Zero-momentum limit of Feynman amplitudes at finite temperature

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In the real-time formalism, we show that, if carefully evaluated, the zero-momentum limit of the real part of the scalar self-energy exists and is unique.

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I. INTRODUCTION

The zero-momentum limit of Feynman amplitudes at finite temperature has generated much discussion in the past few years [1]. As an example, let us summarize the results on the self-energy of a scalar field at finite temperature which has been well studied both in the imaginary-time as well as the real-time formalisms. In the imaginary-time formalism [2] it was shown at one loop [3] that, for the real part of the self-energy, $\mathrm{Re}\pi(0)$ is well defined and corresponds to

$$\mathrm{Re}\pi(0)=\lim_{\mathbf{p}\to 0}\lim_{p^0\to 0}\mathrm{Re}\pi(p)\ . \tag{1}$$

Here the limit $\mathbf{p}\to 0$ is assumed to be taken after the limit $p^0\to 0$ has been taken. It is also known that reversing the order of the limits leads to a different result: namely,

$$\lim_{p^0\to 0}\lim_{\mathbf{p}\to 0}\mathrm{Re}\pi\neq\mathrm{Re}\pi(0)\ . \tag{2}$$

The real part of the self-energy at finite temperature would, therefore, appear to display a nonanalyticity in the sense that it is discontinuous at $p^\mu=0$.

In the real-time formalism [4], on the other hand, it was argued that $\mathrm{Re}\pi(0)$ is not at all defined although the two different limits of vanishing p^μ for $\mathrm{Re}\pi(p)$ exist and coincide with the expressions obtained in the imaginary-time formalism [5]. In the imaginary-time formalism, the limit $p^0\to 0$ is ambiguous since p^0 is defined to take only discrete values in this formalism. A method for computing $\mathrm{Re}\pi(p)$ for small p^μ, in the imaginary-time formalism, was proposed in Ref. [1] which gives an analytic $\mathrm{Re}\pi(p)$. However, there is no compelling reason to accept this proposal over any other within this framework. The nonanalyticity of $\mathrm{Re}\pi(p)$ has also prompted various people [5–8] to postulate additional Feynman rules in the real-time formalism. These rules are, however, quite *ad hoc*.

In this paper we reexamine the calculation of $\mathrm{Re}\pi(p)$ within the framework of the real-time formalism. In the framework of the conventional real-time formalism (namely, without any new *ad hoc* Feynman rules), we show that, when carefully evaluated, $\mathrm{Re}\pi(0)$ is well defined. Furthermore, $\mathrm{Re}\pi(p)$ is analytic at $p^\mu=0$ and, for small p^μ, it almost coincides with the results of Ref. [1].

II. THE CALCULATION

To be specific, as well as for convenience of comparison with other works, let us consider the Lagrangian

$$\mathcal{L}(B,\phi)=\mathcal{L}_0(B)+\tfrac{1}{2}\partial_\mu\phi\partial^\mu\phi-\frac{m^2}{2}\phi^2-\frac{\lambda}{2}B\phi^2\ . \tag{3}$$

Here B and ϕ are two real scalar fields and $\mathcal{L}_0(B)$ represents the free Lagrangian for the field B. For completeness, we note here that our metric is diagonal with the signatures $(+,-,-,-)$. Let us next calculate the self-energy of the B field at one loop and at finite temperature in the real-time formalism. At one loop, the tilde fields of thermofield dynamics do not contribute to this amplitude and, consequently, we have (see Fig. 1)

$$\pi(p)=\frac{i\lambda^2}{2}\int\frac{d^4k}{(2\pi)^4}G_\beta(k)G_\beta(k+p)\ , \tag{4}$$

where the thermal propagator $G_\beta(k)$ is given by

$$\begin{aligned}G_\beta(k)&=\lim_{\epsilon\to 0}\left[\frac{1}{k^2-m^2+i\epsilon}+\sinh^2\theta_k\left[\frac{1}{k^2-m^2+i\epsilon}-\frac{1}{k^2-m^2-i\epsilon}\right]\right]\\&=\lim_{\epsilon\to 0}\left[\frac{1}{k^2-m^2+i\epsilon}-2i\sinh^2\theta_k\frac{\epsilon}{(k^2-m^2)^2+\epsilon^2}\right]\end{aligned} \tag{5}$$

with

$$\sinh^2\theta_k=\frac{1}{e^{\beta|k_0|}-1}\ ,\quad \beta=\frac{1}{kT}\ . \tag{6}$$

