Nonperturbative multiparticle tree graphs in the standard electroweak theory

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I present a lower bound for the cross section at the tree level for $f_R\bar{f}_L \to NZ$ to lowest order in the hypercharge coupling g'. It is seen that the cross section grows like $g^{2N}N!$, and hence violates $J=1$ unitarity for large enough $N \gtrsim (\text{const}/g^2)$. This phenomenon occurs in the kinematic region where coherence among $\sim N!$ graphs is possible: the vector mesons are nonrelativistic and they all have the same or a small fixed number of polarization vectors. The argument is then extended to the process $f_R\bar{f}_L \to ((N/3)W^+, (N/3)W^-, (N/3)Z)$, and finally to the general case of an arbitrary initial state, and a final state containing a large number of nonrelativistic $W^{\pm}$'s and Z's. The result involves only the gauge couplings, and is independent of the size of the quartic scalar coupling. There is qualitative similarity with the result found previously for tree graphs in $\lambda\phi^4$ theory.

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There are indications of a failure of perturbation theory in weakly coupled theories when the number of produced particles N in a high-energy collision becomes of $O(1/\alpha)$. This was first noted in the context of $B+L$ violation in the instanton calculation of Ringwald [1] (and later Espinosa [2]), following some earlier speculations [3]. Subsequently, the $\lambda\phi^4$ theory was examined through the classical field equation with an oscillating source of soft quanta [4] and through a direct evaluation of tree graphs in a high-energy collision $f\bar{f} \to N$ Higgs bosons [5]. In the latter case, it was possible to directly implement the limit $E \to$large, $N \to$large, E/N fixed. In both of these studies, it was found that the tree-level amplitudes near threshold grew like $\sqrt{\lambda}^N N!$. In Ref. [5], the reaction chosen proceeded entirely through the s wave (to lowest order in the Yukawa coupling g_Y), and it was shown that the cross section behaved like

$$\sigma_N \simeq \frac{g_Y^2}{E^2}\left[\frac{\lambda}{16\pi^2 a}\right]^N N! , \qquad (1)$$

where a is a number, so that the s-wave unitarity bound was exceeded at $N \sim 16\pi^2 a/e\lambda$. In Ref. [5] this behavior was also seen to occur in a supersymmetric theory with two scalars, in which λ is proportional to the gauge coupling g^2.

In diagrammatic language, the reinstatement of s-wave unitarity must come from higher-loop corrections. Here once again, the nonperturbative nature of the process becomes apparent: for example, the one-loop correction due to rescattering in the final state introduces an extra factor of λ, which, however, may be compensated by a combinatoric factor $\sim N^2$ to account for the number of ways of rescattering in the final state.

There remains a crucial question: do rescattering corrections lead to an exponential suppression of the resulting cross section, or do they permit the cross section to attain its unitarity bound? This has been discussed by Zakharov [6] and others [7,8] in the context of the Lipatov analysis [9] of $\lambda\phi^4$ theory. The argument goes roughly as follows: consider the unsubtracted dispersion relation for the second derivative of the scalar one-particle irreducible two-point function $\Pi(q^2)$. Modulo constants, it is related to the total annihilation cross section for $q\bar{q}$ into scalars:

$$\Pi''(0) \propto \int \frac{\sigma^{\text{tot}}(s)ds}{s} . \qquad (2)$$

Now suppose the left-hand side of Eq. (2) allows an asymptotic expansion in coupling, such that the nth-order term is $\sim (a\lambda)^n n!$. This term achieves its minimum value of $e^{-\bar{n}}$ for $n=\bar{n}=1/(a\lambda)$, after which the series rapidly diverges. In the case of $\lambda\phi^4$ theory, the Lipatov analysis [9] indicates that the presence of instanton configurations with $\lambda<0$ allows a Borel summation of the rest of the series, and consequently a bounding of the remaining (nonperturbative) contribution by the amount $e^{-\bar{n}}$. Finally, under certain assumptions and with some discussion, one may argue [6,8] that as a result of Eq. (2) this also applies to the n-particle pieces of the cross section. Thus, the claim is that in $\lambda\phi^4$ theory the partial cross sections σ_n for $n \ge \bar{n}$ are exponentially suppressed, whereas for $n<\bar{n}$ they are suppressed by the appropriate large powers of λ.

