Cutoff quantization and the Skyrmion

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The putative classical soliton in the minimal nonlinear σ model (no Skyrme term) is known to be unstable to collapse. We note that the imposition of a short-distance cutoff (which is anyway physically reasonable for a nonrenormalizable model) yields a stable classical soliton. We further suggest that this cutoff, carrying as it does some implicit dynamical information, be treated as a quantized dynamical variable. The resulting one- (experimentally fixed) parameter model agrees with experiment roughly as well as the simple σ model *with* the Skyrme term. We interpret this feature as an indication of the robustness of the description of the nucleon as being dominated by a hedgehog-type meson cloud. It is suggested that the same approach might be useful in some other situations where the long-distance description of the physics is more precisely known than is the short-distance description.

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I. INTRODUCTION

There is now general agreement that the low-energy effective Lagrangian for QCD has the leading term

$$L=-\frac{F_\pi^2}{8}\mathrm{Tr}(\partial_\mu U\partial_\mu U^\dagger)+\cdots \ , \qquad (1.1)$$

where $F_\pi \simeq 132$ MeV is the "pion decay constant" and the 2×2 unitary, unimodular matrix $U(x)$ describes the pions. For simplicity, strange degrees of freedom and the nonzero value of the pion mass are being neglected. Physically, the leading term represents "long-distance" effects. Including terms with more derivatives and additional fields should bring in short-distance dynamics. Remarkably, baryons may be treated as solitons of the purely mesonic L. One makes the classical Skyrme ansatz

$$U_0(\mathbf{x})=\exp[i\hat{\mathbf{x}}\cdot\boldsymbol{\tau}F(r)] \qquad (1.2)$$

with $F(0)=\pi$, $F(\infty)=0$, and treats the baryons as excitations of the variable $A(t)$ defined from

$$U(\mathbf{x},t)=A(t)U_0(\mathbf{x})A^\dagger(t) \qquad (1.3)$$

with $A^\dagger=A^{-1}$. The "profile" $F(r)$ is determined by the minimization of the static Hamiltonian derived from (1.1).

Since the first term in (1.1) is the one which is most definitely established, it is clearly of interest to investigate a possible baryon excitation associated with just this term alone. However, as observed long ago by Skyrme [1], the classical profile $F(r)$ in such a case is unstable to rescaling of the coordinates ("Hobart-Derrick" [2] collapse). To overcome this he added a particular four-derivative term. It seems to us, though, that it is still interesting to pursue the question of solitons in the simple nonlinear σ model (no extra terms). Several reasons can be given.

(i) There is no special reason to expect that the Skyrme four-derivative term is the only additional one. In fact, it is rather likely that many, many others come into play at short distances (not to mention explicit degrees of freedom). One would not like to mortgage one's theory to a particular choice of additional terms.

(ii) The predicted collapse occurs at the classical level. Can quantum fluctuations prevent this collapse as they prevent the collapse of the S-wave state of the classical hydrogen atom?

(iii) The simple σ model may be a paradigm for several other theories of interest. In particular, a topological excitation ("geon") of pure Einstein gravity has similar characteristics [3].

Recently an investigation [4] was made of the effect of quantum fluctuations in a collective scaling variable in the simple nonlinear σ model. It was found that, if one assumed some *a priori* "reasonable" profile (which, of course, could not be a classical solution of the equations of motion), then, at the classical level, the energy functional could be reduced to zero by collapsing this variable, in agreement with expectations. However, quantization of the scaling variable definitely prevents collapse in this mode. This is not the whole story since the field theory contains an infinite number of collective modes and one must check stability for all of them. In fact, it was noted that there is an additional instability at the classical level associated with allowing the profile to persist for large values of r. It was suggested that a confinement-type ansatz might solve this problem at a phenomenological level. At a deeper level one would like to investigate the full quantum theory with more than just the scaling variable quantized. The full problem seems very difficult. In the Appendix we give a heuristic discussion which suggests that, at the level of two additional quantum modes, one still requires a confinement-type ansatz. A similar program was, unbeknownst to the authors of Ref. [4], earlier proposed by Carlson [5] using a path-integral formulation.

In the present paper, we would like to take a slightly different, though closely related track. We start by

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stressing that the simple nonlinear σ model is an effective rather than a fundamental one. Indeed, it is not renormalizable. Hence, it seems rather plausible to work with a short-distance cutoff ϵ in the theory. The static energy functional then takes the form

$$E=\pi F_\pi^2\int_\epsilon^\infty r^2dr\left[\left[\frac{dF}{dr}\right]^2+\frac{2}{r^2}\sin^2F\right], \qquad (1.4)$$

where the ansatz (1.2) was employed. With nonzero ϵ and the boundary conditions $F(\epsilon)=\pi$, $F(\infty)=0$, it is easy to show that there is, in fact, a unique classical solution which minimizes (1.4). This solution does not exist in the $\epsilon\rightarrow 0$ limit, as is well known. We incidentally point out that the analysis can be greatly simplified by noting that a change of variable converts the equation of motion resulting from (1.4) to that of the ordinary damped pendulum (Sec. II). In addition, we explicitly check that the static solution to (1.4) is stable (Sec. III).