We note here that the factor $\frac{1}{2}$ in Eq. (4) is the symmetry factor and that if we use the representations

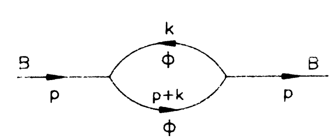

FIG. 1. Graph corresponding to B-field self-energy.

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$$\lim_{\epsilon\to 0}\frac{1}{\pi}\frac{\epsilon}{x^2+\epsilon^2}=\delta(x)\ ,$$
$$\lim_{\epsilon\to 0}\frac{x}{x^2+\epsilon^2}=P\left[\frac{1}{x}\right]\ , \tag{7}$$

then the thermal propagators will take the more familiar form. However, we will continue to work with the form of $G_\beta(k)$ as is given in Eq. (5). The real part of the self-energy can now be easily obtained from Eq. (4) and has the form

$$\mathrm{Re}\pi(p)=\lim_{\epsilon\to 0}-\frac{\lambda^2}{2}\int\frac{d^4k}{(2\pi)^4}\left[\coth\frac{\beta|k_0|}{2}\frac{\epsilon}{(k^2-m^2)^2+\epsilon^2}\frac{(k+p)^2-m^2}{[(k+p)^2-m^2]^2+\epsilon^2}\right.$$
$$\left.+\coth\frac{\beta|k_0+p_0|}{2}\frac{\epsilon}{[(k+p)^2-m^2]^2+\epsilon^2}\frac{k^2-m^2}{(k^2-m^2)^2+\epsilon^2}\right]\ . \tag{8}$$

We can set $p^\mu=0$ in Eq. (8) to obtain

$$\mathrm{Re}\pi(0)=\lim_{\epsilon\to 0}-\lambda^2\int\frac{d^4k}{(2\pi)^4}\coth\frac{\beta|k_0|}{2}\frac{\epsilon}{(k^2-m^2)^2+\epsilon^2}\frac{k^2-m^2}{(k^2-m^2)^2+\epsilon^2}\ . \tag{9}$$

If we take the limit $\epsilon\to 0$ in Eq. (9) and use the formulas in Eq. (7), this can also be written as

$$\mathrm{Re}\pi(0)=-\lambda^2\pi\int\frac{d^4k}{(2\pi)^4}\coth\frac{\beta|k_0|}{2}\delta(k^2-m^2)\frac{1}{k^2-m^2}\ . \tag{10}$$

It is clear that the integrand, in this case, is meaningless and this is the origin of the claim [5] that $\mathrm{Re}\pi(0)$ is not well defined in the real-time formalism.

Let us, however, note here that the proper way to do the integrations in Eqs. (8) and (9) is to take the limit $\epsilon\to 0$ only after doing the integration. The parameter ϵ, indeed, defines a regularization of the quantities being evaluated. Alternately, note that we can write $\mathrm{Re}\pi(0)$ of Eq. (9) also as

$$\mathrm{Re}\pi(0)=\lim_{\epsilon\to 0}-\lambda^2\left[\frac{1}{2}\frac{\partial}{\partial m^2}\right]\int\frac{d^4k}{(2\pi)^4}\coth\frac{\beta|k_0|}{2}\frac{\epsilon}{(k^2-m^2)^2+\epsilon^2}$$
$$=-\frac{\lambda^2\pi}{2}\frac{\partial}{\partial m^2}\int\frac{d^3k}{(2\pi)^4}\int_{-\infty}^{\infty}dk_0\coth\frac{\beta|k_0|}{2}\delta(k^2-m^2)$$
$$=-\frac{\lambda^2}{4}\frac{\partial}{\partial m^2}\int\frac{d^3k}{(2\pi)^3}\frac{1}{\omega_k}\coth\frac{\beta\omega_k}{2}\ , \tag{11}$$

where

$$\omega_k=(\mathbf{k}\cdot\mathbf{k}+m^2)^{1/2}\ . \tag{12}$$

This is, of course, the result also obtained in the imaginary-time formalism. Let us note that we can take the limit $\epsilon\to 0$ before doing the integration in Eq. (9) provided we define the singular integrands using the identities

$$\delta(k^2-m^2)P\left[\frac{1}{k^2-m^2}\right]=\frac{1}{2}\frac{\partial}{\partial m^2}\delta(k^2-m^2)\ . \tag{13}$$

That this relation is true can easily be seen using the representations in Eq. (7) and we note here that such relations have already been discussed earlier in the literature [9,10].

