# Nambu-Goldstone bosons and dynamical symmetry breaking in QCD at high temperatures

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(Received 15 August 1991)

We report on the renormalization-group study of the low-energy effective potential in QCD. We derive the dynamically induced linear $\sigma$ model (DSM), whose self-couplings are not free parameters but are determined in terms of the running coupling of the underlying QCD. In this DSM, the $\sigma$ meson is not a fundamental field, but a genuine quark-antiquark $0^+$ bound state, while the $\pi$ is the quark-antiquark $0^-$ bound state. We show the masslessness of the $\pi$ at zero temperature, so that the $\pi$ is a true Nambu-Goldstone boson coupled to zero-temperature chirality. At high temperatures, the $\pi$ remains massless. Thus the zero-temperature chirality is not restored at high temperatures. The apparent chiral symmetry operative at high temperatures is thus to be distinguished from that at $T = 0$. Comparison is made with the situation in the BCS and other theories where symmetry restoration does indeed take place.

PACS number(s): 12.38.Aw, 11.15.Ex, 11.30.Rd

## I. INTRODUCTION

Chiral symmetry is well known to be broken in QCD at zero temperature. An important signature of this symmetry breaking is the nonvanishing of the ground-state expectation value of $\bar{\psi}\psi$. Current-algebra considerations have long supported this conclusion and recent lattice calculations [1] have shown similar evidence. At nonzero temperatures, however, lattice calculations have seen a critical temperature $T_c$ above which the thermal expectation value vanishes,

$$\langle \mathrm{vac} | \bar{\psi}\, \psi | \mathrm{vac} \rangle_\beta = 0, \quad T \geq T_c, \tag{1}$$

indicating a phase transition. A popular conclusion is that chiral symmetry is restored above $T_c$.

In truth, there is the familiar theorem that follows from the equal-time commutator in the two-flavor case, say,

$$\left[ Q_5^a \, , \, i \int d^3x \; \bar{\psi}\, T^b \gamma_5 \, \psi \right] = i\, \delta^{ab} \int d^3x \; \bar{\psi}\, \psi \tag{2}$$

which shows that when $\langle \mathrm{vac} | \bar{\psi}\, \psi | \mathrm{vac} \rangle \neq 0$ the chiral charge $\mathbf{Q}_5$ *cannot* annihilate the vacuum. Here

$$Q_5^a \equiv 2 \int d^3x \; \psi^\dagger \, T^a \, \gamma_5 \, \psi \tag{3}$$

and $T^a$ is a generator matrix in flavor space. However, there is no converse theorem that the vanishing of $\langle \bar{\psi}\, \psi \rangle_\beta$ by itself *implies* that $\mathbf{Q}_5$ annihilates the vacuum at high temperature.

Indeed, a naive consequence of the chiral-restoration argument would be that the quark at high temperature remains massless. Our earlier calculations [2, 3] have however shown that even though $\langle \mathrm{vac} | \bar{\psi}\, \psi | \mathrm{vac} \rangle_\beta$ vanishes above $T_c$ where

$$T_c = \Lambda_c \, e^{2/3} \tag{4}$$

nevertheless the chiral quark with $m_r \to 0$ moves in the thermal environment as if it had a Lorentz-invariant mass $\mathcal{M}$, where

$$\mathcal{M}^2 \underset{T \to \infty}{\sim} \frac{2\pi^2}{3} \frac{T^2}{\ln \dfrac{T^2}{\Lambda_c^2}} [1 + O((\ln T^2/\Lambda_c^2)^{-2})]. \tag{5}$$

There is thus a conflict between the usual signature of chiral-symmetry breaking and the result of an actual calculation of the physical quark mass at high temperatures. In our earlier paper, we had analyzed the two-point quark Green's function at high temperatures, and shown how the usual signature $\langle \bar{\psi}\, \psi \rangle_\beta$ fails to display the chiral-symmetry breaking in the theory.

In this paper, we have turned to the other signature of chiral symmetry breaking, viz. the masslessness of the Nambu-Goldstone boson [4–6], in order to verify the continued breaking of chiral symmetry at high temperatures.

At zero temperature, the pions indeed do have a small mass, which we attribute to electroweak-symmetry breaking generating a (small) up- and down-quark mass. In the limit of zero current quark masses, the pions will become massless. Theoretically, on the lattice, it has been hard to show the pion mass is strictly zero, although no one doubts the conclusion.

We employ the renormalization-group techniques to study the Nambu-Goldstone bosons in the dynamically broken QCD through the introduction of an effective potential at zero temperature. In the process, we have also proposed a new solution to the old problem of the renormalization property of bilinear composite sources [7], and introduced the effective Lagrangian for the bound states of the quark-antiquark systems in the $0^+$ and $0^-$ states. Much like the eigenvalue conditions introduced earlier in the literature [8], the coupling parameters of this dynamically induced linear $\sigma$ model (DSM) are not free, but are totally determined in terms of the fundamental running coupling constant $(\lambda_r)$ of QCD. In this DSM, the $\pi$ is strictly massless at zero temperature.

We shall show below that the $\pi$ remains massless at high temperatures.

Because our conclusions are startling, especially in view of the experience with BCS theory, we shall end this paper with a comparison with symmetry restoration in other theories. If $M$ is the mass scale of the symmetry breaking at zero temperature, the gap equation at high temperature is typically of the one-loop form [9]

$$M_T^2 = M^2 - a\,g^2T^2, \tag{6}$$

whereupon the conclusion follows that at $T_c = M^2/(a\,g^2)$ the gap vanishes and symmetry is restored. In BCS theory, the finite (and numerically small) Debye cutoff effectively controls the loop integrals, and higher-order corrections are genuinely small, so that the conclusion of a symmetry restoration is stable against those corrections. In QCD, the higher-order contributions cannot be neglected. The parameters $M$ and $g$ are cutoff $(\mu)$ dependent, so that the series on the right-hand side of Eq. (6) must be summed in order to be independent of the cutoff $\mu$ and make sense physically. As we shall see in Sec. IV, the sum over the higher-order diagrams, including analogues of the so-called "daisy" diagrams [10] for QCD, indeed destabilizes the conclusion.

Before embarking on the study of the Nambu-Goldstone boson, we end this section with some further observations related to our earlier work [2] on the pole of the thermal quark propagator. That a Dirac fermion would move in a hot medium with a Lorentz-invariant mass was already noted [11] in the one-loop calculations in QED. Our results agree with theirs at the same one-loop level, with the appropriate substitution of $g_r^2C_f$ for $e_r^2$ in QED. We, however, go beyond one loop and by summing over higher loops convert their result to the renormalization-group-invariant form of Eq. (5). Naively, it would have been expected that a quark with intrinsic mass $m_r$ should behave in the hot environment like a plasmon, with a momentum-dependent mass. That the Dirac fermion behaves like a genuine particle was confirmed by a new QED calculation by Barton [12], who has traced the physical origin of this Lorentz invariance to the momentum independence of the forward Compton scattering as well as pair-annihilation amplitudes in the rest frame of the electron. Since at the one-loop level there is no difference between QED and QCD, his analysis carries over.

