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Two-photon beamstrahlung

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The production of two photons by electrons and positrons in very-high-energy linear colliders is considered. The phase of the amplitude is discussed in detail and the dominant radiation zone is determined. The arguments that the multiphoton radiation is an incoherent process are confirmed. Exact agreement with the Blankenbecler-Drell "incoherent" formula for fractional energy losses is shown. A simple semiclassical explanation of the results is proposed

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I. INTRODUCTION

During the last 3 years the electromagnetic radiation process called "beamstrahlung" has been extensively studied in the context of very-high-energy electron-positron linear colliders working in the TeV energy range.

In several papers general features of beamstrahlung have been discussed. A calculation procedure and approximations have been described in detail in [1–4]. It is based on the distorted-wave Born-approximation approach used together with conventional Feynman-diagram technics. It is worthwhile to note that the basic one-photon result for the energy lot by high-energy electrons has been derived many times before, but in different contexts [5]. An excellent review of the methods, results, and future prospects is presented in [6].

The energy losses by electrons and positrons in very-high-energy linear colliders appear to be unexpectedly high—about 17–20%. Moreover, the photon's spectrum is much harder than in the case of the usually understood bremsstrahlung process. These features allow us to consider beamstrahlung not only as a problem of energy losses, but also as a possible source of the very hard photons which can be used to study such processes as hard photon-photon collisions or pair production.

As results obtained in [1–4] were calculated in the approximation of single-photon emission.

Multiphoton processes were considered first by Blankenbecler and Drell in [7]. They argued that for future colliders in the TeV range, a pulse theoretically can be dropped into thin slices and that radiation from successive slices will be incoherent. Because each slice is sufficiently thin, the probability for radiating more than one photon per slice is negligible. This allows them to treat each slice of the pulse as differentially small and construct the rate equation for the probability of finding an electron and photon with energy fraction x and $1-x$, respectively, at fractional depth within the pulse [see Eqs. (3) and (4) in Ref. [7]]. After solving these equations it is possible to calculate not only δ_1, the fractional energy loss in the single-photon emission approximation, but also δ (total), the fractional energy loss including the effects of all photon emissions. Blankenbecler and Drell also noted that δ (total) is somewhat smaller than δ_1 for the same parameter values. It is understood if we consider the single-photon result not as an average energy loss from scattering events in which only one photon is actually emitted, but rather as an expected energy loss as an electron transverses a very small length dz multiplied by L_b/dz (L_b is a length of the bunch). Thus it is clearly an overestimate of the actual average energy loss, since it fails to take into account the fact that after one photon has been radiated the subsequent energy loss will be somewhat less. In the light of such an interpretation, the multiphoton radiation is not a correction to the one-photon radiation (δ_1) (because in δ_1 the emission of arbitrarily photons is allowed) and δ (total) is more realistic estimate for energy losses.

In the present paper we explicitly demonstrate by straightforward calculation, that multiphoton radiations are in fact incoherent. In other words, our calculations should be considered as a check of the Blankenbecler- and Drell less rigorous argument.

We consider the two-photon radiation amplitude for beamstrahlung in the TeV energy range. We calculate, in the high-energy approximation and ignoring the spin structure of the electron, the radiation rates for two-photon beamstrahlung in a cylindrical bunch with uniform density. The result for energy losses is

$$\delta_2=4.2\left[\frac{\alpha}{\pi}\frac{L_b}{l_c}\right]^2, \tag{1.1}$$

where L_b is the bunch length and l_c the coherent radiation length for beamstrahlung [1], in fair agreement with the Blankenbecler-Drell incoherent probability interpretation.

In the considered approximation, interference is negligible and our calculation of two-photon energy losses can be generalized to the arbitrary number of photons.

The paper is organized as follows. In Sec. II we transform the phase of the two-photon amplitude to the form convenient for determining stationary points in space, calculated in Sec. III. In Sec. IV space integrations are performed. Radiation rates and corresponding energy losses are calculated in Sec. V. In Sec. VI we discuss the obtained results in the light of the incoherent probability interpretation.

II. PHASE

The Feynman diagrams for the two-photon beamstrahlung are presented in Fig. 1. As in [1–3], we have simplified the problem, considering merely the radiation of the highly energetic particle in the external field. It is justified by the small value of the disruption parameter in the considered regime (more detailed discussion is presented in [6]). The corresponding radiation amplitude is

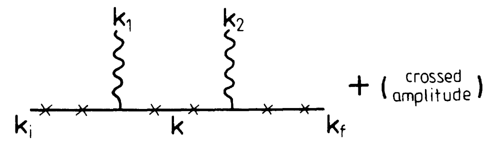

FIG. 1. Feynman graph for the emission of two photons by an electron in the external field.

$$\mathcal{M}=\mathcal{M}_{\mu\nu}\mathcal{E}_1^{\mu}\mathcal{E}_2^{\nu}\ , \tag{2.1}$$

with

$$\mathcal{M}_{\mu\nu}=e^2\int \Psi_i^{\dagger}(\mathbf{x})\overleftrightarrow{\partial}_{\mu\mathbf{x}}G(\mathbf{x},\mathbf{y})\overleftrightarrow{\partial}_{\nu\mathbf{y}}\Psi_f(\mathbf{y})\exp(-i\mathbf{k}_1\cdot\mathbf{x}-i\mathbf{k}_2\cdot\mathbf{y})d^3x\,d^3y+(\text{crossed amplitude , i.e., with } \mathbf{k}_1\leftrightarrow\mathbf{k}_2)\ ,$$

where $\mathbf{k}_1$ ($\mathbf{k}_2$) denotes the emitted photon momentum and $\mathcal{E}_1^{\mu}$ ($\mathcal{E}_2^{\nu}$) is the photon polarization. The wave functions Ψ_i and Ψ_f describe the electron in the presence of the bunch in the initial (i) and final (f) states and $G(\mathbf{x},\mathbf{y})$ is the electron propagator inside the bunch. The wave functions and propagator can be obtained by solving the Dirac equations in the external field.

In this paper we limit our considerations to the spinless case, taking the electron as a scalar particle, which means that we replace the Dirac equation with the Klein-Gordon equation. This gives the dominant part of the process cross section because the spin-flip probability of a high-energy electron is small. We employ the high-energy approximation, i.e., keep only constant and $1/k$ terms, where k is a typical electron energy. All terms proportional to higher powers of $1/k$ are neglected. Under such an approximation it is possible to solve the Klein-Gordon equation for Ψ_i,Ψ_f as well as the Green's function G.