The situation may be very different in the gauge sector of the electroweak theory. In that case, the saddle-point field configurations (negative-g^2 instantons) which would support a Borel summation of the series beyond $n \sim 1/(\text{const})g^2$ do not exist. This makes it difficult to bound the nonperturbative part of the amplitude. Thus, the argument for suppression cannot be carried over in any obvious manner [6], leaving open the possibility of an anomalous contribution to the cross section (either $B+L$ violating or not) which saturates the unitarity limit at energies $\sim m/\alpha_{wk}$. From the analogue of Eq. (2) for the gauge theory, it can be seen that this may generate a contribution to the vector polarization tensor which is *numerically* comparable to a *perturbative* contribution in some low order of α_{wk}. With this motivation, I turn to examine the high-multiplicity tree structure in the gauge sector of the electroweak theory.

The general method of approach, applicable to all situations, is the following:

(1) Consider the scattering from some suitably chosen two-particle initial state i to some N-particle final bosonic state f at a c.m. energy E close to threshold:

$$\frac{E}{Nm}-1\ll 1 \ . \tag{3}$$

(2) Choose the polarization vectors (in the limit $\mathbf{p}_i\to 0$) to belong to some finite number of possibilities (say one for each type of vector meson, or the same for all).

(3) Pick a particular topology for the tree graph—any one will do—which has a well-defined limit as all the three-momenta $\mathbf{p}_i\to 0$, and calculate the amplitude $\overline{\mathcal{M}}_N(0)\sim(g/m)^N$ in this limit for any of the momentum orderings. Since the number of orderings will be of $O(N!)$ and these (by hypothesis) all have the same value $\overline{\mathcal{M}}_N(0)$ as $\mathbf{p}_i\to 0$, the total amplitude (as $\mathbf{p}_i\to 0$) corresponding to such an "s-wave" graph will be

$$\mathcal{M}_N(0)\sim\overline{\mathcal{M}}_N(0)N!\sim\left[\frac{g}{m}\right]^N N! \ . \tag{4}$$

(4) Demonstrate why the $N!$ behavior cannot be remedied through cancellation between graphs of different topology.

(5) Estimate the perturbation on $\overline{\mathcal{M}}_N$ for small $|\mathbf{p}_i|/m$. We shall see below that nonrelativistic phase space introduces a factor $(p_{\max}/m)^{3N}(1/N!)$ multiplying $|\mathcal{M}_N|^2$ in the expression for the cross section. Thus, if coherence (i.e., $N!$ behavior of $\mathcal{M}_N$) is maintained for small *but finite* $p_{\max}/m$ (as $N\to\infty$), then the tree graph in question will violate unitarity at large N.

As the simplest example, I will consider the reaction at c.m. energy E of a right-handed massless fermion and left-handed antifermion

$$f_R\bar{f}_L\to NZ \ , \tag{5}$$

to lowest nonvanishing order in the hypercharge coupling g'. This forces the reaction to proceed initially through a single virtual Z. We may note immediately that the absence of any $W^{\pm}$ in the final state precludes their presence anywhere in any *tree* graph leading to this state. This follows from charge conservation. A representative diagram in the case $N=13$ is shown in Fig. 1(a). Thus, except for the coupling to the initial fermions, the piece of the electroweak Lagrangian relevant for present purposes is (in unitary gauge)

$$\mathcal{L}_{\text{int}}=(\tfrac{1}{8}g^2H^2+\tfrac{1}{2}gmH)Z_\mu Z^\mu \ , \tag{6}$$

where $g^2=g_2^2+g'^2$ and $m\equiv m_Z$.