Now, let us evaluate next $\mathrm{Re}\pi(p)$ for small p^μ using the method of residues. First, let us write

$$\mathrm{Re}\pi(p)=\mathrm{Re}\pi_1(p)+\mathrm{Re}\pi_2(p)\ ,$$

where

$$\mathrm{Re}\pi_1(p)=\lim_{\epsilon\to 0}-\frac{\lambda^2}{2}\int\frac{d^4k}{(2\pi)^4}\coth\frac{\beta|k_0|}{2}\frac{\epsilon}{(k^2-m^2)^2+\epsilon^2}\frac{(k+p)^2-m^2}{[(k+p)^2-m^2]^2+e^2}, \tag{14}$$

$$\mathrm{Re}\pi_2(p)=\lim_{\epsilon\to 0}-\frac{\lambda^2}{2}\int\frac{d^4k}{(2\pi)^4}\coth\frac{\beta|k_0+p_0|}{2}\frac{\epsilon}{[(k+p)^2-m^2]^2+\epsilon^2}\frac{k^2-m^2}{(k^2-m^2)^2+\epsilon^2}\ . \tag{15}$$

Note that, under a redefinition,

$$k_0\leftrightarrow-(k_0+p_0)\ ,\quad \mathrm{Re}\pi_2(p)\leftrightarrow\mathrm{Re}\pi_1(p)\ . \tag{16}$$

Thus we can write

$$\mathrm{Re}\pi(p)=2\,\mathrm{Re}\pi_1(p)\ . \tag{17}$$

The first problem one faces in trying to use the method of residues is that $\coth|x|$ is not an analytic function and, therefore, the method cannot be applied directly. But let us note that we can write

$$\begin{aligned}\mathrm{Re}\pi(p)&=2\,\mathrm{Re}\pi_1(p)\\ &=\lim_{\epsilon\to 0}-\lambda^2\int\frac{d^3k}{(2\pi)^4}\int_{-\infty}^{\infty}dk_0\coth\frac{\beta|k_0|}{2}\frac{\epsilon}{(k^2-m^2)^2+\epsilon^2}\frac{(k+p)^2-m^2}{[(k+p)^2-m^2]^2+\epsilon^2}\\ &=\lim_{\epsilon\to 0}-\lambda^2\int\frac{d^3k}{(2\pi)^4}\int_{0}^{\infty}dk_0\coth\frac{\beta k_0}{2}\frac{\epsilon}{(k^2-m^2)^2+\epsilon^2}\left[\frac{(k_0+p_0)^2-\omega_{k+p}^2}{[(k_0+p_0)^2-\omega_{k+p}^2]^2+\epsilon^2}+\frac{(k_0-p_0)^2-\omega_{k+p}^2}{[(k_0-p_0)^2-\omega_{k+p}^2]^2+\epsilon^2}\right]\ .\end{aligned} \tag{18}$$

The integrand is now analytic in the upper right-hand quadrant of the complex k_0 plane except for isolated singularities. Therefore, we can do the integration using a contour C as shown in Fig. 2. Clearly, the integral vanishes along the arc. However, since $\coth\beta k_0/2$ has a series of poles along the imaginary axis, the integration along this axis will appear to give a nonvanishing contribution. A little analysis, however, shows that the ϵ term would regulate any such contribution to zero in the $\epsilon\to 0$ limit. An alternate and more intuitive way to recognize this is to note that the δ function in the integrand cannot support any contribution from the imaginary axis. Therefore, the integration along the contour gets a contribution only from the real axis which is, of course, the desired result.