In this connection, too, we should add a note about the plasmon state of the quark that was first shown by Weldon [13] and Klimov [14]. A strictly massless electron in a hot environment can also be in a plasmon state. This plasmon state is characterized by a momentum-dependent mass,

$$\omega = \sqrt{k^2 + M_k^2}, \tag{7}$$

where

$$M_k \underset{k\to 0}{\sim} M + \frac{k}{3} - \frac{1}{6}\frac{k^2}{M}, \tag{8}$$

$$M_k \underset{k\to\infty}{\sim} \sqrt{2}M \tag{9}$$

and here $M$ is the Weldon-Klimov mass given by

$$M^2 = \frac{g_r^2}{8}T^2C_f. \tag{10}$$

As Barton [12] has shown, this plasmon state is absent for the case $m_r \neq 0$. Since our treatment of dynamical symmetry breaking starts with $m_r \neq 0$ and we study the system in the critical limit as $m_r \to 0$, our result in Eq. (5) is a genuine dynamical mass state, to be distinguished from the plasmon state of Weldon and Klimov.

Finally, we give the corrections to the physical quark mass at high temperature when the underlying QCD has a primordial $m_r \neq 0$. In this case, the behavior of the quark mass at high temperature is modified to

$$\mathcal{M}^2 \underset{T\to\infty}{\sim} \frac{2\pi^2}{3}\frac{T^2}{\ln\dfrac{T^2}{\Lambda_c^2}}\left(1 - \frac{4\alpha}{b}y_0\left(\frac{2\pi^2T^2}{3\Lambda_c^2e^{1/3}e^{2y_0/b}}\right)^{-1/(2\alpha)}\left(\ln\frac{T^2}{\Lambda_c^2}\right)^{(1-2\alpha)/(2\alpha)}\right), \tag{11}$$

where $m_r$ is given in terms of the positive renormalization-group-invariant parameter $y_0$ by

$$m_r = \Lambda_c e^{1/6}e^{y_0/b}(\lambda y_0)^{\alpha}. \tag{12}$$

Here

$$\alpha = \frac{6C_f}{b}, \tag{13}$$

where $C_f$ is a quadratic Casimir invariant defined for an SU$(N)$ group by

$$C_f = \frac{N^2-1}{2N} \tag{14}$$

and $b$ is the one-loop beta function for the gluon coupling constant, $\lambda_r \equiv g_r^2/16\pi^2$,

$$\mu\frac{d}{d\mu}\lambda_r = -b\lambda_r^2. \tag{15}$$

Note that in the intrinsically broken case, the physical quark mass at high temperature is actually *less* than the quark mass in the dynamically broken QCD. Such a signature would however be hard to see in a lattice calculation.

## II. EFFECTIVE POTENTIAL AT $T = 0$

In this section, we will describe a dimensional regularization approach to the study of the composite generating functional [7]. Let $F(j,\mathbf{j}_5,m)$ be the generating functional in the presence of external composite sources for connected perturbative graphs of the massive QCD theory. These external composite sources are coupled to the fermion bilinears in the following way:

$$\mathcal{L}_J^0 = -\bar{\psi}\psi\, j - 2i\,\bar{\psi}\mathbf{T}\gamma_5\psi\,\cdot \mathbf{j}_5. \tag{16}$$

The flavor-space generator matrices satisfy the normalization

$$\mathrm{Tr}(T^a T^b) = \tfrac{1}{2}\delta^{ab}. \tag{17}$$

For algebraic simplicity, we have chosen to work with two flavors, although the technique used can be generalized to higher numbers of flavors. The mass term that we put in by hand into the QCD Lagrangian is taken to be a scalar in flavor space, so that we do not distinguish between the up- and down-quark masses.

In the chiral-symmetry limit this QCD Lagrangian (with $m_r = 0$) has a global $\mathrm{U}(2)\times\mathrm{U}(2)$ symmetry at the tree level which is broken down by the instanton $\mathrm{U}(1)_A$ anomaly to $\mathrm{SU}(2)_L\times\mathrm{SU}(2)_R\times\mathrm{U}(1)_V$. In the presence of dynamical symmetry breaking, this symmetry is further broken down to $\mathrm{SU}(2)_V\times\mathrm{U}(1)_V$ with the pseudoscalar pion triplet serving as remnants of the chiral SU(2) symmetry.

### A. Infinities from bilinear sources

The generating functional is defined in $n = 4-\epsilon$ dimensions through the integral

$$\int \mathcal{D}\psi\,\mathcal{D}\bar{\psi}\,\mathcal{D}\mathcal{G}_\mu\; e^{i\int d^n x(\mathcal{L}_{\rm QCD}+\mathcal{L}_J^0)} \equiv e^{iF}. \tag{18}$$

The $F$ so generated will have perturbative infinities that are *not* removed by the usual modified minimal subtraction ($\overline{\rm MS}$) QCD counterterms [15] already included in $\mathcal{L}_{\rm QCD}$. These new infinities appear in connected perturbative graphs with one or more external source lines and cannot be absorbed by any wave-function or mass renormalization in QCD. The infinities arise directly from the fact that our sources are associated with bilinear fields.

At the one-loop level, we find that [16]

$$F = -\int d^n x\,\frac{N_c N_f}{8\pi^2}\,\frac{1}{\epsilon}(m^2)^{-\epsilon/2} \times\{\,[(\partial_\mu j)^2 + (\partial_\mu \mathbf{j}_5)^2] + [(j+m)^2 + (\mathbf{j}_5)^2]^2 + \text{finite terms}\,\}. \tag{19}$$

Here we have used dimensional regularization in $4-\epsilon$ dimensions, and $N_c$ denotes the number of colors and $N_f$ the number of flavors in the theory.

Because of these new infinities, the functional $F$ by itself does not make any sense for phenomenological anaylsis. *New* terms will have to be added to the tree-level source Lagrangian in order to achieve a renormalized functional.

### B. Source Lagrangian and renormalizability

These infinities require new counterterms that must be introduced into the tree-level Lagrangian. The needed counterterms, however, involve only external composite sources, as Eq. (19) so clearly shows. But the original tree-level Lagrangian, $\mathcal{L}_J^0$, did not have such source-dependent terms and we are thus in a quandary.