The results for wave functions in external field $U(\mathbf{x})$ obtained in [1–3] are

$$\Psi_i(\mathbf{x})=A_i(\mathbf{x})\exp(i\Phi_i(\mathbf{x}))$$

and the same for $i\to f$, where the phases are given by

$$\Phi_i(\mathbf{x})=k_ix_3-S_i(\mathbf{x})-\frac{1}{2k_i}\int_{-\infty}^{x_3}(\nabla_TS_i)^2dx_3'\ , \tag{2.2a}$$

$$\Phi_f(\mathbf{y})=\mathbf{k}_f\mathbf{y}+S_f(\mathbf{y})+\frac{1}{2k_f}\int_{y_3}^{\infty}[(\nabla_TS_f)^2-2k_{f1}(y_3-y_3')\partial_{y_1}U(\mathbf{y}_T,y_3')]dy_3'\ , \tag{2.2b}$$

and

$$S_i(\mathbf{x})=\int_{-\infty}^{x_3}U(\mathbf{x}_T,x_3')dx_3'\ ,\quad S_f(\mathbf{y})=\int_{y_3}^{\infty}U(\mathbf{y}_T,y_3')dy_3'\ ,$$

with the coordinates $\mathbf{x}_T$ ($\mathbf{y}_T$) transverse and x_3 (y_3) parallel to the initial electron momentum. By $\nabla_T=[\partial_1,\partial_2]$ we denote the gradient in the transverse direction.

The amplitude for initial and final states have the form

$$A_i(\mathbf{x})=1+\frac{1}{2k_i}\left[U(\mathbf{x})+\int_{-\infty}^{x_3}(\nabla_T^2S_i)dx_3'\right]\ , \tag{2.2c}$$

$$A_f(\mathbf{y})=1+\frac{1}{2k_f}\left[U(\mathbf{y})+\int_{y_3}^{\infty}(\nabla_T^2S_f)dy_3'\right]\ . \tag{2.2d}$$

In the following we neglect electron mass, and so

$$E_i\cong k_i=|\mathbf{k}_i|\ ,\quad E_f\cong k_f=|\mathbf{k}_f|\ ,$$

$$k_1=|\mathbf{k}_1|\ ,\quad k_2=|\mathbf{k}_2|\ ,$$

and from energy conservation we have

$$k=k_f+k_2=k_i-k_1\ .$$

The notation and conventions have been adopted from [1,2]. Similarly, we work in the rest system of the bunch and define the XZ plane by the initial and final electron momenta $\mathbf{k}_i$ and $\mathbf{k}_f$ (Fig. 2). The bunch is assumed to be cylindrical with uniform charge density. The quickly varying part of the potential from such a bunch has the form

$$U(\mathbf{x})=U(\mathbf{x}_T,x_3)=\lambda \mathbf{x}_T^2 \quad \text{for } |x_3|<\tfrac{1}{2}L_b \;, \tag{2.3}$$

where $\lambda=n/L_bR_b^2$, $n=Ne^2$ (N is the number of the particles in the bunch); L_b and R_b are the bunch length and radius, respectively.

A detailed calculation of the propagator $G(\mathbf{x},\mathbf{y})$ in the $1/k$ approximation is moved to Appendix A (see Ref. [8]). For the potential (2.3) the general formula for the propagator [Eq. (A5)] gives

$$G(\mathbf{x}_T,\mathbf{y}_T,\bar{z})=G_0(R)\left[1+\frac{\lambda}{3k}R^2\right]\exp\left[-i\tfrac{1}{3}\lambda R(\mathbf{R}_T^2+3\mathbf{x}_T\cdot\mathbf{y}_T)-i\frac{1}{6k}\lambda^2R[(\mathbf{x}_T\times\mathbf{y}_T)^2+\bar{z}^2(\mathbf{x}_T\cdot\mathbf{y}_T+\tfrac{4}{15}\mathbf{R}_T^2)]\right] , \tag{2.4}$$

where

$$\mathbf{R}=(\mathbf{R}_T,\bar{z}) \;, \quad \mathbf{R}_T=\mathbf{y}_T-\mathbf{x}_T \;, \quad \bar{z}=y_3-x_3 \;, \quad R=|\mathbf{R}|$$

and

$$G_0(R)=\frac{1}{4\pi R}e^{ikR}$$

is a Green's function if the Klein-Gordon equation in a vacuum.

Combining (2.2) and (2.4) together with (2.1), we obtain the total phase of the amplitude $\mathcal{M}_{\mu\nu}$:

$$\Phi(\mathbf{x},\mathbf{y})=\Phi_i(\mathbf{x})-\Phi_f(\mathbf{y})-\mathbf{k}_1\cdot\mathbf{x}-\mathbf{k}_2\cdot\mathbf{y}+kR-\tfrac{1}{3}\lambda R\left[\mathbf{R}_T^2+3\mathbf{x}_T\cdot\mathbf{y}_T+\frac{1}{2}\frac{\lambda}{k}[(\mathbf{x}_T\times\mathbf{y}_T)^2+\bar{z}^2(\mathbf{x}_T\cdot\mathbf{y}_T+\tfrac{4}{15}R_T^2)]\right] , \tag{2.5}$$

where

$$\Phi_i(\mathbf{x})=k_ix_3-\lambda\mathbf{x}_T^2(x_3+\tfrac{1}{2}L_b)\left[1+\frac{2}{3k_i}\lambda(x_3+\tfrac{1}{2}L_b)^2\right] ,$$

$$\Phi_f(\mathbf{y})=\mathbf{k}_f\mathbf{y}+\lambda\mathbf{y}_T^2(\tfrac{1}{2}L_b-y_3)\left[1+\frac{2}{3k_f}\lambda(\tfrac{1}{2}L_b-y_3)^2\right]$$
$$+\frac{k_{f1}y_1}{k_f}\lambda(\tfrac{1}{2}L_b-y_3)^2 .$$