As it stands, the present proposal has the advantage that it yields a unique classical solution which can be used as a starting point for the conventional semiclassical quantization. The disadvantage is that it depends on the quantity ϵ. To overcome this problem we further argue that a cutoff generally contains, if only by default, some dynamical information. In the present case it gives a schematic description of the short-distance dynamics we have chosen not to include. As a carrier of dynamical information it seems logical for the cutoff to be treated as a dynamical variable $\epsilon(t)$ and to be quantized. This yields a Schrödinger-like equation similar to the one discussed in Ref. [4], but with a slightly different interpretation: the scaling variable now measures the size of the short-distance region rather than the size of the nucleon. Physical observables are, of course, to be obtained as expectation values with respect to the wave function for the ϵ variable. Since ϵ is now a variable, there is only one parameter F_π in the model. Interestingly, we find that the numerical results for the nucleon as a soliton in this model (Sec. IV) are very similar to those of the two-parameter Skyrme model. We choose to interpret this feature as an indication of the robustness of the concept that a "pion cloud" plays an important role in the low-energy description of the nucleons.

The underlying spirit of the present model for the case of the nucleon is clearly similar to those of the "chiral bag" or "chiral quark" variety [6]. However, our approximation is a cruder one in the sense that we have chosen to ignore the finer details of short-distance dynamics. We feel that the present way of looking at things is rather illuminating and furthermore, may be profitably applied to various other theories of interest (Sec. V).

II. CLASSICAL BEHAVIOR: ANALOGY WITH THE PENDULUM

As we noted in the Introduction, there is a unique classical solution minimizing the energy functional with a short-distance cutoff given in (1.4). The presence of a cutoff prevents collapse and leads to a stable classical solution. In this section we check this fact by exploiting its analogy with the damped pendulum. Though a related, but slightly different, observation has been made before in the literature [7], our approach in the presence of a cutoff leads to a more transparent picture. One clearly sees the existence of a classical solution when there is a cutoff and its absence as the cutoff is removed.

Minimization of (1.4) leads to the following equation for the profile F:

$$\frac{d}{dr}\left[r^2\frac{dF}{dr}\right]=\sin 2F\ . \qquad (2.1)$$

This is a nonlinear differential equation, to be solved respecting the boundary conditions $F(r=\epsilon)=\pi$ and $F(r=\infty)=0$. Near the boundaries it can be linearized, leading to simple differential equations. As $r\rightarrow\infty$, $\sin 2F$ can be approximated by $2F$, while for r near ϵ, it can be approximated by $2(F-\pi)$. This yields the following behavior:

$$F(r\rightarrow\epsilon)\sim\pi-\frac{\alpha}{3}\left[\frac{r}{\epsilon}-\frac{\epsilon^2}{r^2}\right],\quad F(r\rightarrow\infty)\sim\beta\frac{\epsilon^2}{r^2}\ , \qquad (2.2)$$

where α and β are two constants that do not depend on ϵ. To study (2.1) further, we look for an analogous system whose behavior can be inferred from intuition. Introducing new variables $\tau\equiv\ln(r/\epsilon)$ and $\theta=2(\pi-F)$, we can rewrite the above as

$$\frac{d^2\theta}{d\tau^2}+\frac{d\theta}{d\tau}=2\sin\theta. \qquad (2.3)$$

In this form the equation is identical to that of an ordinary damped pendulum. θ is the angle measured from the top, the unstable equilibrium point, of the pendulum's orbit and τ is time. The boundary conditions for the profile are now translated into those of θ as $\theta(\tau=0)=0$ and $\theta(\tau=\infty)=2\pi$. That is, as "time" runs from zero to infinity, θ covers the range from zero to 2π. In other words, the pendulum, pushed from the top, returns to it asymptotically after making one full revolution. We wish to argue that such a behavior is possible for a special value of the initial velocity at which the pendulum is pushed. Before we come to this, we may note that this kind of motion is not typical for the pendulum. Being damped, it will almost always end up losing all of its potential energy and eventually reach stable equilibrium hanging at the bottom of its orbit. Thus, the generic case will rather satisfy the boundary conditions $\theta(\tau=0)=0$ and $\theta(\tau=\infty)=\pi$. In terms of the profile, what we have just observed is that the generic solution of (2.1) approaches $\pi/2$, rather than zero, as r tends to infinity (this is similar to the case of no cutoff, see Ref. [8]). We checked this fact numerically, and the profile is plotted in Fig. 1(a). We find that the profile (or the pendulum) swings past $\pi/2$ (or the bottom of the orbit) before it appears to settle down.

