from the radiation- and matter-dominated eras are indicated. Note that the high-frequency end of the matter- and radiation-era portions of the power spectrum behave as $(f/\rho_{\rm crit})(d\rho_{\rm gr}/df)\propto f^{-1/3}$. The frequencies $f_{\rm peak}$, $f_{\rm obs}$, and $f_{\rm match}$ are shown. The primary contribution to the energy in the frequency interval at $f_{\rm obs}$ is emitted in the matter era. Also shown is the spectra for a $T=0.91$ K blackbody. The frequency and amplitudes to which the first and second generations of LIGO (Laser Interferometric Gravitational Observatory) will be sensitive are shown. The second generation of LIGO might be able to detect the gravitational radiation produced by the cosmic-string spectra shown in this figure [38].

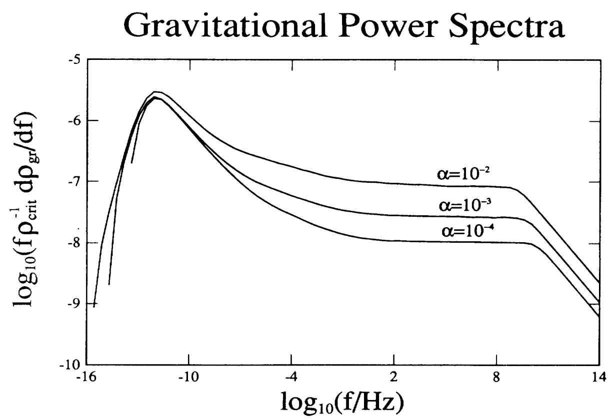

FIG. 4. A family of power spectra with $G\mu/c^2=10^{-6}$ and $\alpha=10^{-2},10^{-3},10^{-4}$.

Two families of power spectra are shown in Figs. 4 and 5. In Fig. 4, three spectra at fixed $G\mu/c^2$ and varying α are shown. The location of the peak in the spectrum is independent of the size of α in this case. This peak corresponds to the $n=1$ mode frequency of the most "dominant population" of loops. Because the rate of loop formation is a rapidly decreasing function of time, the "dominant" population was formed between times $t_B(t_{\rm present})$ and (say) $2t_B(t_{\rm present})$. It is straightforward to estimate the current length of such a loop. A loop formed at time $t_B(t_{\rm present})$ has just finished evaporating at time $t_{\rm present}$: $L(t_{\rm present},t_B(t_{\rm present}))\equiv 0$. A loop formed at

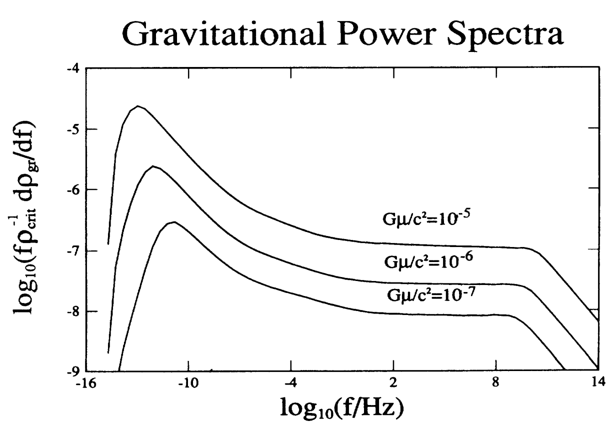

FIG. 5. A family of power spectra with $\alpha=10^{-3}$ and $G\mu/c^2=10^{-5},10^{-6},10^{-7}$.

time $2t_B(t_{\rm present})$ has present length

$$L(t_{\rm present}, 2t_B(t_{\rm present})) = (\alpha + \gamma G\mu/c^2) t_B(t_{\rm present}) = (\gamma G\mu/c^2) t_{\rm present} .$$

Hence, the frequency of this peak is $f_{\rm peak} = 2c^2/\gamma G\mu t_{\rm present}$. The amplitude of the peak in the spectrum diminishes to a limiting value for decreasing α. In Fig. 5, three spectra at fixed α and varying $G\mu/c^2$ are shown. Both the location of the peak in the spectrum, as well as the amplitude of the power spectrum, the energy density in gravitational waves, are strongly affected by the value of μ. It should be clear from these figures that both α and μ play significant roles in determining the amount of energy density within the frequency interval to which the pulsar timing noise is sensitive.

It is straightforward to describe the effects on the power spectrum (Fig. 4) for further decreases in α. Note that the total gravitational-wave energy density produced during the matter era approaches a nonzero limit for $\alpha < \gamma G\mu/c^2$. Hence, if α is decreased below this value, the shape of (and total energy under) the spectrum does not change. However the entire curve moves to the right (up in frequency). This results in a marked increase of the energy density in the frequency interval at $f_{\rm obs}$. Hence, for $\alpha < \gamma G\mu/c^2$, the constraints arising from pulsar timing noise become *more* restrictive than for $\alpha = \gamma G\mu/c^2$.

The constraint imposed by the millisecond pulsar observations serves to restrict the amount of gravitational radiation which may lie within the logarithmic frequency interval at $f_{\rm obs}$. A contour in $\log_{10}(\alpha) - \log_{10}(\mu)$ parameter space of constant gravitational-wave energy density $\Omega_{\rm gr}(f_{\rm obs})$ is shown in Fig. 1. Along this contour, the gravitational radiation produced by the cosmic-string network in the logarithmic frequency interval at $f_{\rm obs}$ is the maximum allowable by the pulsar timing, at the 95% confidence level. (This confidence level is associated with the statistical methods used to analyze the observational pulsar timing data.) As explained above, points which lie below the contour satisfy this observational constraint. For large values of α, the contour lies near the limit produced by Brandenberger and Kung [31]. Notice that our curve yields a weaker constraint than their work. For decreasing loop size α, the contour flattens out. At $\alpha = \gamma G\mu/c^2$, the constraint is $G\mu/c^2 \le 2\times 10^{-6}$. The asymptotic limit to this contour is well below the constraint given by Bennett and Bouchet, in the limit $\alpha \ll \gamma G\mu/c^2$. This is partly due to the different model of gravitational-radiation emission used in our calculation, as will be explained below. We also find that for $\alpha > 8\times 10^{-3}$, $G\mu/c^2 = 10^{-6}$ is inconsistent with observation.

The dramatic tightening of the pulsar bound in comparison to the results of Bennett and Bouchet, as was shown in Fig. 1, comes about for two reasons.