The solution is to introduce *ab initio* additional source-dependent terms into the tree-level Lagrangian in order to absorb these counterterms under renormalization. We write

$$\mathcal{L}_J = \mathcal{L}_J^0 + \frac{N_c N_f}{4\pi^2}\left(-\frac{\mu_\sigma^{-\epsilon}}{2A\lambda_r}[(\partial_\mu j)^2 + (\partial_\mu \mathbf{j}_5)^2] - \frac{\mu_\sigma^{-\epsilon}}{2C\lambda_r}[(j+m)^2+(\mathbf{j}_5)^2]^2\right) + \mathcal{L}_J(\text{counter}) \tag{20}$$

with

$$\mathcal{L}_J(\text{counter}) = -(Z_m Z_2 - 1)\,(\bar{\psi}\,\psi\, j + 2i\,\bar{\psi}\mathbf{T}\gamma_5\psi\cdot\mathbf{j}_5) + \frac{N_c N_f}{4\pi^2}\left(-(Z_m^2 Z_\lambda^{-1}-1)\frac{\mu_\sigma^{-\epsilon}}{2A\lambda_r}[(\partial_\mu j)^2 + (\partial_\mu \mathbf{j}_5)^2] - (Z_m^4 Z_\lambda^{-1}-1)\frac{\mu_\sigma^{-\epsilon}}{2C\lambda_r}[(j+m)^2+(\mathbf{j}_5)^2]^2\right), \tag{21}$$

where $Z_\lambda$ and $Z_m$ are the QCD renormalization constants given to one-loop renormalization-group accuracy by

$$Z_\lambda = (1+b\lambda_r/\epsilon)^{-1}, \tag{22}$$

$$Z_m = (1+b\lambda_r/\epsilon)^{-\alpha} \tag{23}$$

defined in terms of the QCD renormalization scale $\mu$ by

$$Z_\lambda\,\lambda_r\,\mu^\epsilon = \lambda_B, \tag{24}$$

$$Z_m\, m_r = m_B. \tag{25}$$

With the renormalization conditions

$$Z_m j = j_B, \quad Z_m \mathbf{j}_5 = \mathbf{j}_{5B} \tag{26}$$

the total Lagrangian, Eq. (20) with the counterterms in Eq. (21) may be rewritten in terms of the bare sources, assuring renormalization-group invariance.

In Eq. (20), we have introduced the constants $(A, C)$ as well as the renormalization scale $\mu_\sigma$. *A priori*, it need not be identical to the QCD scale $\mu$, but only that it be proportional to $\mu$.

To one-loop renormalization-group accuracy, we seek to counter the leading $1/\epsilon$ singularities. The lowest-order singularities were exhibited in Eq. (19). The higher-order leading singularities go generically as

$$\frac{\lambda_r}{\epsilon^2} + \frac{\lambda_r^2}{\epsilon^3} + \cdots . \tag{27}$$

Our claim is that this series can be summed in terms of the QCD one-loop renormalization-group-invariant constants, $Z_\lambda$ and $Z_m$, as summarized in Eq. (21). We have indeed verified that this is so by explicit two-loop calculation, provided the constants $(A, C)$ are chosen to be

$$\begin{aligned} A &= (12C_f - b), \\ C &= (24C_f - b). \end{aligned} \tag{28}$$

Renormalization-group arguments then show that this is true to all higher orders, to within one-loop renormalization-group accuracy. As far as leading $\epsilon$ singularities are concerned, it is not necessary to know the proportionality constants for the scale factor $\mu_\sigma$. It should perhaps be noted parenthetically that here a two-loop calculation of $\langle\bar{\psi}\psi\rangle$ is really an integration of the *one-loop* fermion propagator, so that it makes physical sense to identify the infinities so induced with functions of the one-loop renormalization constants.

With the $\mathcal{L}_J$ given by Eq. (20), we may now define a new generating functional, $W$, through the integral

$$\int \mathcal{D}\psi \mathcal{D}\bar{\psi} \mathcal{D}G_\mu e^{i\int d^4x(\mathcal{L}_{\rm QCD}+\mathcal{L}_J)} \equiv e^{iW} \tag{29}$$

that satisfies the renormalization relation

$$\mu\frac{d}{d\mu} W(j, \mathbf{j}_5, \lambda_r, m_r, \mu) = 0, \tag{30}$$

where the total differential operator is a short form for

$$\mu\frac{d}{d\mu} \equiv \mu\frac{\partial}{\partial\mu} - b\lambda_r^2\frac{\partial}{\partial\lambda_r} - 6\lambda_r C_f\left(m_r\frac{\partial}{\partial m_r} + j\frac{\partial}{\partial j} + \mathbf{j}_5\cdot\frac{\partial}{\partial\mathbf{j}_5}\right). \tag{31}$$

### C. Dynamical $\sigma$ model

Equation (20) is almost suggestive of a linear $\sigma$ model, but not quite. It is missing a quadratic "mass" term for the scalar and pseudoscalar sources. This fact is related to the curious but important feature of the infinities induced by the quark loops as given in Eq. (19). There are no infinities associated with the quadratic "mass" term

$$(j+m)^2 + (\mathbf{j}_5)^2. \tag{32}$$

Dimensionally, such a divergent term in the functional $F$ would have to be proportional to $m^2$ itself, which would then destroy the invariance of the Lagrangian $\mathcal{L}_{\rm QCD} + \mathcal{L}_J^0$ under the simultaneous shift $j \to j + \Delta$ and $m \to m - \Delta$.

To make the transition to the dynamically induced linear $\sigma$ model, we need such a quadratic mass term, while preserving the invariance under the simultaneous shift. This can be done by introducing a ***negative*** mass-squared term in the tree-level Lagrangian, with $\nu_r$ as the new mass scale. Rewriting our sources in terms of the $\sigma$ and $\boldsymbol{\pi}$ fields,

$$\sigma \equiv \frac{\sqrt{N_c N_f}}{2\pi}\frac{1}{12C_f - b}\frac{1}{\sqrt{\lambda_r}}\, j\, \mu_\sigma^{-\epsilon/2}, \tag{33}$$

$$\boldsymbol{\pi} \equiv \frac{\sqrt{N_c N_f}}{2\pi}\frac{1}{12C_f - b}\frac{1}{\sqrt{\lambda_r}}\, \mathbf{j}_5\, \mu_\sigma^{-\epsilon/2} \tag{34}$$

