PHYSICAL REVIEW D VOLUME 45, NUMBER 8 15 APRIL 1992

The orthogonal circular ensemble

Merle Michael Robinson
Enrico Fermi Institute and Department of Physics, University of Chicago, Chicago, Illinois 60637
(Received 10 July 1991)

The orthogonal circular ensemble is solved exactly in the double scaling limit. The partition function describes unoriented strings with bosonic and fermionic propagators. The differential equations describing multicritical behavior are shown to be identical to those of the symplectic circular ensemble. In the case of the $m=1$ critical point, we find the susceptibility and argue that the orthogonal circular ensemble is unitary.

PACS number(s): 11.17.+y, 02.90.+p, 04.60.+n

I. INTRODUCTION

Large-N matrix integrals have been receiving attention recently because of their connection to string theories and quantum gravity. The graphical expansion of these matrix models is equivalent to a sum over surfaces of arbitrary genus [1,2]. In the past year, a method for taking the continuum limit of these diagrams has been used to examine the nonperturbative features of these models [3–5]. In this limit near criticality, we find differential equations and parametrize the susceptibility with their solutions. Such studies have been completed for the real symmetric [6–8], Hermitian [3–5], and real quaternion self-dual matrix models [9]. The unit-circle counterparts to the latter two of these cases have been solved; these are the unitary circular ensemble [10–13] and the symplectic circular ensemble [14]. The remaining case, the orthogonal circular ensemble, is the subject of this paper.

The partition function for the circular ensembles is

$$Z_N=\int \mu(\mathcal{DU})\exp[\beta\,\mathrm{tr}\,V(\mathcal{U}+\mathcal{U}^T)]\ , \tag{1}$$

where $\mathcal{U}$ is a $2N\times 2N$ symmetric unitary, unitary, or self-dual unitary matrix for the orthogonal, unitary, or symplectic circular ensembles, respectively. The invariant measures for the respective circular ensembles, as defined in Ref. [15], are symbolized here by $\mu(\mathcal{DU})$. We approach the continuum in a double scaling limit so that N/β approaches a critical value while $\beta\to\infty$.

In order to understand the genus expansion, we first expand the matrices [12]

$$\mathcal{U}=\exp[i(2\beta)^{-1/2}\mathcal{R}]\ , \tag{2}$$

where $\mathcal{R}$ is real symmetric, Hermitian, or real quaternion self-dual when $\mathcal{U}$ is symmetric unitary, unitary, or self-dual unitary. By introducing Ψ and $\overline{\Psi}$ as Grassmannian counterparts to $\mathcal{R}$, the Jacobian of the transformation of Eq. (2) can be represented as an integral over these matrices. Ignoring an overall multiplicative constant and substituting $V(\mathcal{U})=\mathcal{U}$, we are left with the integral

$$Z_N=\int \mathcal{DR}\,\mathcal{D}\overline{\Psi}\,\mathcal{D}\Psi\exp\left[\mathrm{tr}\left[-\tfrac{1}{2}\mathcal{R}^2+\overline{\Psi}\Psi+\sum_{j=1}^{\infty}\frac{(-1)^{j+1}}{(2\beta)^j(2j+2)!}\mathcal{R}^{2j+2}+\sum_{j=1}^{\infty}\frac{(-1)^j}{2^{2j}(2\beta)^j(2j+1)}\sum_{k=0}^{2j}\frac{(-1)^k}{k!(2j-k)!}\mathcal{R}^k\overline{\Psi}\mathcal{R}^{2j-k}\Psi\right]\right]\ . \tag{3}$$

The first and third terms in the exponent are just the usual terms in a cosine expansion which results from $V(\mathcal{U})=\mathcal{U}$. The second and fourth terms can be understood by first diagonalizing $\mathcal{R}$ and simultaneously making measure-independent similarity transformations on Ψ and $\overline{\Psi}$. Denoting the eigenvalues of $\mathcal{R}$ as λ_i and ignoring the integration over angular variables, $\mathcal{DR}$ becomes $\prod_i^N d\lambda_i\prod_{k<l}|\lambda_k-\lambda_l|^p$; where p is 1, 2, or 4 if $\mathcal{R}$ is real symmetric, Hermitian, or real quaternion, respectively [15]. Having diagonalized $\mathcal{R}$, the second and fourth terms can be rewritten as a sum over i and j of

$$\overline{\Psi}_{ij}\Psi_{ji}\sin\left[\frac{\lambda_i-\lambda_j}{2\sqrt{2\beta}}\right]\Big/\left[\frac{\lambda_i-\lambda_j}{2\sqrt{2\beta}}\right]+\overline{\Psi}_{ii}\Psi_{ii}\ .$$

When we integrate over the components of the Grassmann matrices, we are left with the factor

$$\prod_{1\le k<l\le N}\left[\sin\left[\frac{\lambda_i-\lambda_j}{2\sqrt{2\beta}}\right]\Big/\left[\frac{\lambda_i-\lambda_j}{2\sqrt{2\beta}}\right]\right]^p .$$

When $\overline{\Psi}_{ij}$ or Ψ_{ji} is real, complex, or quaternion, there are either 1, 2, or 4 independent components, so that p takes the same values as above. Since the $\prod_{1\le k<l\le N}|\lambda_k-\lambda_l|^p$ terms cancel, Eq. (3) provides us with the correct representation of Eq. (1) with $\mathcal{U}$ diagonalized, apart from angular variable integrations and overall multiplicative constants [15].