First, we have used a different model of the emission of gravitational waves. Most previous studies assumed that all power is emitted in the $n=1$ mode of oscillation. In our model the power emitted in each mode of oscillation n behaves as $P_n \propto n^{-4/3}$. In Fig. 6 we reproduce the pulsar constraint of Bennett and Bouchet, and the contour of constant $\Omega_{\rm gr}(f_{\rm obs}) = 4\times 10^{-7}$ for models of the emission of gravitational radiation in which $P_n \propto n^{-q}$ for $q = 7/6, 4/3, 3/2, 5/3, 10$. As q increases, the fraction of energy radiated in the $n=1$ mode approaches unity, and the position of the contour moves upwards. In our model of emission, the first three modes of oscillation roughly span a frequency interval $f \in [f, ef]$. For $q = 7/6, 4/3, 3/2, 5/3, 10$ these first three modes represent $\approx 27\%, 47\%, 60\%, 70\%, 99\%$ of the total radiation emitted. For small values of q, it is a poor approximation to assume that all the power is emitted in the $n=1$ mode.

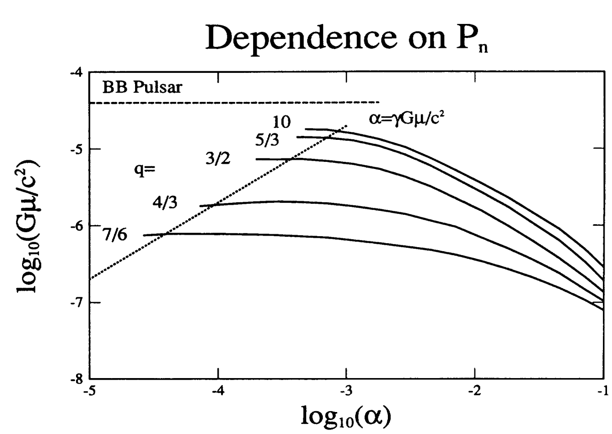

FIG. 6. The pulsar timing constraints for different power-law dependencies of the power coefficients P_n are shown. $P_n \propto n^{-q}$ for $q = 7/6, 4/3, 3/2, 5/3, 10$ are shown. For increasing q, the sum of the P_n's converges more rapidly. Also shown is the constraint produced by Bennett and Bouchet [19] for which $q \to \infty$; all power is emitted in the $n=1$ mode of oscillation.

Second, Bennett and Bouchet obtained their constraint by considering only the contributions to the gravitational radiation produced during the radiation era. The constraints they obtain are of course correct; however our constraints are tighter because we also include the effects of the gravitational radiation produced during the matter era. In fact, when α is slightly larger than 10^{-9} this makes our constraints nearly three orders of magnitude more restrictive. We have verified this explicitly in Fig. 7, where we show the constraint obtained for large q from *only* the gravitational radiation produced in the radiation era: we exactly reproduce the result of Bennett and Bouchet.

Recent (unpublished) pulsar observations and analysis have yielded an updated limit to the gravitational-wave energy density. The current published limits are [5]

$$\Omega_{\rm gr}[f_{\rm obs} = (7.1\ {\rm yr}^{-1})] \le \begin{cases} 4\times 10^{-7} h^{-2} & \text{at 95\% confidence ,} \\ 9\times 10^{-8} h^{-2} & \text{at 68\% confidence .} \end{cases} \qquad (4.1)$$

The new figures, after 8.2 years of observation, are [36,39]

$$\Omega_{\rm gr}[f_{\rm obs} = (8.2\ {\rm yr}^{-1})] \le \begin{cases} 1\times 10^{-7} h^{-2} & \text{at 95\% confidence ,} \\ 2.5\times 10^{-8} h^{-2} & \text{at 68\% confidence .} \end{cases} \qquad (4.2)$$

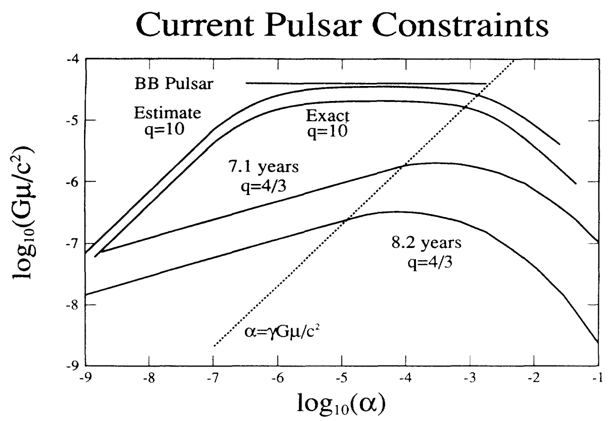

FIG. 7. The constraint curves are shown for all values of α, including those much smaller than $\gamma G\mu/c^2$. Note that these curves all have their maxima near the dotted line $\alpha=\gamma G\mu/c^2$. Various cases for the behavior of the power coefficients $P_n \propto n^{-q}$ are shown. The estimate curve shows the contour obtained by using the analytic expressions derived in Appendix B, with $q=10$; this estimate yields the minimum constraint $G\mu/c^2 \leq 3.6\times10^{-5}$. This is very similar to the limit obtained by Bennett and Bouchet, as the gravitational waves are chiefly emitted in the fundamental, $n=1$, mode. The exact curve shows the contour produced by our numerical calculation for $q=10$; this calculation yields the minimum constraint $G\mu/c^2 \leq 2.2\times10^{-5}$. The back reaction of the cosmic strings and gravitational radiation on the expansion rate results in a slightly tighter constraint than does the analytic approximation. The next two curves show the $q=4/3$,7.1 and 8.2 year observational constraints at the 95% confidence level. The 8.2-yr contour yields the minimum constraint $G\mu/c^2 \leq 3\times10^{-7}$.

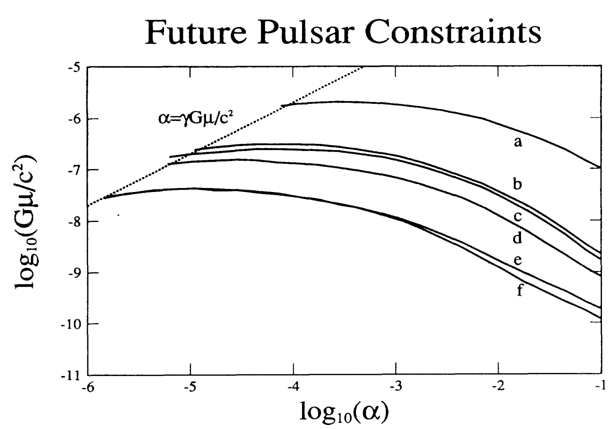

FIG. 8. Present and future constraints to cosmic strings due to pulsar timing are shown. The present contours for 7.1 years of observation (a) and 8.2 years (b) at 95% confidence are shown. The contour for 7.1 years of observation (c) at 68% confidence intersects the line $\alpha=\gamma G\mu/c^2$ at $G\mu/c^2=3\times10^{-7}$. Under the assumption that the pulsar timing limits will decrease with observation time T in proportion to T^{-2} beyond the 8.2-yr limit, the 95% confidence contours after 10 (d) and 15 (e) years of observation are shown. The 10- and 5-yr projections intersect the line $\alpha=\gamma G\mu/c^2$ at $G\mu/c^2=2\times10^{-7}$ and $G\mu/c^2=5\times10^{-8}$. The contour 8.2 years (f) at 68% confidence intersects the line $\alpha=\gamma G\mu/c^2$ at $G\mu/c^2=5\times10^{-8}$. If $G\mu/c^2>10^{-7}$ is required for a successful cosmic-string scenario, then the pulsar timing limit may presently rule out cosmologically interesting cosmic strings.

