Variational calculations of the effective potential with non-Gaussian trial wave functionals

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Variational calculations of the effective potential, going beyond the Gaussian approximation, are discussed in the context of $\lambda\phi^4$ theory. Following Polley and Ritschel we use trial wave functionals obtained by a nontrivial unitary operator $U=e^{-isB}$ acting on a Gaussian wave functional. We discuss in detail two cases in which the operator B has the forms (i) $B=\pi^3$, and (ii) $B=\pi_R\phi_T^2$, where ϕ is the field operator and π is its canonical conjugate. [R and T refer to radial and transverse directions in the O(N)-symmetric case.] We calculate the expectation value of the Hamiltonian in the non-Gaussian trial states thus generated, and obtain the optimization equations for the variational-parameter functions of the ansatz. These can be solved explicitly at $\varphi_c=0$ and lead to a nontrivial correction to the mass renormalization, with respect to the Gaussian case. Numerical results are obtained for the (0+1)-dimensional case, and show a worthwhile quantitative improvement over the Gaussian approximation.

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I. INTRODUCTION

Variational methods using Gaussian trial wave functions have a long history (see [1–4] and references therein). Recent results for the four-dimensional $\lambda\phi^4$ theory are particularly tantalizing, since they challenge the dogma that this theory is "trivial." In fact, they imply the existence of two distinct $\lambda\phi^4$ theories: "precarious" [5,6] and "autonomous" [7,8,9]. It is important to try to improve upon the Gaussian approximation in order to test the validity of these results. One way of doing so is to formulate a quasiperturbative expansion which gives the Gaussian result in its first order [4,10,11]. A complementary approach is to continue using the variational method, but using more elaborate trial wave functionals. Initially this might seem quite impractical since one can essentially only compute Gaussian functional integrals. However, the "method of nonlinear canonical transformations" (NLCT) introduced by Polley and Ritschel [12,13] allows one to circumvent this difficulty. (Related methods have been employed in quantum mechanics [14], field theory [15], and in solid-state physics [16].)

The idea [12,13] is to consider trial wave functionals of the form

$$\Psi=U\Psi_G\ , \tag{1.1}$$

where Ψ_G is normalized Gaussian wave functional, and U is a unitary operator $U=e^{-isB}$. The computation of the expectation value of H in this state,

$$\langle\Psi|H|\Psi\rangle=\langle\Psi_G|U^\dagger HU|\Psi_G\rangle\ , \tag{1.2}$$

reduces to the calculation of the *Gaussian* expectation value of the "transformed Hamiltonian" $\tilde{H}=U^\dagger HU$. Furthermore, $\tilde{H}$ is easily constructed once one has computed the transforms of the field and its canonical conjugate

$$\tilde{\phi}(\mathbf{p})=U^\dagger\phi(\mathbf{p})U,\quad \tilde{\pi}(\mathbf{p})=U^\dagger\pi(\mathbf{p})U\ , \tag{1.3}$$

since the transform of ϕ^2 is just $\tilde{\phi}^2$, etc. The unitary transformation of an operator O is expressible as a multiple commutator series:

$$\tilde{O}=e^{isB}Oe^{-isB}=\sum_{n=0}^{\infty}\frac{i^n s^n}{n!}[B,O]_n\ , \tag{1.4}$$

where $[B,O]_n=[B,[B,O]_{n-1}]$ and $[B,O]_0=O$. With a suitable choice of the Hermitian operator B one can arrange that the series truncates after the first nontrivial term. This then offers the prospect of a feasible calculation [17].

Possible operators B can be constructed from products of ϕ's and π's with various momentum arguments, $\mathbf{p}_1,\ldots,\mathbf{p}_n$. (We use momentum-space arguments rather than x-space arguments for the field; see Appendix A.) The momentum arguments must sum to zero to preserve translation invariance, but apart from this constraint we may include any function $f(\mathbf{p}_1,\ldots,\mathbf{p}_n)$. We call this the "correlator" and treat it as a "variational-parameter function" to be determined, ideally, by minimization of the energy. (The other variational-parameter function we have is the kernel in the Gaussian wave functional.) This procedure will lead to a variational approximation to the effective potential which can represent an improvement upon the Gaussian effective potential (GEP). Thanks to the variational principle we know that any lowering of the energy (for fixed bare parameters) represents an improvement.

The parameter s serves as a convenient bookkeeping device, helping us to identify and organize the non-Gaussian terms. (Notice, however, that s is really redundant, and could always be absorbed into the correlator f.) With s we may write

$$\langle\Psi|H|\Psi\rangle=V_G+sV_1+s^2V_2+\cdots\ , \tag{1.5}$$

where V_G is the GEP. If the linear term V_1 is nonzero we will certainly be able to lower the energy, and hence

obtain an improvement over the GEP.

This approach was used in Refs. [13,18] with operators of the form

$$B=\int_p \int_{q_1} \cdots \int_{q_n} f(\mathbf{p},\mathbf{q}_1,\ldots,\mathbf{q}_n)\pi(\mathbf{p})\phi(\mathbf{q}_1)\cdots\phi(\mathbf{q}_n) \times\bar{\delta}(-\mathbf{p}+\mathbf{q}_1+\cdots+\mathbf{q}_n)\ , \tag{1.6}$$

with a correlator of the form [19]

$$f(\mathbf{p},\mathbf{q}_1,\ldots,\mathbf{q}_n)=g(\mathbf{p},\mathbf{q}_1,\ldots,\mathbf{q}_n)h(\mathbf{p})k(\mathbf{q}_1)\cdots k(\mathbf{q}_n)\ , \tag{1.7}$$

with

$$h(\mathbf{p})k(\mathbf{p})=0\ . \tag{1.8}$$

This constraint ensures that the argument of the π is never the same as the argument of any of the ϕ's, so that they will effectively commute. This then causes the multiple-commutator series, Eq. (1.4), to terminate after the first nontrivial term. However, it is hard to optimize the ansatz [one would need to introduce a Lagrange multiplier function to impose the constraint (1.8)], and so one is reduced to making rather *ad hoc* guesses for the functions h, k, and g. References [13,18] used simple step functions for h and k and took g to be constant.

In this paper we examine other transformations which do not require constraints on the correlator f in order to be tractable. One can then hope to determine, at least approximately, the optimal form of the correlator f. We feel that this adds significantly to the power and elegance of the method. To find suitable operators B we are guided by the following considerations. (i) Operators B, which are only linear or quadratic in ϕ's and π's, do not produce non-Gaussian wave functionals, and so do not give any improvement upon the GEP. Thus, we must go to cubic operators, at least. (ii) In the O(N)-symmetric case, we want B to respect the O($N-1$) symmetry that remains in the presence of a nonzero, constant classical field. (iii) Time-reversal invariance, which implies that the wave functional is real, not complex, requires that B should contain an odd number of π factors. (iv) We want the multiple-commutator series (1.4) to truncate after the first nontrivial term [20]. These considerations leave us with three possible cubic forms for B: π^3, $\pi_R\phi_T^2$, and $\pi_R\pi_T^2$.

In this paper we explore the first two possibilities in detail. We give unrenormalized results for the general $(\nu+1)$-dimensional case, and briefly discuss how to tackle the optimization equations and the mass renormalization problem. We also obtain numerical results for the (0+1)-dimensional case, and compare these to the GEP and to exact results. We are currently working to obtain explicit, renormalized results in the higher-dimensional cases.

The organization of this paper is as follows. Section II reviews the Gaussian effective potential (GEP) approach, and its results in the (0+1)-dimensional case. Section III, supplemented by Appendix B, provides formulas for the expectation value of the $\lambda\phi^4$ Hamiltonian in a trial state $\Psi=U\Psi_G$ for a general transformation U. The π^3 transformation is studied in Sec. IV, and the $\pi_R\phi_T^2$ transformation is considered in Sec. V. In each case we first present results for arbitrary dimensionality, including optimization equations and a discussion of the mass renormalization, and then present numerical results in 0+1 dimensions. We summarize our conclusions in Sec. VI. Appendix A gives some details of our notation.

II. $\lambda\phi^4$ THEORY AND THE GEP

We study the $\lambda\phi^4$ theory as the simplest example of an interacting field theory. Its Hamiltonian density is

$$\mathcal{H}=\tfrac{1}{2}\pi^2+\tfrac{1}{2}(\nabla\phi)^2+\tfrac{1}{2}m_B^2\phi^2+\lambda_B\phi^4\ . \tag{2.1}$$

We begin by briefly outlining the GEP method and results; for a more detailed description see Refs. [4,5]. Some details of our notation are explained in Appendix A.

The effective potential, in Symanzik's definition [21], is obtained by minimizing the expectation value of $\mathcal{H}$ over all possible states which have the expectation value of ϕ equal to the constant classical field φ_c. The GEP is obtained by restricting these states to ones whose Schrödinger-representation wave functionals are of Gaussian form:

$$\Psi_G[\phi]=\exp\left[-\tfrac{1}{2}\int_p[\phi(\mathbf{p})-\phi_0\bar{\delta}(\mathbf{p})] \times W(p)[\phi(-\mathbf{p})-\phi_0\bar{\delta}(\mathbf{p})]\right]\ . \tag{2.2}$$

Thus, the GEP can be viewed as a variational approximation to the effective potential:

$$\bar{V}_G(\varphi_c)=\min_{W(p)}\langle\Psi_G|\mathcal{H}|\Psi_G\rangle\ , \tag{2.3}$$

where, by a straightforward calculation, $\varphi_c \equiv\langle\Psi_G|\phi|\Psi_G\rangle=\phi_0$, and

$$\langle\Psi_G|\mathcal{H}|\Psi_G\rangle=J+\tfrac{1}{2}m_B^2(I_0+\phi_0^2) +\lambda_B(\phi_0^4+6I_0\phi_0^2+3I_0^2)\ , \tag{2.4}$$

where

$$I_N=\tfrac{1}{2}\int_k W(k)^{2N-1},\quad J=\tfrac{1}{4}\int_k\left[W(k)+\frac{\mathbf{k}^2}{W(k)}\right]\ . \tag{2.5}$$

Taking the functional derivative, it is easy to show that the optimum kernel function, $W(p)$, is [2,22]

$$W(p)=\sqrt{\mathbf{p}^2+\Omega^2} \tag{2.6}$$

[so that J reduces to $I_1(\Omega)-\frac{1}{2}\Omega^2 I_0(\Omega)$], with the mass parameter Ω given by [23]

$$\Omega^2=m_B^2+12\lambda_B[I_0(\Omega)+\phi_0^2]\ . \tag{2.7}$$

The I_0 and I_1 integrals are divergent, of course, but the GEP can be renormalized by eliminating the bare parameters m_B and λ_B in favor of two new parameters, m_R and λ_R, defined in terms of derivatives of V_G at the origin [2,5].