As described in the introductory algorithm, I first calculate the amplitude in the limit $\mathbf{p}_i\to 0$ for the symmetric tree topology, exemplified in Fig. 1(a) for the case $N=13$. Also, for simplicity, I will take all the polarization vectors for the Z's to be equal to the same three-vector ϵ (as $\mathbf{p}_i\to 0$). In this limit, all the Z propagators in the tree are given by $i\delta_{ij}/E^2_{\text{int}}$, where E_{int} is the energy carried by some (internal) line in the tree. Also, all the ϵ vectors dot out to unity, except for one which is dotted into a fermion spinor matrix. In this limit, all the trees of this topology formed by reshuffling the particles add coherently, so that the amplitude at threshold for the process (5) proceeding through the symmetric topology of Fig. 1(a) is given by

$$\mathcal{A}_N(0)\simeq\frac{g'\sin\theta_w}{E^2}(v^\dagger\sigma\cdot\epsilon u)\overline{\mathcal{M}}_{(N/3,N/3)}\left[\frac{gm}{4m^2}\right]^{N/3}C_N \ , \tag{7}$$

where $\overline{\mathcal{M}}_{(N/3,N/3)}$ is the subamplitude for the process $Z\to(N/3)H+(N/3)Z$ for any single ordering of the particles. The next factor accounts for the propagation of the penultimate H's and their decay into Z's. Finally, C_N is the combinatoric factor counting the independent permutations of the final state. By examining Fig. 1(a), one may deduce that for large N

$$C_N\simeq\frac{N!}{(2!)^{N/2}} \ . \tag{8}$$

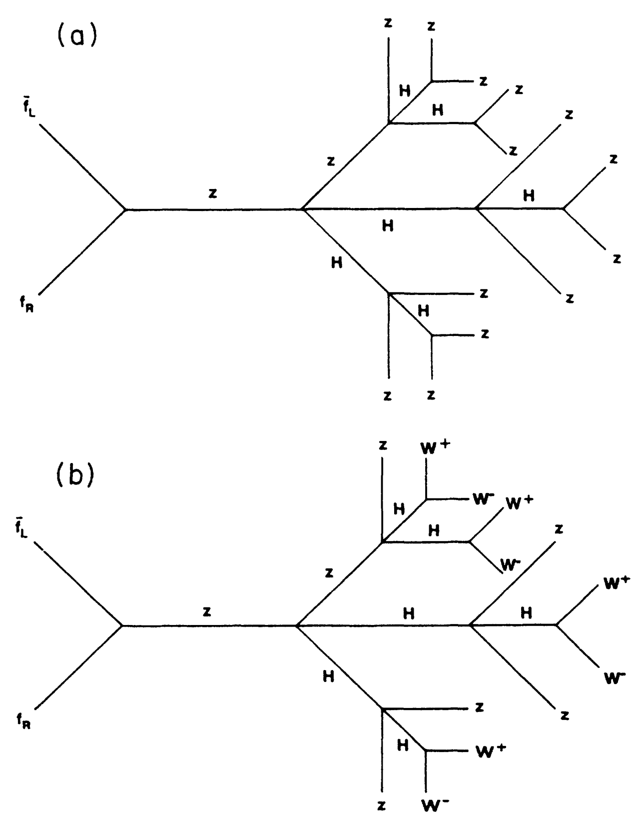

FIG. 1. (a) Symmetric tree graph for $f_R\bar{f}_L\to 13Z$, to lowest order in the hypercharge coupling g'. (b) Similar graph for $f_R\bar{f}_L\to 4W^+4W^-5Z$.