The improvement by a factor of nearly 4 over previous limits is much more than one might expect from the simple scaling discussed at the beginning of Sec. II. The added improvements beyond T^{-4} scaling come mostly from reduction in the "white noise" level of the experiment [39]. The limiting contours for the above four cases are shown in Fig. 8. Already, after 8.2 yr of observation, the cosmic-string scenarios which require $G\mu/c^2 \sim 10^{-6}$ are restricted at the 95% confidence level, and prohibited at the 68% confidence level, under the assumptions of our calculation. (Note that these limits were obtained by statistical analysis, assuming that the energy spectrum of gravitational radiation had a frequency dependence of $\Omega(f)\propto f^0$, whereas our spectrum behaves like $\Omega(f)\propto f^{-1/3}$. This small change in slope should not influence the pulsar timing limits on $\Omega_{\rm gr}(f_{\rm obs})$ by much [39].) It is generally believed that the length scale of structure on the long strings is limited to a fraction $\gamma G\mu/c^2$ of the horizon length. For this reason, one would not expect the size α of loops chopped off the long strings to be much smaller. It is possible that the subsequent fragmentation of loops into smaller ones further decreases α, although numerical studies of this effect by Scherrer, Quashnock, Spergel, and Press [40] suggest that the fragmentation process does not change the order of magnitude of the length scale. However, for completeness we show in Fig. 7 the limits of μ for the entire reasonable range of α. (We assume that the fragmentation occurs after the loops have redshifted away their peculiar velocities.) As described earlier, the constraints tighten considerably for a very small α.

It is interesting to ask how the pulsar timing limit might improve in the future. In the absence of gravitational waves, and barring some unforeseen improvement in the sensitivity of the pulsar timing measurements, it is expected that the gravitational-wave limits will continue to gain in sensitivity only in proportion to T^{-2} (rather than T^{-4} as mentioned in Sec. II). This is because the observations have more or less reached a limit in which "red noise" components with spectra f^{-3} or thereabouts will begin to dominate the timing residuals [39]. Therefore, one might expect the 95% confidence limit to improve with observation time as

$$\Omega_{\rm gr}(f_{\rm obs}=T_{\rm yr}^{-1}) \leq 1\times10^{-7}\left[\frac{8.2}{T_{\rm yr}}\right]^2 h^{-2}\,. \qquad (4.3)$$

The constraining contours after 10 and 15 years of observation are also shown in Fig. 8. After 10 years, under the various assumptions of our calculation, one may expect the pulsar timing limit to prohibit scenarios which require $G\mu/c^2>10^{-7}$. After 15 years, one foresees that $G\mu/c^2>5\times10^{-8}$ will disagree with observation.

In this paper, we have assumed that all the loops cut off the string network at time t have the same length $\alpha l(t)$. This is an idealization: in a realistic string network, the loops would have a distribution of sizes (as was included in [18]). Here, we estimate the possible effects that this distribution of sizes would have on the gravitational-wave background.

For a one-scale model of a cosmic-string network, the distribution of loop sizes can be modeled by a probability distribution function $f(\alpha)$. The probability that a loop is cut off the network with a length in the range $[\alpha l(t),(\alpha+d\alpha)l(t)]$ is given by $f(\alpha)d\alpha$. Since loops are strings shorter than the horizon length $l(t)$ the range of α is $\alpha\in[0,1]$. Because the total probability is unity, one has $\int_0^1 f(\alpha)d\alpha=1$. One may now crudely estimate the contribution to the gravitational-wave energy from a family of loops whose sizes fall in the interval $[\alpha,\alpha+d\alpha]$. The initial energy of the loops is proportional to α, and if $\alpha>\gamma G\mu$ then the lifetime of the loops is also proportional to α. During the lifetime of the loop, the energy density of loops scales like a^{-3}; after the loop has been converted into gravitational waves, the energy density of gravitational waves scales like a^{-4}. Hence the total energy density of gravitational waves contributed by the family is proportional to $\alpha^{3/2}f(\alpha)d\alpha$ during the radiation-dominated era, and $\alpha^{4/3}f(\alpha)d\alpha$ during the matter-dominated era.

Our assumption in this paper, that all the loops cut off the string network at time t have the same length $\alpha l(t)$, will yield the correct radiation-phase energy density in gravitational waves (to within a factor of 2) if there exists a value $\alpha_{\rm mean}$ for which

$$\int_{\alpha_{\rm mean}}^{2\alpha_{\rm mean}}\alpha^{3/2}f(\alpha)d\alpha>\frac{1}{2}\int_0^1\alpha^{3/2}f(\alpha)d\alpha \ . \qquad (4.4)$$

This "mean" value is then the appropriate one to use for the purpose of these calculations.

The numerical simulations done to date have not given clear or conclusive information about $f(\alpha)$. In two of the three simulations (Allen-Shellard, Bennett-Bouchet) $f(\alpha)$ cannot be determined: the loops are all formed at (or quickly fragment to) the smallest size allowed by numerical resolution. In the third simulation (Albrecht-Turok) the loop distribution functions "show evidence of scaling." Our prejudice is that loops probably fragment quickly to very small sizes, and thus that $\alpha\sim\gamma G\mu$ is likely. However, further investigation is required to determine if this is really so.

We have also considered how the presence of small scale structure (kinks) on the long cosmic strings may change the rate of loop production, thus affecting the nucleosynthesis and pulsar timing constraints on cosmic strings. It has been shown that under certain conditions the number of kinks per horizon-sized cosmic string grows rapidly after the formation of the cosmic-string network, reaching a constant value within $\sim 10^0$ e-foldings after t_f. If the loop formation is primarily due to the small scale structure, then at later times one might expect a loop size of order $\alpha\sim\gamma G\mu/c^2$. The effects of such behavior on the nucleosynthesis and pulsar timing constraints is minimal: a network for which $\alpha\sim 1$ initially and then drops rapidly to $\sim\gamma G\mu/c^2$ is not noticeably different from a network which has $\alpha\sim\gamma G\mu/c^2$ at all times.