To make calculations more transparent, let us define dimensionless variables by rescaling all coordinates and momenta in the manner

$$\frac{2\mathbf{x}}{L_b}\to\mathbf{x} \;,$$

$$\frac{4\mathbf{k}_1}{\lambda L_b^2}\to\mathbf{k}_1 \;.$$

Let us also introduce the momentum

$$\mathbf{p}=\mathbf{k}_2+\mathbf{k}_f \;,$$

the small parameters

$$\varepsilon_i=\frac{1}{k} \;, \quad \varepsilon_f=\frac{1}{k_f} \;, \quad \varepsilon=\frac{1}{k} \;,$$

and auxiliary functions

$$\phi_i=2(1+x_3)[1+\tfrac{2}{3}\varepsilon_i(1+x_3)^2] \;, \tag{2.6a}$$

$$\phi_f=2(1-y_3)[1+\tfrac{2}{3}\varepsilon_f(1-y_3)^2] \;. \tag{2.6b}$$

The total phase (2.5) can now be expressed by

$$\Phi=-\tfrac{1}{2}(\phi_i+R)\mathbf{x}_T^2-\tfrac{1}{2}(\phi_f+R)\mathbf{y}_T^2+\tfrac{1}{6}R\,\mathbf{R}_T^2$$
$$-\varepsilon_fk_{f1}y_1(1-y_3)^2$$
$$+\tfrac{1}{6}\varepsilon R\,[(\mathbf{x}_T\times\mathbf{y}_T)^2+\bar{z}^2(\mathbf{x}_T\cdot\mathbf{y}_T+\tfrac{4}{15}R_T^2)]$$
$$+\frac{R}{\varepsilon_i}+\frac{x_3}{\varepsilon}-\mathbf{k}_1\cdot\mathbf{x}-\mathbf{p}\cdot\mathbf{y} \;, \tag{2.7}$$

where Φ is also rescaled according to

$$\frac{8}{\lambda L_b^3}\Phi\to\Phi \;.$$

The form of (2.7) is the most appropriate for our further discussion. It is worthwhile to note here that, in spite of neglecting the spin structure, the considered phase is the same as for the Dirac case. It is a consequence of the high-energy approximation. This is why the discussion and results presented below are valid for spin as well as the spinless case.

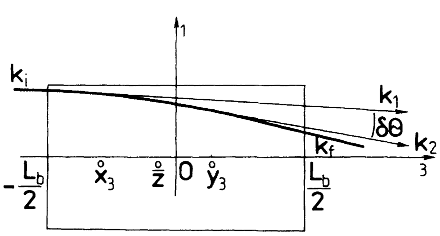

FIG. 2. Radiation of photons by an electron bending inside the bunch. The angle $\delta\vartheta$ between emitted photons is approximately proportional to the distance $\bar{z}$ between vertices. In practice $L_b \gg R_b$ and $R\simeq\bar{z}$.

III. COHERENCE CONDITIONS

A. Transverse directions

In order to obtain the dominant contribution to the radiation amplitude, we now have to find the region of $\mathbf{x}$ and $\mathbf{y}$ where the phase is stationary. First, we impose

$$\nabla_{\mathbf{x}_T}\Phi=0 \quad \text{and} \quad \nabla_{\mathbf{y}_T}\Phi=0 \; . \tag{3.1}$$

Before trying to solve these equations, let us take advantage of the fact, proved in Appendix B, that the stationary transverse separation of the vertices $\mathbf{R}_T$ is of the order of ε. Taking into account that

$$R \propto |\bar{z}| \;, \quad \mathbf{x}_T\times\mathbf{y}_T=\mathbf{x}_T\times\mathbf{R}_T\propto\varepsilon \;, \quad \text{etc.} \;,$$

and discarding terms $O(\varepsilon^2)$, we can simplify considerably the phase (2.7):

$$\Phi=-\tfrac{1}{2}\hat{\phi}_i\mathbf{x}_T^2-\tfrac{1}{2}\hat{\phi}_f\mathbf{y}_T^2+\frac{R_T^2}{2\varepsilon|\bar{z}|}+\frac{|z|}{\varepsilon}+\frac{x_3}{\varepsilon_i}$$
$$+\mathbf{k}_i\cdot\mathbf{x}-\hat{\mathbf{p}}\cdot\mathbf{y} \;, \tag{3.2}$$

where

$$\hat{\mathbf{p}}_T=\mathbf{p}_T+\varepsilon_f(1-y_3)^2\mathbf{k}_{fT}$$

and

$$\hat{\phi}_{i(f)}=\phi_{i(f)}+|\bar{z}|+\tfrac{1}{6}\varepsilon|\bar{z}|^3$$

are x_3 and y_3 dependent only.

Equations (3.1) can now be rewritten as

$$\hat{\phi}_i\mathbf{x}_T+\frac{1}{\varepsilon}\frac{\mathbf{R}_T}{|\bar{z}|}=-\mathbf{k}_{1T} \;,$$
$$\hat{\phi}_f\mathbf{y}_T-\frac{1}{\varepsilon}\frac{\mathbf{R}_T}{|\bar{z}|}=-\hat{\mathbf{p}}_T \;,$$

and solved, giving

$$\mathbf{R}_T\cong\varepsilon\frac{|\bar{z}|}{\hat{\phi}}\mathbf{F}+O(\varepsilon^2) \;,$$

where

$$\hat{\phi}=\hat{\phi}_i+\hat{\phi}_f \;,$$
$$\mathbf{F}=-\hat{\boldsymbol{\Delta}}_T\hat{\phi}_i-\mathbf{k}_{1T}\hat{\phi} \;,$$
$$\hat{\boldsymbol{\Delta}}_T=-(\mathbf{k}_{1T}+\hat{\mathbf{p}}_T) \; .$$

Using this notation, we can write the stationary points in transverse plain as

$$\mathring{\mathbf{x}}=\frac{1}{\hat{\phi}}\left[\hat{\boldsymbol{\Delta}}_T-\varepsilon\frac{\hat{\phi}_f}{\hat{\phi}}|\bar{z}|\mathbf{F}\right] \;,$$
$$\mathring{\mathbf{y}}_T=\frac{1}{\hat{\phi}}\left[\hat{\boldsymbol{\Delta}}_T+\varepsilon\frac{\hat{\phi}_i}{\hat{\phi}}|\bar{z}|\mathbf{F}\right] \; . \tag{3.3}$$

Substituting it into (3.2), we get the phase Φ as a function of x_3 and y_3 only. It has the particularly simple form

$$\Phi(x_3,y_3)=\frac{1}{2}\frac{\hat{\Delta}_T^2}{\hat{\phi}}-\frac{1}{2}\varepsilon|\bar{z}|\frac{\mathbf{F}^2}{\hat{\phi}^2}+\frac{|\bar{z}|}{\varepsilon}+\frac{x_3}{\epsilon_i}$$
$$-k_{13}x_3-p_3y_3 \; . \tag{3.4}$$