the resulting Lagrangian is of the familiar form

$$\begin{aligned} \mathcal{L}_\sigma = &-\frac{1}{2}\left[(\partial_\mu\sigma)^2 + (\partial_\mu\boldsymbol{\pi})^2\right] \\ &- h_r\,(v_0 + \sigma\mu_\sigma^{\epsilon/2})\bar{\psi}\psi - 2i\,h_r\mu_\sigma^{\epsilon/2}\boldsymbol{\pi}\cdot\bar{\psi}\mathbf{T}\gamma_5\psi \\ &- \frac{\kappa_r}{24}\,\mu_\sigma^\epsilon\left[(\sigma + \mu_\sigma^{-\epsilon/2}v_0)^2 + (\boldsymbol{\pi})^2\right]^2 \\ &+ \frac{\nu_r^2}{12}\left[(\sigma + \mu_\sigma^{-\epsilon/2}v_0)^2 + (\boldsymbol{\pi})^2\right] + \mathcal{L}_\sigma(\text{counter}) \end{aligned} \tag{35}$$

with [17]

$$\begin{aligned} \mathcal{L}_\sigma(\text{counter}) = &-\tfrac{1}{2}(Z_\sigma - 1)\left[(\partial_\mu\sigma)^2 + (\partial_\mu\boldsymbol{\pi})^2\right] - (Z_2 Z_h\sqrt{Z_\sigma} - 1)\,h_r\,\mu_\sigma^{\epsilon/2}\left[(\sigma + v_0\mu_\sigma^{-\epsilon/2}\bar{\psi}\psi) + 2i\boldsymbol{\pi}\cdot\bar{\psi}\mathbf{T}\gamma_5\psi\right] \\ &- (Z_\kappa Z_\sigma^2 - 1)\frac{\kappa_r}{24}\,\mu_\sigma^\epsilon\left[(\sigma + \mu_\sigma^{-\epsilon/2}v_0)^2 + (\boldsymbol{\pi})^2\right]^2 + (Z_{\nu^2} Z_\sigma - 1)\frac{\nu_r^2}{12}\left[(\sigma + \mu_\sigma^{-\epsilon/2}v_0)^2 + (\boldsymbol{\pi})^2\right], \end{aligned} \tag{36}$$

where

$$v_0^2 = \frac{\nu_r^2}{\kappa} \tag{37}$$

and the renormalization constants are defined in terms of the QCD constants

$$Z_\sigma = \frac{Z_m^2}{Z_\lambda}, \tag{38}$$

$$Z_h = \sqrt{Z_\lambda}, \tag{39}$$

$$Z_\kappa = Z_\lambda, \quad (40)$$
$$Z_{\nu^2} = Z_\sigma^{-1}. \quad (41)$$

$Z_2$ is the wave-function renormalization constant for the quark field. It is well known to be gauge dependent, while $Z_m$ and $Z_\lambda$ are gauge independent. For convenience, we choose to work in the Landau gauge where $Z_2 = 1$ so that we may drop it from all the quark correlation functions to be considered later.

Among the relations between the renormalization constants of the DSM and those of QCD, Eq. (41) came as a result of the absence of an infinite renormalization of the quadratic mass term. Because of these relations, the coupling constants of this *induced* dynamical $\sigma$ model are not independent but are calculable in terms of the coupling constant of QCD,

$$\frac{h_r^2}{16\pi^2} = \frac{(12C_f - b)}{4N_cN_f}\,\lambda_r, \quad (42)$$
$$\frac{\kappa_r}{16\pi^2} = 12\,\frac{(12C_f - b)^2}{(24C_f - b)}\,\frac{1}{4N_cN_f}\,\lambda_r. \quad (43)$$

The eigenvalue relations [8] given here are accurate to one-loop renormalization-group accuracy. When we go to higher renormalization-group accuracy, the eigenvalues would be modified by higher-order terms in $\lambda_r$. The relations among the renormalization constants also get modified according to

$$Z_\sigma = \frac{Z_m^2}{Z_\lambda}[1 + a_1\lambda_r(Z_\lambda - 1) + a_2\lambda_r^2(Z_\lambda - 1) + \cdots], \quad (44)$$

etc., where $Z_m$ and $Z_\lambda$ are now the appropriate renormalization constants to the same higher-order accuracy.

Note that in Eq. (37) we have implemented the tree-level stability condition for the $\sigma$ field. As is well known, this same condition results in the masslessness of $\boldsymbol{\pi}$. What remains to be shown is whether radiative corrections give rise to a mass for $\boldsymbol{\pi}$.

With the $\mathcal{L}_\sigma$ given by Eq. (35), we may now define the effective potential for the induced $\sigma$ model through the functional integral

$$\int \mathcal{D}\psi\,\mathcal{D}\bar\psi\,\mathcal{D}\mathcal{G}_\mu e^{i\int d^4x(\mathcal{L}_{\rm QCD}+\mathcal{L}_\sigma)} \equiv e^{i\Gamma}. \quad (45)$$

This effective potential is different from that for the usual linear $\sigma$ model in that the $\sigma$ and $\pi$ fields do not propagate. They are really manifestations of the $0^+$ and $0^-$ bound states of the quark-antiquark systems. In the intermediate states, it is the quarks that propagate. The situation is analogous to that of the Nambu–Jona-Lasinio (NJL) model. There, a sum of the fermion bubbles in the $0^-$ channel showed the existence of a massless pole, while the corresponding sum in the $0^+$ channel showed the existence of a massive $\sigma$ particle. Our dynamical $\sigma$ model in effect summarizes the properties of the quark-antiquark bound states.

To distinguish it from the usual linear $\sigma$ model, we shall refer to this as the dynamical $\sigma$ model (DSM).

Note that we do not have a $C\sigma$ term in the linear $\sigma$ model. We are therefore not discussing the case of an extrinsically broken QCD, with the breaking coming from the electroweak sector that is outside of QCD. Our DSM makes contact with the underlying dynamically broken QCD only in the critical limit $m_r \to 0$. We will leave the discussion of a primordial $\boldsymbol{\pi}$ mass to a future discussion.

From the point of view of DSM, the quark mass comes from the spontaneous symmetry breaking due to the negative mass-squared term, with $\nu_r$ setting the scale. $\nu_r$ satisfies the renormalization-group equation

$$\mu\frac{d}{d\mu}\nu_r = \frac{2\alpha - 1}{2}\,\lambda_r b\,\nu_r. \quad (46)$$

This equation is not asymptotically free for the case of interest $2\alpha > 1$ . Strictly speaking, this condition is not satisfied for the case of $SU(3)_c$ with two flavors. We are naturally interested in the physical case of six flavors. We believe, however, that the basic symmetry-breaking mechanisms exhibited in the two-flavor case remains true for the general case. The principal feature of the absence of the infinities in the quadratic mass term is independent of the number of flavors, so that the asymptotically nonfree nature of the equation for $\nu_r$ remains true. The only complications are the additional self-couplings involving the meson multiplet, but they are all calculable in terms of the underlying gluon coupling constant.