The gravitational radiation emitted by the long cosmic strings and loops was examined in the recent numerical simulation work of Allen and Shellard [23]. During the (relatively) short time span of the simulation, they found that the radiation rate per unit physical length of the long strings was fairly constant. This behavior appeared to result from the presence of small-scale wiggles on the long strings, at a fixed physical length scale. If one can extrapolate for several more orders of magnitude of expansion, it indicates that the radiation rate by the long strings might eventually catch up to that of the loops. However the effects of gravitational back reaction on the long string network should prevent this radiation rate from becoming larger than the energy loss rate due to loop production. In our paper, we have chosen to ignore the effects of any radiation directly from the long strings, because this facilitates comparison with the existing literature, and because any such effects should not change the value of $\Omega_{\rm gr}$ by more than a numerical factor of order 2. This is because back reaction limits the minimum length of any structure on the long strings to $\gamma G\mu t/c$, so the effects of long-string radiation are similar to those models in which loops form with small α and then evaporate immediately due to gravitational-radiation emission.

While we have chosen to present our results for a cosmology in which the Hubble parameter $h=1$, we have also considered how the pulsar limits may change for values $h\in[0.5,1]$. The gravitational-radiation energy density during the radiation era is independent of the value of h; the nucleosynthesis constraint is independent of h. The energy density in gravitational waves present in the matter era, however, does depend on h. We find that the gravitational-radiation energy density as a fraction of the critical energy density, present in the logarithmic frequency interval at $f_{\rm obs}$ at the temperature $T=2.74$ K, has the following dependence on h:

$$\frac{f}{\rho_{\rm crit}}\frac{d\rho_{\rm gr}}{df}\propto h^{-k} \ . \qquad (4.5)$$

In this equation, the values of α and μ determine the constant of proportionality and the exponent k, which smoothly decreases in the interval $k\in[1,0]$ as $\alpha\gg\gamma G\mu/v^2$. Now, the pulsar timing bound behaves as $\Omega_{\rm gr}(f_{\rm obs})\propto h^{-2}$. Therefore, the pulsar timing limit on $G\mu/c^2$ weakens for decreasing h. The limit roughly weakens by the factor $\sim 6.5, 2.5$ for $h=0.5, 0.75$, respectively. Therefore, from Fig. 9, at $\alpha=\gamma G\mu/c^2$, the 7.1-yr limit becomes $G\mu/c^2\leq 10^{-5}$ for $h=0.5$, and $G\mu/c^2\leq 5\times 10^{-6}$ for $h=0.75$. The 8.2-yr limit becomes $G\mu/c^2\leq 2\times 10^{-6}$ for $h=0.5$, and $G\mu/c^2\leq 8\times 10^{-7}$ for $h=0.75$.

Uncertainties in the values of the dimensionless parameters $A,\gamma,\langle v^2\rangle$, and f_r which we have used in our calculation will result in an uncertainty in the limits on α and μ. The uncertainties in the parameters are $A_{\rm radiation}=52\pm 10$, $A_{\rm matter}=31\pm 7$, $\gamma=50\pm 15$, $\langle v^2\rangle_{\rm radiation}=0.43\pm 0.02$, $\langle v^2\rangle_{\rm matter}=0.37\pm 0.02$, and $f_r=0.71\pm 0.04$. By examining Eqs. (B2) and (B3) of Appendix B, we can see the leading-order effect of a variation of these parameters. The dominant effect is due to the uncertainty in A and γ; one finds that $\rho_{\rm gr}$ lies in the range $\rho_{\rm gr}\in[0.54,1.6]$. To leading order, $\rho_{\rm gr}\propto[\alpha G\mu/c^2]^{1/2}$. Hence the error bars of our $(\log_{10}(\mu),\log_{10}(\alpha))$-space contours of fixed $\Omega_{\rm gr}$ extend $0.53=-2\log_{10}(0.54)$ below and $0.41=2\log_{10}(1.6)$ above our given contours. The

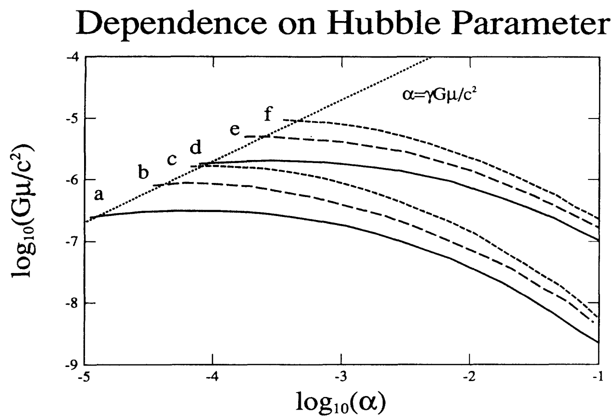

FIG. 9. The dependence of the pulsar timing constraint on the value of the dimensionless Hubble parameter is shown. In this paper, we have chosen to carry out the majority of the calculations using $h=1.0$. For smaller values of h, the pulsar timing constraint weakens. Curves a, b, c show the limiting contours for the 8.2-yr pulsar timing constraint for $h=1.0, 0.75, 0.5$, respectively. Curves d, e, f show the limiting contours for the 7.1-yr pulsar timing constraint for $h=1.0, 0.75, 0.5$, respectively.

cumulative effect of using $h=0.5, 0.75$ and the uncertainty in the dimensionless parameters weakens the pulsar timing limit minimum constraint on $G\mu/c^2$ by factors of $\sim 10, 4$, respectively.

V. CONCLUSION

The principal conclusions of this work are the following. (1) The current model of nucleosynthesis requires $G\mu/c^2 \leq 7\times10^{-6}$ for $\alpha=\gamma G\mu/c^2$. This constraint tightens for larger values of α . (2) The current observations of noise in millisecond pulsar timing require $G\mu/c^2 \leq 2\times10^{-6}$ for $\alpha=\gamma G\mu/c^2$. Newer observations of noise in millisecond pulsar timing require $G\mu/c^2 \leq 3\times10^{-7}$ for $\alpha=\gamma G\mu/c^2$. This constraint also tightens for both larger and *smaller* values of α. (There is no constraint if $\alpha < 10^{-9}$; this value, however, is probably unreasonably small.) The pulsar timing constraint on $G\mu/c^2$ weakens by factors $\sim 6.5, 2.5$ for choices of the dimensionless Hubble parameter $h=0.5, 0.75$. (3) The millisecond pulsar constraint provides the tightest bound on cosmic-string parameters α and μ yet. We have shown that much of this improvement over past calculations is due to our model for the emission of gravitational radiation by oscillating loops. The motivation for this model has been the results of the recent numerical work by Allen and Shellard [23].

There is an additional constraint on cosmic-string networks. The experimental limits on the isotropy of the microwave background radiation constrain the temperature perturbations that the strings would leave imprinted on the microwave sky. These limits have been considered by other authors [41] who obtain the bound $G\mu/c^2 \leq 5\times10^{-6}$. The limits found in this paper are marginally tighter.