We want the pendulum to return to the top in the distant future. If we push the pendulum from the top a little harder, it will swing further past the bottom before it settles down. This suggests that, if we push it with the right velocity, it will swing just enough to reach the top. Then,

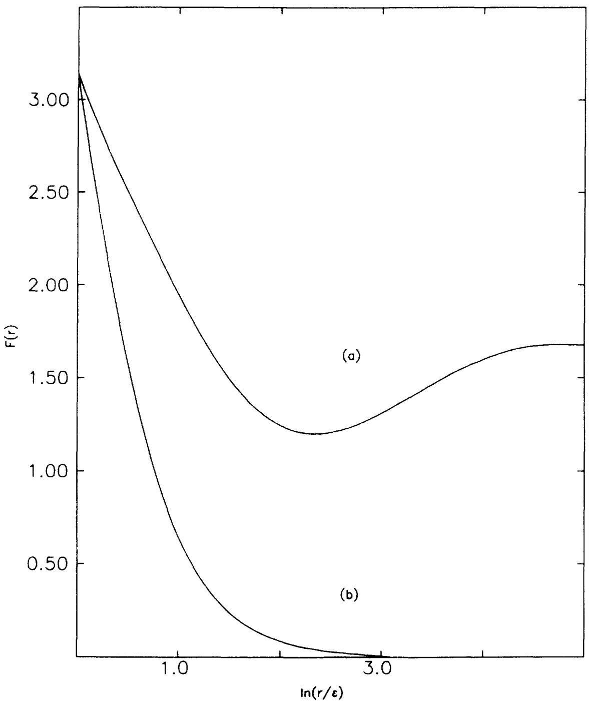

FIG. 1. (a) Behavior of profile for typical initial boundary condition, $F'(\epsilon)$. (b) Profile for the choice of $F'(\epsilon)$ corresponding to $F(r)\to 0$ as $r\to\infty$.

it starts off with the right amount of kinetic energy to be spent against damping and makes use of its potential energy to return to the top after making one full revolution. This will happen asymptotically after infinite time since the pendulum will have infinitesimally small velocity on its return to the top, having spent all of its kinetic energy against damping. This intutive picture suggests that there exists a solution to (2.1) satisfying the required boundary conditions. Further, it is clear that this solution for the profile will be monotonically decreasing. Thus, as long as there is a short-distance cutoff ϵ, the existence of a classical solution is guaranteed. This solution can be obtained numerically and is shown in Fig. 1(b). We find, for the two constants α and β, the values 3.79 and 4.84, respectively. The next highest winding-number solution, for example, would correspond to F going to $-\pi$ at large r; in this case the initial velocity of the pendulum will be such that it makes exactly two revolutions.

The case of $\epsilon=0$ is special. That there is no classical solution in this case is, of course, known from Derrick's theorem. From the point of view of the pendulum, we get a clear picture. In this case, let τ be $\ln(r/a)$, where a is some distance scale. The boundary conditions for F now translate to those of θ as $\theta(\tau=-\infty)=0$ and $\theta(\tau=\infty)=2\pi$. That is, θ starts off from zero in the infinite past and approaches 2π in the infinite future. In other words, the pendulum leaves the top far back in time and returns to it in the distant future. The only place the pendulum can spend infinite time in the past is at its starting point, the top of the orbit. This is possible if it starts off with zero (which corresponds to dF/dr being finite at $r=0$), or rather infinitesimally small, velocity; but then it does not have sufficient energy to overcome damping and reach the top again. This clearly shows what is special about $\epsilon=0$ and why there is no classical solution in this case.

III. CHECK OF STABILITY OF THE CLASSICAL SOLUTION

We rewrite the expression for the static energy functional (1.4) by changing variables to $y\equiv r/\epsilon$. We then obtain

$$E=\epsilon\pi F_\pi^2\int_1^\infty y^2dy\left[\left(\frac{dF}{dy}\right)^2+\frac{2}{y^2}\sin^2F\right] . \tag{3.1}$$

From this expression it is easy to see that the minimizing profile $F(y)$ is universal, i.e., independent of the numerical value of the cutoff ϵ. We also note that, as $\epsilon\to 0$, the energy $E\to 0$, as expected from the classical instability without cutoff. We now show that E is stable against small perturbations $\eta(y)$ of the minimizing profile $F(y)$:

$$F(y)\to F(y)+\eta(y) . \tag{3.2}$$

$\eta(y)$ satisfies the boundary conditions $\eta(1)=0$ and $\eta(\infty)=0$. The second variation in E is then

$$\frac{\delta^2E}{\epsilon\pi F_\pi^2}=\int_1^\infty dy[y^2\eta'^2+2\cos(2F)\eta^2] , \tag{3.3}$$

where the prime denotes differentiation with respect to y. To prove stability of our solution, we must show that δ^2E is positive definite.