After neglecting all terms $O(\varepsilon^2)$, we obtain the final shape of Φ, which is a starting point for our further discussion:

$$\Phi(x_3,y_3)=\frac{1}{8}\frac{\Delta_T^2}{1+\mu}-\frac{1}{24}\frac{\Delta_T^2}{(1+\mu)^2}[\varepsilon_i(1+x_3)^3+\varepsilon_f(1-y_3)^3-\tfrac{1}{4}\varepsilon\bar{z}^3]$$
$$-\frac{1}{4}\frac{1}{1+\mu}\varepsilon_f\Delta_1k_{f1}(1-y_3)^2-\frac{1}{32}\frac{\varepsilon|\bar{z}|}{(1+\mu)^2}\{\boldsymbol{\Delta}_T[|\bar{z}|+2(1+x_3)]+4(1+\mu)\mathbf{k}_{1T}\}^2$$
$$+\frac{|\bar{z}|}{\varepsilon}+\frac{x_3}{\varepsilon_i}-k_{13}x_3-p_3y_3 \;, \tag{3.5}$$

where

$$\mu=\tfrac{1}{2}(|\bar{z}|-\bar{z})=\begin{cases}0 & \text{for } \bar{z}>0 \;,\\ -\bar{z} & \text{for } \bar{z}<0 \;,\end{cases}$$

and the transverse momentum transferred to the bunch is

$$\boldsymbol{\Delta}_T=-(\mathbf{k}_{fT}+\mathbf{k}_{1T}+\mathbf{k}_{2T}) \; .$$

B. Longitudinal direction

We now have to find the region of x_3 and y_3 where the phase changes most slowly. Let us define the energy fractions x_1 and x_2 carried by photons:

$$k_1=x_1k_i \;, \quad k_2=x_2k_i \; .$$

Then

$$k_f=(1-x_1-x_2)k_i=(1-x)k_i \;, \quad k=(1-x_1)k_i \;,$$

and, in the high-energy approximation,

$$k_{13}=x_1k_i-\frac{\mathbf{k}_{1T}^2}{2x_1k_i} \;, \quad k_{23}=x_2k_i-\frac{\mathbf{k}_{2T}^2}{2x_2k_i} \;,$$
$$k_{f3}=(1-x)k_i-\frac{\mathbf{k}_{fT}^2}{2(1-x)k_i} \;, \tag{3.6}$$

where $x=x_1+x_2$. The last four terms in (3.5) can now be rewritten as

$$\frac{|\bar{z}|}{\varepsilon}+\frac{x_3}{\varepsilon_i}-k_{13}x_3-p_3y_3 =2\mu(1-x_3)k_i+\frac{\mathbf{k}_{1T}^2}{2k_ix_1}x_3+\frac{1}{2k_i}\left[\frac{\mathbf{k}_{2T}^2}{x}+\frac{\mathbf{k}_{fT}^2}{1-x}\right]y_3 . \tag{3.7}$$

If we assume that $\mu\neq 0$, the conditions

$$\frac{\partial\Phi}{\partial x_3}=0 \quad\text{and}\quad \frac{\partial\Phi}{\partial y_3}=0 \tag{3.8}$$

cannot be satisfied in the $1/k$ approximation. The reason is that in this case an expression for $\partial\Phi/\partial x_3$ contains one term proportional to incident electron energy k_i, while remaining terms are of the order of $1/k_i$. The presence of such a term in the phase causes strong dumping of the amplitude as a result of quick oscillations. This is why we choose $\mu=0$ and, as a consequence, $\bar{z}>0$. It means that the dominant contribution to the radiation amplitude comes from the region where emission vertices are ordered.

It is worthwhile to note here that, if we would try to "simplify" the problem replacing the propagator G in the amplitude (2.1) by a "free" propagator G_0, the phase (3.5) would contain $-\frac{1}{2}\bar{z}$ factors in place of $1+\mu=1$. Thus such a simplification essentially complicates the problem.

Finally, we obtain

$$\Phi(x_3,y_3)=\tfrac{1}{8}\Delta_T^2+\tfrac{1}{2}\varepsilon_i\left[-\frac{1}{12}\Delta_T^2\left[(1+x_3)^3+\frac{(1-y_3)^3}{1-x}+\frac{1}{4}\frac{\bar{z}^3}{1-x_1}\right]-\frac{1}{2}\frac{(1-y_3)^2}{1-x}\Delta_1k_{f1}\right.$$
$$\left.-\frac{\bar{z}}{16(1-x_1)}[\mathbf{\Delta}_T(2+x_3+y_3)+4\mathbf{k}_{1T}]^2+\frac{\mathbf{k}_{1T}^2}{x_1}x_3+\left[\frac{\mathbf{k}_{2T}^2}{x_2}+\frac{\mathbf{k}_{fT}^2}{1-x}\right]y_3\right] . \tag{3.9}$$

After some algebra, (3.8) can be expressed in the form

$$\frac{\partial\Phi}{\partial x_3}=\frac{\varepsilon_i}{8x_1(1-x_1)}\{[x_1(1+x_3)\Delta_{T1}+2k_{11}]^2+[x_1(1+x_3)\Delta_{T2}+2k_{12}]^2\} , \tag{3.10a}$$

$$\frac{\partial\Phi}{\partial y_3}=\frac{\varepsilon}{8x'(1-x')}\{[x'(1+y_3)\Delta_{T1}+2k_1]^2+[x'(1+y_3)\Delta_{T2}+2k_2]^2\} , \tag{3.10b}$$

where

$$\mathbf{k}_T=x'\mathbf{k}_{1T}+\mathbf{k}_{2T} \quad\text{and}\quad x'=\frac{x_2}{1-x_1} .$$

These results are easy to interpret if we note that x' is the fraction of the intermediate electron energy carried by a second photon and $\mathbf{k}_T$ has the meaning of the transverse momentum of that photon with respect to intermediate electron momentum, so the vertices are completely separated, with the ε_i, x_1, and $\mathbf{k}_{1T}$ variables in (3.10a) replaced by ε, x', and $\mathbf{k}_T$ in (3.10b). Moreover, the structure of the stationary condition for each vertex is similar to that in the one-photon case considered in [1]. This is why the coherence length deduced from (3.10) is strictly the same and the corresponding discussion is also valid here.