In the early work on cosmic strings, it was thought that galaxies and clusters might accrete around string loops. For historical reasons, $G\mu/c^2=10^{-6}$ was considered the minimum value for a cosmologically interesting cosmic-string scenario. We have found that for $\alpha > 8\times10^{-3}$, $G\mu/c^2 \geq 10^{-6}$ disagrees with observations. This indicates that, given our assumptions regarding the one-scale model of cosmic strings, and gravitational radiation by the loops, cosmologically interesting cosmic-string scenarios are ruled out for $\alpha > 8\times10^{-3}$. The current belief is that these loops are too small to play an important role. However the role of the long strings is currently under investigation; wakes formed behind the long strings may serve to initiate gravitational instability and accretion [42,43]. While these wake models of structure formation appear very promising, it remains to be determined what range of values of μ are acceptable. (The value used in recent work on wakes [43] is $G\mu/c^2=4\times10^{-6}$.)

We should also note, in concluding, that cosmic strings with much smaller values of μ may have other interesting cosmological effects, especially if they are superconducting, or associated with baryogenesis. It may even be possible to evade the cosmological constraints we have obtained if (1) cosmic strings decay in some other way than via the generation of gravitational waves—as might occur with global strings, or if (2) they are formed during an inflationary epoch. However, these topics are outside the scope of the present work.

We have also attempted to predict what the pulsar timing constraints on cosmic-string parameters might be over the next decade. Assuming that the residual timing noise remains constant, after ten years of observation, in 1993, for $\alpha=\gamma G\mu/c^2$, $G\mu/c^2 > 2\times10^{-7}$ will be prohibited in order for cosmic strings to be consistent with observation. After fifteen years of observation, in 1998, for $\alpha=\gamma G\mu/c^2$, $G\mu/c^2 > 5\times10^{-8}$ will be prohibited. Therefore, if the assumptions of our calculations are correct, it appears that cosmic-string scenarios which require $G\mu/c^2 > 5\times10^{-8}$ will be ruled out in the near future.

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APPENDIX A

This appendix gives a list of the differential equations which are integrated in our calculation. We also discuss the numerical integration scheme, and the procedure used to determine the number of loops produced by the cosmic-string network. Finally, we present the method used to produce the power spectra of the gravitational radiation emitted by the cosmic strings.

The numerical integration scheme used to solve the following five differential equations was taken from [44]. A fourth-order Runge-Kutta integration routine was used for testing. A Bulirsch-Stoer integration routine was used for production runs. We found both integration methods to be stable; the numerical results were robust. The Runge-Kutta routine produced results which agreed to within a fraction of a percent with the results of the faster Bulirsch-Stoer routine.

The dependent variables of the numerical integration are $a(t)$, $l(t)$, $N_{\rm loops}(t)$, $E_{\rm loops}(t)$ and $E_{\rm gr}(t)$; the independent variable is time t. The following are the differential equations and initial conditions:

$$\dot a(t)=a(t)\sqrt{8\pi G\rho_{\rm tot}(t)/3c^2}\ , \quad \text{(A1)}$$

$$\dot l(t)=l(t)\dot a(t)/a(t)+c\ , \quad \text{(A2)}$$

$$\dot N_{\rm loops}(t)=2\rho_\infty V(t) \times[c/l(t)-\langle v^2\rangle \dot a(t)/a(t)]/[c^2\mu\alpha l(t)]\ , \quad \text{(A3)}$$

$$\dot E_{\rm loops}(t)=c^2\mu f_r\alpha l(t)\dot N_{\rm loops}(t)-\gamma G\mu^2 c C_{\rm loops}(t)\ , \quad \text{(A4)}$$

$$\dot E_{\rm gr}(t)=-E_{\rm gr}(t)\dot a(t)/a(t)+\gamma G\mu^2 c C_{\rm loops}(t)\ , \quad \text{(A5)}$$

$$a(t_f)=1\ , \quad \text{(A6)}$$

$$l(t_f)=2ct_f\ , \quad \text{(A7)}$$

$$N_{\rm loops}(t_f)=0\ , \quad \text{(A8)}$$

$$E_{\rm loops}(t_f)=0\ , \quad \text{(A9)}$$

$$E_{\rm gr}(t_f)=0\ . \quad \text{(A10)}$$

The above differential equations, except for (A2), were derived in Sec. III. Equation (A2) is easily found by taking the derivative of the integral for the horizon radius, which was given before Eq. (3.1). All physical results are independent of the choice of $a(t_f)$. In the above equations, $V(t)$ is the comoving volume $V(t)=a(t)^3L^3$, where L is a constant (all physical results are independent of L). The total energy density of the Universe $\rho_{\rm tot}(t)$ is given in Eq. (3.13). The energy density of the long cosmic strings, $\rho_\infty(t)$, is given in Eq. (3.1). The mean velocity squared of the long cosmic strings is $\langle v^2\rangle$. The variable $C_{\rm loops}(t)$, given in Eq. (3.11), is the total number of loops present at the current time.

The current number of radiating loops, $C_{\rm loops}(t)$, is different than the total number of loops, $N_{\rm loops}(t)$, because loops disappear after radiating all their energy. The total number of loops produced between time t_f and t is $N_{\rm loops}(t)$. In order to easily determine $C_{\rm loops}(t)$, the numerical integration is broken up into successive time intervals $I_i=[t_i,t_{i+1}]$, where $t_{i+1}\equiv t_D(t_i)$ and $t_0=t_f$. As explained in Sec. III, $t_D(t)$ is a function which determines the time of death of a loop formed at time t, and $t_B(t)$ determines the time of birth of a loop which evaporates at time t. The number of new loops formed since the beginning of the interval I_i, at each time step, is stored in an array. We set $N_i(t_i)\equiv 0$. Then, up to the time step at $t\in I_i$, $N_i(t)$ total loops have formed since the beginning of the interval. There are also loops present which were formed in the previous interval I_{i-1}. Thus, the current number of radiating loops at time t is the number of loops formed during $[t_B(t),t_i]$, plus the total number of loops formed during $[t_i t]$. This may be expressed as

$$C_{\rm loops}(t)=\{N_{i-1}(t_i)-N_{i-1}(t_B(t))\}+N_i(t)\ . \quad \text{(A11)}$$

At any given time, as long as the function $\beta(t)$ decreases (this is true for all our calculations), one needs to store only two arrays of N_i: one for the past interval (N_{i-1}) and one for the present interval (N_i).

The arrays of data stored at each time step in the past interval I_{i-1} are $t_D(t')$, $N_{\rm loops}(t')$, and $l(t')$. These are all smooth, monotonic, slowly changing functions; we use rational function and polynomial interpolation routines [44] to find these quantities at any time $t'\in I_{i-1}$. These values are subsequently used to obtain $c_{\rm loops}(t)$ in order to evaluate the RHS of Eqs. (A4) and (A5), and to obtain the length $L(t,t')$ of all surviving, radiating loops in order to calculate the gravitational-wave power spectrum.