Let us next rewrite $\mathrm{Re}\pi(p)$ as

$$\begin{aligned}\mathrm{Re}\pi(p)=\lim_{\mu\to 0^+}\lim_{\epsilon\to 0}\frac{\lambda^2}{4}\int\frac{d^3k}{(2\pi)^4}\frac{\partial}{\partial\epsilon}&\\ \times\int_0^1 dx\int_\mu^\infty dk_0\coth\frac{\beta k_0}{2}&\\ \times\Bigg\{&\frac{1}{k_0+xp_0+\phi_k-\frac{i\epsilon}{2\phi_k}}\frac{1}{k_0+xp_0-\phi_k+\frac{i\epsilon}{2\phi_k}}\\ &+\frac{1}{k_0-xp_0+\phi_k-\frac{i\epsilon}{2\phi_k}}\frac{1}{k_0-xp_0-\phi_k+\frac{i\epsilon}{2\phi_k}}\\ &+\frac{1}{k_0+xp_0+\phi_k+\frac{i\epsilon}{2\phi_k}}\frac{1}{k_0+xp_0-\phi_k-\frac{i\epsilon}{2\phi_k}}\\ &+\frac{1}{k_0-xp_0+\phi_k+\frac{i\epsilon}{2\phi_k}}\frac{1}{k_0-xp_0-\phi_k-\frac{i\epsilon}{2\phi_k}}\\ &+\frac{1}{1-2x}\frac{1}{k_0+xp_0+\phi_k-\frac{i(1-2x)\epsilon}{2\phi_k}}\frac{1}{k_0+xp_0-\phi_k+\frac{i(1-2x)\epsilon}{2\phi_k}}\\ &+\frac{1}{1-2x}\frac{1}{k_0-xp_0+\phi_k-\frac{i(1-2x)\epsilon}{2\phi_k}}\frac{1}{k_0-xp_0-\phi_k+\frac{i(1-2x)\epsilon}{2\phi_k}}\\ &+\frac{1}{1-2x}\frac{1}{k_0+xp_0+\phi_k+\frac{i(1-2x)\epsilon}{2\phi_k}}\frac{1}{k_0+xp_0-\phi_k-\frac{i(1-2x)\epsilon}{2\phi_k}}\\ &+\frac{1}{1-2x}\frac{1}{k_0-xp_0+\phi_k+\frac{i(1-2x)\epsilon}{2\phi_k}}\frac{1}{k_0-xp_0-\phi_k-\frac{i(1-2x)\epsilon}{2\phi_k}}\Bigg\}\ .\end{aligned} \tag{19}$$

Here x is the Feynman parameter and we have defined

$$\phi_k = [(\mathbf{k}+x\mathbf{p})^2 + m^2 - x(1-x)p^2]^{1/2} \tag{20}$$

and we note that, when $p^\mu = 0$,

$$\phi_k = \omega_k \ . \tag{21}$$

The expression in Eq. (21) can now be evaluated by considering the poles enclosed by the contour C and the result is

$$\begin{aligned}
\mathrm{Re}\pi(p) = \lim_{\epsilon\to 0} \frac{i\lambda^2}{8} \int \frac{d^3k}{(2\pi)^3} \frac{\partial}{\partial\epsilon} \Bigg[& \int_0^1 dx \coth\frac{\beta}{2}\left[\phi_k - xp_0 + \frac{i\epsilon}{2\phi_k}\right] \frac{1}{\phi_k + \frac{i\epsilon}{2\phi_k}} \\
& + \int_0^1 dx \coth\frac{\beta}{2}\left[\phi_k + xp_0 + \frac{i\epsilon}{2\phi_k}\right] \frac{1}{\phi_k + \frac{i\epsilon}{2\phi_k}} \\
& + \int_{1/2}^1 dx \frac{1}{1-2x} \coth\frac{\beta}{2}\left[\phi_k - xp_0 - \frac{i(1-2x)\epsilon}{2\phi_k}\right] \frac{1}{\phi_k - \frac{i(1-2x)\epsilon}{2\phi_k}} \\
& + \int_{1/2}^1 dx \frac{1}{1-2x} \coth\frac{\beta}{2}\left[\phi_k + xp_0 - \frac{i(1-2x)\epsilon}{2\phi_k}\right] \frac{1}{\phi_k - \frac{i(1-2x)\epsilon}{2\phi_k}} \\
& + \int_0^{1/2} dx \frac{1}{1-2x} \coth\frac{\beta}{2}\left[\phi_k - xp_0 + \frac{i(1-2x)\epsilon}{2\phi_k}\right] \frac{1}{\phi_k + \frac{i(1-2x)\epsilon}{2\phi_k}} \\
& + \int_0^{1/2} dx \frac{1}{1-2x} \coth\frac{\beta}{2}\left[\phi_k + xp_0 + \frac{i(1-2x)\epsilon}{2\phi_k}\right] \frac{1}{\phi_k + \frac{i(1-2x)\epsilon}{2\phi_k}} \Bigg] \ .
\end{aligned} \tag{22}$$

