# Some comments on supersymmetric grand unification

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We study the constraints imposed on the hypothesis of supersymmetric grand unification by the current coupling-constant data and vice versa.

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Although this was not the way in which it was originally presented, the idea of grand unification [1] could have been motivated in the mid 1970s by noting that the values of the three standard-model coupling constants $\alpha_1,\alpha_2$, and $\alpha_3$, as then known, were consistent with a renormalization-group evolution to a single value at an energy scale a few orders of magnitude below the Planck energy. This grand-unified-theory (GUT) energy scale $M_G$ was high enough that the predicted proton lifetime in the minimal SU(5) unification model was longer than current lower limits. In the intervening years, the values of the coupling constants became more accurately known and the lower limits on the proton lifetime were increased so as to rule out this minimal SU(5). In the mid 1980s for example, it was pointed out that, although some of the best measurements of the standard-model couplings were still consistent with a convergence to a unified value, the resulting unification scale below $10^{14}$ GeV was inconsistent with any grand unification permitting proton decay [2].

The impact of new coupling-constant data from the CERN $e^+e^-$ collider LEP and the well-known result that supersymmetry could delay unification to a higher-energy scale thus significantly retarding proton decay were strikingly emphasized in a recent CERN paper [3]. The work of Ref. [3] treats the three couplings as evolving according to the standard model up to a supersymmetry (SUSY) scale $M_S$ which represents some weighted average of the masses of the SUSY partners and evolving beyond that scale according to the minimal supersymmetric extension of the standard model. The main point of this paper is that current coupling-constant data are consistent with supersymmetric grand unification if $M_S$ is of the order of 1 TeV, well within the range of the next generation of accelerators. In addition the unification scale $M_G$ is sufficiently high to account for the nonobservation of proton decay in the current generation of underground experiments. Recently Arason *et al.* [4] have extended this result to the Yukawa couplings with conclusions concerning the SUSY scale that are consistent with those of Ref. [3].

It is possible that even more precise knowledge of the couplings at the $Z$ scale will correctly predict within narrow bounds the masses of the SUSY particles. On the other hand, as the LEP data become further refined, it is possible that the coupling constants will in turn rule out a unification within a minimal supersymmetric standard model and suggest an alternate high-energy picture.

In this Brief Report, under the assumption of a minimal supersymmetric grand unification, we examine the following questions: (1) To what extent do the coupling constants as given within errors at the LEP energy scale constrain the supersymmetry scale $M_S$? (2) If one insists on a SUSY scale below 1 TeV, how well does the resulting constraint on the value of the strong coupling constant fit the lower-energy data?

The starting point is the renormalization-group equations for the three couplings $\alpha_1,\alpha_2\alpha_3$ corresponding to the standard-model group $U(1)\times SU(2)\times SU(3)$ as a function of the logarithmic scale $t=\ln(q)$:

$$\frac{d\alpha_i^{-1}(q)}{dt}=-b_i-b_{ij}\alpha_j(q)/(4\pi)-\cdots\ , \tag{1}$$

$$b_i=\frac{1}{2\pi}\begin{bmatrix}4N_f/3+N_H/10\\ -22/3+4N_f/3+N_H/6\\ -11+4N_f/3\end{bmatrix}. \tag{2}$$

Here $N_f$ and $N_H$ are the numbers of families and Higgs bosons (defaults: 3 and 1). Above the SUSY scale the coefficients $b_i$ and $b_{ij}$ change to $b_i^s$ and $b_{ij}^S$:

$$\frac{d\alpha_i^{-1}(q)}{dt}=-b_i^s-b_{ij}^s\alpha_j(q)/(4\pi)-\cdots\ , \tag{3}$$

$$b_i^s=\frac{1}{2\pi}\begin{bmatrix}2N_f+3N_H/10\\ -6+2N_f+N_H/2\\ -9+2N_f\end{bmatrix}. \tag{4}$$

The $b_{ij}$ and $b_{ij}^s$ can be read from Ref. [3]. Although the authors of Ref. [3] take into account in their numerical integration the effect of quark thresholds on the $b_i$'s as well as the second-order terms ($b_{ij}$ terms), the resulting $\alpha_i^{-1}$ are extremely linear in $t$ in both the standard-model and SUSY regimes. This allows us to answer the questions posed in the introduction with a simple analytic analysis.

