Reduction of the instantaneous Bethe-Salpeter equation for q-$\bar{q}$ bound states

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We present a straightforward method to reduce the instantaneous Bethe-Salpeter equation to a set of coupled equations for radial wave functions. In the case of positronium, in particular, the results obtained by Cung *et al.* follow immediately; our procedure is, however, much more general and is intended to be applied to mesonic bound states where competing models need to be tested. We also briefly comment on the numerical solution of the ensuing equations.

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I. INTRODUCTION

The Bethe-Salpeter (BS) equation is the correct tool to deal with bound-state problems in quantum field theory [1]. It has been applied with considerable success to QED, where it has allowed an actual computation of the spectrum of positronium [2]. Its use in QCD should, in principle, also do so for the mesonic bound states. However our present knowledge of the interaction kernel between quarks is still very fragmentary. In particular there is now mounting evidence that the long-standing proposal for a scalar confining potential has to be abandoned [3,4]. It would therefore be very useful if we had a simple method allowing us to test alternatives. This paper deals with that question although with an important restriction: the interaction is supposed to be instantaneous in the rest frame of the bound state.

In Sec. II, we introduce the Salpeter equation and derive an expression for the mass of bound states (2.17), which will be at the center of all the subsequent developments. In Sec. III, we discuss several properties of the solutions of the Salpeter equation and present a variational principle for the mass of bound states. The latter is then used to carry out the reduction of the Salpeter equation in Sec. IV and to obtain equations involving only the radial wave functions in Sec. V. Section VI deals with several applications of this formalism and also includes a comment on the numerical solution of the ensuing equations. We finally conclude in Sec. VII. Appendixes A, B, C, and D contain some definitions as well as the formulas necessary to carry out the above-mentioned computations.

II. THE SALPETER EQUATION

Here we are not going to rederive the Bethe-Salpeter [1] equation or its reduction to the Salpeter [5] equation in the case of an instantaneous interaction, since these subjects have been dealt with many times in the literature [6] and have, moreover, received a textbook treatment [7]. Instead, we begin immediately with the Salpeter equation for a fermion-antifermion system, which we write in the center-of-mass frame of the bound state:

$$H_1\chi-\chi H_2+\int\frac{d^3\mathbf{k}'}{(2\pi)^3}V(\mathbf{k}-\mathbf{k}')\times(\Lambda_+^1\Gamma_1\chi'\Gamma_2\Lambda_-^2-\Lambda_-^1\Gamma_1\chi'\Gamma_2\Lambda_+^2)=M\chi\ . \quad (2.1)$$

We here use notation in which the explicit dependence of functions has been dropped. χ', for example, stands for $\chi(\mathbf{k}')$, whereas χ stands for $\chi(\mathbf{k})$. The instantaneous Bethe-Salpeter wave function χ is defined by

$$\chi(\mathbf{k})=\langle 0|\psi_1(\mathbf{k})\psi_2^\dagger(\mathbf{k})|B\rangle\ , \quad (2.2)$$

where $|B\rangle$ represents the studied mesonic bound state. Γ_1 and Γ_2 are two 4×4 matrices characterizing the way in which the fermion and the antifermion couple to the interaction potential. The H_i's are generalized Dirac Hamiltonians

$$H_i(\mathbf{k})=B_i(\mathbf{k})\boldsymbol{\alpha}\cdot\hat{\mathbf{k}}+A_i(\mathbf{k})\beta\ , \quad (2.3)$$

and we will denote the energy of the corresponding fermionic excitations on the vacuum by $E_i(k)$.

If we consider "constituent quarks" of masses m_i, we simply have

$$A_i(\mathbf{k})=m_i,\quad B_i(\mathbf{k})=k,\quad E_i(\mathbf{k})=(\mathbf{k}^2+m_i^2)^{1/2}\ . \quad (2.4)$$

However, (2.3) is more general and allows one to give an adequate treatment of the quark self-energy. This is of particular importance in the case of light quarks, where the phenomenon of spontaneous breaking of chiral symmetry plays a crucial role [8]. The Λ_+^i (Λ_-^i) are projectors on particle (antiparticle) states and are defined in Appendix A. Since these projectors satisfy $\Lambda_+^i+\Lambda_-^i=1$, we can use them to split the BS amplitude into four components:

$$\begin{aligned}\chi_{++}&=\Lambda_+^1\chi\Lambda_+^2\ ,\\ \chi_{+-}&=\Lambda_+^1\chi\Lambda_-^2\ ,\\ \chi_{-+}&=\Lambda_-^1\chi\Lambda_+^2\ ,\\ \chi_{--}&=\Lambda_-^1\chi\Lambda_-^2\ .\end{aligned} \quad (2.5)$$

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It is straightforward to show from the Salpeter equation (2.1) that

$$\chi_{++}(\mathbf{k})=\chi_{--}(\mathbf{k})=0 \ . \tag{2.6}$$

In order to interpret the remaining two components, we will write them using the expansion of the Fermi fields (A11):

$$\chi_{+-}(\mathbf{k})=\sum_{r,s} F_{rs}(\mathbf{k})[u_{1r}(\mathbf{k})v^{\dagger}_{2s}(\mathbf{k})], \qquad F_{rs}(\mathbf{k})=\langle 0|b_{1r}(\mathbf{k})d_{2s}(-\mathbf{k})|B\rangle \ ; \tag{2.7}$$

$$\chi_{-+}(\mathbf{k})=\sum_{r,s} G_{rs}(\mathbf{k})[v_{1r}(\mathbf{k})u^{\dagger}_{2s}(\mathbf{k})] \ , \qquad G_{rs}(\mathbf{k})=\langle 0|d^{\dagger}_{1r}(-\mathbf{k})b^{\dagger}_{2s}(\mathbf{k})|B\rangle \ . \tag{2.8}$$

The first one represents the amplitude of probability that the bound state contains a quark of type 1 and an antiquark of type 2 on top of the vacuum, whereas, in the second, the bound state would contain one $\bar{q}_1$-q_2 pair less than the vacuum. In the case where m_1 or m_2 tends to infinity, the second component should go to zero as the vacuum is then very unlikely to contain q-$\bar{q}$ pairs. However, as we depart from the nonrelativistic limit, the $-+$ component acquires more and more importance, and the bound state (as well as the vacuum) no longer contains a well-defined number of quark-antiquark pairs. This situation culminates for the pion when obtained as a Goldstone boson in which case χ_{-+} and χ_{+-} are indeed of the same magnitude as will be shown later.

The norm of the BS amplitude is defined by [8,9]

$$\|\chi\|^2=\int \frac{d^3\mathbf{k}}{(2\pi)^3}\frac{1}{2}\mathrm{Tr}\left[\chi^{\dagger}\left[\frac{H_1}{E_1}\chi-\chi\frac{H_2}{E_2}\right]\right] \tag{2.9}$$

and is related to the normalization of bound states in the following way:

$$\|\chi\|^2=\frac{1}{(2\pi)^3}\langle B|B\rangle \ . \tag{2.10}$$

It can also be written more explicitly as

$$\|\chi\|^2=\int \frac{d^3\mathbf{k}}{(2\pi)^3}\mathrm{Tr}[\chi_{+-}(\chi_{+-})^{\dagger}-\chi_{-+}(\chi_{-+})^{\dagger}] \ . \tag{2.11}$$

It is interesting to rewrite the Salpeter equation (2.1) in terms of χ_{+-} and χ_{-+}:

$$M\chi_{+-}=(E_1+E_2)\chi_{+-}+V\Lambda^1_+\Gamma_1(\chi'_{+-}+\chi'_{-+})\Gamma_2\Lambda^2_- \ , \tag{2.12}$$

$$M\chi_{-+}=-(E_1+E_2)\chi_{-+}-V\Lambda^1_-\Gamma_1(\chi'_{+-}+\chi'_{-+})\Gamma_2\Lambda^2_+ \ .$$

In the nonrelativistic limit, the kinetic energy dominates the potential energy, and one obtains the solutions

$$M=m_1+m_2, \quad \chi_{+-}\neq 0, \quad \chi_{-+}=0 \ , \qquad M=-(m_1+m_2), \quad \chi_{+-}=0, \quad \chi_{-+}\neq 0 \ . \tag{2.13}$$

The second solution has to be rejected because it would lead to a state of negative squared norm (2.11). We therefore obtain a proof of the fact already mentioned above that only χ_{+-} survives in the nonrelativistic limit.[1] This has encouraged several authors [3,10,11] to consider the so-called reduced Salpeter equation obtained from (2.12) by simply neglecting χ_{-+}:

$$M\chi_{+-}=(E_1+E_2)\chi_{+-}+V\Lambda^1_+\Gamma_1\chi'_{+-}\Gamma_2\Lambda^2_- \ . \tag{2.14}$$

It should not however, be forgotten that this is only a valid approximation when one deals with sufficiently heavy quarks.

In order to obtain from (2.1) an expression for the mass of the bound state in terms of the BS amplitude, we will apply to this equation the successive operations appearing in the right-hand side of (2.9). We then obtain

$$M\|\chi\|^2=\int \frac{d^3\mathbf{k}}{(2\pi)^3}(E_1+E_2)\frac{1}{2}\mathrm{Tr}\left[\chi^{\dagger}\chi-\chi^{\dagger}\frac{H_1}{E_1}\chi\frac{H_2}{E_2}\right] + \frac{1}{2}\int \frac{d^3\mathbf{k}}{(2\pi)^3}\int \frac{d^3\mathbf{k}'}{(2\pi)^3}V(\mathbf{k}-\mathbf{k}')\mathrm{Tr}\left[\chi^{\dagger}\Gamma_1\chi'\Gamma_2-\chi^{\dagger}\frac{H_1}{E_1}\Gamma_1\chi'\Gamma_2\frac{H_2}{E_2}\right] \ . \tag{2.15}$$

This expression can be further simplified by making use of (2.6). Indeed, it is easy to show that these constraints can be rewritten in the form

$$\frac{H_1}{E_1}\chi+\chi\frac{H_2}{E_2}=0 \ , \tag{2.16}$$

and (2.15) therefore becomes

$$M\|\chi\|^2=\int \frac{d^3\mathbf{k}}{(2\pi)^3}(E_1+E_2)\mathrm{Tr}(\chi^{\dagger}\chi)+\int \frac{d^3\mathbf{k}}{(2\pi)^3}\int \frac{d^3\mathbf{k}'}{(2\pi)^3}V(\mathbf{k}-\mathbf{k}')\mathrm{Tr}(\chi^{\dagger}\Gamma_1\chi'\Gamma_2) \ . \tag{2.17}$$

III. PROPERTIES OF THE SOLUTIONS AND VARIATIONAL PRINCIPLE

In the case where Γ_1 and Γ_2 are Hermitian matrices, it is easy to derive from (2.15) and (2.17) several properties of the solutions of the Salpeter equation.