If we would like to take into account quantum fluctuations off the bending plane, we should take into consideration the second derivatives

$$\frac{\partial^2\Phi}{\partial x_3^2}=\frac{\varepsilon_i}{4(1-x_1)}[x_1(1+x_3)\Delta_T^2+2\mathbf{k}_{1T}\cdot\mathbf{\Delta}_T] ,$$

$$\frac{\partial^2\Phi}{\partial y_3^2}=\frac{\varepsilon}{4(1-x')}[x'(1+y_3)\Delta_T^2+2\mathbf{k}_T\cdot\mathbf{\Delta}_T] .$$

The points where these derivatives vanish read

$$\mathring{x}_3=-1-\frac{2}{x_1}\frac{\mathbf{\Delta}_T\cdot\mathbf{k}_{1T}}{\Delta_T^2} ,$$
$$\mathring{y}_3=-1-\frac{2}{x'}\frac{\mathbf{\Delta}_T\cdot\mathbf{k}_T}{\Delta_T^2} . \tag{3.11}$$

Now let us determine the stationary distance between the points of emission:

$$\mathring{\bar{z}}=2(1-x_1)\frac{\mathbf{\Delta}_T\cdot\boldsymbol{\rho}}{\Delta_T^2} , \tag{3.12}$$

where

$$\boldsymbol{\rho}=\frac{\mathbf{k}_{1T}}{x_1}-\frac{\mathbf{k}_{2T}}{x_2} .$$

If we neglect momenta off the bending plane, ρ_1 and, hence, $\mathring{\bar{z}}$ are proportional to the angle $\delta\vartheta$ between emitted photons:

$$\mathring{\bar{z}}\propto\frac{2k_i}{\Delta_{T1}}\delta\vartheta . \tag{3.13}$$

This value can be estimated in a quite different manner. Let us denote electron momenta at the stationary points by $\mathring{\mathbf{k}}_i$ and $\mathring{\mathbf{k}}_f$. Then, from (2.4), it follows that

$$\nabla_T\Phi=\mathring{\mathbf{k}}_{iT}-\mathring{\mathbf{k}}_{fT}-\mathbf{k}_{1T}-\mathbf{k}_{2T}-\mathbf{s}=0 ,$$

where

$$\mathbf{s}=\lambda\mathring{\bar{z}}(\mathring{\mathbf{x}}_T+\mathring{\mathbf{y}}_T)$$

is the momentum transferred to the field from the electron in its classical motion between points of emission. In the first approximation the longitudinal component of this momentum reads

$$s_1=\tfrac{1}{2}\mathring{\bar{z}}\Delta_{T1} .$$

If we now take into account that the angle between directions tangential to the classical electron trajectory is

$$\delta\vartheta\cong\frac{s_1}{k_i} ,$$

we again get (3.13).

C. Stationary phase

In the vicinity of the points (3.11), the total phase can be written in the following form convenient for further integrations:

$$\Phi(t,t')=\omega(a_x t^3+b_x t)+\omega'(a_y t'^3+b_y t') , \tag{3.14}$$

where

$$\omega=\frac{L_b\Delta_T^2}{48k_i} , \quad \omega'=\frac{\omega}{1-x_1} ,$$

$$a_x=\frac{x_1}{1-x_1} , \quad a_y=\frac{x'}{1-x'} ,$$

$$b_x=\frac{12\kappa_1^2}{x_1(1-x_1)} , \quad b_y=\frac{12\kappa^2}{x'(1-x')} ,$$

$$\kappa_i=\frac{(\mathbf{k}_{iT}\times\mathbf{\Delta}_T)_3}{\Delta_T^2} , \quad i=1,2 , \quad \kappa=x'\kappa_1+\kappa_2 ,$$

and we have introduced new longitudinal variables

$$t=x_3-\mathring{x}_3 \quad \text{and} \quad t'=y_3-\mathring{y}_3 .$$

The shape of (3.11) as well as (3.14) suggests a particularly simple generalization of above results to the n-photon emission process.

IV. RADIATION AMPLITUDES

Before we start to calculate space integrals in (2.1), we should extract the leading contribution to the integrand. Let us rewrite (2.1) as

$$\mathcal{M}_{ij}=e^2\int e^{i\Phi}M_{ij}d^3x\, d^3y , \quad i,j=1,2,3 , \tag{4.1}$$

where

$$M_{ij}=\Psi_i^\dagger\partial_{x_i}G\partial_{y_j}\Psi_f-\Psi_i^\dagger\partial_{x_i}\partial_{y_j}G\Psi_f +\partial_{x_i}\Psi_i^\dagger\partial_{y_j}G\Psi_f-\partial_{x_i}\Psi_i^\dagger G\partial_{y_j}\Psi_f . \tag{4.2}$$

Integrating by parts in (4.1) and discarding surface terms, we can replace (4.2) with

$$M_{ij}=G(-4\partial_{x_i}\Psi_i^\dagger\partial_{y_j}\Psi_f+2ik_{1i}\Psi_i^\dagger\partial_{y_j}\Psi_f +2ik_{2j}\partial_{x_i}\Psi_i^\dagger\Psi_f+k_{1i}k_{2j}\Psi_i^\dagger\Psi_f) .$$

After substituting (2.5) and discarding $O(\varepsilon)$ terms, we get

$$M_{ij}=\frac{1}{4\pi\bar{z}}[4\lambda x_i(x_3+\tfrac{1}{2}L_b)+k_{1i}] \times[4\lambda y_j(\tfrac{1}{2}L_b-y_3)+k_{2j}+2k_{fj}] .$$

Let us perform $\mathbf{x}_T$ and $\mathbf{y}_T$ integrations first. The phase Φ can be approximated by

$$\Phi(\mathbf{x},\mathbf{y})\simeq\Phi(x_3,y_3)+\Phi_T(x_T,y_T,x_3,y_3) ,$$

with

$$\Phi_T(x_T,y_T,x_3,y_3)\simeq\tfrac{1}{2}kA(\mathbf{R}_T-\mathring{\mathbf{R}}_T)^2 -\tfrac{1}{2}B_x(\mathbf{x}_T-\mathring{\mathbf{x}}_T)^2-\tfrac{1}{2}B_y(\mathbf{y}_T-\mathring{\mathbf{y}}_T)^2 ,$$

where

$$A=\frac{1}{\bar{z}}+\frac{\lambda}{k}\left[\frac{\bar{z}}{3}-\frac{\mathring{\mathbf{x}}_T^2+\mathring{\mathbf{y}}_T^2}{2\bar{z}}\right] ,$$

$$B_x=\lambda(L_b+\bar{z}+2x_3) ,$$

$$B_y=\lambda(L_b+\bar{z}-2y_3) .$$

All integrals in the transverse plane are of the Gaussian type picked at the $\mathring{\mathbf{x}}_T,\mathring{\mathbf{y}}_T$ points. Calculating them, we get

$$\mathcal{M}_{ij}=e^2\int e^{i\Phi(t,t')}m_{ij}dt\, dt' , \tag{4.3}$$

with

$$m_{ij}=m_0\{[(1+x_3)\Delta_{Ti}+k_{1i}] \times[(1-y_3)\Delta_{Tj}+k_{2j}+2k_{fj}] -2i\delta_{ij}\lambda L_b(x_3+1)(1-y_3)\} ,$$

where

$$m_0=\frac{\pi}{2k\lambda L_b} .$$

If we define for each photon two polarization components in the bending plane and perpendicular to it,

$$\mathcal{E}_{1\parallel}=(1,0,-k_{11}/x_1k_i) , \quad \mathcal{E}_{1\perp}=(0,1,-k_{12}/x_1k_i) ,$$

$$\mathcal{E}_{2\parallel}=(1,0,-k_{21}/x_2k_i) , \quad \mathcal{E}_{2\perp}=(0,1,-k_{22}/x_2k_i) ,$$

we obtain from (4.3) matrix elements for different combinations of polarizations:

$$\begin{aligned} m_{\parallel\parallel}&=-m_0tt' ,\\ m_{\perp\perp}&=-m_0\frac{2\kappa_1}{x_1}\frac{2\kappa}{x'} ,\\ m_{\perp\parallel}&=m_0\frac{2\kappa_1}{x_1}t' ,\\ m_{\parallel\perp}&=m_0t\frac{2\kappa}{x'} . \end{aligned} \tag{4.4}$$