Integrating Eq. (1) from $M_Z$ to $M_S$ and Eq. (3) from $M_S$ to $M_G$ and ignoring for the moment the second-order terms we find

$$\alpha_i^{-1}(M_Z)=\alpha_i^{-1}(M_S)+b_i\ln(M_S/M_Z)\ , \tag{5}$$

 

$$\alpha_i^{-1}(M_S)=\alpha_i^{-1}(M_G)+b_i^s\ln(M_G/M_S) . \quad (6)$$

Assuming a common value for the couplings at $M_G$ $[\alpha_i(M_G)\equiv\alpha_0]$, Eqs. (5) and (6) can be combined to yield

$$\alpha_i^{-1}(M_Z)=\alpha_0^{-1}+(b_i-b_i^s)\ln(M_S/M_Z)+b_i^s\ln(M_G/M_Z) . \quad (7)$$

We have therefore a matrix relationship between the low-energy parameters $\alpha_i^{-1}(M_Z)=(\alpha_1^{-1}(M_Z), \alpha_2^{-1}(M_Z),\alpha_3^{-1}(M_Z))$ and the high-energy parameters $(\alpha_0^{-1},\ln(M_S/M_Z),\ln(M_G/M_Z))$ of the form

$$\begin{bmatrix}\alpha_1^{-1}(M_Z)\\ \alpha_2^{-1}(M_Z)\\ \alpha_3^{-1}(M_Z)\end{bmatrix}=\mathbb{R}\begin{bmatrix}\alpha_0^{-1}\\ \ln(M_S/M_Z)\\ \ln(M_G/M_Z)\end{bmatrix} \quad (8)$$

with

$$\mathbb{R}=\begin{bmatrix}1 & b_1-b_1^s & b_1^s\\ 1 & b_2-b_2^s & b_2^s\\ 1 & b_3-b_3^s & b_3^s\end{bmatrix} . \quad (9)$$

Given any three values of the high-energy parameters, Eq. (8) uniquely determines three low-energy parameters (which may or may not agree with the experimental determination of those parameters, of course). Similarly, given the three experimental values of the low-energy parameters the inverse of $\mathbb{R}$ determines (within errors) three high-energy parameters $\alpha_0, M_S$, and $M_G$. In this sense a unification is trivial although unification with physically sensible values of the high-energy parameters is not guaranteed.[1]

The matrix $\mathbb{R}^{-1}$ is given by

$$\mathbb{R}_{1j}^{-1}=\epsilon_{jki}b_k b_i^s/\det R , \quad (10)$$

$$\mathbb{R}_{2j}^{-1}=\sum_i\epsilon_{jki}b_k^s/\det R , \quad (11)$$

$$\mathbb{R}_{3j}^{-1}=\sum_i\epsilon_{jki}(b_k^s-b_k)/\det R , \quad (12)$$

$$\det R=\sum_i\epsilon_{jki}b_j b_k^s . \quad (13)$$

Thus

$$\ln M_G/M_Z=\sum_i\epsilon_{jki}\alpha_j^{-1}(M_Z)(b_k^s-b_k)/\det R \quad (14)$$

and

$$\ln M_S/M_Z=\sum_i\epsilon_{jki}\alpha_j^{-1}(M_Z)b_k^s/\det R . \quad (15)$$

In order for the left-hand side of Eq. (15) to be much smaller than that of Eq. (14), the coefficients $b_k^s$ and the values of $\alpha_j^{-1}(M_Z)$ must conspire to cancel to a high degree on the right-hand side of Eq. (15). In fact there is significant cancellation between the $\alpha_1^{-1}$ and $\alpha_2^{-1}$ terms with further cancellation coming from the $\alpha_3^{-1}$ term.

For example using the coupling constants quoted in Ref. [3],

$$\alpha_i^{-1}(M_Z)=(59.22\pm0.14,30.10\pm0.23,9.26\pm0.43) , \quad (16)$$

one finds, from Eq. (15),

$$\ln(M_S/M_Z)=-0.19\pm2.29\pm1.99 , \quad (17)$$

where the first error comes from $\sin^2\theta_W$ and the second from $\alpha_3(M_Z)$. It is interesting to note that, in spite of the great accuracy in $\alpha_1$ and $\alpha_2$, their errors are correlated and in fact give a large part of the error in $M_S/M_Z$. It is clear that very small departures from the coupling constants of Eq. (16) can result in large changes in $M_S/M_Z$. For example, a comprehensive recent analysis [5] gives

$$\alpha_i^{-1}(M_Z)=(58.86\pm0.11,29.73\pm0.08,9.43\pm0.54) . \quad (18)$$

Although these values agree well with those of Eq. (16), they would lead to

$$\ln(M_S/M_Z)=2.34\pm1.00\pm2.50 . \quad (19)$$

Because of the extreme sensitivity to the experimental errors on the coupling-constants, effects which would otherwise be negligible, such as the second-order terms in Eqs. (1) and (3), become very important. Because of the observed linearity of the numerical solutions to the renormalization-group equations it is clear that the second-order terms are well approximated by their average contribution. Thus the second-order terms can be taken into account by replacing the $b_i$ and $b_i^s$ by $b_i+\delta b_i$ and $b_i^s+\delta b_i^s$, respectively, where the corrections are

$$\delta b_i=b_{ij}\bar{\alpha}_j/(4\pi)$$
$$=(0.0091,0.0123,-0.0172) , \quad (20)$$

$$\delta b_i^s=b_{ij}^s\bar{\alpha}_j/(4\pi)$$
$$=(0.0181,0.0314,0.0158) . \quad (21)$$