In most discussions of the GEP the form (2.6) is, quite

naturally, assumed from the beginning. However, it is important in the present context to realize that it emerges from an optimization of a general kernel function $W(p)$. When we go on to consider non-Gaussian trial wave functionals $U\Psi_G$, we will find that the optimum $W(p)$ is not necessarily of this form. (We will also find that the classical field φ_c is no longer the same as the shift parameter ϕ_0 in Ψ_G.)

In 0+1 dimensions the "integrals" reduce to

$$I_1=\frac{\Omega}{2},\quad I_0=\frac{1}{2\Omega}\ , \tag{2.8}$$

and the problem reduces to a problem in ordinary quantum mechanics (QM), where one has a lot of intuition and where exact, or very accurate, numerical results are readily available. We study this case in detail to see to what extent one can improve upon the GEP by using the nonlinear-canonical transformation (NLCT) method.

Even though the bare parameters are finite in 0+1 dimensions, it is convenient to introduce a "renormalized mass" m_R and a dimensionless coupling constant $\hat{\lambda}$ through the equations [5]

$$m_B^2=m_R^2(1-6\lambda),\quad \lambda_B=\hat{\lambda}m_R^3\ . \tag{2.9}$$

For the anharmonic-oscillator case ($m_B^2>0$, i.e., $\hat{\lambda}<\frac{1}{6}$), the GEP gives energies to within 2% of the exact results even in the strong-coupling limit. For the double-well potential ($m_B^2<0$) the GEP evolves, as $\hat{\lambda}$ increases, from a single-well shape, through a triple well, to what is effectively a double-well shape. (Actually, the GEP always has a local minimum at the origin, but for large $\hat{\lambda}$ this becomes just a tiny, shallow dip at the top of the central barrier between the two deep wells.) In field theory one would describe this in terms of spontaneous symmetry breaking, with a first-order phase transition from the symmetric vacuum at $\phi_0=0$ to a broken-symmetry vacuum at $\phi_0=\pm v$. The "critical $\hat{\lambda}$" at which the nontrivial minimum becomes deeper than the one at the origin ($\hat{\lambda}_{\rm crit}=1.149$) would be viewed as the onset of the phase transition. In the QM case this language is, strictly speaking, not correct. Because of quantum-mechanical tunneling through the barrier, the two would-be vacua at $\phi_0=\pm v$ can mix, and the true ground state is their symmetric combination. Thus the expectation value of ϕ remains zero, and there is not true spontaneous symmetry breaking. However, once the barrier becomes sufficiently high, the tunneling rate becomes so small that, for all practical purposes, there is symmetry breaking (viewing the system over some long, but finite, time scale). Thus, it is still convenient to talk about a "phase transition" from qualitatively different "single-well" and "double-well" regimes, even though this is a smooth, not a discontinuous, transition. (Above 1+1 dimensions the tunneling becomes suppressed by a volume factor, and then one does have a true phase transition.)

The GEP is very accurate both at small $\hat{\lambda}$ and at large $\hat{\lambda}$ (which is the "extreme double-well limit," where one has two well-separated harmonic wells). The GEP is less accurate in the transition region, where the error in the ground-state energy can be 10–15 %.

Similar results hold for the O(N)-symmetric $\lambda\phi^4$ theory, whose Hamiltonian density is

$$\mathcal{H}=\tfrac{1}{2}\pi_i\pi_i+\tfrac{1}{2}(\nabla\phi_i)(\nabla\phi_i)+\tfrac{1}{2}m_B^2\phi_i\phi_i+\lambda_B(\phi_i\phi_i)^2\ , \tag{2.10}$$

where $i=1,\ldots,N$, and we sum over repeated indices. As explained in [9] the most general Gaussian wave functional will have different kernel functions for the "radial" and "transverse" fields (where the "radial" direction is defined as the direction picked out by the classical field). The GEP is given by minimizing

$$\begin{aligned}\langle\Psi_G|\mathcal{H}|\Psi_G\rangle=&J^R+\tfrac{1}{2}m_B^2I_0^R+(N-1)(J^T+\tfrac{1}{2}m_B^2I_0^T)+\tfrac{1}{2}m_B^2\phi_0^2+\lambda_B\phi_0^4\\&+\lambda_B[3(I_0^R)^2+(N^2-1)(I_0^T)^2+2(N-1)I_0^RI_0^T+6I_0^R\phi_0^2+2(N-1)I_0^T\phi_0^2]\ .\end{aligned} \tag{2.11}$$

Optimizing the kernels leads to

$$W_R(p)=\sqrt{\mathbf{p}^2+\Omega^2},\quad W_T(p)=\sqrt{p^2+\omega^2}\ , \tag{2.12}$$

with

$$\Omega^2=m_B^2+4\lambda_B[(N-1)I_0^T+3I_0^R+3\phi_0^2]\ , \tag{2.13}$$

$$\omega^2=m_B^2+4\lambda_B[(N+1)I_0^T+I_0^R+\phi_0^2]\ . \tag{2.14}$$

III. GENERAL FORMULAS

In this section we consider the NLCT method with a general unitary transformation applied to the O(N)-symmetric theory. As explained in the Introduction, we need to evaluate $\langle\Psi|H|\Psi\rangle=\langle\Psi_G|\tilde{H}|\Psi_G\rangle$, where $\tilde{H}=U^\dagger HU$. Hereafter we shall abbreviate Gaussian expectation values $\langle\Psi_G|A|\Psi_G\rangle$ by $\langle A\rangle$. One first needs to compute the transformed ϕ's and π's. Quite generally, these can be written in the forms

$$\tilde{\phi}_R(\mathbf{p})=\phi_R(\mathbf{p})+\phi_0\bar{\delta}(\mathbf{p})+s\bar{\phi}_R(\mathbf{p})\ , \tag{3.1}$$

$$\tilde{\phi}_T(\mathbf{p})=\phi_T(\mathbf{p})+s\bar{\phi}_T(\mathbf{p})\ , \tag{3.2}$$

$$\tilde{\pi}_R(\mathbf{p})=\pi_R(\mathbf{p})+s\bar{\pi}_R(\mathbf{p})\ , \tag{3.3}$$

$$\tilde{\pi}_T(\mathbf{p})=\pi_T(\mathbf{p})+s\bar{\pi}_T(\mathbf{p}) \tag{3.4}$$

[where it should be understood that ϕ_T, π_T, etc., are O($N-1$) vectors]. In general, the barred quantities will be power series in s, resulting from the multiple commutator series (1.4). For the specific transformations considered later the multiple-commutator series will natural-

ly truncate after, at most, one nontrivial term, and the barred quantities will then either vanish or be a single product of ϕ's and π's. However, in this section we do not assume this simplification, and our formulas apply to a general, unitary transformation.

The untransformed radial field ϕ_R has been redefined to include a shift $\phi_0\bar{\delta}(\mathbf{p})$, so that the Gaussian wave functional for the radial quantum field is now centered on $\phi_R=0$ (i.e., so that $\langle\phi_R\rangle=0$). However, in general, the additional terms generated by the nonlinear transformation will have nonzero Gaussian expectation value, so that the classical field, φ_c, is not ϕ_0. Instead we have

$$\varphi_c\bar{\delta}(\mathbf{p})=\langle\Psi|\phi_R(\mathbf{p})|\Psi\rangle=\langle\tilde{\phi}_R(\mathbf{p})\rangle=\phi_0\bar{\delta}(\mathbf{p})+s\langle\bar{\phi}_R(\mathbf{p})\rangle\ . \tag{3.5}$$

The ϕ_0 parameter no longer has physical meaning and should be eliminated in favor of φ_c. This can be done directly by substituting the last equation back into Eq. (3.1) above.

Having obtained the transformed ϕ's and π's, it is straightforward to construct the transformed Hamiltonian $\tilde{H}$. The Gaussian expectation value of $\tilde{H}$ can then be written in the form (dropping an overall volume factor)

$$\langle\tilde{\mathcal{H}}\rangle=\langle\tilde{\mathcal{H}}_R\rangle+\langle\tilde{\mathcal{H}}_T\rangle+\lambda_B[2(N-1)(I_0^RI_0^T+I_0^T\varphi_c^2)+s\beta_1^{RT}+s^2\beta_2^{RT}+s^3\beta_3^{RT}+s^4\beta_4^{RT}]\ , \tag{3.6}$$

where

$$\langle\tilde{\mathcal{H}}_R\rangle=J^R+s\kappa_1^R+s^2\kappa_2^R+\tfrac{1}{2}m_B^2(I_0^R+s\gamma_1^R+s^2\gamma_2^R+\varphi_c^2)$$
$$+\lambda_B[3(I_0^R)^2+s\beta_1^R+s^2\beta_2^R+s^3\beta_3^R+s^4\beta_4^R+6\varphi_c^2(I_0^R+s^2\gamma_2^R)+\varphi_c^4]\ , \tag{3.7}$$

and

$$\langle\tilde{\mathcal{H}}_T\rangle=(N-1)J^T+s\kappa_1^T+s^2\kappa_2^T+\tfrac{1}{2}m_B^2[(N-1)I_0^T+s\gamma_1^T+s^2\gamma_2^T]+\lambda_B[(N^2-1)(I_0^T)^2+s\beta_1^T+s^2\beta_2^T+s^3\beta_3^T+s^4\beta_4^T]\ , \tag{3.8}$$

where the κ, β, and γ integrals are given in Appendix B. The NLCT effective potential $V_{\rm NLCT}$ is obtained by optimizing $\langle\tilde{H}\rangle$ with respect to the variational-parameter functions W_R, W_T, and the correlator(s) in the operator B.

IV. π^3 TRANSFORMATION

A. General $(\nu+1)$-dimensional case

In this section we consider the π^3 transformation applied to the $N=1$ theory. This is perhaps the simplest nontrivial example of the NLCT method. The unitary operator $U=e^{-isB}$ is given by

$$B=\tfrac{1}{3}\int_p\int_q\int_r f(\mathbf{p},\mathbf{q},\mathbf{r})\pi(\mathbf{p})\pi(\mathbf{q})\pi(\mathbf{r})\bar{\delta}(\mathbf{p}+\mathbf{q}+\mathbf{r})\ . \tag{4.1}$$

This transformation obviously leaves $\pi(\mathbf{p})$ unchanged. The commutator with $\phi(\mathbf{p})$ gives us directly the "$\bar{\phi}$" of Eq. (3.1) as

$$\bar{\phi}(\mathbf{k})=\int_q\int_r f(\mathbf{k},\mathbf{q},\mathbf{r})\pi(\mathbf{q})\pi(\mathbf{r})\bar{\delta}(\mathbf{k}+\mathbf{q}+\mathbf{r})\ . \tag{4.2}$$

There are no higher terms in the multiple commutator series because this expression commutes with B.