[1] This is, in fact, also true if only one of the masses m_1 or m_2 goes to infinity.

(1) The eigenvalues are real.

(2) Wave functions corresponding to different eigenvlaues are orthogonal. Orthogonality is here defined from the expression of the norm (2.9). Two BS amplitudes χ and $\tilde{\chi}$ will be said to be orthogonal if

$$\int \frac{d^3\mathbf{k}}{(2\pi)^3}\frac{1}{2}\mathrm{Tr}\left[\tilde{\chi}^\dagger\left[\frac{H_1}{E_1}\chi-\chi\frac{H_2}{E_2}\right]\right]=0\ . \tag{3.1}$$

(3) For quarks of equal mass and an interaction term symmetrized in the exchange $\Gamma_1\leftrightarrow\Gamma_2$, eigenvalues will come in pairs of eigenvalues of opposite sign.

(4) The Salpeter equation is obtained by making stationary the right-hand side of (2.15) under the double constraint that χ is normalized ($\|\chi^2\|=1$) and that $(H_1/E_1)\chi+\chi(H_2/E_2)=0$.

This last property is very interesting because it establishes a variational principle for the solutions of the Salpeter equation. This will be very useful when we come to the numerical solution of the equation. Unfortunately, bound states do not in general correspond to minima of $M(\chi)$ (the usual proofs of this fact cannot be used here because the spectrum is generally not bounded from below as can be seen from property 3).

In the following sections, we will use the above variational principle to derive the Salpeter equation for various kind of interactions. More precisely, we will use a simplified version in which we first solve the constraint (2.16) before searching for the extremum. The Salpeter equation will then be obtained by variations of (2.17).

We now give the proofs of properties 1 to 4.

(1) The reality of the mass is a direct consequence of the reality of the norm (2.11) and of the left-hand side of (2.17) when Γ_1 and Γ_2 are Hermitian.

(2) Let us consider two solutions of the Salpeter equation, χ and χ', respectively, of mass M and $\tilde{M}$; it is then possible by a method analogous to the one leading to (2.17) to derive the two equations

$$\begin{aligned}&\mathrm{Tr}[(E_1+E_2)\tilde{\chi}^\dagger\chi]+\mathrm{Tr}[V(\mathbf{k}-\mathbf{k}')\tilde{\chi}^\dagger\Gamma_1\chi'\Gamma_2]=M\,\mathrm{Tr}\left[\tilde{\chi}^\dagger\frac{1}{2}\left[\frac{H_1}{E_1}\chi-\chi\frac{H_2}{E_2}\right]\right],\\&\mathrm{Tr}[(E_1+E_2)\chi^\dagger\tilde{\chi}]+\mathrm{Tr}[V(\mathbf{k}-\mathbf{k}')\chi^\dagger\Gamma_1\tilde{\chi}'\Gamma_2]=\tilde{M}\,\mathrm{Tr}\left[\chi^\dagger\frac{1}{2}\left[\frac{H_1}{E_1}\tilde{\chi}-\tilde{\chi}\frac{H_2}{E_2}\right]\right].\end{aligned} \tag{3.2}$$

Taking the Hermitian conjugate of the second and subtracting it from the first, we get (with Γ_1 and Γ_2 Hermitian)

$$0=(M-\tilde{M})\mathrm{Tr}\left[\tilde{\chi}^\dagger\frac{1}{2}\left[\frac{H_1}{E_1}\chi-\chi\frac{H_2}{E_2}\right]\right]. \tag{3.3}$$

For $M\neq\tilde{M}$, we therefore obtain (3.1). Note that the integral used to define orthogonality does not follow from a scalar product as the norm (2.11) is not positive definite. Moreover there are nonvanishing wave functions of zero norm ($\chi_{+-}=\chi_{-+}$).

(3) For quarks of equal mass, the Salpeter equation (2.1) reduces to

$$\frac{1}{2E(k)}\left[H(\mathbf{k}),2E(k)\chi(\mathbf{k})+\int\frac{d^3\mathbf{k}'}{(2\pi)^3}V(\mathbf{k}-\mathbf{k}')\Gamma_1\chi(\mathbf{k}')\Gamma_2\right]=M\chi(\mathbf{k})\ . \tag{3.4}$$

χ being a solution of energy M it is easily shown that χ^+ is a solution of energy $-M$. Indeed taking the Hermitian conjugate of (3.4) we obtain

$$\frac{1}{2E}\left[H,2E\chi^\dagger+\int\frac{d^3\mathbf{k}'}{(2\pi)^3}V(\mathbf{k}-\mathbf{k}')\Gamma_2^\dagger\chi'^\dagger\Gamma_1^\dagger\right]=-M\chi^\dagger\ , \tag{3.5}$$

which is the announced result if one takes into account the assumptions made ($\Gamma_1^+=\Gamma_1$, $\Gamma_2^+=\Gamma_2$, and symmetrization of the interaction in $\Gamma_1\leftrightarrow\Gamma_2$). It is useful to compute the squared norm of this new state:

$$\|\chi^\dagger\|^2=\int\frac{d^3\mathbf{k}}{(2\pi)^3}\mathrm{Tr}\left[\left[\frac{H}{2E},\chi^\dagger\right]\chi\right]=-\int\frac{d^3\mathbf{k}}{(2\pi)^3}\mathrm{Tr}\left[\left[\frac{H}{2E},\chi\right]\chi^\dagger\right]=-\|\chi\|^2\ . \tag{3.6}$$

We see that the square of the norm of the state of energy $-M$ is opposite to the one of the state of energy M. Physically only states of real norm are acceptable (2.10). Those will correspond to a positive mass whenever the left-hand side of (2.17) is positive.

(4) We have to make stationary the right-hand side of (2.15) under the constraints that χ is normalized to 1 and that $[(H_1/E_1)\chi+\chi(H_2/E_2)]=0$. The second constraint (actually an infinity of constraints, one for each $\mathbf{k}$) can be expressed synthetically as

$$\mathrm{Tr}\int \frac{d^3\mathbf{k}}{(2\pi)^3}\left[\frac{H_2}{E_2}\chi^\dagger+\chi^\dagger\frac{H_1}{E_1}\right]\left[\frac{H_1}{E_1}\chi+\chi\frac{H_2}{E_2}\right]=0\ . \tag{3.7}$$

This extremum problem can easily be dealt with by the usual method of introducing Lagrange multipliers $-M$ and λ for the two constraints and then taking variations with respect to M, λ, and $\chi^\dagger$. Apart from the constraints themselves, the final result of this procedure is an equation obtained by variation of (2.15) with respect to $\chi^\dagger$:

$$(E_1+E_2)\left[\chi-\frac{H_1}{E_1}\chi\frac{H_2}{E_2}\right]+V\left[\Gamma_1\chi'\Gamma_2-\frac{H_1}{E_1}\Gamma_1\chi'\Gamma_2\frac{H_2}{E_2}\right]=M\left[\frac{H_1}{E_1}\chi-\chi\frac{H_2}{E_2}\right]\ . \tag{3.8}$$

It is then easily shown that this equation supplemented with the constraint (2.16) is equivalent to (2.1), which establishes the variational principle. In practical applications it is more interesting to first solve the constraint (2.16) thereby introducing "independent components" of the wave function χ. The expression of the Salpeter equation is then obtained by taking variations of (2.17) with respect to these components.

IV. REDUCTION OF THE SALPETER EQUATION

In view of the results of the preceding section, one way to make more explicit the Salpeter equation would be to substitute (2.7) and (2.8) into (2.17) and then take variations with respect to F^*_{rs}, G^*_{rs}. This, however, can lead to lengthy calculations and, in what follows, we will prefer to use another method in which we first expand χ on a basis of 4×4 matrices and then solve the constraint (2.16) before using the variational principle. There is nevertheless one case in which the above method can be useful, and that is when one considers the reduced Salpeter equation obtained by neglecting χ_{--}. The expression of the potential term in (2.17) then reduces to[2]

$$\mathrm{Tr}[V(\chi_{+-})^\dagger\Gamma_1\chi'_{+-}\Gamma_2]=\int\frac{d^3\mathbf{k}}{(2\pi)^3}\int\frac{d^3\mathbf{k}'}{(2\pi)^3}V(\mathbf{k}-\mathbf{k}')F^*_{rs}(\mathbf{k})F_{r's'}(\mathbf{k}')[u^\dagger_{1r}(\mathbf{k})\Gamma_1u_{1r'}(\mathbf{k}')][v^\dagger_{2s'}(\mathbf{k}')\Gamma_2v_{2s}(\mathbf{k})]\ . \tag{4.1}$$

It is clear from the expression (4.1) that this method is equivalent to the one often used in the literature and which consists in deriving an effective Hamiltonian from the expression of the scattering amplitude [12]. Expressing the kinetic energy term and $\|\chi\|^2$ in terms of F_{rs} and carrying the computation to the end, one finally obtains a generalized Breit-Fermi Hamiltonian acting on the wave function ϕ defined by

$$\phi=F_{rs}\xi_r\xi^\dagger_s\ , \tag{4.2}$$

where the ξ_s are the two-component spinors appearing in (A9).