In the above expressions for $m_{\parallel\parallel}$ and $m_{\perp\perp}$ the imaginary part of the amplitude, present in (4.3), has been omitted. It is justified, because the ratio of the real to imaginary parts of $m_{\parallel\parallel}$ is of the order of

$$\frac{\lambda L_b}{\Delta_{T1}} \simeq \frac{1}{n} \simeq 10^{-6} ,$$

and a similar ratio for $m_{\perp\perp}$ can be approximated by

$$\frac{\lambda L_b}{\Delta_{T2}} \simeq \frac{1}{n} \left[\frac{L_b}{l_c} \right]^2 \simeq 10^{-2} .$$

So the integrated amplitude still has a factorized form, which considerably simplifies further discussion.

We now proceed with integrals (4.3). Let us consider in detail the matrix element

$$\mathcal{M}_{\parallel\parallel} = -2m_0 \int \int_{-\infty}^{+\infty} tt' e^{i\Phi(t,t')} \times \vartheta(t'-t-\mathring{y}_3+\mathring{x}_3)dt\,dt' , \qquad (4.5)$$

where the ϑ function takes into account that we limit the space region to $\bar{z}>0$. With the use of the integral representation of ϑ [Eq. (4.5)] can be converted to

$$\mathcal{M}_{\parallel\parallel} = -m_0 \frac{1}{\pi i} \int_{-\infty}^{+\infty} d\tau \frac{e^{i\mathring{z}\tau}}{\tau - i\varepsilon} \mathrm{Ai}'(u^2-\alpha_x\tau)\mathrm{Ai}'(v^2-\alpha_y\tau) , \qquad (4.6)$$

where

$$\alpha_x = \tfrac{1}{3}(\omega a_x)^{-1/3} , \quad \alpha_y = \tfrac{1}{3}(\omega' a_y)^{-1/3} ,$$

$$u = 2\kappa_1 \left[\frac{\omega}{(1-x_1)x_1^2} \right]^{1/3} ,$$

$$v = 2\kappa \left[\frac{\omega}{(1-x')x'^2} \right]^{1/3} ,$$

and Ai' are the derivative of Airy functions connected to MacDonald functions by the relation

$$\mathrm{Ai}'(u^2) = \sqrt{3}u^2 K_{2/3}(2u^3) .$$

The integral (4.6) is different from zero for $\mathring{z}>0$ only. In light of the above discussion of (3.12), this means that combinations of $\mathbf{k}_{1T}$, $\mathbf{k}_{2T}$, x_1, and x_2 values, which correspond to $\mathring{z}<0$, give a negligible contribution to the matrix element. This fact is of a crucial importance—it causes a strong reduction of the phase space for photons.

From (4.6) we obtain

$$\mathcal{M}_{\parallel\parallel} = \tfrac{1}{3}\mathcal{I}(x_1,x_2)\mathrm{Ai}'(u^2)\mathrm{Ai}'(v^2) ,$$

where

$$\mathcal{I}(x_1,x_2) = \frac{32\pi^2\alpha}{\lambda}\omega^{-1/3} \left[\frac{1-x}{x_1x_2} \right]^{2/3} (1-x_1)^{1/3} .$$

The remaining elements have the form

$$\mathcal{M}_{\perp\perp} = -3\mathcal{I}(x_1,x_2)u\,\mathrm{Ai}(u^2)v\,\mathrm{Ai}(v^2) ,$$
$$\mathcal{M}_{\perp\parallel} = -i\mathcal{I}(x_1,x_2)u\,\mathrm{Ai}(u^2)\mathrm{Ai}'(v^2) , \qquad (4.7)$$
$$\mathcal{M}_{\parallel\perp} = -i\mathcal{I}(x_1,x_2)\mathrm{Ai}'(u^2)v\,\mathrm{Ai}(v^2) ,$$

where Ai are the Airy functions defined through

$$\mathrm{Ai}(u^2) = \frac{u^2}{\sqrt{3}} K_{1/3}(2u^2) .$$

V. RADIATION RATES

Let us concentrate first on the radiation rate for two photons with polarization in the bending plane. It can be written as

$$I_{\parallel\parallel} = \frac{1}{32k_i^2(2\pi)^7} \int |\mathcal{M}_{\parallel\parallel}+\mathcal{M}^c_{\parallel\parallel}|^2 d^2k_{1T}\,d^2k_{2T}\,k_{f1}\,dk_{f1} \times \frac{dx_1\,dx_2}{x_1x_2(1-x_1-x_2)} , \qquad (5.1)$$

where $\mathcal{M}^c_{\parallel\parallel}$ denotes the crossed amplitude, i.e., an amplitude with four-momenta and polarizations of final photons interchanged. In (5.1) we have introduced an additional $\frac{1}{2}$ factor due to the identity of produced particles. Both integrations over k_{12} and k_{22} can be performed after changing variables to u and v. The Airy functions have a strongly limited range, and therefore expansion of the integration limits to the whole plane is justified. Using the relation

$$\int_0^{\infty} du\,\mathrm{Ai}'(u^2) = \frac{3\pi^2}{16} \frac{2^{1/3}}{\Gamma(\frac{1}{3})} ,$$

and normalizing to the bunch cross section πR_b^2, we obtain, for the $|\mathcal{M}_{\parallel\parallel}|^2$ part of (5.1),

$$I^0_{\parallel\parallel} = \frac{\pi}{16^2} \frac{2^{2/3}}{\Gamma^2(\frac{1}{3})} \frac{\alpha^2}{\lambda^2 k_i^2} \times \int \frac{\Delta_{T1}^2}{\omega^{4/3}} dk_{11}\,dk_{21}\,dk_{f1} \times \frac{(1-x)^{2/3}(1-x_1)^{1/3}}{(x_1x_2)^{5/3}} dx_1\,dx_2 . \qquad (5.2)$$