For the $N=1$ case there are no transverse fields, so the effective potential can be obtained just from Eq. (3.7) for $\langle\tilde{\mathcal{H}}_R\rangle$, and we may drop the R superscript. Evaluating the Gaussian matrix elements involved [see Eqs. (B1)–(B8) in Appendix B], we obtain

$$\kappa_2=\tfrac{1}{4}\int_p\int_q\int_r\mathbf{p}^2f^2(\mathbf{p},\mathbf{q},\mathbf{r})W(q)W(r)\bar{\delta}(\mathbf{p}+\mathbf{q}+\mathbf{r})\ , \tag{4.3}$$

$$\gamma_2=\tfrac{1}{2}\int_p\int_q\int_r f^2(\mathbf{p},\mathbf{q},\mathbf{r})W(q)W(r)\bar{\delta}(\mathbf{p}+\mathbf{q}+\mathbf{r})\ , \tag{4.4}$$

$$\beta_1=-2\varphi_c\alpha_1\ , \tag{4.5}$$

$$\beta_2=6I_0\gamma_2\ , \tag{4.6}$$

$$\beta_3=4\varphi_c\alpha_3\ , \tag{4.7}$$

$$\beta_4=3(\gamma_2^2+\alpha_4)\ , \tag{4.8}$$

where the α's stand for

$$\alpha_1=\int_p\int_q\int_r f(\mathbf{p},\mathbf{q},\mathbf{r})\bar{\delta}(\mathbf{p}+\mathbf{q}+\mathbf{r})\ , \tag{4.9}$$

$$\alpha_3=\int_{p_1}\int_{p_2}\int_{p_3}f(\mathbf{p}_1+\mathbf{p}_2,-\mathbf{p}_1,-\mathbf{p}_2)f(\mathbf{p}_1+\mathbf{p}_3,-\mathbf{p}_1,-\mathbf{p}_3)f(\mathbf{p}_2-\mathbf{p}_3,-\mathbf{p}_2,\mathbf{p}_3)W(p_1)W(p_2)W(p_3)\ , \tag{4.10}$$

$$\alpha_4=\int_{p_1}\int_{p_2}\int_{p_3}\int_{p_4}f(\mathbf{p}_1+\mathbf{p}_2,-\mathbf{p}_1,-\mathbf{p}_2)f(\mathbf{p}_1+\mathbf{p}_3,-\mathbf{p}_1,-\mathbf{p}_3)$$
$$\times f(\mathbf{p}_2+\mathbf{p}_4,-\mathbf{p}_2,-\mathbf{p}_4)f(\mathbf{p}_3+\mathbf{p}_4,-\mathbf{p}_3,-\mathbf{p}_4)W(p_1)W(p_2)W(p_3)W(p_4)\ . \tag{4.11}$$

Thus, $\langle\tilde{\mathcal{H}}\rangle$ can finally be written as

$$\langle\tilde{\mathcal{H}}\rangle=\langle\mathcal{H}\rangle_G+s^2\kappa_2+\tfrac{1}{2}m_B^2s^2\gamma_2+\lambda_B[-2\varphi_cs\alpha_1+6s^2(I_0+\varphi_c^2)\gamma_2+4\varphi_cs^3\alpha_3+3s^4(\gamma_2^2+\alpha_4)]\ , \tag{4.12}$$

where the GEP part $\langle \mathcal{H} \rangle_G$ is given by Eq. (2.4).

Note that in writing these expressions we have assumed that $f(\mathbf{p},\mathbf{q},\mathbf{r})$ is symmetric in its three arguments. There is obviously no loss of generality in so doing since only the symmetric part of f contributes in Eq. (4.1). However, when attempting to optimize f, one should take care to explicitly symmetrize the above equations before taking a functional derivative with respect to f. (The point is that the functional derivative considers an *arbitrary* variation of f, and does not necessarily stay within the subspace of symmetric functions.) Also note that f must be invariant under an overall rotation or inversion of our coordinate system. Therefore, in particular, $f(\mathbf{p},\mathbf{q},\mathbf{r})=f(-\mathbf{p},-\mathbf{q},-\mathbf{r})$.

Taking functional derivatives of Eq. (4.12) with respect to the kernel function $W(p)$ and the correlator $f(\mathbf{p},\mathbf{q},\mathbf{r})$ yields

$$\frac{\bar{\delta}\langle \tilde{\mathcal{H}} \rangle}{\delta W(k)} = \frac{\bar{\delta}\langle \mathcal{H} \rangle_G}{\delta W(k)} + \tfrac{1}{2}\int_p \int_q s^2 f^2(\mathbf{p},\mathbf{q},\mathbf{k})[(\mathbf{p}^2+\Omega^2)W(q)]\bar{\delta}(\mathbf{p}+\mathbf{q}+\mathbf{k})$$
$$+12s^3\lambda_B\varphi_c\int_q\int_r \hat{f}(\mathbf{k},\mathbf{r})\hat{f}(\mathbf{q},-\mathbf{r})\hat{f}(\mathbf{k},\mathbf{r})W(q)W(r)+6s^4\lambda_B\gamma_2\int_q \hat{f}^2(\mathbf{q},\mathbf{k})W(q)$$
$$+12s^4\lambda_B\int_{p_1}\int_{p_2}\int_{p_3}\hat{f}(\mathbf{p}_1,\mathbf{p}_2)\hat{f}(\mathbf{p}_1,\mathbf{p}_3)\hat{f}(\mathbf{p}_2,\mathbf{k})\hat{f}(\mathbf{p}_3,\mathbf{k})W(p_1)W(p_2)W(p_3)\,, \tag{4.13}$$

$$\frac{\bar{\delta}\langle \tilde{\mathcal{H}} \rangle}{\delta f(\mathbf{p},\mathbf{q},\mathbf{r})} = \Big[-2s\lambda_B\varphi_c+\frac{s^2}{6}f(\mathbf{p},\mathbf{q},\mathbf{r})[(\mathbf{p}^2+\Omega^2)W(q)W(r)+(\mathbf{q}^2+\Omega^2)W(r)W(p)+(\mathbf{r}^2+\Omega^2)W(p)W(q)]$$
$$+4\lambda_B\varphi_c s^3\int_t W(t)[\hat{f}(-\mathbf{q},\mathbf{t})W(q)\hat{f}(\mathbf{r},\mathbf{t})W(r)+\hat{f}(-\mathbf{p},\mathbf{t})W(p)\hat{f}(\mathbf{r},\mathbf{t})W(r)+\hat{f}(-\mathbf{p},\mathbf{t})W(p)\hat{f}(\mathbf{q},\mathbf{t})W(q)]$$
$$+2s^4\lambda_B\gamma_2 f(\mathbf{p},\mathbf{q},\mathbf{r})[W(q)W(r)+W(r)W(p)+W(p)W(q)]$$
$$+4s^4\lambda_B\int_{t_1}\int_{t_2}\hat{f}(\mathbf{t}_1,\mathbf{t}_2)W(t_1)W(t_2)[\hat{f}(\mathbf{q},\mathbf{t}_1)W(q)\hat{f}(\mathbf{r},\mathbf{t}_2)W(r)+\hat{f}(\mathbf{p},\mathbf{t}_1)W(p)\hat{f}(\mathbf{r},\mathbf{t}_2)W(r)$$
$$+\hat{f}(\mathbf{p},\mathbf{t}_1)W(p)\hat{f}(\mathbf{q},\mathbf{t}_2)W(q)]\Big]\bar{\delta}(\mathbf{p}+\mathbf{q}+\mathbf{r})\,, \tag{4.14}$$

where

$$\hat{f}(\mathbf{q},\mathbf{r})=\int_p f(\mathbf{p},\mathbf{q},\mathbf{r})\bar{\delta}(\mathbf{p}+\mathbf{q}+\mathbf{r}) \tag{4.15}$$

and

$$\Omega^2=m_B^2+12\lambda_B(I_0+\varphi_c^2)\,. \tag{4.16}$$

Setting these derivatives to zero yields optimization equations that determine the optimal kernel and correlator. The product sf is proportional to φ_c near the origin, so one can solve the equations explicitly at $\varphi_c=0$. In (4.13) only the Gaussian term survives, so that

$$\overline{W}_0(p)=\sqrt{p^2+\Omega_0^2}\,, \tag{4.17}$$

where the zero subscript indicates a quantity evaluated at $\varphi_c=0$. In (4.14) the first two terms dominate as $\varphi_c\to 0$, so that

$$s=12\lambda_B\varphi_c\,, \tag{4.18}$$

$$\bar{f}_0(\mathbf{p},\mathbf{q},\mathbf{r})=[(\mathbf{p}^2+\Omega_0^2)\overline{W}_0(q)\overline{W}_0(r)+(\mathbf{q}^2+\Omega_0^2)\overline{W}_0(r)\overline{W}_0(p)+(\mathbf{r}^2+\Omega_0^2)\overline{W}_0(p)\overline{W}_0(q)]^{-1}$$
$$=\{\overline{W}_0(p)\overline{W}_0(q)\overline{W}_0(r)[\overline{W}_0(p)+\overline{W}_0(q)+\overline{W}_0(r)]\}^{-1}\,. \tag{4.19}$$

(Recall that only the product sf is really meaningful: We have simply chosen to normalize f in a convenient fashion, letting s carry the other factors.) Stepping away from the origin, one could proceed to solve the equations iteratively, obtaining $\bar{f}$ and $\overline{W}$ as power series in φ_c.