Equation (3.3) can also be looked upon as the expectation value of a one-dimensional Hamiltonian operator which gives rise to the Schrödinger-like equation

$$-u''+\frac{2}{y^2}\cos[2F(y)]u=\lambda u , \tag{3.4}$$

where $u\equiv y\eta$ is a normalizable wave function and λ is the "energy" eigenvalue. The problem is then equivalent to showing that there are no zero or negative-energy solutions to (3.4). Offhand, it might appear that negative-energy bound states are possible in the potential $V(y)=(2/y^2)\cos[2F(y)]$. However, this turns out not to be the case.

Rather than studying the eigenvalue problem (3.4), we use an indirect technique given in Ref. [9] to prove positivity of δ^2E. We rewrite (3.3) by adding the total derivative term

$$\int_1^\infty\frac{d}{dy}(y\sigma\eta^2)dy , \tag{3.5}$$

which does not change δ^2E provided that $y\sigma\eta^2$ vanishes at the boundaries. Then (3.3) becomes

$$\frac{\delta^2E}{\epsilon\pi F_\pi^2}=\int_1^\infty dy[y^2\eta'^2+2y\sigma\eta\eta'+(2\cos2F+y\sigma'+\sigma)\eta^2] . \tag{3.6}$$

The integrand will be a perfect square if there exists an auxiliary function $\sigma(y)$ obeying the Ricatti equation

$$y\sigma' = \sigma^2 - \sigma - 2\cos 2F . \quad (3.7)$$

Hence, finding any suitable σ will guarantee stability. We note that, since $\cos 2F \to 1$ both as $y \to 1$ and as $y \to \infty$, one has, near the boundaries,

$$\sigma \to \frac{2+Cy^3}{1-Cy^3} , \quad (3.8)$$

where the constant C depends on the boundary. Hence, we will look numerically for possible solutions of (3.7) obeying

$$\sigma(1) = \text{finite}, \quad \sigma(\infty) = -1 . \quad (3.9)$$

It is easy to find that a family of such solutions exist, a few of which are shown in Fig. 2. Thus, we have checked stability for classical solutions in the cutoff model with the ansatz (1.2).

A further remark must be added if the variation η goes to zero at infinity as $1/\sqrt{y}$ or slower, since, in that event, the added term (3.5) is nonzero. However, from (3.9) the added term is seen to be negative, so that $\delta^2 E$ is still positive as required for stability.

It might be amusing to note that the function $u(y)$ defined by

$$\sigma(y) = -y\frac{d}{dy}\ln[u(y)] , \quad (3.10)$$

converts the Ricatti equation (3.7) into the Schrödinger equation (3.4), but with $\lambda = 0$. At first, one might therefore think that we are finding a zero-"energy" eigenvalue of (3.4) by obtaining a nonzero σ. However, it can be seen that the $u(y)$ obtained by integrating (3.10) and using the obtained $\sigma(y)$ is not normalizable and, hence, is not a suitable eigenfunction of the Schrödinger operator in (3.4).

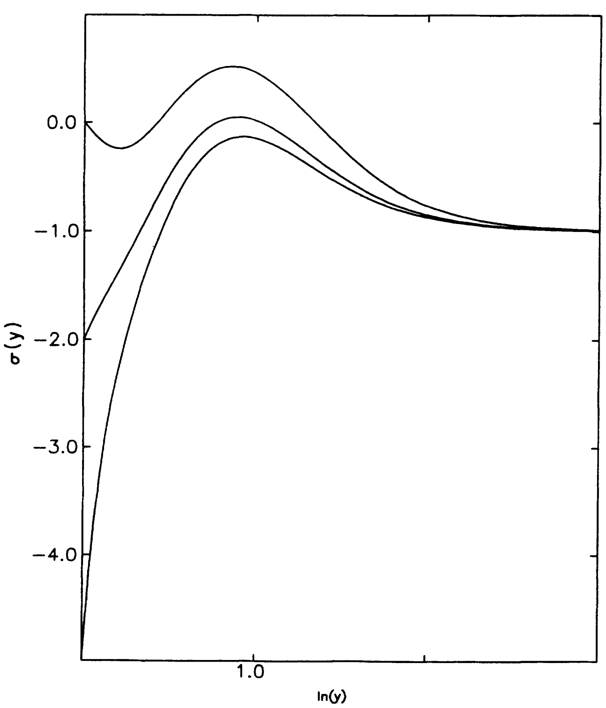

FIG. 2. Some solutions for the auxiliary function σ for different choices of initial conditions.