The $\epsilon \to 0$ limit can be taken after taking the derivative with respect to ϵ and the result can be simplified to

$$\mathrm{Re}\pi(p) = -\frac{\lambda^2}{4}\int \frac{d^3k}{(2\pi)^3}\frac{\partial}{\partial m^2}\left[\int_0^{1/2} dx \frac{1}{\phi_k}\coth\frac{\beta}{2}(\phi_k - xp_0) + \int_0^{1/2} dx\frac{1}{\phi_k}\coth\beta_2(\phi_k + xp_0)\right] . \tag{23}$$

It is obvious from Eq. (23) that

$$\mathrm{Re}\pi(0) = -\frac{\lambda^2}{4}\frac{\partial}{\partial m^2}\int\frac{d^3k}{(2\pi)^3}\frac{1}{\omega_k}\coth\frac{\beta\omega_k}{2} \tag{24}$$

which is the same as Eq. (11). Furthermore, we can Taylor expand $\mathrm{Re}\pi(p)$ for small p^μ and up to order p^2, it has the form

$$\begin{aligned}
\mathrm{Re}\pi(p) = & -\frac{\lambda^2}{4}\frac{\partial}{\partial m^2}\int\frac{d^3k}{(2\pi)^3}\frac{1}{\omega_k}\coth\frac{\beta\omega_k}{2} + \frac{\lambda^2}{24}\left[\frac{\partial}{\partial m^2}\right]^2\int\frac{d^3k}{(2\pi)^3}p^2\frac{1}{\omega_k}\coth\frac{\beta\omega_k}{2} \\
& -\frac{\lambda^2}{96}\frac{\partial}{\partial m^2}\int\frac{d^3k}{(2\pi)^3}(p^0)^2\frac{1}{\omega_k}\frac{\partial^2}{\partial\omega_k^2}\coth\frac{\beta\omega_k}{2} + O(p^3) \ .
\end{aligned} \tag{25}$$

It is obvious now that $\mathrm{Re}\pi(p)$ is analytic at $p^\mu = 0$ and its value there is equal to $\mathrm{Re}\pi(0)$. We also note that our result, namely, Eq. (23) has the same form as the result of Ref. [1] [see their Eq. (3.33)] except for the limits of x integration. This corresponds to the fact that our result in Eq. (25) agrees with that of Ref. [1] [see their Eq. (3.24)] except for the coefficient of the last term. This difference, however, disappears for values of $p^0 = i2\pi n/\beta$, and agrees with the usual imaginary-time result, as can be checked by making the change of variables $x \to 1-x$ and

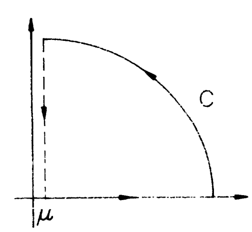

FIG. 2. Contour in the complex k^0 plane used in the integration.

$\mathbf{k} \to -\mathbf{k}-\mathbf{p}$ in the second term of Eq. (23) and performing the x integration, following Ref. [1].

III. CONCLUSION

We have shown within the framework of the conventional real-time formalism (namely, without any new *ad hoc* Feynman rules) that, when evaluated carefully and consistently, the real part of the self-energy for a scalar field is well defined at $p^{\mu}=0$. We have nothing to say about the singularity structure of $\mathrm{Im}\pi(p)$ at the present time.

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