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BRIEF REPORTS

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Trapped surfaces on a spherically symmetric initial data set

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Based on the initial-value constraints we derive an integral relation involving arbitrary spherically symmetric initial data. As a consequence of this relation an integral inequality is derived which describes the interior content of a spherical trapped surface. This inequality exhibits clearly the possible role played by the trace of the extrinsic curvature. It can enhance or suppress the presence of trapped surfaces.

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INTRODUCTION

The concept of a trapped surface, introduced in a fundamental paper by Penrose [1], plays an extremely useful role in the description of relativistic collapse. Subject to the satisfaction of some reasonable conditions which are in agreement with the classical description of matter, null geodesic incompleteness must take place once trapped surfaces form [1]. In addition, such surfaces lead to "gravitational confinement", as a recent theorem proven by Israel [2] indicates. Roughly, this new theorem states that an initially trapped surface of fixed area "will act as a wall that permanently seals off its interior content from casual influence of the environment [. . .] at least as long as it remains non singular." In view of this, it is important to understand the special state of the gravitational field that triggers the formation of trapped surfaces. Intuitively, one expects that whenever matter or gravitational waves are sufficiently compressed, a trapped surface ought to form. However, what is missing is a precise formulation (if one exists) of the term "sufficient compression." Clarification of this notion could provide considerable insight into collapsing configurations. For example, suppose one assigns on an initial slice data intending to describe the collapsing phase of a star. Recall that, in principle, one can always recognize the presence of such surfaces on the initial slice. If, for instance, the slice is placed at a moment of a time symmetry, trapped surfaces may be located once minimal surfaces are spotted. However, irrespective of whether the data are arbitrary or time symmetric again we lack necessary and sufficient conditions on the data that will guarantee existence of trapped surfaces. If such criteria can be determined, then one may appeal to Israel's confinement theorem to conclude that the resulting spacetime singularity is necessarily shielded [subject to the condition that no catastrophical explosion (such as shock waves) originating from the spacetime singularity can take place].

Early attempts to formulate appropriate criteria are described in [3] but these criteria involve characteristic initial data and are concerned with the existence of averaged trapped surfaces. An important alternative is described in recent work by Schoen and Yau [4]. This attempt involves quite sophisticated mathematical machinery, and the authors present necessary and sufficient conditions upon initial data implying existence of apparent horizons. However, the conditions are rather mathematical in nature and may not be easily adapted to physical problems. More recently a new step has been taken by Ó Murchadha and collaborators [5]. They have formulated some inequalities involving spherically symmetric initial data upon maximal slices. In turn, those inequalities act as necessary and sufficient conditions for the existence of trapped surfaces. Although their results are very important, the maximal property of the slice appears rather restrictive, particularly in view of some recent results by Witt [6] which suggest that "most spacetimes" with sources obeying the dominant-energy condition do not admit maximal slices.

The purpose of the present report is to present a derivation of an integral relation involving spherically symmetric initial data on arbitrary initial slices. Based on this relation an inequality is derived that characterizes the interior content of a trapped, spherically symmetric surface. Although one expects the energy density to be dominant and very high in the interior of a trapped surface, our inequality shows that this may not always be the case. This is so because of the role played by a nonvanishing trace of the extrinsic curvature in the initial slice.

Let us first derive this inequality. For that, let (Σ,γ,k,ρ,J) be an initial data set [7]. By definition they satisfy

$$R-k^{\alpha\beta}k_{\alpha\beta}+k^2=16\pi\rho \ , \tag{1a}$$

$$D_\alpha(k^{\alpha\beta}-\gamma^{\alpha\beta}k)=-8\pi J^\beta \ . \tag{1b}$$

R is the scalar curvature of γ_{ij}, D_α the metric-compatible covariant derivative, and $k=k^\alpha_\alpha$. The assumption of spherical symmetry requires that γ_{ij} and k_{ij} must obey [8]

$$L_{\xi_{(i)}}\gamma_{ij}=L_{\xi_{(i)}}k_{ij}=0,\quad (i)=1,2,3\ , \tag{2}$$

where $\xi_{(i)}$ are the familiar infinitesimal generators of SO(3) symmetry. Because of the existence of this symmetry, Σ can be naturally foliated by a sequence of nested two-spheres, i.e., the group orbits. Consequently, if n_α is the outward pointing unit normal of each orbit, then