### D. Tadpole condition

The effective potential [Eq. (45)] has to be expanded around the quantum vacuum where $\langle \mathrm{vac}|\sigma|\mathrm{vac}\rangle$ vanishes. As is well known, this is achieved by performing an additional shift in the $\sigma$ field

$$\sigma + v_0\,\mu_\sigma^{-\epsilon/2} \to \sigma + v\,\mu_\sigma^{-\epsilon/2}, \quad (47)$$

where, perturbatively,

$$v = v_0 + v_1 + v_2 + \cdots \quad (48)$$

and requiring that the $\sigma$ tadpole contribution (including all counterterms) vanish

$$\Gamma_\sigma^{(1)} = -Z_h\sqrt{Z_\sigma}h_r\mu_\sigma^{\epsilon/2}\langle\bar\psi\,\psi\rangle - \tfrac{1}{6}Z_\kappa Z_\sigma^2\kappa\,v^3\mu_\sigma^{-\epsilon/2} + \tfrac{1}{6}\nu_r^2 v\,\mu_\sigma^{-\epsilon/2} \quad (49)$$
$$= 0. \quad (50)$$

As a result of the full shift in Eq. (48), the relation between the quark mass $m_r$ and the parameters of DSM is no longer the tree-level one as implied by the tree-level Lagrangian [Eq. (35)], but is given perturbatively by

$$m_r = h_r(v_0 + v_1 + v_2 + \cdots). \quad (51)$$

The additional shift in $\sigma$ is just the right one to bridge between the asymptotically free renormalization group equation for $m_r$

$$\mu\frac{d}{d\mu}m_r = -\alpha\lambda_r b\;m_r \quad (52)$$

and the one for $\nu_r$. Perturbatively,

$$h_r v_1 = \frac{4\alpha - 1}{4}\lambda_r b\, h_r v_0 \left(\ln\frac{m_r^2}{\mu_\sigma^2} - 1\right). \tag{53}$$

Since $\langle\bar\psi\,\psi\rangle$ is determined in terms of $m_r$, while the counterterm and vacuum shift terms are determined in terms of the parameters of DSM, this quantum requirement in Eq. (49) leads to a relation between $m_r$ and $\nu_r$.

$$m_r = \sqrt{\frac{(4\alpha - 1)}{12\,(2\alpha - 1)}}\nu_r\,(\lambda_r y_0)^\alpha\,(\lambda_r y_\nu)^{(2\alpha-1)/2} \tag{54}$$

or, conversely,

$$m_r\,(\lambda_r y_0)^{-\alpha} = \sqrt{\frac{(4\alpha - 1)}{12\,(2\alpha - 1)}}\nu_r\,(\lambda_r y_\nu)^{(2\alpha-1)/2}, \tag{55}$$

where $y_0$ and $y_\nu$ are renormalization-group invariants. Equation (55) is the key that relates the dynamical $\sigma$ model to the underlying QCD. It relates the renormalization-group invariant of DSM ($y_\nu$) to the corresponding renormalization-group invariant of QCD ($y_0$). The $y_\nu$ depends on the proportionality factor between $\mu_\sigma$ and $\mu$. We choose

$$\mu_\sigma = \mu\, e^{-1/3} \tag{56}$$

so that the perturbative expansions for $y_0$ and $y_\nu$ are identical

$$y_\nu = \frac{1}{\lambda_r} + \frac{b}{2}\left(\ln\frac{m_r^2}{\mu^2} - \frac{1}{3}\right) + O(\lambda_r^2). \tag{57}$$

With this, the critical limit of QCD ($m_r \to 0$) is very simply reflected in the converse limit in DSM of $\nu_r \to \infty$ for all $\mu > \Lambda_c$, in the case of interest where $2\alpha > 1$. This curious feature of the DSM shows that, at $T = 0$, the system is locked in the spontaneous-symmetry-breaking phase. The Nambu-Goldstone boson is guaranteed to be massless, and in spite of the alarming infinity in $\nu_r$, the physical mass of the $\sigma$ meson remains finite and calculable.

### E. Two-point Green's functions

The two-point Green's function for the $\sigma$ field reads perturbatively

$$\Gamma_\sigma^{(2)} = -iz_\sigma^{-1}\left(p^2 + \frac{\nu_r^2}{3}(\lambda_r y_\sigma)^{(2\alpha-1)}\right) \tag{58}$$

$$= -iz_\sigma^{-1}\left(p^2 + \mathcal{M}_\sigma^2\right), \tag{59}$$

where $z_\sigma^{-1}$ is the finite wave-function renormalization, and $y_\sigma$ is a renormalization-group invariant given iteratively by

$$y_\sigma = \frac{1}{\lambda_r} + \frac{b}{2}\left(\ln\frac{\sqrt{\frac{4\alpha-1}{4(2\alpha-1)}}\mathcal{M}_\sigma^2}{\mu^2} - \frac{4\alpha+1}{3(2\alpha-1)}\right). \tag{60}$$

In the critical limit, $y_\sigma \to 0$, and we find the mass relation for the $\sigma$ meson at $T = 0$

$$\mathcal{M}_\sigma^2 = \Lambda_c^2\,\frac{4(2\alpha-1)}{4\alpha-1}\,e^{(4\alpha+1)/[3(2\alpha-1)]}. \tag{61}$$

The two-point function for the $\pi$ is also easily computed. Perturbatively, it is given by

$$\Gamma_\pi^{(2)} = -i\delta^{ab}[\,p^2 + (Z_\sigma - 1)p^2 + \tfrac{1}{6}(Z_\kappa Z_\sigma^2 - 1)\kappa_r v^2 + \tfrac{1}{6}(\kappa_r v^2 - \nu_r^2) + \Sigma_\pi], \tag{62}$$

where $\Sigma_\pi$ is the one-loop integral

$$-i\delta^{ab}\Sigma_\pi(p) = 4h_r^2\,Z_m^2\,\mu_\sigma^\epsilon \int \frac{d^n x}{(2\pi)^n}\langle\bar\psi(x)\gamma_5 T^a\psi(x)\;\bar\psi(0)\gamma_5 T^b\psi(0)\rangle e^{ip\cdot x}. \tag{63}$$

By Eq. (49), the counterterm contributions in the two-point $\pi$ Green's function may be cast in the form

$$\Gamma_\pi^{(2)} = -i\delta^{ab}\left(p^2 + (Z_\sigma - 1)p^2 - \frac{h_r Z_m \mu_\sigma^\epsilon}{v}\langle\bar\psi\,\psi\rangle + \Sigma_\pi(p)\right). \tag{64}$$