IV. QUANTIZING THE CUTOFF

In the Introduction, we suggested that the cutoff parameter ϵ be treated as a dynamical variable $\epsilon(t)$. The profile $F(r)$ will then depend on t implicitly through $\epsilon(t)$. If we now follow Ref. [4], using the previously obtained classical solution for the profile and taking account of the angular variables $A(t)$, we obtain the following collective-coordinate Lagrangian from (1.1):

$$L_{\rm col} = a\dot{x}^2 - bx^{2/3} + cx^2 \mathrm{Tr}(\dot{A}\dot{A}^\dagger) , \quad (4.1)$$

where the dot refers to differentiation with respect to t and x is related to ϵ as $x = \epsilon^{3/2}$. The coefficients a, b, and c are given by

$$a = \frac{4\pi}{9}F_\pi^2 \int_1^\infty y^4 dy \left[\frac{dF}{dy}\right]^2 ,$$

$$b = \pi F_\pi^2 \int_1^\infty y^2 dy \left[\left[\frac{dF}{dy}\right]^2 + \frac{2}{y^2}\sin^2 F(y)\right] , \quad (4.2)$$

$$c = \frac{4\pi}{3}F_\pi^2 \int_1^\infty y^2 dy \, \sin^2 F(y) ,$$

where $y = r/\epsilon$ as in the previous section. Numerically, we find their values to be 1.46, 0.78, and 0.91 GeV^2, respectively. As in Ref. [4], we may now make a rough estimate for the energy using the uncertainty principle. Neglecting the angular variables, we minimize the resulting effective energy

$$\frac{1}{4a\bar{x}^2} + b\bar{x}^{2/3} \quad (4.3)$$

with respect to the mean value $\bar{x}$. We obtain for the ground-state energy a value 0.94 GeV that is quite close to the mass of the nucleon. The cutoff ϵ, given by $\bar{x}^{2/3}$, turns out to be 0.18 fm; too small to agree with the size of the nucleon (0.72 fm). This is, however, what we should expect since the cutoff is a carrier of short-distance dynamics. The main contribution to the size of the nucleon should come from the "size" of the pion cloud. If we estimate the isoscalar charge-squared radius from the isoscalar density $B(r) = -(2/\pi)F'\sin^2 F$, using the classical F, as

$$\langle r^2 \rangle = \int_\epsilon^\infty r^2 dr \, B(r)$$

$$= -\frac{2}{\pi}\int_1^\infty y^2 dy \frac{dF}{dy}\sin^2 F(y)\epsilon^2 = 3.42\epsilon^2 , \quad (4.4)$$

we get, for the size of the nucleon, $\langle r^2 \rangle^{1/2} = 0.33$ fm, which is still small. More careful analysis gives better agreement, as we will see.

To obtain reliable results, one should solve the Schrödinger equation for the problem at hand. The Hamiltonian operator derived from the Lagrangian (4.1) reads [4]

$$H = -\frac{1}{4ax^3}\frac{\partial}{\partial x}\left[x^3\frac{\partial}{\partial x}\right] + bx^{2/3} + \frac{I(I+1)}{2cx^2} , \quad (4.5)$$

where the isospin I results from quantizing the angular variables. This gives the Schrödinger equation $H\psi=E\psi$, where the wave function ψ is normalized with respect to the measure x^3dx. In terms of a new wave function $\phi\equiv x^{3/2}\psi$, normalized with respect to the standard measure dx, we obtain the Schrödinger equation

$$\left[-\frac{1}{4a}\frac{\partial^2}{\partial x^2}+bx^{2/3}+\frac{3}{16ax^2}+\frac{I(I+1)}{2cx^2}\right]\phi=E\phi\ . \quad (4.6)$$

We have solved this eigenvalue problem numerically and obtained three low-lying states for the cases $I=0$, $\frac{1}{2}$, and $\frac{3}{2}$. The $I=0$ case is not interesting physically, but lets us compare our earlier rough estimates with the exact results. The ground-state energy in this case comes out to be 1.27 GeV, which is comparable to the rough estimate of 0.94 GeV. The next two excited states have energies 1.77 and 2.15 GeV.

In the case of $I=\frac{1}{2}$, which is interesting physically because it corresponds to the nucleon, we obtain the three low-lying solutions at 1.54, 1.96, and 2.32 GeV. These should be compared to the experimentally observed particles $N(940)$, $N(1440)$, and $N(1710)$. We observe that the *splittings* are in reasonable agreement with experiment. The normalized wave functions are as shown in Figs. 3(a)–3(c). When $I=\frac{3}{2}$, the three low-lying energy eigenvalues turn out to be 1.98, 2.32, and 2.63 GeV. The observed particles to be compared are $\Delta(1232)$, $\Delta(1600)$, and $\Delta(1920)$. Again, the predicted $(\frac{3}{2})^+$ splittings agree with experiment. The $\Delta(1232)-N(940)$ difference turns out to be a bit too large, however.