From the partition function in Eq. (3), we now complete the propagators. As $\beta\rightarrow\infty$, we are only left with the first two terms of the exponent. Under these conditions, the propagators in the circular orthogonal ensemble are

$$\begin{aligned}
\langle \mathcal{R}_{ij}\mathcal{R}_{kl}\rangle &= \tfrac{1}{2}(\delta_{ik}\delta_{jl}+\delta_{il}\delta_{jk}) , \\
\langle \Psi_{ij}\overline{\Psi}_{kl}\rangle &= \tfrac{1}{2}(\delta_{ik}\delta_{jl}+\delta_{il}\delta_{jk}) , \\
\langle \mathcal{R}_{ij}\Psi_{kl}\rangle &= \langle \mathcal{R}_{ij}\overline{\Psi}_{kl}\rangle = 0 .
\end{aligned} \tag{4}$$

Thus the symmetric unitary matrix model, like the real symmetric matrix model, contains crossed propagators. Contributions from oriented and nonoriented surfaces exist, but some of the symmetric unitary model diagrams contain a fermion double-line propagator. Similarly, the self-dual unitary matrix model is a theory over unoriented strings. The propagators are [18]

$$\begin{aligned}
\langle \mathcal{R}_{ij}\mathcal{R}_{kl}\rangle &= \tfrac{1}{2}(\delta_{il}\delta_{jk}+\mathcal{T}_{ik}\mathcal{T}_{jl}) , \\
\langle \Psi_{ij}\overline{\Psi}_{kl}\rangle &= \tfrac{1}{2}(\delta_{il}\delta_{jk}+\mathcal{T}_{ik}\mathcal{T}_{jl}) , \\
\langle \mathcal{R}_{ij}\Psi_{kl}\rangle &= \langle \mathcal{R}_{ik}\overline{\Psi}_{kl}\rangle = 0 ,
\end{aligned} \tag{5}$$

where $\mathcal{T}^T=-\mathcal{T}$, $\mathcal{T}^2=-1$, and $\mathcal{T}^*=\mathcal{T}$. In Ref. [8], propagators are employed to compare diagram contributions for the real symmetric and real quaternion self-dual matrix models by comparing the effect of crossed propagators. Diagram contributions for the latter are obtained from the former by the replacement of N by $-N$. Equation (3) contains additional Grassmannian matrices, but the corresponding propagators are identical to the non-Grassmannian counterparts. Since the original argument is based solely on the form of the propagators, we conclude that diagram contributions for the self-dual unitary matrices are obtained from those contributions for the symmetric unitary matrices by the substitution of $-N$ for N. To better illustrate the point, we end with some examples. Knowing that $N\sim\beta$ in the scaling limit, the first contributions are proportional to

$$\begin{aligned}
\frac{1}{N}\langle \mathrm{tr}\mathcal{R}^4\rangle &= 16N^2\pm 20N+10 , \\
\frac{1}{N}\langle \mathrm{tr}\mathcal{R}^2\overline{\Psi}\Psi\rangle &= 8N^2\pm 8N+2 , \\
\frac{1}{N}\langle \mathrm{tr}\mathcal{R}\overline{\Psi}\mathcal{R}\Psi\rangle &= \pm 4N+6 , \\
\frac{1}{N}\langle \mathrm{tr}\overline{\Psi}\mathcal{R}^2\Psi\rangle &= 8N^2\pm 8N+2 .
\end{aligned} \tag{6}$$

The upper and lower signs arise in the orthogonal circular ensemble and symplectic circular ensemble, respectively. In the unitary case, where there are no nonorientable surfaces, the terms of odd order in N are absent.

In the next section we diagonalize the symmetric unitary matrix identities to write the partition function as a determinant over polynomials of the eigenvalues. Rows and columns of this matrix are then added to eliminate matrix elements and produce a recursion relation for the partition function. From the recursion relations for the polynomials and the partition function, we derive a set of differential equations in the double scaling limit. Finally we expand the susceptibility in terms of solutions to the differential equations. One of the two branches of the expansion has alternating signs between terms of odd and even powers of N. This branch corresponds to the symplectic circular ensemble because the same pattern appears in the diagram contributions. The orthogonal circular ensemble is identified with the branch that has the same sign for all terms but the first.

II. Z_N AS A DETERMINANT

Our objective is the susceptibility in the scaling limit. In the process, we follow previous work [6,14] to obtain a recursion relation for the partition function and the resulting scaling-limit equations.