The problem of calculating the power spectrum produced by the oscillating cosmic-string loops may be simply stated: "how much energy does each loop radiate into a particular logarithmic frequency interval?" Now, a loop of length L will radiate the power $P_n G\mu^2 c$ at the frequency $f_n=2n/L$. As was described in Sec. III, the power coefficients are given by $P_n\propto n^{-4/3}$. We approximate the sum of the P_n's from mode n_1 to n_2:

$$\sum_{n=n_1}^{n_2} P_n\approx\gamma\int_{n_1}^{n_2} n^{-4/3}dn \Big/ \int_1^\infty n^{-4/3}dn\ . \quad \text{(A12)}$$

Because the loops formed at any instant have sizes distributed about the mean, replacing the discrete sum by a smooth integral more realistically models the frequency distribution of the resulting radiation. One may now calculate the power emitted into any frequency interval $[f_1,f_2]$, such as the logarithmic interval bounded by $f_{\rm obs}$. We store any array of 150 bins, where each bin contains the energy in the logarithmic frequency interval $[f_0e^m,f_0e^{m+1}]$. [With forethought, we choose $f_0(t)e^{80}=f_{\rm obs}a(t_{\rm present})/a(t)$ so that the $m=80$ bin, near the center of the 150 bins, will contain the energy density used in calculating the pulsar timing limit.] At each time step, the frequency f and the energy in each bin are redshifted, and the energy in gravitational waves emitted by all loop modes in each frequency interval is added to the respective bins. Each bin at frequency f then contains $f(d\rho_{\rm gr}/df)$. The total energy in the bins is in excellent agreement with $E_{\rm gr}$, obtained by integrating Eq. (A5). We follow this procedure until $T=2.74$ K, then normalize the energy in each bin by dividing by $\rho_{\rm crit}$ in order to obtain

$$\frac{f}{\rho_{\rm crit}}\frac{d\rho_{\rm gr}}{df}\ .$$

APPENDIX B

In this appendix, we derive analytic estimates for the energy density in gravitational radiation produced by a cosmic-string network. These analytic expressions yield results comparable to the results of the numerical integration which comprise the core of this paper. These analytic estimates support the conclusions based on our numerical calculations. These estimates are also useful for showing the rough dependence of the energy density in gravitational radiation on the parameters α and μ, as well as on other parameters which we have not varied, such as A and γ.

The one-scale model of a cosmic-string network, described in Sec. III, allows us to put together simply an analytic expression for the total energy density in cosmic-string-produced gravitational radiation. In the following calculation, we will set $l(t)=2ct$ in the radiation era and $l(t)=3ct$ in the matter era, such that $\beta=[1+f_r\alpha cl(t)/\gamma G\mu t]^{-1}$ is time independent. Then, $t_B(t)=\beta t$, and $t_D(t)=t/\beta$. Thus, from Eq. (3.7) for the rate of loop formation, we may write the expression

$$E_{\rm gr}(t)=\int_{t_f}^{t_B(t)}\frac{dN_{\rm loops}}{dt'}dt'\int_{t'}^{t_D(t')}\gamma G\mu^2 c\frac{a(t'')}{a(t)}dt'' + \int_{t_B(t)}^{t}\frac{dN_{\rm loops}}{dt'}dt'\int_{t'}^{t}\gamma G\mu^2 c\frac{a(t'')}{a(t)}dt'' . \quad \text{(B1)}$$

The first term integrates over the loops which were formed between times $t_f<t'<t_B(t)$, and are no longer present at the time t. The second term integrates over all the loops formed after time $t_B(t)$ which are still present at time t. It is very simple to evaluate this expression at the time of nucleosynthesis:

$$\rho_{\rm gr}\rho_{\rm crit}^{-1}|_{\rm nuc}=\frac{8\pi}{9}A\gamma\frac{(G\mu)^2}{\alpha c^4}[1-\langle v^2\rangle] \times[(\beta^{-3/2}-1)\ln(\beta t_{\rm nuc}/t_f) + \tfrac{2}{3}(\beta^{-3/2}-1)+\ln\beta] . \quad \text{(B2)}$$

It is important to remember that β depends on μ and $f_r\alpha$. The leading, logarithmic term of this expression gives the results obtained by Brandenberger *et al.* and Davis, for the calculation of the nucleosynthesis constraint. The full dependence of the constraint on μ and α due to this analytic expression is shown in Fig. 10. In computing this contour, we have included a numerical correction for the change in the number of degrees of freedom from the time t_f to the time $t_{\rm nuc}$. Thus, the RHS of Eq. (B2) is reduced by the factor $[g(T_f)/g(T_{\rm nuc})]^{1/3}=(106.75/10.75)^{1/3}=2.15$. We find that the analytic contour displays the same behavior, and lies very close to the numerically determined contour. The remaining differences arise from the effects of the back reaction of the cosmic strings and gravitational radiation on the cosmological fluid, the cosmological scale factor, and the horizon radius.

A similar calculation may be carried out to determine the energy density in gravitational radiation in the logarithmic frequency interval at $f_{\rm obs}$. It will be necessary to calculate the contributions to the spectrum of gravitational radiation in both the radiation- and matter-dominated eras. Thus, in addition to an expression of the form of (B1), there will be a similar term for the matter era, in which t_f is replaced by $t_{\rm eq}$ and $t_{\rm nuc}$ is replaced by $t_{\rm present}$. Making these changes, and integrating Eq. (B1), the contribution from the matter era is

$$\rho_{\rm gr}\rho_{\rm crit}^{-1}|_{\rm matter}=\frac{4\pi}{45}A\gamma\frac{(G\mu)^2}{\alpha c^4}[1-2\langle v^2\rangle] \times\{\tfrac{3}{2}(\beta^{-5/3}-1)[\beta^{2/3}-(t_{\rm eq}/t_{\rm present})^{2/3}] +\beta^{-1}-1-\tfrac{3}{2}(1-\beta^{2/3})\} . \quad \text{(B3)}$$

In order to sum the contributions to the gravitational-radiation energy density from both eras, we must evaluate Eq. (B2) at the time of equal matter and radiation, redshifted to the present. Therefore,

$$\rho_{\rm gr}\rho_{\rm crit}^{-1}|_{\rm present}=\rho_{\rm gr}\rho_{\rm crit}^{-1}|_{\rm matter}+\rho_{\rm gr}\rho_{\rm crit}^{-1}|_{\rm eq}/(Z_{\rm eq}+1) \quad \text{(B4)}$$

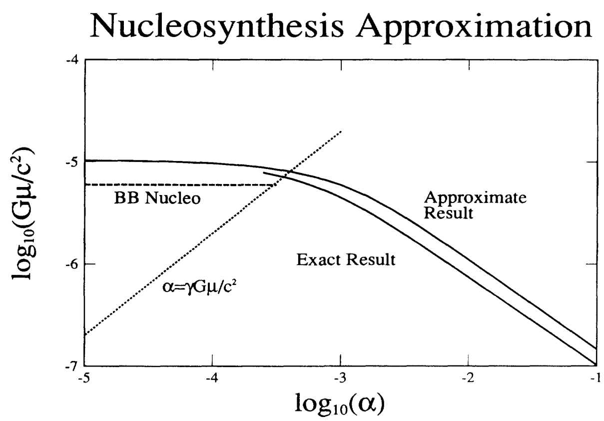

FIG. 10. An analytic approximation to the nucleosynthesis constraint, as described in Appendix B, is shown. This approximation displays the same behavior as the exact, numerical constraint, differing only by a constant numerical factor caused by the back reaction of the cosmic strings on the cosmological expansion rate. The constraint produced by Bennett and Bouchet [19], which is valid in the limit $\alpha\leq\gamma G\mu/c^2$, is shown.