The second derivative of the effective potential, $V=\min\langle \mathcal{H} \rangle$, at the origin can be used to define a renormalized mass:

$$m_R^2=\left.\frac{d^2V}{d\varphi_c^2}\right|_{\varphi_c=0}\,. \tag{4.20}$$

This is most conveniently obtained by *first* eliminating the redundant parameter s, setting it to be $12\lambda_B\varphi_c$, thereby making $\langle \tilde{\mathcal{H}} \rangle$ manifestly an even function of φ_c. One can then obtain m_R^2 from a partial first derivative with respect to φ_c^2:

$$m_R^2=2\left.\frac{dV}{d(\varphi_c^2)}\right|_{\varphi_c=0}=2\left.\frac{\partial\langle \tilde{H} \rangle}{\partial(\varphi_c^2)}\right|_{\varphi_c=0}\,. \tag{4.21}$$

Only the partial derivative is needed because the (functional) derivatives with respect to f and W vanish by virtue of the optimization equations. This gives

$$m_R^2=\Omega_0^2+2(12\lambda_B)^2(\kappa_2+\tfrac{1}{2}\Omega_0^2\gamma_2-\tfrac{1}{6}\alpha_1)\,, \tag{4.22}$$

but from the f equation (4.14), multiplied by $f(\mathbf{p},\mathbf{q},\mathbf{r})$ and integrated, we obtain the identity

$$\kappa_2+\tfrac{1}{2}\Omega_0^2\gamma_2=\tfrac{1}{12}\alpha_1 \quad (\text{at } \varphi_c=0) \ . \tag{4.23}$$

Therefore, the relationship between m_R^2 and m_B^2 is given by

$$m_R^2=\Omega_0^2-24\lambda_B^2\alpha_1|_{\varphi_c=0} \ , \tag{4.24}$$

$$\Omega_0^2=m_B^2+12\lambda_B I_0(\Omega_0) \ . \tag{4.25}$$

In the Gaussian case we would have had $m_R^2=\Omega_0^2$. The extra term involves the α_1 integral of (4.9) evaluated with the $\varphi_c=0$ forms of f and W above, which can be rewritten as

$$\alpha_1|_0=\int_0^\infty dt\int d^\nu x\int_p\int_q\int_r e^{-i\mathbf{x}\cdot(\mathbf{p}+\mathbf{q}+\mathbf{r})}\frac{1}{\overline{W}_0(p)\overline{W}_0(q)\overline{W}_0(r)}e^{-[\overline{W}_0(p)+\overline{W}_0(q)+\overline{W}_0(r)]t} \ , \tag{4.26}$$

where we have used the Fourier-transform form of $\overline{\delta}(\mathbf{p}+\mathbf{q}+\mathbf{r})$. This integral reduces to

$$\alpha_1|_0=4\int d^{\nu+1}x\,[G(x)]^3 \ , \tag{4.27}$$

where

$$G(x)=\int\frac{d^{\nu+1}p}{(2\pi)^{\nu+1}}\frac{e^{-ip^\mu x_\mu}}{p^\mu p_\mu+\Omega^2}$$

$$=\int_p\frac{e^{-i\mathbf{x}\cdot\mathbf{p}}e^{-\omega|t|}}{2\omega} \quad [\text{with } \omega=\overline{W}_0(p)] \tag{4.28}$$

is the (Euclidean) x-space propagator. In this form we recognize (4.24) as the same modification of the mass renormalization that one finds in second order in the post-Gaussian expansion [11]. In fact, $\alpha_1|_0$ corresponds to $\frac{2}{3}I^{(3)}$, the integral arising from the "barred circle" vacuum diagram. This is finite in $1+1$ dimensions, logarithmically divergent in $2+1$ dimensions (whereas the Gaussian term in the bare mass is linearly divergent), and quadratically divergent (like the Gaussian term) in $3+1$ dimensions.

B. Numerical results in 0+1 dimensions

In quantum mechanics, with $f=1$ and $W(p)=\Omega$, $\langle\tilde{\mathcal{H}}\rangle$ becomes

$$\langle\tilde{\mathcal{H}}\rangle=\langle\mathcal{H}\rangle_G+\tfrac{1}{4}m_B^2s^2\Omega^2$$

$$+\lambda_B\left[-2\varphi_c s+3s^2\Omega^2\left[\frac{1}{2\Omega}+\varphi_c^2\right]\right.$$

$$\left.+4\varphi_c s^3\Omega^3+\tfrac{15}{4}s^4\Omega^4\right] \ . \tag{4.29}$$

The effective potential is obtained by minimizing this expression with respect to s and Ω, for each value of φ_c. We have carried out this procedure numerically and in Fig. 1 we show the resulting potential, compared to the GEP, for two illustrative cases. As expected, the curves lie below the corresponding GEP curves. The two curves coincide at the origin because there the optimal value of s is zero. As $|\varphi_c|$ increases, the optimal s becomes positive, and the optimal Ω changes a little from its GEP value, resulting in a lowering of the effective potential.

As a quantitative measure of the degree of improvement we can examine the ground-state energy estimate, obtained from the value at the global minimum of the potential. This can be compared with the corresponding GEP estimate and with the exact result. For m_B^2 positive (the anharmonic-oscillator case) V_{NLCT} changes only insignificantly from the GEP and its minimum remains at the origin, yielding the same ground-state energy as the GEP. This value is within 2% of the exact result. For m_B^2 negative (the double-well case) there can be more significant changes in the shape and in the ground-state energy. A quantitative comparison is given in Table I (with "exact" results taken from the numerical calculations of [24]). As in Ref. [4] we use the dimensionless variable

$$\zeta^2=\frac{m_B^2}{(2\lambda_B)^{2/3}} \ , \tag{4.30}$$

and quote the energies, relative to the minimum of the classical potential, in units of $|m_B^2|$.

The NLCT calculation starts to yield lower ground-state energies than the GEP once $\zeta^2<-1.412\,78$, because then the global minimum moves away from the origin. For example, at $\zeta^2=-2$ [Figs. 1(a) and (b)] the GEP

TABLE I. Comparison of π^3 NLCT and GEP energies to exact results [24] for (0+1)-dimensional $\lambda\phi^4$ theory.

ζ^2	E_0(GEP)	$E_0(\pi^3)$	E_0(exact)	% Error (GEP)	% Error (π^3)
−2	0.441 942	0.439 712	0.402 268	9.86	9.31
−4	0.651 360	0.631 827	0.572 412	13.79	10.38
−5	0.669 565	0.656 809	0.635 011	5.44	3.43
−7	0.685 542	0.678 318	0.676 348	1.36	0.29
−10	0.694 821	0.690 719	0.690 392	0.64	0.05
−50	0.706 043	0.705 688	0.705 686	0.05	3×10^{-4}

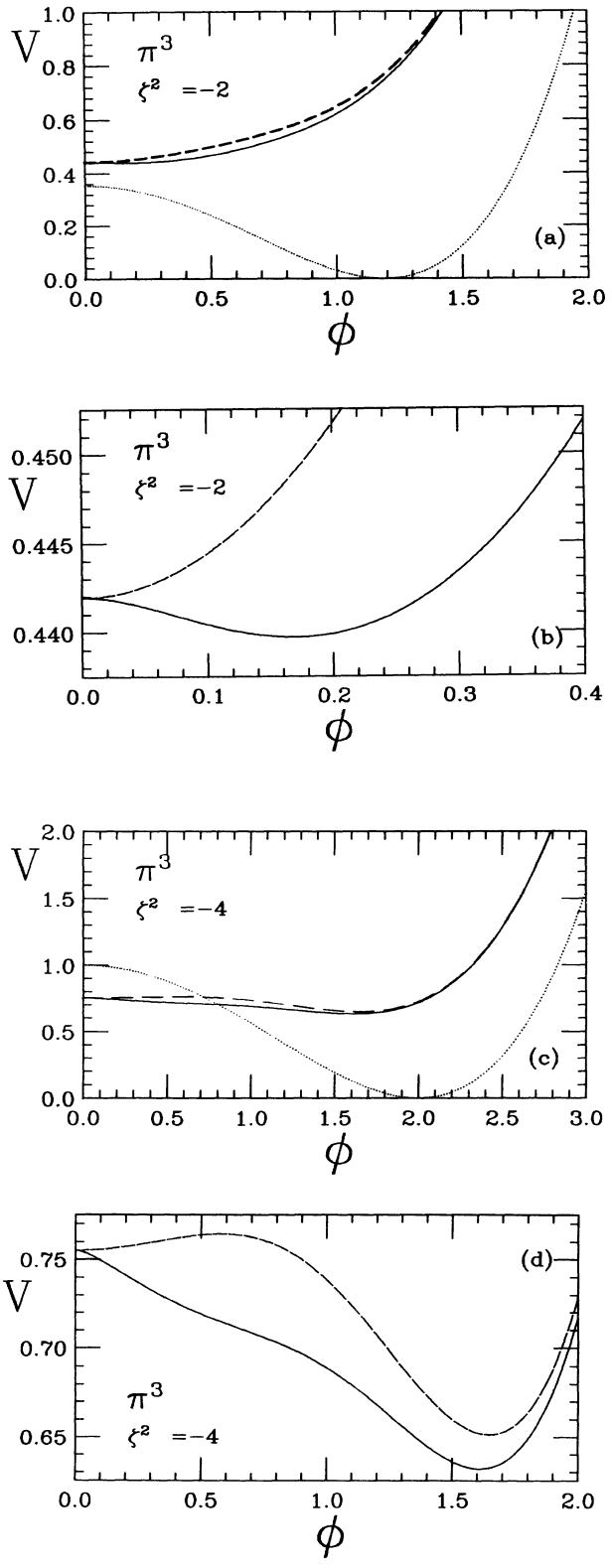

FIG. 1. (a) The effective potential for the NLCT π^3 (solid line) compared to the GEP (dashed line) and the classical potential (dotted line) in 0+1 dimensions, for $\zeta^2 = -2$. A closeup of the difference between the two effective potentials is shown in (b). Corresponding plots for $\zeta^2 = -4$ are shown in (c) and (d). (Units: $|m_B| = 1$.)

still has a single-well shape, but V_{NLCT} has developed lower minima away from the origin. For $\zeta^2 = -4$ [Figs. 1(c) and (d)] both the GEP and V_{NLCT} have a double-well shape, but the latter has a lower minimum value. The GEP value is 13.79% too large [25]; the NLCT calculation improves this, but only by an additional 3%. However, as we get into the "deep-double-well" region ($\zeta^2 < -10$) the percentage error is reduced by an order of magnitude or more. Thus, the moral is that the NLCT method provides corrections to the GEP results in the same sense that a "second-order" calculation can be expected to improve a "first-order" one: That is, in the "transition region," where the GEP itself is least accurate, the changes are large in absolute terms but small as a percentage of the actual error: In the "deep-double-well" region, where the GEP itself is quite accurate, the changes are small in absolute terms, but dramatically reduce the percentage error.