Here we have used, as average values,

$$\bar{\alpha}_i=(0.02,0.04,0.06) . \quad (22)$$

Except in the case of $b_2^s$, the second-order correction are only a few percent. Nevertheless they cause major corrections to $\ln(M_S/M_Z)$:

$$\delta\ln(M_S/M_Z)\sim\sum_{ijk}\epsilon_{jki}\delta b_k^s\alpha_j^{-1}(M_Z)/\det R\sim3.8 . \quad (23)$$

(The second-order corrections to $\det R$ are of order 1%.) Adding this to the first-order value of Eq. (17) or (19) one finds

$$\ln(M_S/M_Z)=3.6\pm3.1 \text{ (couplings from Ref. [3])} \quad (24)$$

or

$$\ln(M_X/M_Z)=6.1\pm2.7 \text{ (couplings from Ref. [5])} . \quad (25)$$

In each case the errors are dominated by the first-order

[1]The author is indebted to Pierre Ramond for a discussion on this point.

results although the major part of the central values come from the second-order terms. The importance of two-loop effects to the running of the coupling constant has been noted before [6]. The sensitivity of the scale $M_S$ to small effects has also been recently noted in other numerical studies [7]. The conclusion of the present analysis is that the current knowledge of the $Z$ scale couplings does not pin down the SUSY scale to within a factor of 10 nor does it necessarily predict the SUSY threshold to be within the Superconducting Super Collider (SSC) energy range. The full range of uncertainty in $M_S$ implied by Eqs. (24) and (25) is greater than three orders of magnitude. In addition the importance of the second-order corrections makes it clear that other small effects could contribute non-negligibly to $M_S/M_Z$. For instance, it is known [8] that there is a significant threshold correction to the SU(2) coupling at the scale of $2M_W$ which could lower the effective value of $\alpha_2^{-1}$ by 3.7% above this energy. If, as an approximation, one lowers $\alpha_2^{-1}(M_Z)$ by this amount in Eqs. (15) and (23), the effect is to raise $M_S/M_Z$ appreciably. Similarly for large top-quark mass the second-order effect due to Higgs-boson exchange [5] could also appreciably perturb the SUSY scale. This effect was not considered in Ref. [3] but is presumably part of the analysis of Ref. [4].

Finally, we turn to the second question in the introduction. If one requires a minimal supersymmetric unification with a SUSY threshold in the SSC range ($M_S < 10^4$ GeV), what are the constraints on $\alpha_3^{-1}(M_Z)$ and $\sin^2\theta_W$?

Including the average second-order corrections, Eq. (15) can be put into the form

$$\ln(M_S/M_Z) = 263 - 1290\sin^2\theta_W + 4.61\alpha_3^{-1}(M_Z) . \quad (26)$$

Requiring a SUSY threshold below 10 TeV implies

$$\alpha_3^{-1}(M_Z) < 280[\sin^2\theta_W(M_Z) - 0.200] . \quad (27)$$

Similarly Eq. (15) can be written

$$\ln(M_G/M_Z) = 5.09 + 171\sin^2\theta_W - 1.44\alpha_3^{-1}(M_Z) . \quad (28)$$

Requiring that $M_G > 10^{15}$ GeV for proton stability implies a further though somewhat weaker constraint between $\alpha_3$ and $\sin^2\theta_W$. For example, from Eq. (27), a precise value for $\sin^2\theta_W(M_Z)$ of 0.233 would require $\alpha_3(M_Z) > 0.108$. Although this is consistent with some QCD analyses, there is a large body of data [2] including heavy-quarkonium decays and jet mass measurements that suggest significantly smaller values. The data from $\Upsilon$ decay taken by itself is consistent with the LEP value of $\alpha_3^{(10)}$ although it might suggest a SUSY scale at the upper end of the region of interest. The lighter quarkonia favor even smaller values of $\alpha_3$. The situation bears watching closely for further clues to the high-energy content of the ultimate theory. On the other hand, if the high values of $\alpha_3$ are confirmed, the narrowness of the quarkonium states requires an explanation beyond asymptotic freedom.

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