For the example of a $\gamma_0\otimes\gamma_0$ interaction, one obtains

$$H=E_1+E_2+H_0+H_1 i(\hat{\mathbf{k}}\times\hat{\mathbf{k}}')\cdot(\mathbf{S}_1-\mathbf{S}_2)$$
$$+H_2 i(\hat{\mathbf{k}}\times\hat{\mathbf{k}}')\cdot\mathbf{S}-H_3(\hat{\mathbf{k}}\times\hat{\mathbf{k}}')\cdot\mathbf{S}(\hat{\mathbf{k}}\times\hat{\mathbf{k}}')\cdot\mathbf{S} \tag{4.3}$$

with $\mathbf{S}=\mathbf{S}_1+\mathbf{S}_2$ and

$$H_0=\tfrac{1}{2}V[(\sin\varphi\sin\varphi'+\cos\theta\cos\theta')$$
$$+(\cos\varphi\cos\varphi'+\sin\theta\sin\theta')\hat{\mathbf{k}}\cdot\hat{\mathbf{k}}']\ , \tag{4.4}$$

$$H_1=\tfrac{1}{2}V(\sin\theta\cos\varphi'+\cos\varphi\sin\theta')\ , \tag{4.5}$$

$$H_2=\tfrac{1}{2}V[(\cos\varphi\cos\varphi'+\sin\theta\sin\theta')$$
$$+(\cos\theta-\sin\varphi)(\cos\theta'-\sin\varphi')\hat{\mathbf{k}}\cdot\hat{\mathbf{k}}']\ , \tag{4.6}$$

$$H_3=\tfrac{1}{2}V(\cos\theta-\sin\varphi)(\cos\theta'-\sin\varphi')\ , \tag{4.7}$$

where

$$\varphi=\frac{\varphi_1+\varphi_2}{2}\ ,\quad \theta=\frac{\varphi_2-\varphi_1}{2}\ , \tag{4.8}$$

and the angle φ_i are defined in (A14).

We now come to the method we use to deal with the full Salpeter equation. We first have to expand χ on a basis of 4×4 matrices. Following Le Yaouanc *et al.* [8] we choose 16 Hermitian matrices of square 1, the coefficients of the expansion being complex. We then have

$$\chi=\mathcal{L}_0+\mathcal{L}_i\rho_i+\mathcal{N}_0\cdot\boldsymbol{\sigma}+\mathcal{N}_i\cdot\rho_i\boldsymbol{\sigma}\ , \tag{4.9}$$

where the matrices ρ_i and σ_i are defined in Appendix A.

It is easily shown that the constraint (2.16) can be solved by expressing the 16 components of χ in terms of eight new functions $(L_1,L_2,\mathbf{N}_1,\mathbf{N}_2)$ in the following way:

$$\begin{aligned}
&\mathcal{L}_0=\sin\theta(\hat{\mathbf{k}}\cdot\mathbf{N}_2)\ ,\\
&\mathcal{L}_1=-\sin\varphi L_1\ ,\\
&\mathcal{L}_2=i\cos\theta L_2\ ,\\
&\mathcal{L}_3=-\cos\varphi(\hat{\mathbf{k}}\cdot\mathbf{N}_1)\ ,\\
&\mathcal{N}_0=\sin\theta L_2\hat{\mathbf{k}}+i\cos\varphi(\hat{\mathbf{k}}\times\mathbf{N}_2)\ ,\\
&\mathcal{N}_1=\sin\varphi\hat{\mathbf{k}}(\hat{\mathbf{k}}\cdot\mathbf{N}_1)-\cos\theta\hat{\mathbf{k}}\times(\hat{\mathbf{k}}\times\mathbf{N}_1)\ ,\\
&\mathcal{N}_2=i[\cos\theta\hat{\mathbf{k}}(\hat{\mathbf{k}}\cdot\mathbf{N}_2)-\sin\varphi\hat{\mathbf{k}}\times(\hat{\mathbf{k}}\times\mathbf{N}_2)]\ ,\\
&\mathcal{N}_3=\cos\varphi L_1\hat{\mathbf{k}}-i\sin\theta(\hat{\mathbf{k}}\times\mathbf{N}_1)\ .
\end{aligned} \tag{4.10}$$

[2]By contrast the general expression appearing in the Salpeter equation would require four such terms involving the various factors $u^\dagger\Gamma_iu', u^\dagger\Gamma_iv', v^\dagger\Gamma_iu', v^\dagger\Gamma_iv'$.

In this formalism, the separation of the components χ_{+-} and χ_{-+} occurs through the transformation

$$L_1=L_{(L)}+L_{(S)},\quad \mathbf{N}_1=\mathbf{N}_{(L)}+\mathbf{N}_{(S)}\ , \tag{4.11}$$

$$L_2=L_{(L)}-L_{(S)},\quad \mathbf{N}_2=\mathbf{N}_{(L)}-\mathbf{N}_{(S)}\ . \tag{4.12}$$

The indices (L) and (S) correspond, respectively, to "large" (χ_{+-}) and "small" (χ_{-+}) components. In the nonrelativistic limit only χ_{+-} (as already mentioned) survives, and one has $L_1=L_2=L_{(L)}$, $\mathbf{N}_1=\mathbf{N}_2=\mathbf{N}_{(L)}$.

Having solved the constraint, it is easy to get the Salpeter equation, one simply has to set (4.10) into (2.17) and to take variations with respect to the independent variables $(L_1,L_2,\mathbf{N}_1,\mathbf{N}_2)$. We will therefore rewrite (2.17) in terms of these variables. For the square of the norm of the BS amplitude, we have

$$\|\chi\|^2=4\int\frac{d^3\mathbf{k}}{(2\pi)^3}[L_2^*(\mathbf{k})L_1(\mathbf{k})+L_1^*(\mathbf{k})L_2(\mathbf{k})+\mathbf{N}_2^*(\mathbf{k})\cdot\mathbf{N}_1(\mathbf{k})+\mathbf{N}_1^*(\mathbf{k})\cdot\mathbf{N}_2(\mathbf{k})]\ . \tag{4.13}$$

In the same manner, the kinetic-energy term becomes

$$\int\frac{d^3\mathbf{k}}{(2\pi)^3}[E_1(k)+E_2(k)]\mathrm{Tr}[\chi^\dagger(\mathbf{k})\chi(\mathbf{k})]$$
$$=4\int\frac{d^3\mathbf{k}}{(2\pi)^3}[E_1(k)+E_2(k)]\times[L_1^*(\mathbf{k})L_1(\mathbf{k})+L_2^*(\mathbf{k})L_2(\mathbf{k})+\mathbf{N}_1^*(\mathbf{k})\cdot\mathbf{N}_1(\mathbf{k})+\mathbf{N}_2^*(\mathbf{k})\cdot\mathbf{N}_2(\mathbf{k})]\ . \tag{4.14}$$

The expression of the potential energy is more involved and depends on the type of interaction considered through the matrices Γ_1 and Γ_2. In Appendix D, we give the necessary formulas for several interesting cases. Here we will consider as an example a charge-charge interaction ($\gamma_0\otimes\gamma_0$ component of a vector interaction) that corresponds to $\Gamma_1=\Gamma_2=1$ in (2.1). Using (D1) and (4.10) we obtain

$$\int\frac{d^3\mathbf{k}}{(2\pi)^3}\int\frac{d^3\mathbf{k}'}{(2\pi)^3}V(\mathbf{k}-\mathbf{k}')\mathrm{Tr}[\chi^\dagger(\mathbf{k})\chi(\mathbf{k}')]$$
$$=4\int\frac{d^3\mathbf{k}}{(2\pi)^3}\int\frac{d^3\mathbf{k}'}{(2\pi)^3}V(\mathbf{k}-\mathbf{k}')\{(S_\varphi S'_\varphi+C_\varphi C'_\varphi\hat{\mathbf{k}}\cdot\hat{\mathbf{k}}')L_1^*L'_1+(C_\theta C'_\theta+S_\theta S'_\theta\hat{\mathbf{k}}\cdot\hat{\mathbf{k}}')L_2^*L'_2$$
$$+S_\theta S'_\theta(\hat{\mathbf{k}}\times\mathbf{N}_1^*)\cdot(\hat{\mathbf{k}}'\times\mathbf{N}_1^*)\cdot(\hat{\mathbf{k}}'\times\mathbf{N}'_1)+C_\varphi C'_\varphi(\hat{\mathbf{k}}\times\mathbf{N}_2^*)\cdot(\hat{\mathbf{k}}'\times\mathbf{N}'_2)$$
$$-C_\varphi S'_\theta L_1^*\hat{\mathbf{k}}\cdot(\hat{\mathbf{k}}'\times\mathbf{N}'_1)-S_\theta C'_\varphi L_2^*\hat{\mathbf{k}}\cdot(\hat{\mathbf{k}}'\times\mathbf{N}'_2)$$
$$-S_\theta C'_\varphi(\hat{\mathbf{k}}\times\mathbf{N}_1^*)\cdot\mathbf{k}'L'_1+C_\varphi S'_\theta(\hat{\mathbf{k}}\times\mathbf{N}_2^*)\cdot\mathbf{k}'L'_1)$$
$$+C_\varphi C'_\varphi(\mathbf{N}_1^*\cdot\hat{\mathbf{k}})(\hat{\mathbf{k}}'\cdot\mathbf{N}'_1)+S_\theta S'_\theta(\mathbf{N}_2^*\cdot\hat{\mathbf{k}})(\hat{\mathbf{k}}'\cdot\mathbf{N}'_2)$$
$$+(S_\varphi-C_\theta)(S'_\varphi-C'_\theta)[(\mathbf{N}_1^*\cdot\hat{\mathbf{k}})(\hat{\mathbf{k}}\cdot\hat{\mathbf{k}}')(\hat{\mathbf{k}}'\cdot\mathbf{N}'_1)+(\mathbf{N}_2^*\cdot\hat{\mathbf{k}})(\hat{\mathbf{k}}\cdot\hat{\mathbf{k}}')(\hat{\mathbf{k}}'\cdot\mathbf{N}'_2)]$$
$$+C_\theta C'_\theta(\mathbf{N}_1^*\cdot\mathbf{N}'_1)+S_\varphi S'_\varphi(\mathbf{N}_2^*\cdot\mathbf{N}'_2)$$
$$+(S_\varphi-C_\theta)C'_\theta(\mathbf{N}_1^*\cdot\mathbf{k})(\mathbf{k}\cdot\mathbf{N}'_1)+(C_\theta-S_\varphi)S'_\varphi(\mathbf{N}_2^*\cdot\hat{\mathbf{k}})(\hat{\mathbf{k}}\cdot\mathbf{N}_2)$$
$$+C_\theta(S'_\varphi-C'_\theta)(\mathbf{N}_1^*\cdot\hat{\mathbf{k}}')(\hat{\mathbf{k}}'\cdot\mathbf{N}'_1)+S_\varphi(C'_\theta-S'_\varphi)(\mathbf{N}_2^*\cdot\hat{\mathbf{k}}')(\hat{\mathbf{k}}'\cdot\mathbf{N}'_2)\}\ , \tag{4.15}$$