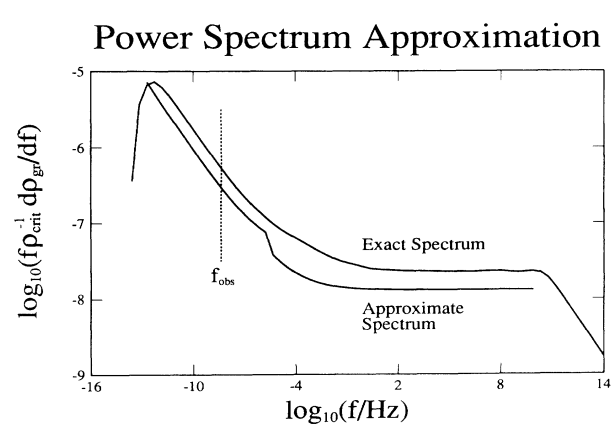

FIG. 11. An analytic approximation to the gravitational-wave power spectrum is shown, using the cosmic-string parameters $\alpha=10^{-4}$ and $G\mu/c^2=3.2\times10^{-6}$.

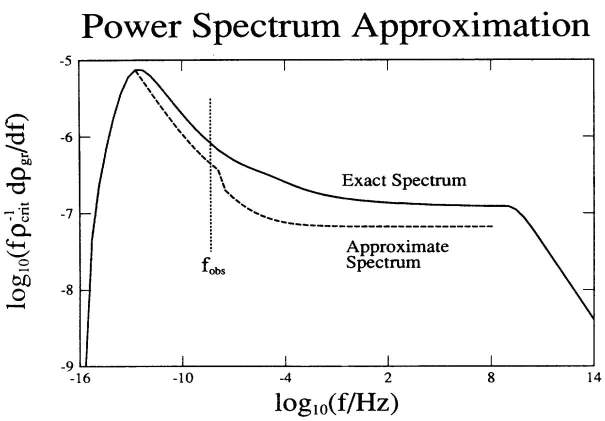

FIG. 12. An analytic approximation to the gravitational-wave power spectrum is shown, using the cosmic-string parameters $\alpha=10^{-2}$ and $G\mu/c^2=3.2\times10^{-6}$.

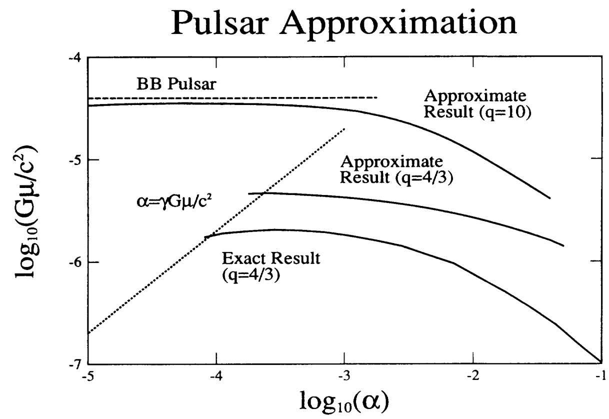

FIG. 13. Analytic approximations to the pulsar timing constraint, for both $q=10$ and $q=4/3$, as described in Appendix B, are shown. The constraint determined by Bennett and Bouchet [19], valid in the limit $\alpha\leq\gamma G\mu/c^2$, is shown. Overall, the analytic expressions yield a poor approximation to the exact, numerical result.

gives the total energy in gravitational radiation as observed at present, as a fraction of the critical energy density, as observed today.

In order to determine the fraction of the total energy density in gravitational waves which lies in the logarithmic frequency interval at $f_{\rm obs}$, the radiation- and matter-era contributions are scaled to fit a power spectrum. As seen in Fig. 3, the radiation-era contribution is limited roughly to the frequency interval $f\in[f_{\rm peak},f_{\max}]$. Here

$$f_{\max}=2\frac{c}{f_r\alpha l(t_f)}\frac{a(t_f)}{a(t_{\rm present})}$$

is the frequency of emission of loops at time t_f in the $n=1$ mode of oscillation, as observed today. The bounds of this interval are the lowest and highest frequencies for which the radiation-era contribution fits approximately to a flat spectrum. The matter-era contribution is fitted to a peaked spectrum which behaves as

$$\frac{f}{\rho_{\rm crit}}\frac{d\rho_{\rm gr}}{df}\propto f^{-1/3}\,,$$

also seen in Fig. 3. Thus, given μ and α, an estimated analytic spectrum may be constructed using the equation

$$\frac{f}{\rho_{\rm crit}}\frac{d\rho_{\rm gr}}{df}=\frac{1}{3}\rho_{\rm gr}\rho_{\rm crit}^{-1}|_{\rm matter}(f_{\rm peak}/f)^{1/3}+\rho_{\rm gr}\rho_{\rm crit}^{-1}|_{\rm rad}\times[(Z_{\rm eq}+1)\ln(f_{\max}/f_{\rm peak})]^{-1}\,. \quad ({\rm B5})$$

A sample gravitational-radiation power spectrum produced in this way is shown in Figs. 11 and 12 for different values of α and μ. In the neighborhood of $f\sim(7.1\ {\rm yr})^{-1}$, the approximate and exact power spectra do not agree very well. The dip in the power spectrum at frequencies just above $f_{\rm obs}$ is due to the change in the number of particle degrees of freedom which occur before nucleosynthesis. For the parameters $\alpha=10^{-4}$ and $G\mu/c^2=3.2\times10^{-6}$ used in Fig. 11, the approximate spectrum mimics the shape, but is not a very good fit to the exact, numerical spectrum at the frequency $f_{\rm obs}$. For larger values of α, as was discussed in Sec. IV, the slope of the peak in the spectrum softens rapidly near the flat portion of the spectrum. The approximate and exact spectra do not agree as well for large α. For the parameters $\alpha=10^{-2}$ and $G\mu/c^2=3.2\times10^{-6}$ used in Fig. 12, the approximate spectrum is a poor fit to the exact, numerical spectrum at the frequency $f_{\rm obs}$.