$$\hat{\gamma}_{\alpha\beta}=\gamma_{\alpha\beta}-n_\alpha n_\beta\ , \tag{3}$$

$$\hat{k}_{ij}=\frac{1}{2}L_n\hat{\gamma}_{ij}=\frac{\hat{k}}{2}\hat{\gamma}_{ij},\quad \hat{k}=k_i^i \tag{4}$$

are the induced metric and second fundamental form (or extrinsic curvature) describing the embedding of the orbit within Σ. The Gauss-Codazzi equation then implies

$$R={}^{(2)}R+2R_{\alpha\beta}n^\alpha n^\beta+\hat{k}^{\alpha\beta}\hat{k}_{\alpha\beta}-\hat{k}^2 \tag{5}$$

while for later use note that the Gauss-Bonnet theorem implies

$$\int {}^{(2)}R\, da=8\pi\ . \tag{6}$$

Integrating (5) from the center of symmetry up to an orbit labeled by r_0 and using (6) we obtain

$$\int R\sqrt{\gamma}\,dV=8\pi L+\int(\hat{k}_{\alpha\beta}\hat{k}^{\alpha\beta}-\hat{k}^2+2R_{\alpha\beta}n^\alpha n^\beta)\sqrt{\gamma}\,dV. \tag{7}$$

where L is the proper distance of the orbit r_0 from the center of symmetry. In view of (7), performing the same integration on the Hamiltonian constraint (1a) leads us to

$$16\pi\int\rho\sqrt{\gamma}\,d^3V =8\pi L+\int(2R_{\alpha\beta}n^\alpha n^\beta+\hat{k}^{\alpha\beta}\hat{k}_{\alpha\beta} -\hat{k}^2-k^{\alpha\beta}k_{\alpha\beta}+k^2)\sqrt{\gamma}\,d^3V\ . \tag{8}$$

It is helpful to perform the subsequent analysis in geodesic coordinates. Thus, without loss of generality, the intrinsic metric on Σ can be written

$$ds^2=dr^2+B(r)(d\theta^2+\sin^2\theta d\phi^2) \tag{9}$$

while the field k_{ij} can be taken as follows:

$$k^\alpha_\beta=L^\alpha_\beta+\frac{1}{3}\delta^\alpha_\beta k =f(r)\delta^\alpha_r\delta^r_\alpha-\frac{f(r)}{2}(\delta^\alpha_\theta\delta^\theta_\beta+\delta^\alpha_\phi\delta^\phi_\beta)+\frac{1}{3}\delta^\alpha_\beta g(r)\ . \tag{10}$$

Note also that in this coordinate gauge

$$R_{\alpha\beta}n^\alpha n^\beta=\frac{1}{2B^2}\left[\left[\frac{dB}{dr}\right]^2-2B\frac{d^2B}{dr^2}\right]\ . \tag{11}$$

Utilizing (9) and substituting (10) into the momentum constraint (1b) leads to the following expression for the momentum density:

$$8\pi J_\beta=\left[\frac{d}{dr}\left[\frac{2}{3}g-f\right]-\frac{3}{2}\frac{f}{B}\frac{dB}{dr}\right]\delta^r_\beta \tag{12}$$

while in the above coordinate gauge (8) takes the form

$$16\pi\int\rho\sqrt{\gamma}\,dV=8\pi L-8\pi\frac{dB}{dr}\bigg|_0^{r_0} +4\pi\int_0^{r_0}\frac{1}{2B}\left[\left[\frac{dB}{dr}\right]^2-3f^2B^2 +\frac{4}{3}g^2B^2\right]dr\ . \tag{13}$$

An analogous integral relation can also be derived utilizing the momentum constraint (1b). To do so, we first form $J_\beta n^\beta$ and then integrate as done previously. After a trivial integration by parts we arrive at

$$16\pi\int J_\beta n^\beta\sqrt{\gamma}\,dV=8\pi\left[\frac{2}{3}g-f\right]B\bigg|_0^{r_0} -8\pi\int_0^{r_0}\left[\left[\frac{2}{3}g-f\right]\frac{dB}{dr} +\frac{3}{2}f\frac{dB}{dr}\right]dr\ . \tag{14}$$

Combining (13) and (14) we get

$$16\pi\int(\rho-J_\beta n^\beta)\sqrt{\gamma}\,dV=8\pi L-8\pi B\left[\frac{1}{B}\frac{dB}{dr}+\frac{2}{3}g-f\right]\bigg|_0^{r_0}+2\pi\int_0^{r_0}B\left[\frac{1}{B}\frac{dB}{dr}-f+\frac{2}{3}g\right]\left[\frac{1}{B}\frac{dB}{dr}+3f+2g\right]dr\ . \tag{15}$$

To extract the consequences of this equation, let us imagine that the data have evolved into the spacetime (M,g). By construction, Σ is an embedded spacelike hypersurface in (M,g). Denote by $\partial/\partial t$ its future-pointing timelike unit vector.