where we have used the notation

$$C_\varphi=\cos\varphi\ ,\quad S_\varphi=\sin\varphi\ , \tag{4.16}$$

$$C_\theta=\cos\theta,\quad S_\theta=\sin\theta\ . \tag{4.17}$$

We can then get an explicit form of the Salpeter equation by putting (4.13), (4.14), and (4.15) into (2.17) and making variations with respect to $L_1^*,L_2^*,\mathbf{N}_1^*,\mathbf{N}_2^*$ (considered as independent of the variables $L_1,L_2,\mathbf{N}_1,\mathbf{N}_2$). However, this would not be very interesting at this stage, and we will wait to write down these equations until the next section where we reexpress everything in terms of the radial wave functions only. Nevertheless, we will consider here the case of the reduced Salpeter equation in order to show that we then reobtain the results of Cung *et al.* [10]. This equation is obtained by taking $L_1=L_2=L$ and $\mathbf{N}_1=\mathbf{N}_2=\mathbf{N}$ in (2.17) and then making variations with respect to L^* and $\mathbf{N}^*$. Using (4.13), (4.14), and (4.15) we then get

$$(E_1+E_2)L+H_0L'-H_1(\hat{\mathbf{k}}\times\hat{\mathbf{k}}')\cdot\mathbf{N}'=ML\ ,$$
$$(E_1+E_2)\mathbf{N}+H_0\mathbf{N}'+H_1(\hat{\mathbf{k}}\times\hat{\mathbf{k}}')L'+\tfrac{1}{2}V(\cos\varphi\cos\varphi'+\sin\theta\sin\theta')(\hat{\mathbf{k}}\hat{\mathbf{k}}'-\hat{\mathbf{k}}'\hat{\mathbf{k}})\cdot\mathbf{N}'$$
$$+H_3[2\hat{\mathbf{k}}(\hat{\mathbf{k}}\cdot\hat{\mathbf{k}}')\hat{\mathbf{k}}'-\hat{\mathbf{k}}\hat{\mathbf{k}}-\hat{\mathbf{k}}'\hat{\mathbf{k}}']\cdot\mathbf{N}'=M\mathbf{N}\ . \tag{4.18}$$

It is easily shown that this result is equivalent to the equations obtained by Cung *et al.* That is, for the states ${}^{[1-3]}J_J$ (4.18) reduces to [Eqs. (3.9a) and (3.9b)] of Ref. [10], whereas for the states ${}^3(J_\pm 1)_J$ it gives back [Eq. (3.28)] of Ref. [10].[3] We would like to remark that our method allowed us to obtain immediately equations involving Hermitian operators, whereas in Ref. [10] several transformations were required to achieve this goal. As mentioned in [10] it is possible, by using the representation of the spin operator on the wave functions L and $\mathbf{N}$, to reexpress (4.18) in terms of an effective Hamiltonian. Not surprisingly, one then reobtains (4.3), but the computations are awkward and the method we have used to arrive at (4.3) is certainly better suited for the derivation of effective Hamiltonians.

V. EQUATIONS IN TERMS OF THE RADIAL WAVE FUNCTIONS

Now we would like to go further and to get systems of equations for the various radial wave functions contributing to a state of given J^{PC}. For that purpose we need to reexpress (2.17) in terms of the functions $L_i(k)$, $N_{i0}(k)$, $N_{i+}(k)$, and $N_{i-}(k)$ through (B15) and (B16). For the square of the norm of the BS amplitude (4.13) and for the kinetic energy (4.14) this is easily done using the orthogonality of the (vector-) spherical harmonics. As before the potential-energy term requires a little more work. Within the examples given in Appendix D we have essentially three types of integrations to compute:

$$I_1=\int\frac{d^3\mathbf{k}}{(2\pi)^3}\int\frac{d^3\mathbf{k}'}{(2\pi)^3}V(\mathbf{k}-\mathbf{k}')\mathcal{L}^*(\mathbf{k})\mathcal{L}(\mathbf{k}')\ , \tag{5.1}$$

$$I_2=\int\frac{d^3\mathbf{k}}{(2\pi)^3}\int\frac{d^3\mathbf{k}'}{(2\pi)^3}V(\mathbf{k}-\mathbf{k}')\mathcal{N}^*(\mathbf{k})\cdot\mathcal{N}(\mathbf{k}')\ , \tag{5.2}$$

$$I_3=\int\frac{d^3\mathbf{k}}{(2\pi)^3}\int\frac{d^3\mathbf{k}'}{(2\pi)^3}V(\mathbf{k}-\mathbf{k}')[\mathcal{N}^*(\mathbf{k})\cdot\hat{\mathbf{q}}][\hat{\mathbf{q}}\cdot\mathcal{N}(\mathbf{k}')]\ . \tag{5.3}$$

The formulas allowing one to perform the integrations over the angles are given in Appendix C, and we obtain, respectively,

$$I_1=\int_0^\infty\frac{dk\,k^2}{(2\pi)^3}\int_0^\infty\frac{dk'k'^2}{(2\pi)^3}(2\pi)V_J(k,k')\mathcal{L}^*(k)\mathcal{L}(k')\ , \tag{5.4}$$

$$I_2=\int_0^\infty\frac{dk\,k^2}{(2\pi)^3}\int_0^\infty\frac{dk'k'^2}{(2\pi)^3}(2\pi)[V_{J-1}(k,k')\mathcal{N}^*_-(k)\mathcal{N}_-(k')+V_J(k,k')\mathcal{N}^*_0(k)\mathcal{N}_0(k')-V_{J+1}(k,k')\mathcal{N}^*_+(k)\mathcal{N}_+(k')]\ , \tag{5.5}$$

$$\begin{aligned}I_3=\int_0^\infty\frac{dk\,k^2}{(2\pi)^3}\int_0^\infty\frac{dk'k'^2}{(2\pi)^3}(2\pi)(&-\mathcal{N}^*_-V^{(1)}_{J-1}\mathcal{N}'_--\mathcal{N}^*_0V^{(1)}_J\mathcal{N}'_0-\mathcal{N}^*_+V^{(1)}_{J+1}\mathcal{N}'_+\\&+\mu^2\mathcal{N}^*_-V^{(2)}_{J-1,J-1}\mathcal{N}'_-+\nu^2\mathcal{N}^*_+V^{(2)}_{J+1,J+1}\mathcal{N}'_+\\&+\mu\nu\mathcal{N}^*_+V^{(2)}_{J+1,J-1}\mathcal{N}'_-+\mu\nu\mathcal{N}^*_-V^{(2)}_{J-1,J+1}\mathcal{N}'_+)\ .\end{aligned} \tag{5.6}$$

The quantities V_L and $V_{L,L'}$ as well as the potentials $V^{(1)}$ and $V^{(2)}$ appearing in (5.6) are defined in Appendix C. The expressions (5.4), (5.5), and (5.6) are not yet in the final form. They contain the variables $\mathcal{L}$ and $\mathcal{N}$, and we need to reexpress everything in terms of $L,\mathbf{N}$. Again, the necessary formulas can be found in Appendix C (C10).

Putting everything together and taking, now, the variations with respect to the radial wave functions we obtain two systems of equations which correspond, respectively, to states with $P=(-1)^{J+1}$ and $P=(-1)^J$. For the example of a $\gamma_0\otimes\gamma_0$ interaction they are

$$\begin{aligned}ML_1&=(E_1+E_2)L_2+C_\theta V_JC'_\theta L'_2+S_\theta(\mu^2V_{J-1}+\nu^2V_{J+1})S'_\theta L'_2+\mu\nu S_\theta(V_{J-1}-V_{J+1})C'_\varphi N'_{20}\ ,\\ML_2&=(E_1+E_2)L_1+S_\varphi V_JS'_\varphi L'_1+C_\varphi(\mu^2V_{J-1}+\nu^2V_{J+1})C'_\varphi L'_1-\mu\nu C_\varphi(V_{J-1}-V_{J+1})S'_\theta N'_{10}\ ,\\MN_{10}&=(E_1+E_2)N_{20}+S_\varphi V_JS'_\varphi N'_{20}+C_\varphi(\nu^2V_{J-1}+\mu^2V_{J+1})C'_\varphi N'_{20}+\mu\nu C_\varphi(V_{J-1}-V_{J+1})S'_\theta L'_2\ ,\\MN_{20}&=(E_1+E_2)N_{10}+C_\theta V_JC'_\theta N'_{10}+S_\theta(\nu^2V_{J-1}+\mu^2V_{J+1})S'_\theta N'_{10}-\mu\nu S_\theta(V_{J-1}-V_{J+1})C'_\varphi L'_1\ ,\end{aligned} \tag{5.7}$$

and

$$\begin{aligned}Mn_{1(+)}&=(E_1-E_2)n_{2(+)}+C_\varphi V_JC'_\varphi n'_{2(+)}+S_\varphi(\nu^2V_{J-1}+\mu^2V_{J+1})S'_\varphi n'_{2(+)}+\mu\nu S_\varphi(V_{J-1}-V_{J+1})C'_\theta n'_{2(-)}\ ,\\Mn_{2(+)}&=(E_1+E_2)n_{1(+)}+S_\theta V_JS'_\theta n'_{1(+)}+C_\theta(\nu^2V_{J-1}+\mu^2V_{J+1})C'_\theta n'_{1(+)}+\mu\nu C_\theta(V_{J-1}-V_{J+1})S'_\varphi n'_{1(-)}\ ,\\Mn_{1(-)}&=(E_1+E_2)n_{2(-)}+S_\theta V_JS'_\theta n'_{2(-)}+C_\theta(\mu^2V_{J-1}+\nu^2V_{J+1})C'_\theta n'_{2(-)}+\mu\nu C_\theta(V_{J-1}-V_{J+1})S'_\varphi n'_{2(+)}\ ,\\Mn_{2(-)}&=(E_1+E_2)n_{1(-)}+C_\varphi V_JC'_\varphi n'_{1(-)}+S_\varphi(\mu^2V_{J-1}+\nu^2V_{J+1})S'_\varphi n'_{1(-)}-\mu\nu S_\varphi(V_{J-1}-V_{J+1})C'_\theta n'_{1(+)}\ .\end{aligned} \tag{5.8}$$

[3]For the determination of the quantum numbers associated with the various components of the wave function, see Appendix B and formulas (C5)–(C9).