The wave functions, computed above for $I=\frac{1}{2}$, can be used to obtain the mean values of various parameters that are of interest. For instance, the average value of the cutoff can be obtained from

$$\langle\epsilon\rangle=\int_0^\infty dx\,\phi^2(x)x^{2/3}\ , \quad (4.7)$$

which yields a value of 0.29 fm, close to our rough estimate. The root-mean-squared cutoff can be similarly estimated; it turns out to be nearly the same, $\langle\epsilon^2\rangle^{1/2}=0.30$ fm. As for the isoscalar charge-squared radius, we use the isoscalar density $B(r,x)=-(2/\pi)F'\sin^2F$ as before [see Eq. (4.4)] but now compute

$$\langle r^2\rangle=\int_0^\infty dx\,\phi^2(x)\int_\epsilon^\infty r^2dr\,B(r,x)$$
$$=3.42\langle\epsilon^2\rangle\ , \quad (4.8)$$

giving 0.55 fm for the size of the nucleon. This is reasonably close to the experimental value of 0.72 fm.

The average profile function F_{av} can be obtained from

$$F_{av}(r)=\int_0^\infty dx\,\phi^2(x)F(r,x)\ , \quad (4.9)$$

with the natural assumption that $F(r,x)$ stays at π for $r<\epsilon$, (i.e., $x>r^{3/2}$). We have plotted this profile in Fig. 4. From the behavior near infinity,

$$F_{av}(r)\sim\int_0^\infty dx\,\phi^2(x)\frac{\beta\epsilon^2}{r^2}=\frac{\beta\langle\epsilon^2\rangle}{r^2}=\frac{10.97\ \mathrm{GeV}^{-2}}{r^2} \quad (4.10)$$

(β, defined in Sec. II, has the value 4.84), we can find [10] the axial-vector coupling constant of neutron β decay,

$$g_A=2\pi F_\pi^2(10.97\ \mathrm{GeV}^{-2})=1.20\ , \quad (4.11)$$

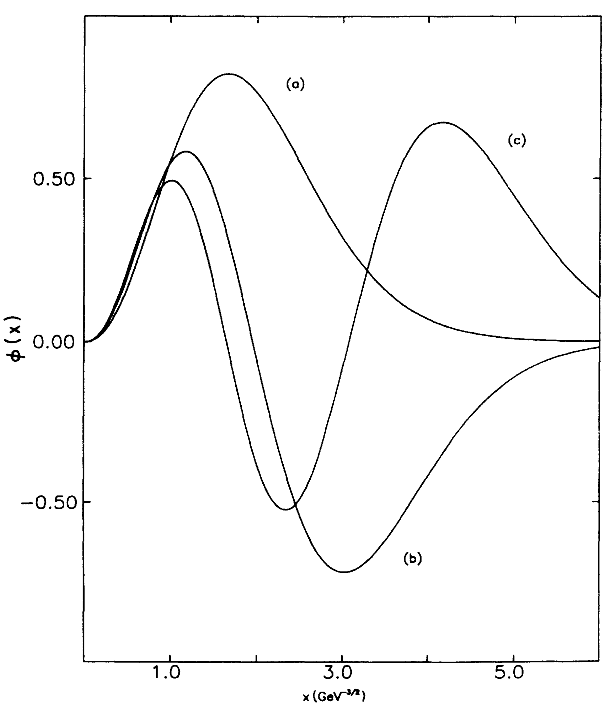

FIG. 3. Normalized x-space wave functions for the three lowest-lying $I=\frac{1}{2}$ states. The order of increasing energy is a, b, c.

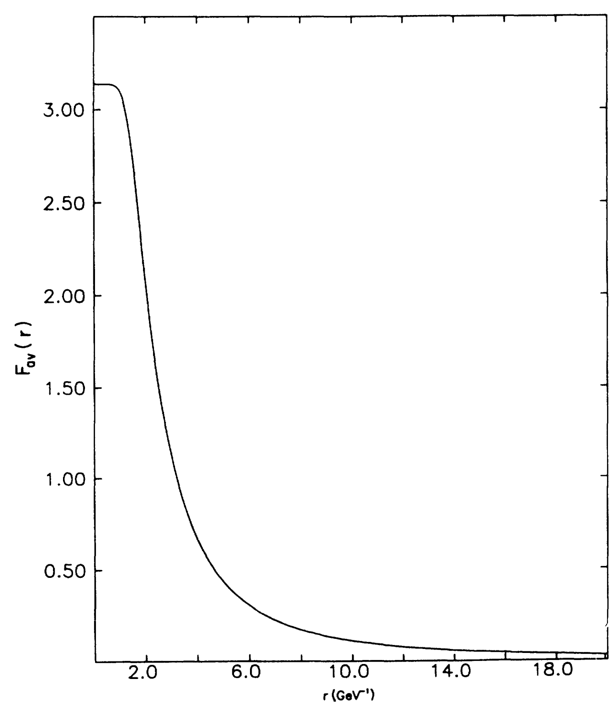

FIG. 4. Average profile function.