Next we discuss the changes in the shape of the potential. For this purpose, it is convenient to use the "renormalized parameters" m_R, $\hat{\lambda}$ defined in Eqs. (2.9). As $\hat{\lambda}$ increases the effective potential changes from a single-well to a double-well shape. This can be seen in Fig. 2, which compares the GEP and the π^3 effective potential. The GEP always has a local minimum at the origin, whereas for the π^3 potential, the origin becomes a local maximum once $\hat{\lambda}$ exceeds $1/\sqrt{8}$, and a pair of shallow minima de-

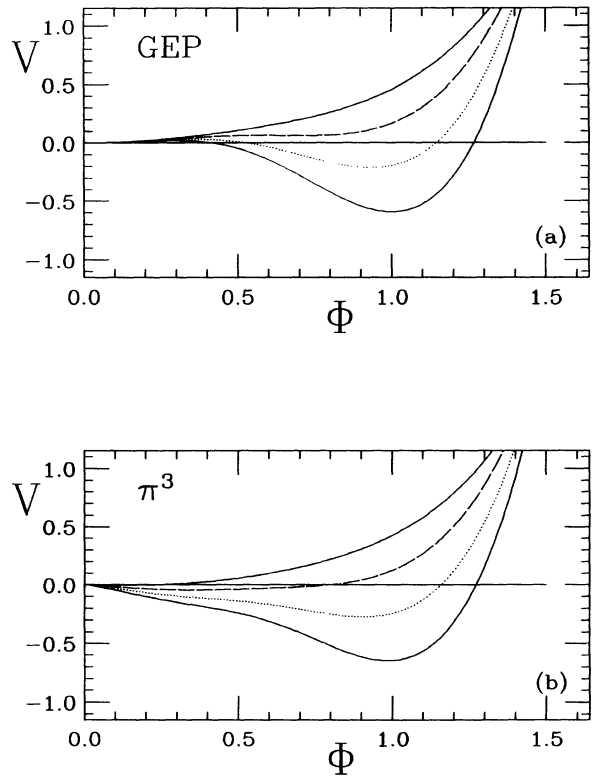

FIG 2. The GEP (a) and the effective potential for the π^3 transformation (b) in 0+1 dimensions for $\hat{\lambda} = 0.5$, 1.0, 1.5, 2.0 (from top to bottom). This plot utilizes the "renormalized parameters" of Eq. (2.9).

velop on either side of the origin. If we were in higher dimensions, this could be described as a "second-order phase transition." However, as $\hat{\lambda}$ is increased further, a second pair of minima appear, farther out from the origin, and these soon become deeper than the original pair of minima. This "first-order transition" happens at $\hat{\lambda}=1.119$, very close to the "critical $\hat{\lambda}$" obtained in the GEP case.

This behavior can be seen in Fig. 3, which plots $\phi_{\min}$, the value of $|\varphi_c|$ (in units of m_R) at which the global minimum of the effective potential occurs. (If we were in higher dimensions, $\phi_{\min}$ would be an "order parameter" for a spontaneous-symmetry-breaking phase transition.) The general features of Fig. 3, that the transition occurs "earlier" (at a lower $\hat{\lambda}$), and is "softer" (involves a smoother variation of $\phi_{\min}$) than the GEP would imply, can be expected to be true in higher dimensions as well. Indeed, this is what Polley and Ritschel have found in their (1+1)-dimensional NLCT calculations (see Fig. 2 of Ref. [13]).

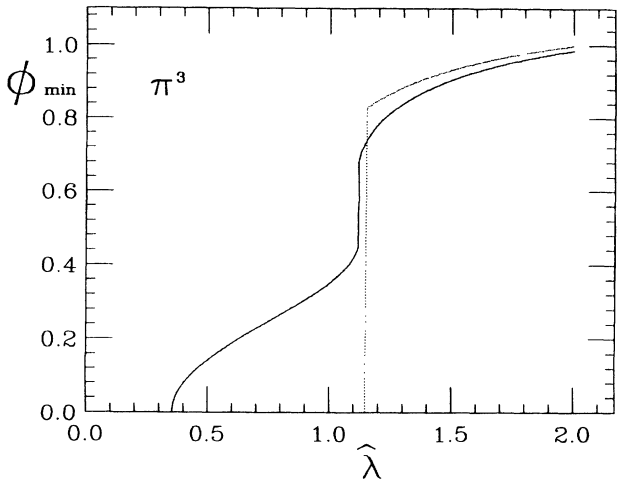

FIG. 3. The value $\phi_{\min}$ of φ_c at the global minimum of the effective potential as a function of $\hat{\lambda}$ for the π^3 ansatz (solid) and the GEP (dotted). The ansatz shows a "second-order phase transition" at $\hat{\lambda}=1/\sqrt{8}$ and a "first-order transition" at $\hat{\lambda}=1.119$.

V. $\pi_R\phi_T^2$ TRANSFORMATION

A. General $(\nu+1)$-dimensional case

In the O(N)-symmetric case ($N\geq 2$) there is a nice form of the operator B similar to Ref. [13],

$$B=\int_p\int_q f(\mathbf{p},\mathbf{q})\pi_R(\mathbf{p}+\mathbf{q})\phi_T(\mathbf{p})\cdot\phi_T(\mathbf{q})\ , \tag{5.1}$$

for which the multiple commutator series (1.4) automatically terminates, since π_R commutes with ϕ_T. The correlator f needs no restrictions, other than overall momentum conservation, already used to write f as a function of two variables instead of three, and invariance under overall rotations or inversions of the coordinate system. We may also take $f(\mathbf{p},\mathbf{q})$ to be symmetric in its two arguments, without loss of generality. Thus, in particular,

$$f(\mathbf{k},l)=f(l,\mathbf{k})=f(-\mathbf{k},-l)\ . \tag{5.2}$$

Taking the commutators of the fields and their canonical conjugate quantities with B gives

$$\begin{aligned}\bar{\phi}_R(\mathbf{k})&=\int_p\int_q f(\mathbf{p},\mathbf{q})\phi_T(\mathbf{p})\cdot\phi_T(\mathbf{q})\bar{\delta}(\mathbf{p}+\mathbf{q}-\mathbf{k})\ ,\\ \bar{\pi}_T(\mathbf{k})&=-2\int_p f(\mathbf{p},\mathbf{k})\pi_R(\mathbf{p}+\mathbf{k})\phi_T(\mathbf{p})\ ,\\ \bar{\pi}_R(\mathbf{k})&=0,\quad \bar{\phi}_T(\mathbf{k})=0\ .\end{aligned} \tag{5.3}$$

Since each of these commute with B, there are no further terms in the multiple commutator series. Inserting these into the general formulas of Sec. III, all the radial κ terms, along with all transverse γ and β terms, will vanish. The remaining terms, employing a notation "$\chi_1,\ldots,\chi_7$" parallel to Polley and Ritschel's [13], are

$$\kappa_2^T=\chi_7=\tfrac{1}{4}(N-1)\int_p\int_q(\mathbf{p}+\mathbf{q})^2f^2(\mathbf{p},\mathbf{q})F_T(p)F_T(q)+\tfrac{1}{4}(N-1)\int_p\int_q f^2(\mathbf{p},\mathbf{q})[F_T(p)+F_T(q)]/F_R(|\mathbf{p}+\mathbf{q}|)\ , \tag{5.4}$$

$$\chi_2^R=\chi_2=\tfrac{1}{2}(N-1)\int_p\int_q f^2(\mathbf{p},\mathbf{q})F_T(p)F_T(q)\ , \tag{5.5}$$

$$\beta_3^R=4\varphi_c\chi_4=4\varphi_c(N-1)\int_p\int_q\int_r f(\mathbf{p},\mathbf{q})f(\mathbf{p},\mathbf{r})f(\mathbf{q},-\mathbf{r})F_T(p)F_T(q)F_T(r)\ , \tag{5.6}$$

$$\beta_4^R=3(\chi_2^2+\chi_6)=3\chi_2^2+3(N-1)\int_p\int_q\int_r\int_s f(\mathbf{p},\mathbf{q})f(\mathbf{p},\mathbf{r})f(\mathbf{q},\mathbf{s})f(\mathbf{r},\mathbf{s})F_T(p)F_T(q)F_T(r)F_T(s)\ , \tag{5.7}$$

$$\beta_1^{RT}=4\varphi_c\chi_3=4\varphi_c\tfrac{1}{2}(N-1)\int_p\int_q f(\mathbf{p},\mathbf{q})F_T(p)F_T(q)\ , \tag{5.8}$$

$$\beta_2^{RT}=2\chi_5=2(N-1)\int_p\int_q\int_r f(\mathbf{p},\mathbf{q})f(\mathbf{q},\mathbf{r})F_T(p)F_T(q)F_T(r)\ , \tag{5.9}$$

where

$$F_R=1/W_R,\quad F_T=1/W_T\ . \tag{5.10}$$

This leaves us with

$$\langle\tilde{\mathcal{H}}\rangle=\langle\mathcal{H}\rangle_G+s^2(\chi_7+\tfrac{1}{2}m_B^2\chi_2)+\lambda_B\{4\varphi_c s\chi_3+s^2[6I_0^R+2(N-1)I_0^T+6\varphi_c^2]\chi_2+2s^2\chi_5+4\varphi_c s^3\chi_4+3s^4(\chi_6+\chi_2^2)\}\ , \tag{5.11}$$

where $\langle\mathcal{H}\rangle_G$ is the Gaussian result, given by Eq. (2.4) and

$$I_0^R=\tfrac{1}{2}\int_p F_R(p),\quad I_0^T=\tfrac{1}{2}\int_p F_T(p)\ . \tag{5.12}$$

Functional differentiation with respect to the (inverse) kernel functions $F_R(p)$, $F_T(p)$, and the correlator $f(\mathbf{p},\mathbf{q})$ yields

$$\frac{\bar{\delta}\langle\tilde{\mathcal{H}}\rangle}{\delta F_R^{-1}(k)}=\frac{\bar{\delta}\langle\mathcal{H}\rangle_G}{\delta F_R^{-1}(k)}+\frac{1}{2}s^2(N-1)\int_p f^2(\mathbf{p},\mathbf{k}-\mathbf{p})F_T(p)-3\lambda_B s^2\chi_2 F_R^2(k)\ , \tag{5.13}$$

$$\begin{aligned}\frac{\bar{\delta}\langle\tilde{\mathcal{H}}\rangle}{\delta F_T^{-1}(k)}=\frac{\bar{\delta}\langle\mathcal{H}\rangle}{\delta F_T^{-1}(k)}-\tfrac{1}{2}(N-1)F_T^2(k)\Bigg[&s^2\int_p(\mathbf{p}+\mathbf{k})^2f^2(\mathbf{p},\mathbf{k})F_T(p)+s^2\int_p f^2(\mathbf{p},\mathbf{k})/F_R(|\mathbf{p}+\mathbf{k}|)+s^2\Omega^2\int_p f^2(\mathbf{p},\mathbf{k})F_T(p)\\&+2\lambda_B\Bigg[4\varphi_c s\int_p f(\mathbf{p},\mathbf{k})F_T(p)+s^2\chi_2\\&+2s^2\int_p\int_q[2f(\mathbf{p},\mathbf{q})f(\mathbf{q},\mathbf{k})+f(\mathbf{p},\mathbf{k})f(\mathbf{k},\mathbf{q})]F_T(p)F_T(q)\\&+12\varphi_c s^3\int_p\int_q f(\mathbf{p},\mathbf{q})f(\mathbf{p},\mathbf{k})f(\mathbf{q},-\mathbf{k})F_T(p)F_T(q)\\&+12s^4\int_p\int_q\int_r f(\mathbf{p},\mathbf{q})f(\mathbf{p},\mathbf{r})f(\mathbf{q},\mathbf{k})f(\mathbf{r},\mathbf{k})F_T(p)F_T(q)F_T(r)\\&+6s^4\chi_2\int_p f^2(\mathbf{p},\mathbf{k})F_T(p)\Bigg]\Bigg]\ ,\end{aligned} \tag{5.14}$$