We have chosen to write this last system in terms of the variables $n_{i(\pm)}$ defined by (C3) because it gives rise to simpler expressions. One can, however, easily go back to the variables $N_{i\pm}$ which have a more direct physical interpretation.

The form factors appearing in the potential-energy terms are all expressed as the sine or cosine of two function $\varphi(k)$ and $\theta(k)$.[4] This already constitutes an improvement compared to the various parametrizations used in the literature [3,10,11,13-15]. Moreover it allows us to deal very quickly with several interesting limiting cases. For example, for quarks of equal "mass" one would have $\theta=0$ and $\varphi=\varphi_1=\varphi_2$. In this case, the system of equations (5.7) decouples in two subsystems for $\{L_1,L_2\}$ and $\{N_{10},N_{20}\}$. For identical quarks this is the expression of charge-conjugation invariance and corresponds to the decoupling of ${}^1J_J$ and ${}^3J_J$ states, which have the same P but opposite C. As an example that will be useful later, we give the equations obtained for $\{L_1,L_2\}$ in this case:

$$\begin{aligned} &2EL_2+\frac{1}{(2\pi)^2}\int_0^\infty dk'k'^2V_JL_2'=ML_1\ , \\ &2EL_1+\frac{1}{(2\pi)^2}\int_0^\infty dk'k'^2 \\ &\quad\times[S_\varphi V_JS_\varphi'+C_\varphi(\mu^2V_{J-1}+\nu^2V_{J+1})C_\varphi']L_1'=ML_2\ . \end{aligned} \tag{5.9}$$

Another interesting limiting case is the one in which one of the quarks becomes infinitely heavy. As an example, for $m_2\to\infty$, we would have $\varphi_2=\pi/2$ and therefore $\sin\theta=\cos\varphi$, $\cos\theta=\sin\varphi$ so that we can forget about θ and express everything in terms of φ. In this case, it is also seen that the solutions satisfy

$$L_1=L_2,\quad N_{10}=N_{20}\ , \tag{5.10}$$

$$n_{1(+)}=n_{2(+)},\quad n_{1(-)}=n_{2(-)}\ . \tag{5.11}$$

It should however be noted that the equations so obtained are not equivalent to the Dirac equation due to the absence of crossed diagrams in the kernel of the BS equation [10].

VI. APPLICATIONS

We now come to the applications of our formalism and the comparison with previously obtained results. Most of the existing computations in the literature only concern the reduced Salpeter equation, whereas our method applies to the full Salpeter equation. It is precisely one advantage of our technique that the amount of work necessary to deal with the Salpeter equation is not much greater than what is required to treat the reduced equation. Moreover, the generalization to the full Salpeter equation is mandatory when one is interested in light-quark mesons. As a particularly striking example, we consider the pion in a situation where chiral symmetry is spontaneously broken. The pion is the lowest-energy $J=0$ solution of (5.9), and in this case it can be shown to be massless (for zero current quark mass) as a consequence of the Goldstone theorem [8]. The corresponding solution is

$$M=0,\quad L_1=0,\quad L_2=\sin\varphi\ . \tag{6.1}$$

Clearly, this cannot be obtained from the reduced Salpeter equation, which is an approximation only valid when L_1 and L_2 are nearly equal (or equivalently when the bound state contains only one $q\bar{q}$ pair). The passage to the full Salpeter equation is therefore of practical importance. For example, we have reconsidered and generalized several models treated in the literature. The first is the model of a vector-plus-scalar potential investigated by Gara and co-workers [3] and concerning which several explicit formulas have already been given in the present paper. Comparison with their results is most easily effected in the formalism presented at the beginning of Sec. IV. Indeed upon correcting the formulas of Ref. [3] in accordance with the comments of Lucha, Rupprecht, and Schöberl [15], we have, in fact, established agreement between our formulation and that of Ref. [3]. In addition, results concerning the full Salpeter equation (5.7) and (5.8) have been used in our study of the scalar potential [4].

As a more involved example we have also considered the model of Jacobs, Olsson, and Suchyta [11], which includes the effects of retardation up to order v^2/c^2. Again, we have confirmed their findings and generalized them to the full Salpeter equation. Moreover, our results are expressed in a much more synthetic way. They are reduced to two closed expressions, respectively, valid for the states $(0^-,1^+,2^-,3^+,\ldots)$ and $(0^+,1^-,2^+,3^-,\ldots)$. Also our expression of the form factors in terms of θ and φ is simpler than the parametrization used in [11].

As mentioned in the Introduction, our main motivation for this analysis of the Bethe-Salpeter equation was to establish a framework for discussing alternatives to the model of scalar confinement. Our results concerning this vast question are discussed in a separate publication [16]. If some of them can be obtained by a simple analysis of the form of the equations, acceptance or rejection of an hypothesis in general depends on a numerical computation. We will therefore briefly comment on this topic here.

Several techniques to numerically solve the reduced Salpeter equation have been presented in the literature [17-19] and it would be very interesting if one could generalize them to the full Salpeter equation. However, this can only be done up to a certain extent. In our own computations we have used a generalization of the method of Ref. [18]; that is, we transform the Salpeter equation into a matrix equation by expanding the components of the BS amplitude on an orthonormal basis for $\mathcal{L}_2(R^3)$. Of course, for the numerical computation this matrix has to be truncated and at a given order one will obtain approximate solutions depending on the choice of the basis. The difference with the reduced case [18] is that the approximate eigenvalues so obtained are no longer upper bounds for the exact result. Indeed as mentioned in Sec. III,

[4]As can be seen from (4.10) this will be a general property as long as Γ_1 and Γ_2 do not contain their own form factors.

bound states are in general simply stationary points of $M(\chi)$, not minima. We were, however, able to obtain approximate solutions by searching for extrema of $M(\alpha)$, where α is a variational parameter characterizing the basis of trial function.

The results so obtained can be checked and (in some cases) have been checked by using an iterative method. The latter is, however, much more time consuming and much more difficult to implement then the variational one. For example, because the iterative process naturally converges to nodeless wave functions, excited states can only be obtained by orthogonalizing at each step with respect to lower mass states, whereas the matrix diagonalization simultaneously gives approximations for the ground state and the first few excited states. On the other hand, finding an inflection point in $M(\alpha)$ is not always an easy task, and an iterative method based on a discretization may be more systematic and more rigorous. However, when tested together both methods gave compatible results, and we have therefore generally opted for the variational technique because of its ease of use.

VII. CONCLUSION

We have presented a general and straightforward method to reduce the Bethe-Salpeter equation to a set of coupled equations for radial wave functions. The only restriction is that the interaction be instantaneous. For unequal mass fermions and given angular momentum, the final result consists of two systems of four coupled equations corresponding, respectively, to states of opposite parity. These equations involve form factors, which can all be expressed as the sine or cosine of two functions $\theta(k)$ and $\varphi(k)$. These results simplify in several limiting cases such as equal masses ($m_1=m_2$) or $m_1\to\infty$. Supplemented by a technique for numerical resolution our procedure allows the testing of several models of interaction and is therefore of great importance for the study of mesonic bound states in QCD.

APPENDIX A: DIRAC MATRICES AND SPINORS

1. Dirac Matrices

The representation of the Dirac matrices that we have used is

$$1=\begin{bmatrix}1&0\\0&1\end{bmatrix},\quad \rho_1=\gamma_5=\begin{bmatrix}0&1\\1&0\end{bmatrix},$$

$$\rho_2=-i\gamma_0\gamma_5=\begin{bmatrix}0&-i1\\i1&0\end{bmatrix},\quad \rho_3=\gamma_0=\beta=\begin{bmatrix}1&0\\0&-1\end{bmatrix},$$

$$\sigma_k=\frac{i}{2}\epsilon_{ijk}\gamma^i\gamma^j=\begin{bmatrix}\tau_k&0\\0&\tau_k\end{bmatrix},\quad \rho_1\sigma_k=\alpha^k=\begin{bmatrix}0&\tau_k\\\tau_k&0\end{bmatrix},$$

$$\rho_2\sigma_k=-i\gamma^k\begin{bmatrix}0&-i\tau_k\\i\tau_k&0\end{bmatrix},\quad \rho_3\sigma_k=\gamma^k\gamma_5=\begin{bmatrix}\tau_k&0\\0&-\tau_k\end{bmatrix}.$$

These matrices are all Hermitian and of square 1. Moreover, they satisfy

$$[\rho_1,\sigma_k]=0\ , \tag{A1}$$

$$[\rho_i,\rho_j]=2i\epsilon_{ijk}\rho_k,\quad \{\rho_i,\rho_j\}=2\delta_{ij},\quad \rho_i\rho_j=\delta_{ij}+i\epsilon_{ijk}\rho_k\ , \tag{A2}$$

$$[\sigma_i,\sigma_j]=2i\epsilon_{ijk}\sigma_k,\quad \{\sigma_i,\sigma_j\}=2\delta_{ij},\quad \sigma_i\sigma_j=\delta_{ij}+i\epsilon_{ijk}\sigma_k\ . \tag{A3}$$

2. Basis spinors and expansion of the fermionic field

We consider a generalized Dirac Hamiltonian

$$H(\mathbf{p})=B(p)\boldsymbol{\alpha}\cdot\hat{\mathbf{p}}+A(p)\beta,\quad p=|\mathbf{p}|\ . \tag{A4}$$