which is in good agreement with the observed value of 1.23.

It is clear that the predictions of this one-parameter model are of similar quality to the ones obtained in the usual [10] two-parameter Skyrme model with the Skyrme term. As in that model, the nucleon mass is predicted somewhat too high when the physical value of F_π is used. Multiplying F_π by 0.61 would bring the $N(940)$ and $\Delta(1232)$ into agreement, but there is no special logical reason to do so.

V. DISCUSSION

Our intention has not been to propose a model which gives a better numerical fit than the standard Skyrme model, but rather to demonstrate that the main results of the Skyrme model are not very sensitive to details of the short-distance physics. Certainly, the addition of detailed models for the "core" region containing various higher-derivative terms, vector and perhaps other mesons or explicit quarks involves new parameters which can be expected to enable one to fine-tune some of the predictions. In common with other models which do not include explicit quarks in the core region, the nucleon mass comes out substantially higher than experiment. Physically [11], explicit quarks can occupy negative-energy levels and overcome this problem. This might be mocked up by an overall energy subtraction. From a practical point of view, the present model has some similar features to those [12] in which the breathing mode is quantized—the "radial" wave function now refers to the cutoff or "core" size rather than to the actual nucleon size. It should be remarked that the only numerical parameter in the present model is F_π, which is taken from experiment. One might regard the choice $F(\epsilon)=\pi$ as a specific value of $F(\epsilon)$, considered as a parameter; however, it is certainly a natural choice.

The present approach may be useful in other problems where, either for simplicity or lack of knowledge, one wants to suppress information about the physics at short distances. There are many possible examples: One might like to include vectors in a simplified way in the chiral Lagrangian, thereby presumably pushing the unknown region [i.e., $\langle \epsilon \rangle$ in Eq. (4.7)] to still smaller distances. One might want to investigate higher winding-number solitons with the quantized cutoff. One might like to study possible solitons in the electroweak theory wherein the quantized cutoff could describe something like technicolor physics. As already remarked in Sec. I, the technique might enable one to study the dynamics of gravitational solitons, ϵ representing either an unspecified short-distance modification of Einstein gravity or a nonperturbative effect of that theory which would enforce stability.

In this paper we have discussed a concept related to, but not identical with, quantum stability for solitons in the minimal nonlinear σ model. We did, however (see the Appendix), give a heuristic discussion of the "pure" quantum stability problem in which *two* parameters characterizing the profile were treated as quantum variables. The suggestion was that a "confinement"-type ansatz was still required for stability.

ACKNOWLEDGMENTS

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APPENDIX

Here we present a brief heuristic discussion of the quantum stability [4,7,8,13,14] of the soliton for the simple nonlinear σ model [first term of (1.1)] without cutoff, but where *two* quantum degrees of freedom are taken into account. In Ref. [4], it was shown that a "reasonable" profile leads to a soliton whose classical collapse would be prevented by quantum fluctuations of a scaling variable. But, if the profile were allowed to develop a tail behaving like $r^{-\delta-3/2}$ for small δ, it was shown that the soliton could be collapsed. However, δ was treated as a classical parameter rather than as a quantum variable. It was speculated that a "confinement" ansatz restricting the profile size might lead to a satisfactory picture. In Ref. [13], it was shown that a different parameter C could be varied to collapse the soliton. Again, this was not a true test of quantum stability, since C was not quantized.