It is simple to determine the pulsar timing constraint to cosmic strings in this analytic approximation; one need only evaluate Eq. (B5) at $f\approx(7.1\ {\rm yr})^{-1}$. The constraint on μ and α due to this analytic approximation is shown in Fig. 13. For small values of α, the analytic estimate is seen to be excellent. However, for larger values of α the limiting contour does not agree as well with the numerical result. Thus, this analytic approximation has only limited use.

[1] A. Vilenkin, Phys. Rep. **121**, 263 (1985).
[2] R. Brandenberger, J. Phys. G. **15**, 1 (1989).
[3] *The Formation and Evolution of Cosmic Strings,* edited by G. Gibbons, S. Hawking, and T. Vachaspati (Cambridge University Press, Cambridge, England, 1990).
[4] *Cosmic Strings: The Current Status,* edited by F. Accetta and L. Krauss (World Scientific, Singapore, 1988).
[5] D. R. Stinebring, M. F. Ryba, J. H. Taylor, and R. W. Romani, Phys. Rev. Lett. **65**, 285 (1990).
[6] J. Yang, M. Turner, G. Steigman, D. Schramm, and K. Olive, Astrophys. J. **281**, 493 (1984).
[7] K. Olive, D. Schramm, G. Steigman, and T. Walker, Phys. Lett. B **236**, 454 (1990).
[8] A. Boesgaard and G. Steigman, Annu. Rev. Astron. Astrophys., **23**, 319 (1985).
[9] T. Walker, G. Steigman, D. Schramm, K. A. Olive, and Ho-Skik Kang, Astrophys. J. **376**, 757 (1991).
[10] E. Kolb and M. Turner, *The Early Universe* (Addison-

Wesley, New York, 1990).
[11] S. Weinberg, *Gravitation and Cosmology* (Wiley, New York, 1971).
[12] A. Vilenkin, Phys. Lett. **107B**, 47 (1981).
[13] C. Hogan and M. J. Rees, Nature (London) **311**, 109 (1984).
[14] T. Vachaspati and A. Vilenkin, Phys. Rev. D **31**, 3052 (1985).
[15] R. Davis, Phys. Lett. **161B**, 285 (1985).
[16] R. Brandenberger, A. Albrecht, and N. Turok, Nucl. Phys. **B277**, 605 (1986).
[17] D. Bennett, Phys. Rev. D **33**, 872 (1986); **34**, 1235(E) (1986); **34**, 3592 (1986); **34**, 3932(E) (1986).
[18] F. Accetta and L. Krauss, Phys. Lett. B **233**, 93 (1989); Nucl. Phys. **B319**, 747 (1989).
[19] D. Bennett and F. Bouchet, Phys. Rev. D **41**, 720 (1990); **43**, 2733 (1991).
[20] A. Albrecht and N. Turok, Phys. Rev. D **40**, 973 (1990).
[21] N. Sanchez and M. Signore, Phys. Lett. B **214**, 14 (1988); **241**, 332 (1990); M. Signore and N. Sanchez, Report No. MEUDON-90103, 1990 (unpublished).
[22] M. Quiros, Phys. Lett. B **267**, 27 (1991).
[23] B. Allen and E. P. S. Shellard, Phys. Rev. D **45**, 1898 (1992).
[24] R. Zimmerman and R. Hellings, Astrophys. J. **241**, 475 (1980).
[25] B. Carr, Astron. Astrophys. **89**, 6 (1980).
[26] A. Albrecht and N. Turok, Phys. Rev. Lett. **54**, 1868 (1985).
[27] D. Bennett and F. Bouchet, Phys. Rev. Lett. **60**, 257 (1988).
[28] H. Kurki-Suonio, R. Matzner, K. Olive, and D. Schramm, Astrophys. J. **353**, 406 (1990).
[29] C. Alcock, G. Fuller, and G. Mathews, Astrophys. J. **320**, 439 (1987).
[30] D. Bennett and F. Bouchet, Phys. Rev. Lett. **60**, 257 (1988); **63**, 2776 (1989); Astrophys. J. **354**, L41 (1990); in *The Formation and Evolution of Cosmic Strings,* Proceedings of the Symposium, Cambridge, England, 1989, edited by G. W. Gibbons, S. W. Hawking, and T. Vachaspati (Cambridge University Press, Cambridge, England, 1989); Phys. Rev. D **41**, 2408 (1990).
[31] R. Brandenberger and J. H. Kung, in *The Formation and Evolution of Cosmic Strings* [30].
[32] B. Allen and E. P. S. Shellard, Phys. Rev. Lett. **64**, 119 (1990); E. P. S. Shellard and B. Allen, in *The Formation and Evolution of Cosmic Strings* [30].
[33] B. Allen and R. R. Caldwell, Phys. Rev. D **43**, 3173 (1991).
[34] A. Everett, Phys. Rev. D **24**, 858 (1981).
[35] A. Vilenkin, Phys. Rev. D **43**, 1060 (1991).
[36] M. F. Ryba, Ph.D. thesis, Princeton University, 1991.
[37] L. Krauss and P. Romanelli, Astrophys. J. **358**, 47 (1990).
[38] R. Weiss (private communication).
[39] M. F. Ryba and J. H. Taylor (private communication).
[40] R. J. Scherrer, J. M. Quashnock, D. N. Spergel, and W. H. Press, Phys. Rev. D **42**, 1908 (1990).
[41] F. R. Bouchet, D. P. Bennett, and A. Stebbins, Nature (London) **335**, 410 (1988).
[42] J. Silk and A. Vilenkin, Phys. Rev. Lett. **53**, 1700 (1984); T. Vachaspati, *ibid.* **57**, 1655 (1986); M. J. Rees, Mon. Not. R. Astron. Soc. **222**, 21 (1986); A. Stebbins, S. Veeraraghavan, R. Brandenberger, J. Silk, and N. Turok, Astrophys. J. **322**, 1 (1987); T. Hara and S. Miyoshi, Prog. Theor. Phys. **81**, 1187 (1990); L. Perivolaropoulos, R. Brandenberger, and A. Stebbins, Phys. Rev. D **41**, 1764 (1990); L. Perivolaropoulos, R. Brandenberger, and A. Stebbins, Int. J. Mod. Phys. A **5**, 1633 (1990).
[43] T. Vachaspati and A. Vilenkin, Phys. Rev. Lett. **67**, 1057 (1991); D. Vollick, Phys. Rev. D **45**, 1884 (1992).
[44] W. Press, S. Teukolsky, B. Flannery, and W. Vetterling, *Numerical Recipes: The Art of Scientific Programming* (Cambridge University Press, Cambridge, England 1988).