The eigenvectors of H will be denoted by $u_s(\mathbf{p})$ and $v_s(\mathbf{p})$ (the index s taking two values) and will be, respectively, solutions of

$$H(\mathbf{p})u_s(\mathbf{p})=E(p)u_s(\mathbf{p}),\quad H(\mathbf{p})v_s(\mathbf{p})=-E(p)v_s(\mathbf{p}) \tag{A5}$$

with

$$E^2(p)=A^2(p)+B^2(p)\ . \tag{A6}$$

Explicitly, we have

$$u_s(\mathbf{p})=\frac{1}{\sqrt{2E(E+A)}}\begin{bmatrix}(E+A)\xi_s\\B\boldsymbol{\tau}\cdot\hat{\mathbf{p}}\xi_s\end{bmatrix},$$
$$v_s(\mathbf{p})=\frac{1}{\sqrt{2E(E+A)}}\begin{bmatrix}-B\boldsymbol{\tau}\cdot\hat{\mathbf{p}}\xi_s\\(E+A)\xi_s\end{bmatrix}, \tag{A7}$$

where the $\xi_s(\mathbf{p})$ are two independent normalized spinors and the τ_i represent the Pauli matrices. If we choose the $\xi_s(\mathbf{p})$ as eigenvectors of $\boldsymbol{\tau}\cdot\hat{\mathbf{p}}$, the $u_s(\mathbf{p})$ and $v_s(\mathbf{p})$ will also be eigenvectors of the helicity $\boldsymbol{\sigma}\cdot\hat{\mathbf{p}}$. In this case, we have

$$\xi_+(\mathbf{p})=\frac{1}{\sqrt{2(1+\hat{\mathbf{p}}_3)}}\begin{bmatrix}1+\hat{\mathbf{p}}_3\\\hat{\mathbf{p}}_+i\hat{\mathbf{p}}_2\end{bmatrix},$$
$$\xi_-(\mathbf{p})=\frac{1}{\sqrt{2(1+\hat{p}_3)}}\begin{bmatrix}-\hat{\mathbf{p}}_1+i\hat{\mathbf{p}}_2\\1+\hat{\mathbf{p}}_3\end{bmatrix}. \tag{A8}$$

Alternatively, we can take the $\xi_s(\mathbf{p})$ as eigenstates of the third component of the spin ($S_i=\tau_i/2$). In this case, however, the $u_s(\mathbf{p})$ and $v_s(\mathbf{p})$ will not be spin eigenstates (except in the nonrelativistic limit):

$$\xi_+(\mathbf{p})=\begin{bmatrix}1\\0\end{bmatrix},\quad \xi_-(\mathbf{p})=\begin{bmatrix}0\\1\end{bmatrix}. \tag{A9}$$

In both cases the $\xi_s(\mathbf{p})$ are chosen to satisfy $\xi_s^\dagger(\mathbf{p})\xi_{s'}(\mathbf{p})=\delta_{ss'}$ and the u,v's are therefore normalized in the following way:

$$u_s^\dagger(\mathbf{p})u_{s'}(\mathbf{p})=\delta_{ss'},\quad v_s^\dagger(\mathbf{p})v_{s'}(\mathbf{p})=\delta_{ss'},\quad u_s^\dagger(\mathbf{p})v_{s'}(\mathbf{p})=0\ . \tag{A10}$$

We can use these four spinors as a basis to expand the fermionic field. We then have

$$\psi(\mathbf{x})=\int \frac{d^3\mathbf{k}}{(2\pi)^{3/2}} e^{i\mathbf{k}\cdot\mathbf{x}} \sum_s [b_s(\mathbf{k})u_s(\mathbf{k})+d_s^\dagger(-\mathbf{k})v_s(\mathbf{k})] \ . \tag{A11}$$

The $b_s^\dagger(\mathbf{k})$ $[d_s^\dagger(\mathbf{k})]$ are then creation operators for a particle [antiparticle] of momentum $\mathbf{k}$ and helicity s. It is also possible to define projection operators Λ_+ (Λ_-) on particle (antiparticle) states as

$$\Lambda_+(\mathbf{k})=\sum_s u_s(\mathbf{k})u_s^\dagger(\mathbf{k}), \quad \Lambda_-(\mathbf{k})=\sum_s v_s(\mathbf{k})v_s^\dagger(\mathbf{k}) \tag{A12}$$

for these we obtain the explicit expressions

$$\Lambda_\pm(\mathbf{k})=\frac{E(k)\pm H(\mathbf{k})}{2E(k)}=\frac{E(k)\pm B(k)\boldsymbol{\alpha}\cdot\hat{\mathbf{k}}\pm A(k)\beta}{2E(k)}$$
$$=\frac{1}{2}[1\pm\cos\varphi(k)\boldsymbol{\alpha}\cdot\hat{\mathbf{k}}\pm\sin\varphi(k)\beta] \ , \tag{A13}$$

where we have defined the function $\varphi(p)$ through

$$\frac{A}{E}=\sin\varphi \ , \frac{B}{E}=\cos\varphi \tag{A14}$$

in accordance with (A6).

3. Parity, charge conjugation, and angular momentum

In second quantization, the parity ($\mathcal{P}$) and charge-conjugation ($\mathcal{C}$) operators are defined by

$$\mathcal{P}\psi(\mathbf{x},t)\mathcal{P}^\dagger=\eta_P\gamma^0\psi(-\mathbf{x},t) \tag{A15}$$

and

$$\mathcal{C}\psi(\mathbf{x},t)\mathcal{C}^\dagger=\eta_C\gamma^2[\psi^\dagger(\mathbf{x},t)]^T \ , \tag{A16}$$

where η_P and η_C are arbitrary phases.

In the same context, the generator of rotation is given by

$$\mathbf{J}=\int d^3\mathbf{r}\,\psi^\dagger(\mathbf{r})(\mathbf{r}\times\mathbf{p}+\tfrac{1}{2}\boldsymbol{\sigma})\psi(\mathbf{r}) \ . \tag{A17}$$

APPENDIX B: IDENTIFICATION OF THE QUANTUM NUMBERS OF THE BOUND STATE

In general, q-$\bar{q}$ bound states are characterized by the quantum numbers J^P. However, if the quark and the antiquark are of the same type charge conjugation is also a good quantum number and the bound state is denoted by J^{PC}. We give in this appendix the necessary formulas to establish the correspondence between the expression of the wave function and the quantum numbers of the associated bound state. We will analyze in turn parity, charge conjugation, and angular momentum.

1. Parity and charge conjugation

Making use of the action of the parity operator $\mathcal{P}$ on Fermi fields (A15) it is easy to derive

$$P_B\chi(\mathbf{k})=\gamma^0\chi(-\mathbf{k})\gamma^0 \ , \tag{B1}$$

where P_B is the parity of the bound state (the parity of the vacuum is conventionally taken to be $+1$).

Analogously, for the charge conjugation $\mathcal{C}$ we obtain, from (A16),

$$C_B\chi(\mathbf{k})=\gamma^2\chi^T(-\mathbf{k})\gamma^2 \ . \tag{B2}$$

Using (4.9) and (4.10) we then get explicitly

$$\begin{aligned} &P_BL_1(\mathbf{k})=-L_1(-\mathbf{k}), \quad C_BL_1(\mathbf{k})=L_1(-\mathbf{k}) \ , \\ &P_BL_2(\mathbf{k})=-L_2(-\mathbf{k}), \quad C_BL_2(\mathbf{k})=L_2(-\mathbf{k}) \ , \\ &P_B\mathbf{N}_1(\mathbf{k})=-\mathbf{N}_1(-\mathbf{k}), \quad C_B\mathbf{N}_1(\mathbf{k})=-\mathbf{N}_1(-\mathbf{k}) \ , \\ &P_B\mathbf{N}_2(\mathbf{k})=-\mathbf{N}_2(-\mathbf{k}) \ , \quad C_B\mathbf{N}_2(\mathbf{k})=-\mathbf{N}_2(-\mathbf{k}) \ . \end{aligned} \tag{B3}$$

2. Angular momentum

Under an infinitesimal rotation in the space of states the BS amplitude χ is modified by

$$\delta\chi=\langle 0|[\psi\psi^\dagger,\mathbf{J}]|\beta\rangle \ , \tag{B4}$$

where $\mathbf{J}$ has been defined in (A17). Computing the commutator in (B4) we find that the action of $\mathbf{J}$ on $\chi(\mathbf{x})$ reduces to

$$\mathbf{J}\chi(\mathbf{x})=\mathbf{x}\times(-i\nabla)]\chi(\mathbf{x})+\tfrac{1}{2}[\boldsymbol{\sigma},\chi(\mathbf{x})] \ , \tag{B5}$$

which can also be expressed as $\mathbf{J}=\mathbf{L}+\mathbf{S}$, where $\mathbf{L}$ is the usual orbital angular momentum operator and where the spin operator $\mathbf{S}$ is defined by

$$\mathbf{S}\chi=\tfrac{1}{2}[\boldsymbol{\sigma},\chi] \ . \tag{B6}$$

Consequently, the eigenstates of $\mathbf{J}^2$ and J_Z can be obtained by working out the L-S coupling. The eigenstates of $\mathbf{L}^2$ and L_Z are the spherical harmonics $Y_{Lm}(\theta,\varphi)$, whereas those of $\mathbf{S}^2$ and S_z are easily obtained from (B6):

$$\mathbf{S}_2=0 \quad S_Z=0 \quad 1 \quad \rho_i \tag{B7}$$

$$\mathbf{S}^2=1\begin{cases} S_Z=+1 & -\frac{1}{\sqrt{2}}(\sigma_1+i\sigma_2) & -\frac{1}{\sqrt{2}}(\sigma_1+i\sigma_2)\rho_i \\ S_Z=0 & \sigma_3 & \sigma_3\rho_i \\ S_Z=-1 & \frac{1}{\sqrt{2}}(\sigma_1-i\sigma_2) & \frac{1}{\sqrt{2}}(\sigma_1-i\sigma_2)\rho_i \ . \end{cases} \tag{B8}$$

$\mathbf{S}^2=0$ or 1 evidently correspond to the two possible couplings for a quark and an antiquark of spin $\frac{1}{2}$.