It is also apparent that an expression for $\overline{\mathcal{M}}_{(N/3,N/3)}$ may be obtained from the quartic scalar analysis of Ref. [5] with the substitutions in Eq. (3) of that reference $\lambda_4 \to \frac{1}{2}g^2, N \to \frac{2}{3}N$. Thus

$$\overline{\mathcal{M}}_{(N/3,N/3)} \simeq E^3 \left[\frac{\sqrt{\frac{1}{2}g^2}}{3\sqrt{3}} \frac{\frac{2}{3}N}{E} \right]^{2N/3} = E^3 \left[\sqrt{\tfrac{2}{3}\tfrac{1}{9}} \right)^{2N/3} \left[\frac{g}{m} \right]^{2N/3}, \tag{9}$$

and, from Eq. (7),

$$\mathcal{A}_N(0) \simeq g' \sin\theta_w E (v^\dagger \sigma \cdot \epsilon u) a^N \left[\frac{g}{m} \right]^N N! , \tag{10}$$

where $a = (\frac{3}{2})^{1/3}/(9\sqrt{2}) \simeq \frac{1}{11}$.

Suppose now that Eq. (10) were a reasonable approximation for the amplitude $\mathcal{A}_N$ when $|\mathbf{p}_i|/m \lesssim p_{\max}/m \equiv \varepsilon \ll 1$. Then the cross section may be computed:

$$\sigma(f_R \bar{f}_L \to NZ(\epsilon)) \simeq |\mathcal{A}_N|^2 \rho_N / (\text{flux}) . \tag{11}$$

Here ρ_N is the nonrelativistic phase-space factor [10]

$$\rho_N \simeq \frac{1}{N!} \frac{1}{E^4} m^{2N} \varepsilon^{3N} \left[\frac{b}{16\pi^2} \right]^N , \tag{12}$$

with $b = \sqrt{8\pi}(e/3)^{2/3} = 4.69$. Thus, if $\varepsilon \not\to 0$ as $N \to \infty$, then

$$\sigma(f_R \bar{f}_L \to NZ(\epsilon)) \simeq \frac{g'^2 \sin^2\theta_w}{E^2} (a^2 b)^N \left[\frac{g^2 \varepsilon^3}{16\pi^2} \right]^N N! \tag{13}$$

in violation of $J=1$ unitarity when $N \geq (\text{const})(16\pi^2/g^2\varepsilon^3)$.

What about cancellation between graphs of different topology? In the present case, the absence of such cancellation is easy to see. First, every graph which has only quartic vertices (except for the final branching $H \to ZZ$) has the same sign. Second, every quartic $HHZZ$ vertex may be replaced by a pair of cubic HZZ vertices separated by a Z propagator. One verifies via explicit calculation that these graphs also add coherently to the previous ones.

At this point, I turn to address the more difficult, and crucial, item (5) above—the effect of finite $\mathbf{p}_i/m$. I will demonstrate that the result derived above holds for small but finite $p_{\max}/m$ as $N \to \infty$.

The continuation away from $\mathbf{p}_i \simeq 0$ introduces a number of complexities into the analysis: (1) the polarization vectors can no longer be taken as purely three dimensional, nor can they all be equal. This tends to introduce incoherence in a manner to be discussed. (2) The nonvanishing three-momenta and the consequent presence of timelike components for the polarization vectors will activate contributions from the $k_\mu k_\nu/m^2$ pieces of the vector propagators. These contributions will depend on the external $\mathbf{p}_i$, they are not of one sign, and will promote incoherence. Can these effects be controlled for finite $|\mathbf{p}_i|/m$ as $N \to \infty$? Let us examine them in turn.