$$\begin{aligned}\frac{\bar{\delta}\langle\tilde{\mathcal{H}}\rangle}{\delta f(\mathbf{p},\mathbf{q})}=&s^2\tfrac{1}{2}(N-1)f(\mathbf{p},\mathbf{q})F_T(p)F_T(q)\left\{(\mathbf{p}+\mathbf{q})^2+\frac{1}{F_R(|\mathbf{p}+\mathbf{q}|)}\left[\frac{1}{F_T(p)}+\frac{1}{F_T(q)}\right]+\Omega^2\right\}\\&+2\lambda_B(N-1)F_T(p)F_T(q)\Bigg\{\varphi_c s+s^2\int_r[f(\mathbf{p},\mathbf{r})+f(\mathbf{q},\mathbf{r})]F_T(r)\\&+6\varphi_c s^3\int_r f(\mathbf{r},\mathbf{p})f(\mathbf{q},-\mathbf{r})F_T(r)+3s^4\chi_2 f(\mathbf{p},\mathbf{q})\\&+6s^4\int_r\int_s f(\mathbf{r},\mathbf{s})f(\mathbf{r},\mathbf{p})f(\mathbf{s},\mathbf{q})F_T(r)F_T(s)\Bigg\}\ ,\end{aligned} \tag{5.15}$$

where

$$\Omega^2=m_B^2+4\lambda_B[(N-1)I_0^T+3I_0^R+3\varphi_c^2]\ . \tag{5.16}$$

The optimal values for the kernels and f are found by setting these expressions equal to zero. As with the π^3 transformation, the optimization equations greatly simplify at the origin. The (inverse) kernels reduce to the GEP solutions

$$F_R(p)=F_T(p)=\frac{1}{\sqrt{\mathbf{p}^2+\Omega_0^2}} \tag{5.17}$$

and the correlator obeys the integral equation

$$\begin{aligned}\bar{f}_0(\mathbf{p},\mathbf{q})\left\{(\mathbf{p}+\mathbf{q})^2+\Omega_0^2+\frac{1}{F_R(|\mathbf{p}+\mathbf{q}|)}\left[\frac{1}{F_T(p)}+\frac{1}{F_T(q)}\right]\right\}\\=1-4\lambda_B\int_r[\bar{f}_0(\mathbf{p},\mathbf{r})+\bar{f}_0(\mathbf{q},\mathbf{r})]F_T(r)\ ,\end{aligned} \tag{5.18}$$

where

$$\Omega_0^2=m_B^2+4\lambda_B(N+2)I_0(\Omega_0)\ , \tag{5.19}$$

and we have fixed s to be

$$s=-4\lambda_B\varphi_c\ , \tag{5.20}$$

so that $\bar{f}$ is normalized conveniently. The solution can be written as

$$\bar{f}_0(\mathbf{p},\mathbf{q})=R(\mathbf{p},\mathbf{q})[1+g(p)+g(q)]\ , \tag{5.21}$$

where

$$R(\mathbf{p},\mathbf{q})=\frac{1}{\sqrt{(\mathbf{p}+\mathbf{q})^2+\Omega_0^2}}\times\frac{1}{\sqrt{(\mathbf{p}+\mathbf{q})^2+\Omega_0^2}+\sqrt{p^2+\Omega_0^2}+\sqrt{q^2+\Omega_0^2}} \tag{5.22}$$

is similar to the correlator used in Ref. [19], and $g(p)$ is determined by a linear integral equation in one variable:

$$g(p)[1-h(p)]=h(p)-4\lambda_B\int_r R(\mathbf{p},\mathbf{r})F_T(r)g(r)\ , \tag{5.23}$$

where

$$h(p)=-4\lambda_B\int_r R(\mathbf{p},\mathbf{r})F_T(r) \tag{5.24}$$

is a known function of p. For dimensional reasons $g(p)$ behaves as $\lambda_B/p^{3-\nu}$ at large p, so that, below $3+1$ dimensions, it can be neglected in (5.21) as far as leading ultraviolet divergences are concerned.

As explained earlier in the context of the π^3 transformation, we can obtain the renormalized mass from a partial differentiation with respect to φ_c^2 at the origin. This gives

$$m_R^2=\Omega_0^2+2(4\lambda_B)^2(\chi_7+\tfrac{1}{2}\Omega^2\chi_2+2\lambda_B\chi_5-\chi_3)|_{f=\bar{f}_0}\ , \tag{5.25}$$

but the optimization equation for f implies the identity

$$\chi_7+\tfrac{1}{2}\Omega_0^2\chi_2+2\lambda_B\chi_5=\tfrac{1}{2}\chi_3 \quad (\text{at } \varphi_c=0)\ , \tag{5.26}$$

so that

$$m_R^2=\Omega_0^2-16\lambda_B^2\chi_3|_{\varphi_c=0}\ , \tag{5.27}$$

with Ω_0^2 given by (5.19) above. If we take $f\simeq R$ we can reduce χ_3, by the same tricks as in the π^3 case, to

$$\chi_3=2(N-1)\int d^{\nu+1}x[G(x)]^3\ , \tag{5.28}$$

so that the same $I^{(3)}$ integral emerges. In 2+1 dimensions the neglect of the $g(p)$ terms is justified in that it would only affect the m_R^2, m_B^2 relation by a finite term. However things will definitely be more complicated in 3+1 dimensions.

B. Numerical results in (0+1) dimensions

In quantum mechanics the correlation function f can be set to unity and the nontrivial transformations reduce to

$$\begin{aligned}\tilde{\phi}_R&=\phi_R+\varphi_c-s(N-1)I_0^T+s\phi_T^2\ ,\\ \tilde{\pi}_T&=\pi_T-2s\pi_R\phi_T\ .\end{aligned} \tag{5.29}$$

The (inverse) kernel functions in the Gaussian become single variables: $F_R\to 1/\Omega$, $F_T\to 1/\omega$, and it is also convenient to make the change of variable

$$s\to\sigma\varphi_c/\langle\bar{\phi}_R\rangle=2\sigma\varphi_c\omega\ . \tag{5.30}$$

The expectation value of $\tilde{\mathcal{H}}$ can then be written as

$$\begin{aligned}\langle\tilde{\mathcal{H}}\rangle=&\langle\mathcal{H}\rangle_G+2\sigma^2\varphi_c^2(N-1)\Omega\omega+m_B^2\sigma^2\varphi_c^2(N-1)\\&+\lambda_B(N-1)\left[\varphi_c^2\frac{1}{\omega}[4\sigma+4\sigma^2(N+3)]+6\sigma^2\varphi_c^2\frac{1}{\Omega}+\varphi_c^4[12\sigma^2+32\sigma^3+12\sigma^4(N+3)]\right]\ .\end{aligned} \tag{5.31}$$

Optimizing $\langle\tilde{\mathcal{H}}\rangle$ with respect to Ω, ω, and σ gives the effective potential. The three optimization equations can be solved numerically, beginning at the origin, where the solutions coincide with the GEP ones. The $\bar{\sigma}$ equation is cubic, but only one of its roots is real. At the origin σ is given by $\bar{\sigma}_0=-2\hat{\lambda}/(3+8\hat{\lambda})$ [with $\hat{\lambda}$ defined by Eq. (5.32) below]. However, s, being proportional to φ_c, is zero at the origin, so the potential coincides with the GEP there. The optimal values of Ω and ω remain quite close to their GEP values near the origin and for large φ_c ($\varphi_c>\phi_{\min}$, where $\phi_{\min}$ is the position of the global minimum of $V_{\rm NLCT}$). The NLCT mass ω is always less than or equal to the GEP ω. The NLCT Ω starts off bigger than the GEP Ω but at about $\frac{1}{2}\phi_{\min}$ it becomes smaller until they practically agree for $\varphi_c>\phi_{\min}$. The parameter s is negative and its magnitude grows to a maximum (whose height is proportional to $\hat{\lambda}$) located near $\phi_{\min}$. Beyond $\phi_{\min}$, s goes rapidly to zero and the two effective potentials match closely.

The quantitative comparison with the GEP and exact ground-state energy eigenvalues is very similar to the π^3 transformation case, so we shall not discuss it in detail. The corrections are moderately large in the "transition region," but give only a modest decrease in the percentage error. In the single-well and deep-double-well regions the changes are very small, but they provide a substantial reduction in the percentage error.

One can gain insight into how the NLCT improves upon the GEP by looking at the form of the new wave function Ψ. To illustrate this we show in Fig. 4 a contour plot of the optimized $|\Psi|^2$ versus ϕ_R and ϕ_T ($N=2$ case) for $\hat{\lambda}=2.0$ ($\zeta^2=-5.95$) and for $\varphi_c=\phi_{\min}$. A corresponding plot of the Gaussian wave function would show concentric, elliptical contours with major axes aligned in the transverse direction. Thus, the unitary transformation is lowering the energy by bending the wave function around into a "banana shape," so that it fits better into the circular valley of the potential.

To discuss the changes in the shape of the effective potential it is again useful to use the "renormalized variables," which in the O(N) case are given by

$$\begin{aligned}m_B^2&=m_R^2[1-2(N+2)\hat{\lambda}]\ ,\\ \lambda_B&=\hat{\lambda}m_R^3\ .\end{aligned} \tag{5.32}$$

In Fig. 5 we show a comparison of the GEP with the NLCT effective potential for a range of $\hat{\lambda}$ values for the $N=2$ case. As with the π^3 transformation, we see that the "phase transition" occurs earlier and is "softened."

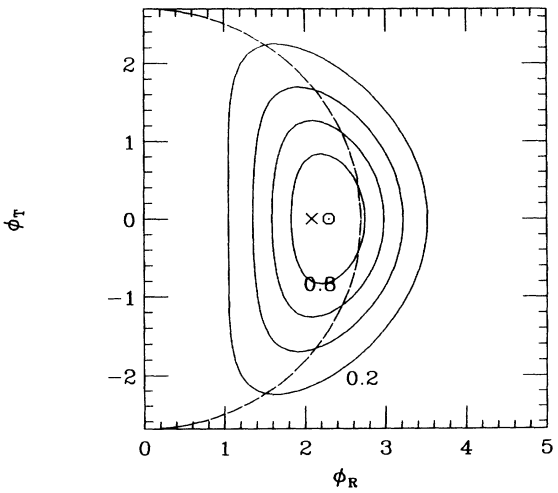

FIG. 4. Contour plot of $|\Psi|^2$ for the (optimized) wave function Ψ produced by the $\pi_R\phi_T^2$ transformation. This example is for the $N=2$ case with $\hat{\lambda}=2.0$ ($\zeta^2=-5.95$) for φ_c at the minimum of the NLCT effective potential. The Ω, ω, and σ parameters have been appropriately optimized. The cross corresponds to the position of φ_c (i.e., $\langle\phi\rangle$), while the dotted half-circle indicates the minimum of the classical potential.