We introduce the notation

$$\begin{aligned} &\rho_\mu=(1,\rho_i) \ , \\ &\sigma_+=-\frac{1}{\sqrt{2}}(\sigma_1+i\sigma_2), \quad \sigma_0=\sigma_3, \quad \sigma_-=\frac{1}{\sqrt{2}}(\sigma_1-i\sigma_2) \ . \end{aligned} \tag{B9}$$

We find the following generic wave functions for states of given J and M:

$$\mathbf{S}^2=0, \quad \chi_{JM}(\mathbf{k})=R_J(k)Y_{JM}(\mathbf{k})\rho_\mu \ , \tag{B10}$$

$$\mathbf{S}^2=1, \quad \chi_{JLM}(\mathbf{k})=R_{JL}(k)\sum_{m,q}(L,m;1,q|J,M) \times Y_{Lm}(\hat{\mathbf{k}})\sigma_{-q}\rho_\mu \ , \tag{B11}$$

where L can take the values $J-1, J$ or $J+1$. The components with $\mathbf{S}^2=1$ can also be written in terms of the vector spherical harmonics defined by

$$\mathbf{Y}_{JLM}(\hat{\mathbf{k}})=\sum_{m,q}(L,m;1,-q|J,M)Y_{Lm}(\hat{\mathbf{k}})\mathbf{e}_{-q} , \quad \text{(B12)}$$

where the $\mathbf{e}_{-q}$ represent the spherical basis of vectors:

$$\mathbf{e}_{+}=-\frac{1}{\sqrt{2}}(\mathbf{e}_1+i\mathbf{e}_2), \quad \mathbf{e}_0=\mathbf{e}_3, \quad \mathbf{e}_{-}=\frac{1}{\sqrt{2}}(\mathbf{e}_1-i\mathbf{e}_2) . \quad \text{(B13)}$$

We then have

$$\mathbf{S}^2=1, \quad \chi_{JLM}(\hat{\mathbf{k}})=R_{JL}(k)\mathbf{Y}_{JLM}(\hat{\mathbf{k}})\cdot\sigma\rho_{\mu} . \quad \text{(B14)}$$

Therefore, for a state of given J and M, the components $L_1, L_2, \mathbf{N}_1, \mathbf{N}_2$ of the BS amplitude will be written as $(i=1,2)$

$$L_i(\mathbf{k})=L_i(k)Y_{JM}(\hat{\mathbf{k}}) \quad \text{(B15)}$$

and

$$\mathbf{N}_i(\mathbf{k})=N_{i-}(k)\mathbf{Y}_{J(J-1)M}(\hat{\mathbf{k}})+N_{i0}(k)\mathbf{Y}_{JJM}(\hat{\mathbf{k}}) +N_{i+}(k)\mathbf{Y}_{J(J+1)M}(\hat{\mathbf{k}}) . \quad \text{(B16)}$$

We can finally summarize our data concerning the J^{PC} quantum numbers in the following table where each component is associated with the quantum numbers of the state that it represents:

$$\begin{array}{llll} P=(-1)^{J+1} & C=(-1)^{J} & L_1,L_2 & {}^1J_J \\ P=(-1)^{J+1} & C=(-1)^{J+1} & N_{10},N_{20} & {}^3J_J \\ P=(-1)^{J} & C=(-1)^{J} & N_{1+},N_{1-},N_{2+},N_{2-} & {}^3(J\pm1)_J \end{array} \quad \text{(B17)}$$

In the last column, we have also made the link with the usual spectroscopic notations.

APPENDIX C: REDUCTION OF THE BS EQUATION TO A SYSTEM OF EQUATIONS FOR THE RADIAL WAVE FUNCTIONS

We give in this appendix the essential of the formulas necessary to carry out the reduction of the BS equation to a system of equations for the radial wave functions following the method presented in the text. We first define some notation. The components of the BS wave function will be written as

$$L(\mathbf{k})=L(k)Y_{JM}(\hat{\mathbf{k}}) , \quad \text{(C1)}$$

$$\mathbf{N}(\mathbf{k})=N_{-}(k)\mathbf{Y}_{-}(\hat{\mathbf{k}})+N_0(k)\mathbf{Y}_0(\hat{\mathbf{k}})+N_{+}(k)\mathbf{Y}_{+}(\hat{\mathbf{k}}) , \quad \text{(C2)}$$

where it is understood that we are interested in a state of angular momentum quantum numbers JM and where the indices $-,0,+$ correspond, respectively, to components with $L=J-1, J$ and $J+1$. We also introduce the functions $n_{(+)}$ and $n_{(-)}$ defined by

$$\begin{bmatrix} n_{(+)} \\ n_{(-)} \end{bmatrix}=\begin{bmatrix} \mu & \nu \\ -\nu & \mu \end{bmatrix}\begin{bmatrix} N_{+} \\ N_{-} \end{bmatrix} \quad \text{(C3)}$$

with

$$\mu=\left[\frac{J}{2J+1}\right]^{1/2}, \quad \nu=\left[\frac{J+1}{2J+1}\right]^{1/2} . \quad \text{(C4)}$$

We then obtain using the properties of the spherical and vector spherical harmonics:

$$\hat{\mathbf{k}}L(\mathbf{k})=L(k)[\mu\mathbf{Y}_{-}(\hat{\mathbf{k}})-\nu\mathbf{Y}_{+}(\hat{\mathbf{k}})] , \quad \text{(C5)}$$

$$\hat{\mathbf{k}}\cdot\mathbf{N}(\mathbf{k})=n_{(-)}(k)Y_{JM}(\hat{\mathbf{k}}) , \quad \text{(C6)}$$

$$\hat{\mathbf{k}}[\hat{\mathbf{k}}\cdot\mathbf{N}(\mathbf{k})]=n_{(-)}(k)[\mu\mathbf{Y}_{-}(\hat{\mathbf{k}})-\nu\mathbf{Y}_{+}(\hat{\mathbf{k}})] , \quad \text{(C7)}$$

$$\hat{\mathbf{k}}\times\mathbf{N}(\mathbf{k})=-in_{(+)}(k)\mathbf{Y}_0(\hat{\mathbf{k}}) -iN_0(k)[\mu\mathbf{Y}_{+}(k)+\nu\mathbf{Y}_{-}(\hat{\mathbf{k}})] , \quad \text{(C8)}$$

$$\hat{\mathbf{k}}\times[\hat{\mathbf{k}}\times\mathbf{N}(\mathbf{k})]=-n_{(+)}(k)[\nu\mathbf{Y}_{-}(\hat{\mathbf{k}})+\mu\mathbf{Y}_{+}(\hat{\mathbf{k}})] -N_0(k)\mathbf{Y}_0(\hat{\mathbf{k}}) . \quad \text{(C9)}$$

Those formulas allows us to rewrite the relation between the components $\mathcal{L}_{\mu}, \mathcal{N}_{\mu}$ and the functions $L_i, \mathbf{N}_i$ $(i=1,2)$ in terms of radial wave functions. Indeed, using (C5)–(C9) in (4.10), we obtain

$$\begin{aligned} \mathcal{L}_0&=\sin\theta n_{2(-)}Y_{JM} , \\ \mathcal{L}_1&=-\sin\varphi L_1 Y_{JM} , \\ \mathcal{L}_2&=i\cos\theta L_2 Y_{JM} , \\ \mathcal{L}_3&=-\cos\varphi n_{1(-)}Y_{JM} , \\ \mathcal{N}_0&=\cos\varphi n_{2(+)}\mathbf{Y}_0+(\nu\cos\varphi N_{20}+\mu\sin\theta L_2)\mathbf{Y}_{-} +(\mu\cos\varphi N_{20}-\nu\sin\theta L_2)\mathbf{Y}_{+} , \\ \mathcal{N}_1&=\cos\theta N_{10}\mathbf{Y}_0+(\mu\sin\varphi n_{1(-)}+\nu\cos\theta n_{1(+)})\mathbf{Y}_{-} +(-\nu\sin\varphi n_{1(-)}+\mu\cos\theta n_{1(+)})\mathbf{Y}_{+} , \\ \mathcal{N}_2&=i[\sin\varphi N_{20}\mathbf{Y}_0+(\nu\sin\varphi n_{2(+)}+\mu\cos\theta n_{2(-)})\mathbf{Y}_{-} +(\mu\sin\varphi n_{2(+)}-\nu\cos\theta n_{2(-)})\mathbf{Y}_{+}] , \\ \mathcal{N}_3&=-\sin\theta n_{1(+)}\mathbf{Y}_0+(\mu\cos\varphi L_1-\nu\sin\theta N_{10})\mathbf{Y}_{-} +(-\nu\cos\varphi L_1-\mu\sin\theta N_{10})\mathbf{Y}_{+} . \end{aligned} \quad \text{(C10)}$$

In order to transform (2.17) into an expression involving only radial wave functions we have to compute several integrations over the angles. We list below all the integrals needed to deal with the various interactions considered in Appendix D:

$$\int d\Omega\, Y^{*}_{JM}(\hat{\mathbf{k}})Y_{J'M'}(\hat{\mathbf{k}})=\delta_{JJ'}\delta_{MM'} , \quad \text{(C11)}$$

$$\int d\Omega\, \mathbf{Y}^{*}_{JLM}(\hat{\mathbf{k}})\cdot\mathbf{Y}_{J'L'M'}(\hat{\mathbf{k}})=\delta_{JJ'}\delta_{LL'}\delta_{MM'} , \quad \text{(C12)}$$

$$\int d\Omega\int d\Omega' Y^{*}_{JM}V(q)Y_{J'M'}(\hat{\mathbf{k}}')=\delta_{JJ'}\delta_{MM'}(2\pi)V_J(k,k') , \quad \text{(C13)}$$

$$\int d\Omega\int d\Omega'\mathbf{Y}^{*}_{JLM}(\hat{\mathbf{k}})V(q)\cdot\mathbf{Y}_{J'L'M'}(\hat{\mathbf{k}}') =\delta_{JJ'}\delta_{LL'}\delta_{MM'}(2\pi)V_L(k,k') , \quad \text{(C14)}$$

$$\int d\Omega \int d\Omega' Y^*_{JLM}(\hat{\mathbf{k}})\cdot\hat{\mathbf{q}}V(q)\hat{\mathbf{q}}\cdot\mathbf{Y}_{J'L'M'}(\hat{\mathbf{k}}')$$
$$=-\delta_{JJ'}\delta_{LL'}\delta_{MM'}(2\pi)V_L^{(1)}(k,k')$$
$$+\delta_{JJ'}\delta_{MM'}(\mu\delta_{L,J-1}+\nu\delta_{L,J+1})(2\pi)$$
$$\times V^{(2)}_{L,L'}(\mu\delta_{L',J-1}+\nu\delta_{L',J+1}) , \quad \text{(C15)}$$

where in the last integral $\mathbf{q}$ stands for $\mathbf{k}$-$\mathbf{k}'$. The quantities V_L and $V_{L,L'}$ are defined by

$$V_L(k,k')=8\pi\int_0^\infty dr\, r^2V(r)j_L(kr)j_L(k'r)$$
$$=\int_{-1}^{+1}dx\, P_L(x)V(|\mathbf{k}-\mathbf{k}'|) , \quad \text{(C16)}$$
$$V_{L,L'}(k,k')=8\pi\int_0^\infty dr\, r^2V(r)j_L(kr)j_{L'}(k'r) . \quad \text{(C17)}$$