If $A=4\pi B$ stands for the proper area of an SO(3) orbit in Σ, then the Lie derivative of A along $\partial/\partial t$ is given by $L_{\partial/\partial t}A=wA$, where

$$w=\hat{\gamma}^{\alpha\beta}k_{\alpha\beta}=(\gamma^{\alpha\beta}-n^\alpha n^\beta)k_{\alpha\beta}=-f+\tfrac{2}{3}g \tag{16}$$

is the trace of the second fundamental of the orbit as embedded in the spacetime. Combined with $L_{\partial/\partial n}A=\hat{k}A$ we obtain

$$(L_{\partial/\partial t}+L_{\partial/\partial n})A=(\hat{k}+w)A =\left[\frac{1}{B}\frac{dB}{dr}-f+\frac{2}{3}g\right]A\ , \tag{17}$$

$$(L_{\partial/\partial t}-L_{\partial/\partial n})A=(-\hat{k}+w)A = \left[-\frac{1}{B}\frac{dB}{dr}-f+\frac{2}{3}g\right]A \ . \qquad (18)$$

On the other hand $l=\partial/\partial t+\partial/\partial n$ and $n=\frac{1}{2}(\partial/\partial t-\partial/\partial n)$ obey

$$l^\alpha l_\alpha = n^\alpha n_\alpha = 0, \quad l^\alpha n_\alpha = -1 \qquad (19)$$

and stand for the future-pointing outgoing and ingoing null vector fields tangent to the null geodesic orthogonal to S, while $\hat{k}+w$, $-\hat{k}+w$ represent the expansion, contraction of the light rays emitted orthogonally from S. In light of these remarks, (15) can be written as follows:

$$16\pi\int\left[\rho-J_\alpha n^\beta-\frac{k}{8\pi}(\hat{k}+w)\right]\sqrt{\gamma}\,dV=8\pi L-2A(\hat{k}+w)\Big|_0^r + \frac{1}{2}\int(\hat{k}+w)(\hat{k}-w)\sqrt{\gamma}dV+\int(k_{\alpha\beta}n^\alpha n^\beta-k)(\hat{k}+w)\sqrt{\gamma}dV \ . \qquad (20)$$

This is our final expression. It is a fully covariant integral identity derived using the constraint equations and involves only initial data. However, our considerations so far have been quite general. To extract information from the integral relation (20), let us restrict our considerations to a collapsing spherical star. In this case the free functions f, g, and B must be chosen to make ρ different than zero within the star interior. Collapse to the future of Σ will be implemented by the choice $w(r)<0$ at least for the part of Σ occupied by matter [9]. On the other hand, regularity of the three-geometry in the vicinity of $r=0$ requires $\hat{k}(r)>0$ [10]. Let $r=r_0$ be the first root of the equation $\hat{k}(r)+w(r)=0$. By construction, the outgoing (and respectively ingoing) beam of light emitted orthogonally from spherical orbits within $r=r_0$ initially propagates outwards (and respectively inwards). The $r=r_0$ orbit is special as it marks the first trapped surface (actually marginally trapped). Furthermore, any interior orbit obeys $(\hat{k}+w)(\hat{k}-\omega)>0$ and $(k^{\alpha\beta}n_\alpha n_\beta-k)=-w>0$; thus the integral relation (20) implies

$$16\pi\int\left[\rho-J_\beta n^\beta-\frac{k}{8\pi}(\hat{k}+w)\right]\sqrt{\gamma}\,dV =8\pi L+\frac{1}{2}\int(\hat{k}+w)(\hat{k}-w)\sqrt{\gamma}\,dV +\int(k^{\alpha\beta}n_\alpha n_\beta-k)(\hat{k}+w)\sqrt{\gamma}\,dV \ . \qquad (21)$$