The "critical $\hat{\lambda}$" is 0.84, compared to 1.05 for the GEP, and at $\hat{\lambda}_{\rm crit}$ the potential is flatter and the nontrivial minima are closer to the origin. Furthermore, while the GEP always has a local minimum at the origin, $V_{\rm NLCT}$ has a local maximum at the origin once $\hat{\lambda} > \hat{\lambda}_s = 1.29$.

Similar behavior is seen for higher N values. The "phase transition" is again "earlier" and "softer." This can be seen clearly in Fig. 6, which plots $\phi_{\rm min}$, the value of $|\varphi_c|$ at which the global minimum of the potential occurs, as a function of $\hat{\lambda}$. For $N \leq 5$ the "transition" is still "first order," involving a discontinuous jump in $\phi_{\rm min}$, and it occurs at the $\hat{\lambda}_{\rm crit}$ value tabulated in Table II. At a somewhat larger $\hat{\lambda}$ value, $\hat{\lambda}_s$, the origin becomes a local maximum. As N increases $\hat{\lambda}_{\rm crit}$ and $\hat{\lambda}_s$ come together, until for $N \geq 6$ the "transition" becomes "second order." The "transition" then occurs at $\hat{\lambda}_s$:

$$\hat{\lambda}_s = \frac{2+\sqrt{6N-2}}{4(N-1)} \tag{5.33}$$

which is the value of $\hat{\lambda}$ at which the second derivative of V at the origin vanishes.

Interestingly, the GEP and the $\pi_R \phi_T^2$ potential do not coincide in the "extreme double-well limit," $\hat{\lambda} \to \infty$. In the NLCT case the energy is lowered in the region between the origin and the minimum of the potential, producing a minimum which is much flatter on its inward side, and reducing the value of $\phi_{\rm min}$. (The effect becomes more pronounced for larger values of N.) However, the *depth* of the minimum is hardly changed, in agreement with the fact that the GEP result for the ground-state energy becomes exact as $\hat{\lambda} \to \infty$. Moreover the shape differences would be wiped away if one performed a Maxwell construction to produce a convex effective potential.

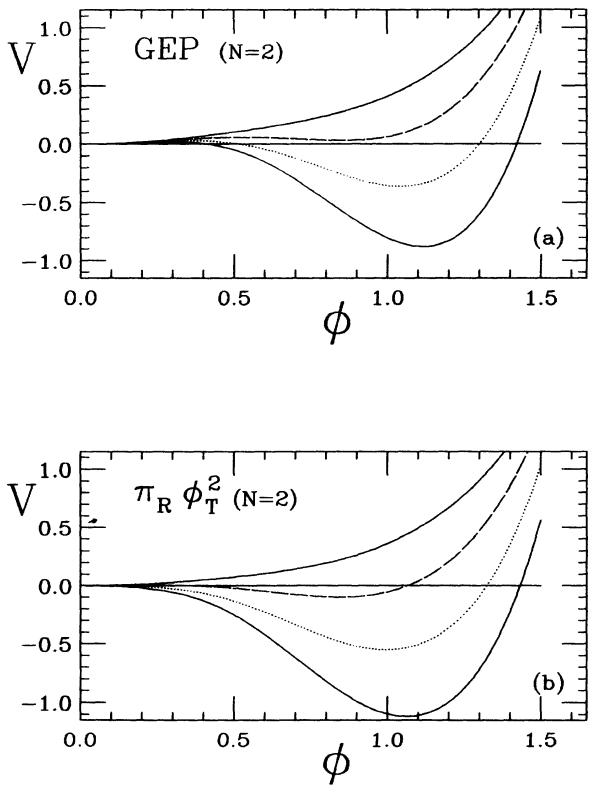

FIG. 5. The GEP (a) and the effective potential for the $\pi_R \phi_T^2$ transformation (b) for $N=2$ in $0+1$ dimensions for $\hat{\lambda}=0.5$, 1.0, 1.5, 2.0 (from top to bottom) in terms of the "renormalized parameters" of Eq. (5.32).

TABLE II. $\hat{\lambda}_{\rm crit}$ and $\hat{\lambda}_s$ for different N for the $\pi_R \phi_T^2$ NLCT in $0+1$ dimensions. For $N \leq 5$ the theory undergoes a "first-order phase transition" at $\hat{\lambda}_{\rm crit}$, while for $N \geq 6$ it undergoes a "second-order phase transition" at $\hat{\lambda}_s$.

N	$\hat{\lambda}_{\rm crit}$	$\hat{\lambda}_s$
2	0.85	1.29
3	0.66	0.75
4	0.54	0.56
5	0.45	0.46
6		0.39
7		0.35
10		0.27

VI. SUMMARY AND CONCLUSIONS

In this work we have studied non-Gaussian variational calculations of the effective potential of $\lambda\phi^4$ theory. The non-Gaussian trial wave functional was generated by a nontrivial unitary operator $U=e^{-isB}$ acting upon a Gaussian [13]. Expressions for the general case were given in Sec. III. Two specific transformations $B=\pi^3$ and $B=\pi_R \phi_T^2$ were investigated in detail.

In $0+1$ dimensions we obtained numerical results for both transformations, and these share the following common features. (i) Both transformations improve the GEP upperbound on the ground-state energy of the double-well potential. The improvement has the characteristics of a "second-order" correction to a "first-order" result: When the GEP is accurate the changes are small in absolute terms, but they substantially improve the percentage error: When the GEP is less accurate the changes are

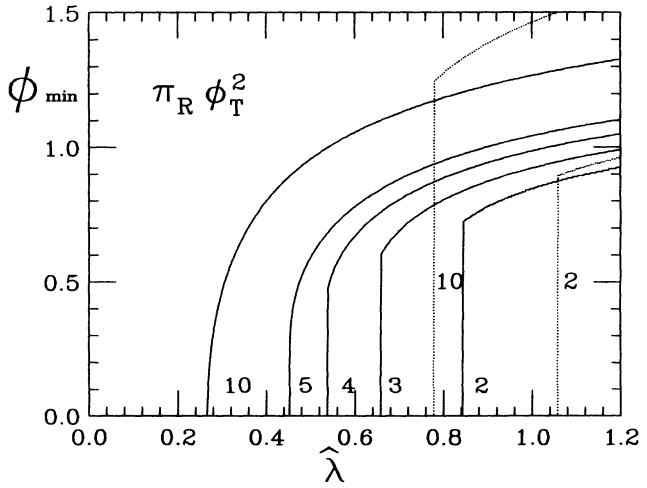

FIG. 6. The value $\phi_{\rm min}$ of φ_c at the global minimum of the effective potential as a function of $\hat{\lambda}$ for the $\pi_R \phi_T^2$ ansatz (solid) and the GEP (dotted) for various N.

larger, absolutely, but they only modestly reduce the percentage error. (ii) Both transformations produce effective potentials whose shape differs noticeably from the GEP in the transition region from single- to double-well behavior: The "phase transition" occurs at lower $\hat{\lambda}$ values and is "softer" and in several cases becomes "second order."

In higher dimensions we have obtained the unrenormalized effective potential for both transformations, as well as the optimization equations which determine the variational-parameter functions. We have solved these equations at the origin, and thereby obtained a correction to the Gaussian mass renormalization. Interestingly, this correction corresponds to that found in the second order of the post-Gaussian δ expansion [11]. It should be possible to solve the optimization equations iteratively, as a series in φ_c^2. The next iteration will determine the coupling-constant renormalization, necessary in $3+1$ dimensions. The way is open, therefore, to obtain explicitly renormalized effective potentials from these non-Gaussian variational calculations.

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APPENDIX A

It is convenient to work in momentum space, employing Fourier transforms of the field $\phi(\mathbf{x})$ and its conjugate $\pi(\mathbf{x})$:

$$\phi(\mathbf{p})=\int d^{\nu}x\, e^{-i\mathbf{p}\cdot\mathbf{x}}\phi(\mathbf{x}) ,$$
$$\pi(\mathbf{p})=\int d^{\nu}x\, e^{i\mathbf{p}\cdot\mathbf{x}}\pi(\mathbf{x}) . \tag{A1}$$

The signs are such that

$$[\phi(\mathbf{p}),\pi(\mathbf{k})]=i\bar{\delta}(\mathbf{p}-\mathbf{k}) , \tag{A2}$$

and we employ the convention that $(2\pi)^{\nu}$ factors are to be associated with momentum-space integrations, delta functions, and functional derivatives involving fields with momentum-space arguments ($\nu\equiv$No. of spatial dimensions):

$$\bar{\delta}(\mathbf{p})\equiv(2\pi)^{\nu}\delta(\mathbf{p}) , \tag{A3}$$

$$\int_p\equiv\int\frac{d^{\nu}p}{(2\pi)^{\nu}} , \tag{A4}$$

$$\pi(\mathbf{p})=\frac{-i\bar{\delta}}{\delta\phi(\mathbf{p})}=-i(2\pi)^{\nu}\frac{\delta}{\delta\phi(\mathbf{p})} . \tag{A5}$$

The Hamiltonian $H=\int d^{\nu}x\,\mathcal{H}$ can be written as

$$H=\int_p\left[\tfrac{1}{2}\pi(\mathbf{p})\pi(-\mathbf{p})+\tfrac{1}{2}(\mathbf{p}^2+m_B^2)\phi(\mathbf{p})\phi(-\mathbf{p})+\lambda_B\int_q\int_k\int_r\phi(\mathbf{p})\phi(\mathbf{q})\phi(\mathbf{k})\phi(\mathbf{r})\bar{\delta}(\mathbf{p}+\mathbf{q}+\mathbf{k}+\mathbf{r})\right] . \tag{A6}$$