Let us now consider an integral over k_{11},k_{21}. The phase space is defined by the relations

$$\mathring{x}_3 \in (-1,1) , \quad \mathring{y}_3 \in (-1,1) , \quad \mathring{z} = \mathring{y}_3 - \mathring{x}_3 > 0 .$$

The corresponding region in the k_{11},k_{21} plane is shown in Fig. 3, and its area is equal to

$$\frac{1}{2}x_1x' = \frac{1}{2} \frac{x_1x_2}{1-x_1} . \qquad (5.3)$$

Changing the variable k_{f1} into Δ_{T1} and integrating from 0 to $2n/R_b$, we obtain

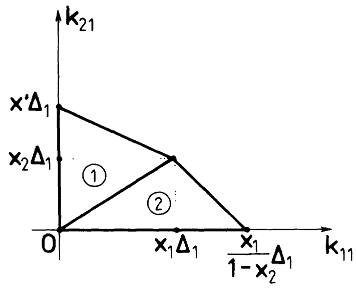

FIG. 3. Phase space for photon transverse momenta in the bending plane: triangle 1, for the graph from Fig. 1; triangle 2, for the crossed graph.

$$\int \frac{\Delta_{T1}^2}{\omega^{4/3}} dk_{11}\, dk_{21}\, dk_{f1} = \frac{3}{20}\left[\frac{48k_i}{L_b}\right]^{4/3}\left[\frac{2n}{L_b}\right]^{10/3}\frac{x_1x_2}{1-x_1} ,$$

and finally

$$I_{\parallel\parallel}^0 = \frac{\zeta\tilde{\alpha}}{(x_1x_2)^{2/3}}\left[\frac{1-x}{1-x_1}\right]^{2/3} ,$$

where the numerical factor is

$$\zeta = \frac{9\pi^2}{40}\frac{6^{1/3}}{\Gamma^2(\frac{1}{3})}$$

and effective coupling constant

$$\tilde{\alpha} = \frac{\alpha}{\pi}\frac{L_b}{l_c} .$$

The coherent radiation length l_c is the same here as in the one-photon radiation case, i.e.,

$$l_c = \left[\frac{R_b^2 L_b^2 k_i}{n^2}\right]^{1/3} .$$

Similarly, for the crossed term $|\mathcal{M}_{\parallel\parallel}^c|^2$, we get

$$I_{\parallel\parallel}^c = \frac{\zeta\tilde{\alpha}^2}{(x_1x_2)^{2/3}}\left[\frac{1-x}{1-x_2}\right]^{2/3} .$$

The remaining interference term $\mathcal{M}_{\parallel\parallel}\mathcal{M}_{\parallel\parallel}^c$ has a structure similar to (5.2), but phase space vanishes in this case. It is because $\hat{z}$ changes its sign when the photon momenta are interchanged. The lack of interference is intuitively obvious. The dominant part of radiation goes into angles much smaller than the angle of classical bending of an electron between emission vertices. Therefore, interchanging the photon momenta leads to the process with a vanishing amplitude.

Similar calculations for the remaining radiation rates give

$$I_{\perp\perp} = \tfrac{1}{9}I_{\parallel\parallel} , \quad I_{\parallel\perp} = I_{\perp\parallel} = \tfrac{1}{3}I_{\parallel\parallel} ,$$

and the total radiation rate can be written as

$$I(x_1,x_2) = \frac{1.0\tilde{\alpha}^2}{(x_1x_2)^{2/3}}\left[\left[\frac{1-x}{1-x_1}\right]^{2/3} + \left[\frac{1-x}{1-x_2}\right]^{2/3}\right] . \tag{5.4}$$

The last step in the calculation of energy losses is the numerical integration over x_1 and x_2:

$$\delta_2 = \int_0^1 dx_1 \int_0^{1-x_1} dx_2\, I(x_1,x_2)(x_1+x_2) .$$

Finally, we arrive at a particularly simple expression

$$\delta_2 = 4.2\tilde{\alpha}^2 . \tag{5.5}$$

For the "super" accelerators considered in [1–3], $L_b/l_c \simeq 60$, the total energy loss is of the order of 8%.

VI. CONCLUSIONS

In this section we consider the two-photon beamstrahlung once again using the probability approach considered in [7] and also presented in [6].

For a single-photon emission approximation, the fractional energy loss δ_1 can be written as

$$\delta_1 = \int_{-L_b/2}^{L_b/2} dz \int_0^1 dx\, xP(x,z) , \tag{6.1}$$

where $P(x,z)$ is the differential probability of emitting a photon which carries fractional energy x. For a uniform cylindrical bunch, $P(x,z)$ is independent of z.

If multiple-photon radiation is incoherent as claimed by Blankenbecler and Drell, then the two-photon energy loss should be

$$\delta_2 = \int_{-L_b/2}^{L_b/2} dz_1 \int_{z_1}^{L_b/2} dz_2 \int_0^1 dx_1 \int_0^1 dx' [x_1 + (1-x_1)x'] P(x_1,z_1) P'(x_1,x',z_2) . \tag{6.2}$$

Here $P'(x_1,x',z)$ denotes the probability of emitting the second photon, given that the electron has already lost a fractional energy x_1 to the first photon. x' is a fraction of the electron's intermediate energy that the second photon carries off, and so the quantity in square brackets is the total fractional energy carried off by both photons [$x' = x_2/(1-x_1)$; see Eq. (3.10)]. Using the Blankenbecler-Drell results for P and P' [formulas (1)–(3) in Ref. [7] or (5.1) and (5.15) in Ref. [6]], it is easy to have

$$\delta_2 = 4.27\tilde{\alpha}^2 , \tag{6.3}$$

which is exactly our result (5.5).

In light of this agreement, we conclude that the claim of Blankenbecler and Drell is confirmed.

ACKNOWLEDGMENTS

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APPENDIX A

A detailed calculation of the propagator $G(\mathbf{x},\mathbf{y})$ inside the bunch in the $1/k$ approximation is presented in [8]. Here we quote the simplest way to obtain formula (2.4).