At $p = 0$, $\Sigma_\pi$ may be expressed directly in terms of the momentum dependent mass of the two-point quark Green's function in the Landau gauge

$$S^{-1}(p) = \gamma\cdot p - iM_p, \tag{65}$$

where $M_q$ includes all the higher-order radiative corrections. For consider the Ward identity

$$q_\mu\Gamma_\mu^5 = S^{-1}(p+q)\gamma_5 + \gamma_5 S^{-1}(p) + 2im_r\Gamma^5, \tag{66}$$

where in an obvious notation, $\Gamma^5$ is the quark's dressed $\gamma_5$ vertex for the emission of a zero four-momentum pion. By taking the limit $q \to 0$, we see that the dressed $\gamma_5$ vertex is

$$\Gamma^5(p) = \frac{M_p}{m_r}\gamma_5. \tag{67}$$

In our DSM, the pion is not a fundamental field, so it does not itself propagate in the innards of the effective field theory. Therefore, there is no Nambu-Goldstone pole in $\Gamma_\mu^5$. The Nambu-Goldstone pole appears when the quark lines are contracted to form a bubble with another axial-vector vertex.

By writing down the Schwinger-Dyson equation for $\Sigma_\pi(0)$ in terms of the dressed vertex and propagators, it is easy to show that (see Fig.1)

$$\Sigma_\pi(p=0) = h_r^2\,Z_m\,\mu_\sigma^\epsilon(4N_cN_f)\int\frac{d^n q}{(2\pi)^n}\,\frac{iM_q/m_r}{q^2+M_q^2} \tag{68}$$

$$= +\,\frac{h_r^2\,Z_m\,\mu_\sigma^\epsilon}{m_r}\,\langle\bar\psi\,\psi\rangle \tag{69}$$

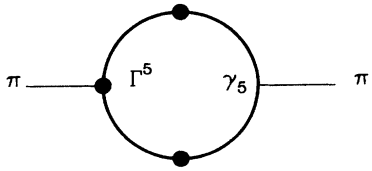


FIG. 1. $\Sigma_\pi(0)$ in terms of dressed propagators and vertex.

so that, as is well known, it is exactly cancelled by the quark-loop contribution to the tadpole. The cancellation is seen to work to all orders, as a result of our use of the Ward identities. The resulting two-point function for the $\pi$ is

$$\Gamma_\pi^{(2)}(p) = -i\left(1 - \frac{2\alpha-1}{2}\lambda_r b \,\ln\frac{m^2}{\mu_\sigma^2} + O(\lambda_r^2)\right)p^2. \tag{70}$$

Before we end this section, we remark on the crucial point in the proof of the masslessness of the pion. In order to be able to use the vacuum-stability condition, Eq. (49), and substitute it in Eq. (62), it was important that $v \neq 0$. Since we are studying dynamical symmetry breaking by taking a nonzero $m_r$ to begin with, where $m_r = h_r v$, we are justified in the step that leads up to Eq. (64). The pion is massless for all $m_r \neq 0$ and remains so in the critical limit.

## III. NAMBU-GOLDSTONE BOSON AT HIGH $T$

In the preceding section, we saw the pion self-energy correction, $\Sigma_\pi(p=0)$ being cancelled by the tadpole contribution at $T=0$. In this section, we will show that this cancellation persists at high temperatures. In the presence of a heat bath, we continue to require [18] that $\langle\text{vac}|\sigma|\text{vac}\rangle_\beta = 0$, so that Eq. (49) is valid in the thermal vacuum. The thermal Green's function for the pion now reads

$$\Gamma_\pi^{(2)}\Big|_T = -i\delta^{ab}\left(p^2 + (Z_\sigma - 1)p^2 - \frac{h_r\, Z_m\, \mu_\sigma^\epsilon}{v}\langle\bar\psi\,\psi\rangle_\beta + \Sigma_\pi(p)\Big|_T\right). \tag{71}$$

At $p=0$, the same analysis holds for high $T$. For the full thermal fermion propagator, even in the Landau gauge, has the form

$$S_\beta^{-1}(p) = \gamma\cdot\tilde p - i\tilde M_{p,\beta}, \tag{72}$$

where $\tilde p$ is a non–Lorentz-invariant function of $(\mathbf{p}, p_0)$ as a result of the thermal radiative corrections and $\tilde M_{p,\beta}$ is the chiral-flip mass including all the higher-order radiative corrections, and is now a function of the temperature of the heat bath. This chiral-flip component of the mass is not, however, to be confused with the physical mass of the quark at high temperature. By the same Ward identity as before, Eq. (66), we find in the limit of zero pion momentum $(q \to 0)$

$$\Gamma_\beta^5(p) = \frac{\tilde M_{p,\beta}}{m_r}\gamma_5 \tag{73}$$

so that we have

$$\Sigma_\pi(p=0)|_T = h_r^2\, Z_m\, \mu_\sigma^\epsilon \int \frac{d^n q}{(2\pi)^n}\frac{i\tilde M_{q,\beta}/m_r}{\tilde q^{\,2} + \tilde M_{q,\beta}^2}\Bigg|_\beta^{\text{Fermi}} \tag{74}$$

$$= +\,\frac{h_r^2\, Z_m\, \mu_\sigma^\epsilon}{m_r}\langle\bar\psi\,\psi\rangle_\beta \tag{75}$$

and the pion mass remains zero at high $T$.

For $p \neq 0$, the thermal Green's function for the pion takes the form

$$\Gamma_\pi^{(2)}|_T \underset{T\to\infty}{\sim} -ip^2\left[1 - \frac{2\alpha-1}{2}\lambda_r b\left(\ln\frac{T^2\pi^2}{\mu_\sigma^2} - 2\gamma\right) + O(\lambda_r^2)\right], \tag{76}$$

where $\gamma$ is the Euler constant $= 0.577215\ldots$.

In the linear $\sigma$ model, the same conclusion was reached by Mohan in Ref. [18] that the pion remains massless as $T$ rises from zero. The difference is that at the critical temperature, $T_c$, the vacuum shift, $v$, *vanishes*. The system makes a phase transition and stays on the new fixed point with $v=0$. When that happens, the tadpole contribution decouples from the pion Green's function. The term $\frac{1}{6}(\kappa_r v^2 - \nu_r^2)$ is simply fixed at $-\frac{1}{6}\nu_r^2$ and the delicate cancellation between the tadpole and the pion self-energy correction is lost. Since the high-temperature behavior of $\Sigma_\pi$ is proportional to $T^2$, the conclusion for the linear $\sigma$ model is that the pion becomes massive for $T > T_c$.