APPENDIX B

The κ, γ, and β integrals of Sec. III are

$$\kappa_1^R=\tfrac{1}{2}\langle\bar{\pi}_R\pi_R+\pi_R\bar{\pi}_R\rangle+\tfrac{1}{2}\int_p\mathbf{p}^2[\langle\bar{\phi}_R(\mathbf{p})\phi_R(-\mathbf{p})\rangle+\langle\phi_R(\mathbf{p})\bar{\phi}_R(-\mathbf{p})\rangle] , \tag{B1}$$
$$\kappa_2^R=\tfrac{1}{2}\langle\bar{\pi}_R\bar{\pi}_R\rangle+\tfrac{1}{2}\int_p\mathbf{p}^2[\langle\bar{\phi}_R(\mathbf{p})\bar{\phi}_R(-\mathbf{p})\rangle-\langle\bar{\phi}_R(\mathbf{p})\rangle\langle\bar{\phi}_R(-\mathbf{p})\rangle] , \tag{B2}$$
$$\gamma_1^R=\langle\text{“}\phi_R\bar{\phi}_R\text{”}\rangle , \tag{B3}$$
$$\gamma_2^R=\langle\bar{\phi}_R^2\rangle-\langle\bar{\phi}_R\rangle^2 , \tag{B4}$$
$$\beta_1^R=\langle\text{“}\phi_R^3\bar{\phi}_R\text{”}\rangle+4\varphi_c\langle\text{“}\phi_R^2\bar{\phi}_R\text{”}\rangle-12\varphi_c\langle\bar{\phi}_R\rangle\langle\phi_R^2\rangle+6\varphi_c^2\langle\text{“}\phi_R\bar{\phi}_R\text{”}\rangle , \tag{B5}$$
$$\beta_2^R=\langle\text{“}\phi_R^2\bar{\phi}_R^2\text{”}\rangle-4\langle\bar{\phi}_R\rangle\langle\text{“}\phi_R^2\bar{\phi}_R\text{”}\rangle+6\langle\bar{\phi}_R\rangle^2\langle\phi_R^2\rangle-12\varphi_c\langle\bar{\phi}_R\rangle\langle\text{“}\phi_R\bar{\phi}_R\text{”}\rangle+4\varphi_c\langle\text{“}\phi_R\bar{\phi}_R^2\text{”}\rangle , \tag{B6}$$
$$\beta_3^R=\langle\text{“}\phi_R\bar{\phi}_R^3\text{”}\rangle-4\langle\bar{\phi}_R\rangle\langle\text{“}\phi_R\bar{\phi}_R^2\text{”}\rangle+6\langle\bar{\phi}_R\rangle^2\langle\text{“}\phi_R\bar{\phi}_R\text{”}\rangle+4\varphi_c\langle\bar{\phi}_R^3+2\langle\bar{\phi}_R\rangle^3-3\langle\bar{\phi}_R\rangle\bar{\phi}_R^2\rangle , \tag{B7}$$
$$\beta_4^R=\langle\bar{\phi}_R^4\rangle+6\langle\bar{\phi}_R\rangle^2\langle\bar{\phi}_R^2\rangle-4\langle\bar{\phi}_R\rangle\langle\bar{\phi}_R^3\rangle-3\langle\bar{\phi}_R\rangle^4 , \tag{B8}$$

where quotation marks denote all permutations of the enclosed operators: e.g.,

$$\langle\text{“}\phi_R^3\bar{\phi}_R\text{”}\rangle=\langle\phi_R^3\bar{\phi}_R+\phi_R^2\bar{\phi}_R\phi_R+\phi_R\bar{\phi}_R\phi_R^2+\bar{\phi}_R\phi_R^3\rangle . \tag{B9}$$

For the transverse fields the expectation value $\langle\bar{\phi}_T\rangle$ will vanish, provided that the transformation preserves the $O(N-1)$ symmetry among the transverse fields, and hence the formulas are simpler:

$$\kappa_1^T=\tfrac{1}{2}\langle\bar{\pi}_T\cdot\pi_T+\pi_T\cdot\bar{\pi}_T\rangle+\tfrac{1}{2}\int_p\mathbf{p}^2[\langle\bar{\phi}_T(\mathbf{p})\cdot\phi_T(-\mathbf{p})\rangle+\langle\phi_T(\mathbf{p})\cdot\bar{\phi}_T(-\mathbf{p})\rangle] , \tag{B10}$$
$$\kappa_2^T=\tfrac{1}{2}\langle\bar{\pi}_T\cdot\bar{\pi}_T\rangle+\tfrac{1}{2}\int_p\mathbf{p}^2\langle\bar{\phi}_T(\mathbf{p})\cdot\bar{\phi}_T(-\mathbf{p})\rangle , \tag{B11}$$

$$\gamma_1^T=\langle \phi_T\cdot\bar{\phi}_T+\bar{\phi}_T\cdot\phi_T\rangle\ , \tag{B12}$$

$$\gamma_2^T=\langle \bar{\phi}_T^2\rangle\ , \tag{B13}$$

$$\beta_1^T=\langle \phi_T^2\phi_T\cdot\bar{\phi}_T+\phi_T^2\bar{\phi}_T\cdot\phi_T+\phi_T\cdot\bar{\phi}_T\phi_T^2+\bar{\phi}_T\cdot\phi_T\phi_T^2\rangle\ , \tag{B14}$$

$$\beta_2^T=\langle \phi_T^2\bar{\phi}_T^2+\bar{\phi}_T^2\phi_T^2+(\phi_T\cdot\bar{\phi}_T)^2+(\bar{\phi}_T\cdot\phi_T)^2+\phi_T\cdot\bar{\phi}_T\bar{\phi}_T\cdot\phi_T+\bar{\phi}_T\cdot\phi_T\phi_T\cdot\bar{\phi}_T\rangle\ , \tag{B15}$$

$$\beta_3^T=\langle \phi_T\cdot\bar{\phi}_T\bar{\phi}_T^2+\bar{\phi}_T^2\phi_T\cdot\bar{\phi}_T+\bar{\phi}_T\cdot\phi_T\bar{\phi}_T^2+\bar{\phi}_T^2\bar{\phi}_T\cdot\phi_T\rangle\ , \tag{B16}$$

$$\beta_4^T=\langle \bar{\phi}_T^2\bar{\phi}_T^2\rangle\ . \tag{B17}$$

The β^{RT} integrals are given by

$$\begin{aligned}\beta_1^{RT}=&\langle \bar{\phi}_R\phi_R\phi_T^2+\phi_R\bar{\phi}_R\phi_T^2+\phi_R^2\bar{\phi}_T\cdot\phi_T+\phi_R^2\phi_T\cdot\bar{\phi}_T\rangle-2\langle\bar{\phi}_R\rangle\langle\phi_R\phi_T^2\rangle-2\varphi_c\langle\bar{\phi}_R\rangle\langle\phi_T^2\rangle\\&+2\varphi_c\langle\bar{\phi}_R\phi_T^2+\phi_R\bar{\phi}_T\cdot\phi_T+\phi_R\phi_T\cdot\bar{\phi}_T\rangle+\varphi_c^2\langle\bar{\phi}_T\cdot\phi_T+\phi_T\cdot\bar{\phi}_T\rangle+\text{c.c.}\ ,\end{aligned} \tag{B18}$$

$$\begin{aligned}\beta_2^{RT}=&\langle \bar{\phi}_R^2\phi_T^2+\bar{\phi}_R\phi_R\bar{\phi}_T\cdot\phi_T+\bar{\phi}_R\phi_R\phi_T\cdot\bar{\phi}_T+\phi_R\bar{\phi}_R\bar{\phi}_T\cdot\phi_T+\phi_R\bar{\phi}_R\phi_T\cdot\bar{\phi}_T+\phi_R^2\bar{\phi}_T^2\rangle\\&-2\langle\bar{\phi}_R\rangle\langle\bar{\phi}_R\phi_T^2+\phi_R\bar{\phi}_T\cdot\phi_T+\phi_R\phi_T\cdot\bar{\phi}_T\rangle+\langle\bar{\phi}_R\rangle^2\langle\phi_T^2\rangle\\&+2\varphi_c[\langle\bar{\phi}_R\bar{\phi}_T\cdot\phi_T+\bar{\phi}_R\phi_T\cdot\bar{\phi}_T+\phi_R\bar{\phi}_T\cdot\bar{\phi}_T\rangle-\langle\bar{\phi}_R\rangle\langle\bar{\phi}_T\cdot\phi_T+\phi_T\cdot\bar{\phi}_T\rangle]+\varphi_c^2\langle\bar{\phi}_T^2\rangle+\text{c.c.},\end{aligned} \tag{B19}$$

$$\begin{aligned}\beta_3^{RT}=&\langle \bar{\phi}_R^2\bar{\phi}_T\cdot\phi_T+\bar{\phi}_R^2\phi_T\cdot\bar{\phi}_T+\bar{\phi}_R\phi_R\bar{\phi}_T^2+\phi_R\bar{\phi}_R\bar{\phi}_T^2\rangle-2\langle\bar{\phi}_R\rangle\langle\bar{\phi}_R\bar{\phi}_T\cdot\phi_T+\bar{\phi}_R\phi_T\cdot\bar{\phi}_T+\phi_R\bar{\phi}_T^2\rangle\\&+\langle\bar{\phi}_R\rangle^2\langle\bar{\phi}_T\cdot\phi_T+\phi_T\cdot\bar{\phi}_T\rangle+2\varphi_c[\langle\bar{\phi}_R\bar{\phi}_T\cdot\bar{\phi}_T\rangle-\langle\bar{\phi}_R\rangle\langle\bar{\phi}_T^2\rangle]+\text{c.c.},\end{aligned} \tag{B20}$$

$$\beta_4^{RT}=\langle \bar{\phi}_R^2\bar{\phi}_T^2\rangle-2\langle\bar{\phi}_R\rangle\langle\bar{\phi}_R\bar{\phi}_T^2\rangle+\langle\bar{\phi}_R\rangle^2\langle\bar{\phi}_T^2\rangle+\text{c.c.} \tag{B21}$$

Momentum arguments have been suppressed in the above formulas. Written out in full we would have, for example,

$$\langle\phi\bar{\phi}\phi\cdots\rangle=\int_{p_1}\int_{p_2}\cdots\int_{p_n}\langle\phi(\mathbf{p}_1)\bar{\phi}(\mathbf{p}_2)\phi(\mathbf{p}_3)\cdots\rangle\bar{\delta}(\mathbf{p}_1+\mathbf{p}_2+\cdots+\mathbf{p}_n)\ ,$$

$$\langle\bar{\phi}\rangle^2\langle\phi\phi\cdots\rangle=\int_{p_1}\int_{p_2}\cdots\int_{p_n}\langle\bar{\phi}(\mathbf{p}_1)\rangle\langle\bar{\phi}(\mathbf{p}_2)\rangle\langle\phi(\mathbf{p}_3)\phi(\mathbf{p}_4)\cdots\rangle\bar{\delta}(\mathbf{p}_1+\mathbf{p}_2+\cdots+\mathbf{p}_n)\ .$$

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