(1) *Inequality of the ϵ_i when $\mathbf{p}_i \neq 0$.* The presence of the polarization vectors in the amplitude will be manifest as a string of $\sim N/2$ dot products $(\epsilon_1 \cdot \epsilon_2)(\epsilon_3 \cdot \epsilon_4) \cdots +$permutations for the Z's coming from the penultimate $H \to ZZ$ branchings, as well as other factors such as $\epsilon_5 \cdot (p_1 + \cdots + p_4)$ for the Z's not coming from such branchings. A representative estimate of the incoherence introduced for $\mathbf{p}_i \neq 0$ among the $N!$ permutations due to the inequality of the ϵ_i may be obtained via the following construction: take each $\epsilon_i = (0, \boldsymbol{\epsilon})$ independent of i in the rest frame of the corresponding Z_i. Then boost to the lab frame, and calculate $-\epsilon_i \cdot \epsilon_j = (\boldsymbol{\epsilon}_i \cdot \boldsymbol{\epsilon}_j - \epsilon_{i0}\epsilon_{j0})_{\text{lab}}$. Taking $|\mathbf{p}_i| = |\mathbf{p}_j| = p$ for calculational simplicity, I find to $O(p^2/m^2)$ that

$$-\epsilon_i \cdot \epsilon_j = 1 + \tfrac{1}{2}(p^2/m^2)(\cos\theta_i - \cos\theta_j)^2 + \cdots = 1 + \varepsilon_{ij} \tag{14}$$

with $|\varepsilon_{ij}| \leq 2(p_{\max}/m)^2 = 2\varepsilon^2$. Thus, for small but finite ε the effect of $\mathbf{p}_i \neq 0$ from this source will be a replacement

$$\sum_{\text{perms}} 1 \to \sum_{\text{perms}} \prod_{\text{pairs}} (1 + \varepsilon_{ij}) \geq (1 - 2\varepsilon^2)^{N/2} N! . \tag{15}$$

For small ε this does not affect the $N!$ behavior. In the relativistic limit, one finds that

$$-\epsilon_i \cdot \epsilon_j \simeq (p^2/m^2)\cos\theta_i \cos\theta_j [1 - \cos(\theta_i - \theta_j)] ,$$

and the $\sim N!$ strings of products do not add coherently in any manner.

(2) *The activation of the $k_\mu k_\nu/m^2$ pieces of the vector meson propagators.* We first note that because of this piece of the propagator, up to $\frac{1}{3}$ of the final-state polarization vectors may now be dotted into sums of a few external momenta. Such kinematic factors destroy coherence. In the interior of the graph, $k_\mu k_\nu/m^2$ can introduce large terms $\sim (E_{\text{int}}/m)^2$ in the numerators of the propagators. An upper bound on this effect may be obtained by replacing *every* Z propagator used in the previous estimate of $\overline{\mathcal{M}}_{(N/3,N/3)}$ by $(E_{\text{int}}/m)^2/E_{\text{int}}^2 = (1/m^2)$, and multiplying by the appropriate external factors $(\epsilon p)^{N/3} \sim (p_{\max}/m)^{N/3}$. Since about half the propagators in the tree are Z propagators, we may obtain the effect of this replacement from Eq. (3) of the scalar tree result [5] by substituting for every factor E^{-2} the geometric mean $(Em)^{-1}$ of the Higgs and the "revised" Z propagator $1/m$. After some algebra, I obtain an upper bound on the magnitude of the incoherent piece of the amplitude (before permutations), corresponding to $\overline{\mathcal{M}}_{(N/3,N/3)}$:

$$|\overline{\mathcal{M}}^{(\text{incoh})}_{(N/3,N/3)}| \simeq |\overline{\mathcal{M}}^{(\text{coh})}_{(N/3,N/3)}| \left[\frac{9\sqrt{3}}{2} \frac{p_{\max}}{m} \right]^{N/3} . \tag{16}$$

Clearly, we can choose a small *but finite* $p_{\max}/m$ such that coherence will be maintained as $N \to \infty$.