For our DSM, as we shall show, the vacuum shift does not vanish as $T$ increases. Before the critical limit, $m_r \neq 0$, and the pion was massless for all $T$. As $m_r \to 0$, the pion remains massless for all $T$.

### A. Tadpole at high $T$

To study the vacuum shift at high $T$, we return to Eq. (49) and find the perturbative result

$$h_r v = h_r v_0\left[1 + \frac{4\alpha-1}{4}\lambda_r b\left(\ln\frac{T^2\pi^2}{\mu_\sigma^2} - \frac{2\pi^2}{3}\frac{T^2}{m_r^2} - 2\gamma\right) + O(\lambda_r^2)\right]. \tag{77}$$

From the known renormalization-group property of analysis [19], the series sums up to the form

$$h_r v = h_r v_0\,(\lambda_r y_v)^{(4\alpha-1)/2}, \tag{78}$$

where $y_v$ is a renormalization-group invariant given by

$$y_v = \frac{1}{\lambda_r} + \frac{b}{2}\left(\ln\frac{T^2\pi^2}{\mu_\sigma^2} - \frac{2\pi^2}{3}\frac{T^2}{\mathcal{M}_T^2} - 2\gamma\right) \tag{79}$$

and $\mathcal{M}_T$ is the temperature-dependent physical mass of the quark. Note that in performing the sum over higher loops here one has included the analogues of the so-called "daisy" graphs, and "superdaisy" graphs first named by Dolan and Jackiw [10] and as a result the $m_r$ becomes

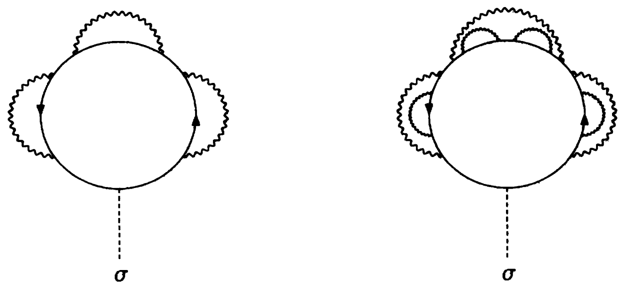


FIG. 2. Higher-order "daisy" and "superdaisy" graphs that are included in the sum in Eq. (78).

promoted to a temperature-dependent mass. (See Fig. 2.)

In spite of the appearance of the temperature-dependent quantities on the right-hand side, $y_v$ is temperature independent. This is because on the left-hand side, we have $h_r v = m_r$ and $m_r$ is the original, temperature-independent, parameter in the theory. Indeed $y_v$, with our choice of $\mu_\sigma = \mu e^{-1/3}$, is equal to $y_0$, as noted earlier. This can then be used to in turn determine the temperature dependence of the physical mass of the quark. The result is the same as that quoted in Eq. (5).

## IV. GAP EQUATION

Finally, we turn to a comparison of our result with the earlier work on symmetry restoration and seek to give a feeling as to where exactly our startling conclusion gets to differ from conventional wisdom. It is well known that symmetry breaking may be ascribed to some gap parameter in the theory that is spontaneously nonvanishing. In the pioneering work of Bardeen, Cooper, and Schrieffer (BCS) [20] the gap parameter, $\Delta$, plays the role of a mass for the quasiparticle excitations formed out of paired electrons of opposite spin. The gap parameter at zero temperature is determined self-consistently by the equation

$$\Delta = \frac{1}{2} g \int \frac{d^3k}{(2\pi)^3} \frac{\Delta}{\sqrt{\xi_k^2 + \Delta^2}}, \tag{80}$$

where $\xi_k$ is the energy of the electron as measured from the Fermi surface ($\epsilon = \mu$),

$$\xi_k \equiv \epsilon_k - \mu. \tag{81}$$

Superficially, the integral appears to be a quadratically divergent one. Solid-state-physics considerations however render it a finite one, being logarithmically dependent on the only "cutoff" in the theory, the Debye frequency $\omega_D$. The cutoff comes about through the observation by Debye that vibrations of the solid as a whole involve only wavelengths that are longer than the lattice spacings. For atomic-vibration frequencies above $\omega_D$, the collection of atoms no longer behaves as a coherent solid. The integral in Eq. (80) is thus to be taken over the range of energies $|\xi_k| \leq \hbar\omega_D$.

For most materials, the Debye temperature is of order $10^2$K, as opposed to the Fermi temperature of the metals being of order $10^4$K. The integral in Eq. (80) thus involves a range of $\xi_k$ that is small relative to the Fermi surface. In that approximation, the gap equation reduces to [21]

$$\Delta = gN(0) \int_0^{\hbar\omega_D} d\xi \frac{\Delta}{\sqrt{\xi^2 + \Delta^2}}, \tag{82}$$

where $N(0)$ is the density of states at the Fermi surface,

$$N(0) = \frac{1}{2\pi^2} \left( k^2 \frac{dk}{d\epsilon_k} \right)_{\epsilon_k = \mu}. \tag{83}$$

At finite temperature, the gap equation takes the form

$$\Delta = gN(0) \int_0^{\hbar\omega_D} d\xi \frac{\Delta}{\sqrt{\xi^2 + \Delta^2}} \tanh \frac{\sqrt{\xi^2 + \Delta^2}}{2T}. \tag{84}$$

At $T = T_c$, $\Delta$ vanishes, and the system makes a phase transition and stays at the fixed point $\Delta = 0$ for $T > T_c$.

A point that will be at issue in our later discussion is the question whether the phase transition temperature is destabilized by higher-order corrections to the gap equation. In BCS, the higher-order corrections are controlled by $gN(0)$ and are genuinely small. Since the $T_c$ is low in BCS, higher-order corrections do not alter the physics conclusions.

In 1961, Nambu and Jona-Lasinio (NJL) [4] generalized the BCS gap equation to relativistic field theory and wrote down the NJL gap equation, where the gap parameter here is the physical mass of the fermion,

$$M = \frac{1}{i} g \int \frac{d^4p}{(2\pi)^4} \frac{M}{p^2 + M^2}. \tag{85}$$

Just like its progenitor, the gap equation here is quadratically divergent. Following BCS, NJL simply introduced a cutoff $\Lambda$ into the theory and wrote the gap equation in the cutoff-dependent form

$$M = \frac{g\Lambda^2}{16\pi^2} M \left[ 1 - \frac{M^2}{\Lambda^2} \ln \left( \frac{\Lambda^2 + M^2}{M^2} \right) \right], \tag{86}$$

where, in order that there be a nonzero dynamical mass, they assume that

$$\frac{g\Lambda^2}{16\pi^2} > 1. \tag{87}$$

For $T \neq 0$, the gap equation becomes

$$M = M \frac{g\Lambda^2}{16\pi^2} \left( 1 - \frac{M^2}{\Lambda^2} \ln \frac{\Lambda^2 + M^2}{T^2} - \frac{2\pi^2}{3} \frac{T^2}{\Lambda^2} \right) \tag{88}$$

or

$$\frac{16\pi^2}{g\Lambda^2} = 1 - \frac{M^2}{\Lambda^2} \ln \frac{\Lambda^2 + M^2}{T^2} - \frac{2\pi^2}{3} \frac{T^2}{\Lambda^2} \tag{89}$$

and we see the beginning of a $T_c$ that comes from the root of this equation.