Because of the positive-definite character of the last two terms we therefore infer

$$\int\left[\rho-J_\beta n^\beta-\frac{k}{8\pi}(\hat{k}+w)\right]\sqrt{\gamma}dV>\frac{L}{2} \ , \qquad (22)$$

i.e., an integral inequality valid for the interior of the first trapped surface. The right-hand side expresses the proper distance of the trapped surface from the center of the star, and in a sense it is a measure of the size of the trapped surface. The left-hand side is more interesting. It provides us with valuable insight into the trapped interior region. Although one expects that within $r\le r_0$ the energy density ρ must be very high and in some sense dominant, inequality (22) implies that this may not always be so. In fact, it is entirely consistent that the $r\le r_0$ region be dominated by the presence of high negative extrinsic curvature while the matter data play a secondary role. The situation is dramatically opposite at the other extreme, i.e., in cases where data are placed on slices characterized by positive, definite extrinsic curvature. In this instance, if in Σ there exists a trapped surface, then according to (22) the energy density and possible inward matter flow must be very high and dominant.

We may point out that one may derive more inequalities as consequences of (20). For example, by inspection of (20) one may easily verify that

$$\int\left[\rho-J_\beta n^\beta-\frac{1}{16\pi}(k_{\alpha\beta}n^\alpha n^\beta+k)(\hat{k}+w)\right]\sqrt{\gamma}dV>\frac{L}{2} \qquad (23)$$

(it is understood that the volume of integration, as in (22), is again the interior of the first trapped surface). However, we feel that (22) expresses more clearly the role of k rather than the above inequality. It is also clear that neither (22) nor (23) by themselves imply the existence of trapped regions (although they are naturally implied by trapped surfaces). Let us further point out a connection between the integral relation (20) and Hawking's quasilocal mass-energy formula. Recall that some time ago, Hawking [11] proposed a quasilocal expression for the gravitational mass within every regular closed two-surface S. It is given by the following integral:

$$m(S)=\left[\frac{A}{16\pi}\right]^{1/2}\left[1-\frac{1}{2\pi}\int_S\rho\mu\,da\right] \ . \qquad (24)$$

A is the proper area of S; ρ and μ are two of the Newman-Penrose spin coefficients representing (half) of the convergence/divergence respectively of the outgoing/ingoing null geodesics orthogonal to S. Specializing the above formula to the case where S is an SO(3) orbit on the initial slice Σ, it is not difficult to prove that $m(S)$ can be written as follows:

$$m(S)=\sqrt{A/16\pi}\left[1-\frac{(\hat{k}-w)(\hat{k}+w)}{16\pi}A\right] \ , \qquad (25)$$

while with a bit more effort the above expression can be brought into a more recognizable form [12]:

$$m(S)=\sqrt{A/16\pi}\left[1-\frac{g^{\mu\nu}\nabla_\mu A\nabla_\nu A}{16\pi A}\right] \ , \qquad (26)$$

where now $g^{\mu\nu}$ is the full spacetime metric. From (25) we see that we can eliminate one of the integrands in the right-hand side of (20) in favor of Hawking's mass formula. In this form it is clearer how (20) might be generalized to situations lacking spherical symmetry. This point is under investigation and will be discussed in a future paper.

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[7] Recall that this means a three-manifold Σ, a positive-definite metric γ, a symmetric second-rank tensor k, a scalar field ρ, and a vector field J. Furthermore, it is assumed that they will be smooth (i.e., all of them are C^∞, Σ is free of singularities), and ρ and J obey the local energy condition $\rho \geq (J_\alpha J^\alpha)^{1/2}$. Perhaps it is worth pointing out that the results of the present paper have been derived without imposing this inequality.

[8] By the constraint equations ρ and J are also spherically symmetric. For a discussion of the mechanism generating spacetime symmetries from an initial-value point of view, see V. Moncrief, J. Math. Phys. **15**, 1963 (1974); B. K. Berger, *ibid.* **17**, 1269 (1976); T. Zannias, Phys. Rev. D **44**, 2397 (1991).

[9] At least one class of "Lagrangian geodesic observers" will feel that the area of SO(3) orbits is squeezed.

[10] In fact, the geometry should be locally flat in the vicinity of $r=0$; thus $\hat{k}(r) \to \infty$ as $r \to 0$.

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