I now turn to discuss the more general case of a final state containing an equal mixture of W^+, W^-, and Z. For the moment, I will still consider the incoming state to consist of $f_R \bar{f}_L$, so that the tree begins with a single Z. The simplest graph to discuss is the analogue of Fig. 1(a), shown in Fig. 1(b), in which there are only H's and Z's until the final branching of the penultimate H's, $H \rightarrow W^+W^-$. Except for a modification of the combinatoric factor by a factor of 3^N [roughly, $N! \rightarrow (N/3)!^3$], the amplitude from this graph can be obtained from the pure Z case discussed above, and will again be of order $a^N (g/m)^N N!$. As in the previous case, there are many other topologies, and each topology which does not vanish as $\mathbf{p}_i \rightarrow 0$ will contribute an amplitude of magnitude $\sim N!$. However, in contrast to the previous case, the presence of the non-Abelian quartic and cubic vertices of vector mesons will permit variation in signs of graphs of differing topologies. Nevertheless, I will discuss why it is not conceivable that they cancel to the extent of restoring tree-level unitarity.

For orientation, I note that the number of different topologies is well approximated by $(\mathrm{const})^N$; for example, in scalar theory with both cubic and quartic coupling, I find that the number of topologically distinct tree graphs for $1 \rightarrow N$ is approximately given by $(3.26)^N$ for $N \geq 50$. Thus, we have $(\mathrm{const})^N$ different amplitudes, each of magnitude $\sim N!$, and we require the sum to be of $O(\sqrt{N!})$ or smaller (in order to satisfy unitarity at the tree level). Thus, the sum of the coefficients of $N!$ for the different amplitudes must cancel to within $O(1/\sqrt{N!})$ of zero, *but not to zero,* for all $N \gg 16\pi^2/g^2$. This seems impossible to arrange. For example, the hypothetical cancellation would need to occur *for every value of the Higgs mass* (which is arbitrary), since corrections even as small as $M_H^2/E^2 \gtrsim 1/N^2$ would reinstate the $N!$ [or $(N-2)!$] behavior of the amplitude. All of this discussion is also clearly applicable to multiboson production from any initial state, although the delineation of the unitarity bound requires a little more discussion.

Thus I come to my concluding remarks.

(1) As was the case with scalars, it has been shown that the tree-level cross section for the process $f_R \bar{f}_L \rightarrow NZ$ violates unitarity for $N \gtrsim (\mathrm{const})/g^2$ in the nonrelativistic region $E/Nm - 1 = \frac{1}{2}\varepsilon^2$, $\varepsilon \ll 1$ but finite as $N \rightarrow \infty$. The condition on the final-state polarization vectors is that they constitute a finite set as $N \rightarrow \infty$. The bound was then extended to the most general initial and final states, consistent with the above restrictions on kinematics and polarization vectors.

(2) In perturbative language, unitarity will be restored through higher loops. The argument made by Zakharov [6] in the scalar case that the restoration of unitarity in fact leads to an exponential suppression, is not applicable to the non-Borel-summable vector case (see introductory discussion). Thus, at energies above $O(m/g^2 \sim E_{\mathrm{sph}})$, possible strong scattering to states of high multiplicity, in a process which does *not* violate baryon number, is on the same footing as the $B+L$-violating case discussed in Refs. [1] and [2]. In both situations there would be nonperturbative corrections to vector polarization tensors which may be numerically comparable to low-order perturbative corrections.

(3) It has been remarked by Casher and Nussinov [7] and Zakharov [11] that an intuitive argument against the production of a coherent state in the scalar case is that the final-state interaction is repulsive, leading to instability of such a state. The case of the vectors has no such argument: in general, the interactions are of both signs. In the case with only Z's, the $ZZ \rightarrow ZZ$ interaction (via s-, t-, and u-channel exchange of a Higgs boson) is *attractive* unless $1.6m_Z \leq m_H \leq 2.0m_Z$.

(4) With the amplitude used for illustration in this paper, the breakdown of unitarity at the tree level occurs at a very large value of N [$>(16\pi^2/g^2\varepsilon^3)$]. In order to observe experimentally the consequences of these coherent states, one may hope that higher loops or the inclusion of more diagrams will substantially reduce this number.

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