The NJL gap equation is manifestly cutoff dependent. While as a fundamental theory it is nonrenormalizable, one can argue that perhaps the cutoff is a physical one, just like in condensed-matter physics, and discuss only the physics for energies less than the cutoff. In that spirit, then, there is no question about the higher-order corrections to the NJL gap equation.

In 1984, we [22] applied the NJL formulation of the self-consistent generation of mass to the renormalizable QCD. If $M$ is the effective quark mass at zero four momentum, then the gap equation reads

$$M = \frac{3g^2C_f}{i}\int \frac{d^4p}{(2\pi)^4}\frac{M}{p^2(p^2+M^2)} + O(g^4). \qquad (90)$$

This gap equation is now logarithmically divergent rather than quadratically. To render it finite, a cutoff $\mu$ may be introduced. In contrast with the condensed-matter case, however, this cutoff does not have physical significance. In a renormalizable field theory, we must attempt to sum over all higher-order terms and insist that the answer in the end is independent of the cutoff $\mu$ that we introduced to define the integrals. In Ref. [22], we showed how to generalize the gap equation (90) to a sum over higher orders in the leading-log approximation. Working in dimensional regularization, with $n = 4-\epsilon$, the gap equation for $M$ becomes

$$M = -3\lambda_r C_f\, M\left(\ln\frac{M^2}{\mu^2}-\frac{1}{3}\right) + O(\lambda_r^2), \qquad (91)$$

where we have again used the symbol $\lambda_r$ for the combination $g_r^2/16\pi^2$ with $g_r$ being the running coupling constant of SU(3). The higher-order terms, in the leading-log approximation, may be summed and they convert the gap equation into the simple form

$$M\left[1+\frac{\lambda_r b}{2}\left(\ln\frac{M^2}{\mu^2}-\frac{1}{3}\right)\right]^{\alpha} = 0. \qquad (92)$$

In a way, we have converted a naive solution of the gap equation

$$1 = -3\lambda_r C_f\left(\ln\frac{M^2}{\mu^2}-\frac{1}{3}\right) \qquad (93)$$

to the renormalization-group invariant one

$$\frac{1}{\lambda_r} = -\frac{b}{2}\left(\ln\frac{M^2}{\mu^2}-\frac{1}{3}\right). \qquad (94)$$

At high temperatures, the gap equation for QCD takes the form

$$M = -3\lambda_r C_f M\left(\ln\frac{T^2}{\mu^2}-\frac{2\pi^2}{3}\frac{T^2}{M^2}\right) + O(\lambda_r^2). \qquad (95)$$

If we take this equation at face value, it is easy to find that the dynamical mass at high $T$ is given by

$$M^2 \underset{T\to\infty}{\sim} 2\pi^2\lambda_r C_f T^2. \qquad (96)$$

Such an answer would however not be admissible since it depends on the renormalization point $\mu$. We have summed over the higher loops and find the renormalization-group-invariant form of the gap equation

$$M\,(\lambda_r y_T)^{\alpha} = 0 \qquad (97)$$

with

$$y_T = \frac{1}{\lambda_r} + \frac{b}{2}\left(\ln\frac{T^2}{\mu^2}-\frac{2\pi^2}{3}\frac{T^2}{M^2}\right) = 0 \qquad (98)$$

and a solution that is now genuinely independent of $\mu$.

Finally, we return to the gap equation for the vacuum shift in the DSM. As Eq. (77) shows, a perturbative calculation of the gap might stop at the one-loop level and claim that

$$h_r v = h_r v_0\left[1+\frac{4\alpha-1}{4}\lambda_r b\left(\ln\frac{T^2\pi^2}{\mu_\sigma^2}-\frac{2\pi^2}{3}\frac{T^2}{m_r^2}-2\gamma\right)\right] \qquad (99)$$

and conclude that the vacuum shift $v$ vanishes at the critical temperature given by the root of the equation,

$$T_c^2 = m_r^2\frac{6}{\pi^2(4\alpha-1)\lambda_r b} \qquad (100)$$

and conclude forthwith that the pion becomes massive above that temperature. As we have repeatedly stressed however for a renormalizable field theory, we must sum over the higher orders and make certain that the result is independent of the cutoff $\mu$ we have introduced to define the integrals. In the present case, the higher orders as summarized by Eqs. (78) and (79) destabilize the simple root. The next term in the leading-log series [Eq. (78)] gives a positive $\lambda_r^2 T^4/m_r^2$ contribution that opposes the negative $\lambda_r T^2$ contribution. And when we have summed it all up, the entire series is constrained to be independent of $T$. The critical temperature given by the perturbative root in Eq. (100) is thus an artifact of first-order perturbation theory which is destabilized by higher-order terms. The full shift $v$ does not vanish at higher temperatures, and the pion remains massless as $T \to \infty$.

Most considerations of symmetry restoration in particle physics have been based on one-loop arguments and are subject to further investigations of renormalization-group consistency. The example we have quoted here is a good instance of how the one-loop restoration temperature is an artifact of perturbation theory.

## V. CONCLUSION

In this paper, we have reported on the renormalization-group study of chiral-symmetry breaking in QCD. Our techniques are applicable to other theories as well. It is an open question as to whether in the electroweak or other unified theories, symmetry breaking could persist at high temperatures, and if it does whether there are cosmological consequences that follow therefrom. Even in QCD itself, it would be of interest to extend our calculation beyond two flavors and analyze the parameters of the full $\sigma$ model for the five or six flavors that we believe exist in nature. We hope to come back to these and other questions in a future publication.

## ACKNOWLEDGMENTS

This work was written while one of us (N.P.C.) was visiting Brookhaven National Laboratory and he wishes to thank the Theory Group for the warm hospitality and stimulating discussions. L.N.C. was supported in part by Grant No. DE-AS05-80ER10713, while N.P.C. was supported in part by Grant No. PHY-89-07680, and by a grant from PSC-BHE.



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