Let us start from the standard Klein-Gordon equation for the Green's function:

$$[k^2+\nabla^2-2kU(\mathbf{x})+U^2(\mathbf{x})]G(\mathbf{x},\mathbf{y})=-\delta(\mathbf{x}-\mathbf{y}) \ , \qquad \text{(A1)}$$

where $k=|\mathbf{k}|\simeq E$ is the energy of the electron and $U(\mathbf{x})$ is a potential generated by the bunch. The following representation of the propagator can be assumed:

$$G(\mathbf{x},\mathbf{y})=G_0(R)S_0(\mathbf{x},\mathbf{y})\left[1+\frac{1}{k}\Psi(\mathbf{x},\mathbf{y})\right] , \qquad \text{(A2)}$$

where

$$G_0(R)=\frac{1}{4\pi R}e^{ikR}$$

is a propagator for the "free" Klein-Gordon equation

$$(k^2+\nabla^2)G_0(R)=-\delta(\mathbf{R}) \ . \qquad \text{(A3)}$$

In the following we will look for two auxiliary functions S_0 and Ψ instead of $G(\mathbf{x},\mathbf{y})$. These functions should obey the conditions

$$\begin{aligned} &S_0(\mathbf{x},\mathbf{y})=S_0(\mathbf{y},\mathbf{x}) \ , \\ &\Psi(\mathbf{x},\mathbf{y})=\Psi(\mathbf{y},\mathbf{x}) \ , \end{aligned} \qquad \text{(A4)}$$

and we additionally assume

$$\Psi(\mathbf{x},\mathbf{x})=0 \ .$$

Now, when we put the assumed form of $G(\mathbf{x},\mathbf{y})$ into (A1) and use (A3), we obtain the equation

$$\begin{aligned} &\nabla^2S_0+\frac{1}{k}\Psi\nabla^2S_0+\frac{2}{k}\nabla S_0\cdot\nabla\Psi+\frac{1}{k}S_0\nabla^2\Psi \\ &\quad+2ik\hat{\mathbf{n}}\cdot\nabla S_{+2i}\Psi\hat{n}\cdot\nabla S_0+2iS_0\hat{\mathbf{n}}\cdot\nabla\Psi+U^2S_0 \\ &\quad+\frac{1}{k}U^2\Psi S_0-2U\Psi S_0-2kUS_0-\frac{2}{R}\hat{\mathbf{n}}\cdot\nabla S_0\left[1+\frac{1}{k}\Psi\right] \\ &\quad-\frac{2}{Rk}S_0\hat{\mathbf{n}}\cdot\nabla\Psi=0 \ , \end{aligned}$$

where

$$\hat{\mathbf{n}}=\frac{\mathbf{R}}{R} \quad \text{and} \quad \hat{\mathbf{n}}\cdot\nabla=\frac{\partial}{\partial R} \ .$$

Neglecting in this equation terms proportional to $1/k$ and $1/k^2$, we find

$$\begin{aligned} &\nabla^2S_0+2k(i\hat{\mathbf{n}}\cdot\nabla-U)S_0+U^2S_0+2iS_0\hat{\mathbf{n}}\cdot\nabla\Psi \\ &\quad-2US_0\Psi+2\left[i\Psi-\frac{1}{R}\right]\hat{\mathbf{n}}\cdot\nabla S_0=0 \ . \end{aligned}$$

It can be rewritten as a set of two equations:

$$\begin{aligned} &i\hat{\mathbf{n}}\cdot\nabla S_0=US_0 \ , \\ &\nabla^2S_0+U^2S_0-\frac{2}{R}\hat{\mathbf{n}}\cdot\nabla S_0+2iS_0\hat{\mathbf{n}}\cdot\nabla\Psi=0 \ . \end{aligned}$$

The solution of these equations obeying (A4) can be rewritten as

$$\begin{aligned} &S_0(\mathbf{x},\mathbf{y})=\exp\left[-i\int_0^R U(\hat{\mathbf{n}}s+\mathbf{x})ds\right] , \\ &\Psi(\mathbf{x},\mathbf{y})=\tfrac{1}{2}[U(\mathbf{y})-U(\mathbf{x})]+\frac{i}{2}\int_0^R S_0^{-1}\nabla_t^2S_0(\hat{\mathbf{n}}s+\mathbf{x})ds \ , \end{aligned} \qquad \text{(A5)}$$

where

$$\nabla^2=(\hat{\mathbf{n}}\cdot\overline{\nabla})^2+\frac{2}{R}\hat{\mathbf{n}}\cdot\overline{\nabla}+\nabla_t^2 \ .$$

For the potential $U(\mathbf{x})=\lambda\mathbf{x}_T^2$, this solution gives the desired formula for the electron propagator inside the positron bunch.

APPENDIX B

Let us consider the following set of equations defining the transverse location of the radiation [we use the same notation as in Sec. V and discard terms of $O(1/k)$]:

$$\begin{aligned} &2(1+x_3)\mathbf{x}_T=-\tau\hat{\mathbf{n}}_T-R\mathbf{y}_T-\mathbf{k}_{1T} \ , \\ &2(1-y_3)\mathbf{y}_T=\tau\hat{\mathbf{n}}_T-R\mathbf{x}_T-\mathbf{p}_T \ , \end{aligned} \qquad \text{(B1)}$$

where

$$\tau=\frac{1}{\varepsilon}-\frac{1}{2}(\mathbf{x}_T^2+\mathbf{y}_T^2+R^2+\tfrac{1}{3}z^2) \ , \quad \hat{\mathbf{n}}_T=\frac{\mathbf{R}_T}{R} \ .$$

Adding up these two equations, we get

$$\not{f}_i\mathbf{x}_T+\not{f}_f\mathbf{y}_T=\nabla_T \ .$$

Hence $\mathbf{x}_T$ and $\mathbf{y}_T$ can be expressed in the form

$$\mathbf{y}_T=\frac{\not{f}_i\mathbf{R}_T+\nabla_T}{\not{f}} \ , \quad \mathbf{x}_T=\frac{-\not{f}_f\mathbf{R}_T+\nabla_T}{\not{f}} \ ,$$

where $\not{f}=\not{f}_i+\not{f}_f$.

Substituting this into (B1) and introducing a new vector $\mathcal{J}$ defined by

$$\not{f}=-\Delta_T\not{f}_i-\mathbf{k}_{1T}\not{f} \ ,$$

we obtain an equation

$$\mathbf{R}_T[(\tau+R^2)\not{f}-R\not{f}_i\not{f}_f]=R\mathcal{J} \ . \qquad \text{(B2)}$$

Let us now consider the asymptotically large k. The left side of (B2) has an asymptotic form $\mathbf{R}_T\cdot(1/\varepsilon)$, while the right-hand side tends to be constant. The separation of the vertices cannot grow up like $1/\varepsilon$ because we assume that both vertices are inside the bunch and, hence, $R\leq 2$. Moreover, for increasing R, the amplitude will be damped out by the factor $1/R$ from the propagator. Therefore, we obtain

$$\mathbf{R}_T=\mathbf{v}\varepsilon+O(\varepsilon^2) \ ,$$

where $\mathbf{v}$ is constant.

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