We now consider the profile shown in Fig. 5 characterized by the variables R_1 and R_2. Note that the length of the straight line hinged at $F(0)=\pi$ is taken to be π. This choice enables us to investigate the possibility that the profile itself may collapse to the two axes (when $R_1 \to 0$). Explicitly,

$$F(r)=\begin{cases} \pi-\dfrac{r}{R_1}(\pi^2-F_\pi^2 R_1^2)^{1/2}, & r<R_1 , \\ \dfrac{r-R_2}{R_1-R_2}[\pi-(\pi^2-F_\pi^2 R_1^2)^{1/2}], & R_1<r<R_2 , \\ 0, & r>R_2 . \end{cases} \tag{A1}$$

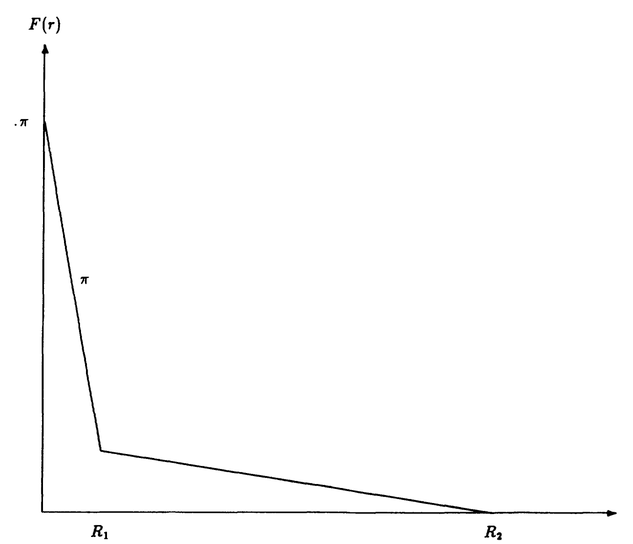

FIG. 5. Profile for the heuristic discussion of the two quantum variables case. r is in units of F_π^{-1}.

Making the approximations

$$R_1 \ll \frac{\pi}{F_\pi}, \quad R_1 \ll R_2 \tag{A2}$$

(which clearly do not interfere with a possible collapse of the profile itself) and substituting into the Lagrangian

$$L = \pi F_\pi^2 \int_0^\infty r^2 dr \left[\left(\frac{dF}{dt} \right)^2 - \left(\frac{dF}{dr} \right)^2 - \frac{2}{r^2}\sin^2 F \right], \tag{A3}$$

yields

$$\frac{L}{\pi F_\pi^2} \approx \left[\frac{\pi^2}{5} R_1 + \frac{F_\pi^4 R_1^2 R_2^3}{30\pi^2} \right] \dot{R}_1^2 + \frac{F_\pi^4 R_1^4 R_2}{20\pi^2} \dot{R}_2^2 + \frac{F_\pi^4 R_1^3 R_2^2}{20\pi^2} \dot{R}_1 \dot{R}_2 - \left[\frac{\pi^2}{3} + 1 \right] R_1 - \frac{F_\pi^4}{4\pi^2} R_1^4 R_2 . \tag{A4}$$

We are neglecting the "rotational" (spin, isospin) degrees of freedom here. The canonical momenta are computed from (A4) in the usual way, as $P_i = \partial L / \partial \dot{R}_i$. In order to obtain a heuristic treatment of the effects of quantum fluctuations in R_1 and R_2, we simply substitute

$$P_1 \to \frac{1}{R_1}, \quad P_2 \to \frac{1}{R_2} \tag{A5}$$

in natural units. This type of substitution was noted in Ref. [4] to yield the correct order of magnitude in the one variable case. With the neglect of the $\pi^2 R_1 \dot{R}_1^2/5$ term in (A4) (which can be *a posteriori* justified), it amounts to setting

$$\dot{R}_1 \to \frac{12\pi}{R_1^3 R_2^3 F_\pi^6}, \quad \dot{R}_2 \to \frac{4\pi}{R_1^4 R_2^2 F_\pi^6} . \tag{A6}$$

Finally, using this in the Hamiltonian arising from (A4), we find an expression for the "quantum energy"

$$E_{\rm qu}(R_1, R_2) \approx \frac{8\pi}{R_1^4 R_2^3 F_\pi^6} + \pi F_\pi^2 \left[\left(\frac{\pi^2}{3} + 1 \right) R_1 + \frac{F_\pi^4 R_1^4 R_2}{4\pi^2} \right] . \tag{A7}$$

For orientation, note that, if either R_1 or R_2 is held fixed, there will be a stable nonzero minimum of $E_{\rm qu}$ with respect to the other variable. This more or less corresponds to the old result. The crucial question is whether one can reduce $E_{\rm qu}$ to zero when both R_1 and R_2 are allowed to vary. Inspection of (A7) shows that it is possible to do so along paths in $R_1 - R_2$ space satisfying

$$R_2 = \frac{\text{const}}{R_1^k}, \quad \tfrac{4}{3} < k < 4 , \tag{A8}$$

and taking $R_1 \to 0$. For such paths, $R_2 \to \infty$ in the limit. Thus, the profile itself collapses to the axes in this model. It is amusing to note that a "confinement" hypothesis would keep R_2 finite by fiat and prevent the complete collapse. Of course, the above result on quantum stability must be considered highly tentative, both because only two out of the infinite number of variables in the problem were studied and because the two which were retained were quantized heuristically.

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