The integrations involving the potential (C13)–(C15), were computed by using the Fourier transform of V as follows:

$$\mathbf{V}(\mathbf{k})=\int d^3\mathbf{r}\,\mathbf{V}(\mathbf{r})e^{i\mathbf{k}\cdot\mathbf{r}} . \quad \text{(C18)}$$

Expanding $e^{i\mathbf{k}\cdot x}$ and $e^{i\mathbf{k}'\cdot\mathbf{x}}$ in terms of spherical harmonics then allowed the integrals over Ω and Ω' to be worked out. The last integral (C15) is a little bit more involved and requires that we introduce a new potential $\tilde{V}$ such that $V(q)=q^2\tilde{V}(q)$. We then have

$$\hat{\mathbf{q}}V(q)\hat{\mathbf{q}}=\mathbf{q}\tilde{V}(q)\mathbf{q}=-\int d^3\mathbf{x}\,e^{-i\mathbf{q}\cdot\mathbf{x}}\vec{\nabla}\tilde{V}(\mathbf{x})\overleftarrow{\nabla} . \quad \text{(C19)}$$

Computing the right-hand side still gives rise to two new potentials $V^{(1)}$ and $V^{(2)}$:

$$\partial_i\partial_j\tilde{V}=\delta_{ij}V^{(1)}-\hat{\mathbf{r}}_i\hat{\mathbf{r}}_jV^{(2)} , \quad \text{(C20)}$$

where

$$V^{(1)}=\frac{1}{r}\frac{d\tilde{V}}{dr}=-\frac{1}{r^3}\int_0^r dx\, x^2V(x) , \quad \text{(C21)}$$
$$V^{(2)}=\frac{1}{r}\frac{d\tilde{V}}{dr}-\frac{d^2\tilde{V}}{dr^2}=V-\frac{3}{r^3}\int_0^r dx\, x^2V(x) . \quad \text{(C22)}$$

A particularly interesting example is the one of powerlike potentials. In this case, we have

$$V=r^\alpha,\quad V^{(1)}=-\frac{1}{\alpha+3}V,\quad V^{(2)}=\frac{\alpha}{\alpha+3}V . \quad \text{(C23)}$$

As another example, we consider the case where $V(q)=q^2(dV/dq^2)$, which has been used in [11] and for which we have

$$V=-\frac{3}{2}U-\frac{1}{2}r\frac{dU}{dr},\quad V^{(1)}=\frac{1}{2}U ,\quad V^{(2)}=-\frac{1}{2}r\frac{dU}{dr} . \quad \text{(C24)}$$

APPENDIX D: EXPRESSION OF THE POTENTIAL-ENERGY TERM

As already mentioned in the text, the expression of the potential-energy term in (2.17) depends on the choice of the matrices Γ_1 and Γ_2 so that each case must be treated separately. We give below, for several phenomenologically interesting cases, the expansion of $\frac{1}{4}\mathrm{Tr}(\chi^\dagger\Gamma_1\chi'\Gamma_2)$:

$$\tfrac{1}{4}\mathrm{Tr}(\chi^+\chi')=\mathcal{L}_0^*\mathcal{L}_0'+\mathcal{L}_1^*\mathcal{L}_1'+\mathcal{L}_2^*\mathcal{L}_2'+\mathcal{L}_3^*\mathcal{L}_3'+\mathcal{N}_0^*\cdot\mathcal{N}_0'+\mathcal{N}_1^*\cdot\mathcal{N}_1'+\mathcal{N}_2^*\cdot\mathcal{N}_2'+\mathcal{N}_3^*\cdot\mathcal{N}_3' , \quad \text{(D1)}$$
$$\tfrac{1}{4}\mathrm{Tr}(\chi^\dagger\gamma_0\chi'\gamma_0)=\mathcal{L}_0^*\mathcal{L}_0'-\mathcal{L}_1^*\mathcal{L}_1'-\mathcal{L}_2^*\mathcal{L}_2'+\mathcal{L}_3^*\mathcal{L}_3'+\mathcal{N}_0^*\cdot\mathcal{N}_0'-\mathcal{N}_1^*\cdot\mathcal{N}_1'-\mathcal{N}_2^*\cdot\mathcal{N}_2'+\mathcal{N}_3^*\cdot\mathcal{N}_3' , \quad \text{(D2)}$$
$$\delta_{ij}\tfrac{1}{4}\mathrm{Tr}(\chi^\dagger\alpha_i\chi'\alpha_j)=3(\mathcal{L}_0^*\mathcal{L}_0'+\mathcal{L}_1^*\mathcal{L}_1'-\mathcal{L}_2^*\mathcal{L}_2'-\mathcal{L}_3^*-\mathcal{L}_3')+(-\mathcal{N}_0^*\cdot\mathcal{N}_0'-\mathcal{N}_1^*\cdot\mathcal{N}_1'+\mathcal{N}_2^*\cdot\mathcal{N}_2'+\mathcal{N}_3^*\cdot\mathcal{N}_3') , \quad \text{(D3)}$$
$$\frac{q_iq_j}{q^2}\frac{1}{4}\mathrm{Tr}(\chi^\dagger\alpha_i\chi'\alpha_j)=\mathcal{L}_0^*\mathcal{L}_0'+\mathcal{L}_1^*\mathcal{L}_1'-\mathcal{L}_2^*\mathcal{L}_2'-\mathcal{L}_3^*\mathcal{L}_3'-\mathcal{N}_0^*\cdot\mathcal{N}_0'-\mathcal{N}_1^*\cdot\mathcal{N}_1'+\mathcal{N}_2^*\cdot\mathcal{N}_2'+\mathcal{N}_3^*\cdot\mathcal{N}_3'$$
$$+2[\mathcal{N}_0^*\cdot\hat{\mathbf{q}})(\hat{\mathbf{q}}\cdot\mathcal{N}_0')+(\mathcal{N}_1^*\cdot\hat{\mathbf{q}})(\hat{\mathbf{q}}\cdot\mathcal{N}_1')-(\mathcal{N}_2^*\cdot\hat{\mathbf{q}})(\hat{\mathbf{q}}\cdot\mathcal{N}_2')-(\mathcal{N}_3^*\cdot\hat{\mathbf{q}})(\hat{\mathbf{q}}\cdot\mathcal{N}_3')] , \quad \text{(D4)}$$
$$\tfrac{1}{4}\mathrm{Tr}[\chi^+(\boldsymbol{\alpha}\cdot\hat{\mathbf{k}}')\chi'(\boldsymbol{\alpha}\cdot\hat{\mathbf{k}}')]=\mathcal{L}_0^*\mathcal{L}_0'+\mathcal{L}_1^*\mathcal{L}_0'-\mathcal{L}_2^*\mathcal{L}_2'-\mathcal{L}_3^*\mathcal{L}_3'-\mathcal{N}_0^*\cdot\mathcal{N}_0'-\mathcal{N}_1^*\cdot\mathcal{N}_1'+\mathcal{N}_2^*\cdot\mathcal{N}_2'+\mathcal{N}_3^*\cdot\mathcal{N}_3'$$
$$+2[(\mathcal{N}_0^*\cdot\hat{\mathbf{k}}')(\hat{\mathbf{k}}'\cdot\mathcal{N}_0')+(\mathcal{N}_1^*\cdot\hat{\mathbf{k}}')(\hat{\mathbf{k}}'\cdot\mathcal{N}_1')-(\mathcal{N}_2^*\cdot\hat{\mathbf{k}}')(\hat{\mathbf{k}}'\cdot\mathcal{N}_2')-(\mathcal{N}_3^*\cdot\mathbf{k}')(\mathbf{k}'\cdot\mathcal{N}_3')] , \quad \text{(D5)}$$
$$\tfrac{1}{4}\mathrm{Tr}(\chi^+\tfrac{1}{2}[\boldsymbol{\alpha}\cdot\mathbf{k}',\chi')]=-i\mathcal{L}_2^*(\hat{\mathbf{k}}'\cdot\mathcal{N}_3')+i\mathcal{L}_3^*(\hat{\mathbf{k}}'\cdot\mathcal{N}_2')+i\mathcal{N}_0^*(\hat{\mathbf{k}}'\times\mathcal{N}_1')+i\mathcal{N}_1^*\cdot(\hat{\mathbf{k}}'\times\mathcal{N}_0')$$
$$+i\mathcal{N}_3^*\cdot\hat{\mathbf{k}}'\mathcal{L}_2'-i\mathcal{N}_2^*\cdot\hat{\mathbf{k}}'\mathcal{L}_3' . \quad \text{(D6)}$$

It should be remembered that, because we are using a wave function constructed with $\psi\psi^\dagger$ (2.2) instead of $\psi\bar{\psi}$, the Lorentz structure of the interaction will only be obtained after multiplication by γ_0. Therefore, using Γ_1 and Γ_2 in (2.17) indeed corresponds to an interaction of the type $\gamma_0\Gamma_1\otimes\gamma_0\Gamma_2$. (D1), for example, corresponds to the component 0 of a vector interaction; (D2) represents a scalar potential; (D3) and (D4) allow the treatment of transverse gluons, whereas (D5) and (D6) appear in the model of electric confinement.

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