Diagonalizing $N\times N$ symmetric unitary matrices in Eq. (1), we obtain an integral over the complex eigenvalues [15]

$$Z_N=\int\prod_{l=1}^{N}\frac{dz_l}{2\pi i z_l}\prod_{k<l}|z_k-z_l| \times\exp\left[-\sum_{j=1}^{N}\beta V(z_j+z_j^{-1})\right] . \tag{7}$$

We shall remove the absolute norm in the measure by integrating over alternate eigenvalues z_{2j+1}. Since the eigenvalues are complex, the integral is first rearranged using the identity

$$\prod_{1\le k<l\le N}|e^{i\theta_k}-e^{i\theta_l}|=i^{N(N-1)/2}\det[\exp(ip\theta_j)] , \tag{8}$$

where $p=-\frac{1}{2}(N-1),-\frac{1}{2}(N-3),\ldots,\frac{1}{2}(N-3),\frac{1}{2}(N-1)$ and $j=1,2,\ldots,N$ with θ_j ordered $(-\pi\le\theta_1\le\theta_2\le\cdots\le\theta_N\le\pi)$. If $z_m=e^{i\theta_m}$ then the partition function is rewritten as

$$Z_N=i^{N(N-1)/2}N!\int\prod_{l=1}^{N}d\mu_1(z_l)\det[z_j^p] , \tag{9}$$

where

$$d\mu_1(z)=\frac{dz}{2\pi i z}\exp\left[-\sum_{j=1}^{N}\beta V(z+z^{-1})\right]$$

and p is defined as before with the z's "ordered."

For $N=2M$, p are all half-odd integers. A set of polynomials (over $d\mu_2(z)=(dz/2\pi i z)\exp[-2\beta V(z+z^{-1})]$) can be defined [9]:

$$\begin{aligned}
s_k(z)&=z^{k+1/2}-z^{-k-1/2}+\sum_{j=0}^{k-1}A_{k,j}(z^{j+1/2}-z^{-j-1/2}) , \\
c_k(z)&=z^{k+1/2}+z^{-k-1/2}+\sum_{j=0}^{k-1}\overline{A}_{k,j}(z^{j+1/2}+z^{-j-1/2}) ,
\end{aligned} \tag{10}$$

which satisfy $\int d\mu_2\, c_k c_l=\overline{h_k}\delta_{k,l}$, $\int d\mu_2\, s_k s_l=h_k\delta_{k,l}$, and $\int d\mu_2\, c_k s_l=0$. Now,

$$\det[z_j^p]=\det\begin{bmatrix} s_{M-1}(z_1) & \cdots & s_0(z_1) & c_0(z_1) & \cdots & c_{M-1}(z_1) \\ s_{M-1}(z_2) & \cdots & s_0(z_2) & c_0(z_2) & \cdots & c_{M-1}(z_2) \\ \vdots & \cdots & \vdots & \vdots & \cdots & \vdots \\ s_{M-1}(z_{2M}) & \cdots & s_0(z_{2M}) & c_0(z_{2M}) & \cdots & c_{M-1}(z_{2M}) \end{bmatrix} . \tag{11}$$

We use the notation $z(\theta)=e^{i\theta}$; z_1 is integrated from $z(-\pi)$ to z_2, z_3 from z_2 to $z_4, \ldots,$ and $z_{N/2-1}$ from $z_{N/2-2}$ to z_N. The top odd rows are then added to all the lower odd rows to make the lower and upper limits: $z(-\pi)$ and z_{2j}. Now the integral is symmetric in the variables z_{2j}, and the range of integration can be switched from $z_{2(j-1)}<z_{2j}<z_{2(j+1)}$ to $z(-\pi)<z_{2j}<z(\pi)$ if we divide by $1/(M!)$.[1]

The integral can now be written in terms of the Pfaffian of an antisymmetric matrix [15]. Using the notation

$$\int\int a_i b_j \equiv \int_{z(-\pi)}^{z(\pi)} d\mu_1(x)\int_{z(-\pi)}^{x} d\mu_1(y)[a_i(y)b_j(x) - b_j(y)a_i(x)] , \tag{12}$$

we write

$$Z_{2M}=C_{2M}\left[\det\begin{bmatrix} \int\int c_i c_j & \int\int c_i s_j \\ \int\int s_i c_j & \int\int s_i s_j \end{bmatrix}\right]^{1/2} , \tag{13}$$

where $C_{2M}\equiv i^{(2M)(2M-1)/2}(2M)!=i^{-M}(2M)!$. Since $c_i\to c_i$ and $s_i\to -s_i$ as $\theta\to-\theta$, the upper left and lower right sections of this matrix are zero. Thus

$$Z_N=|C_N\det[g_{ij}]| , \tag{14}$$

if $g_{ij}=2\int\int c_i s_j$.

Before we compute the elements of the matrix g_{ij}, we further discuss the properties of the polynomials in the next section.

III. THE POLYNOMIALS

In this section, we basically reiterate material from Myers and Periwal [14] for the sake of clarity. We even use the same notation. Notice, however, that β has been replaced by 2β. This stems from the difference in the measure. In this case, the polynomials are orthogonal with respect to $d\mu_2$, whereas in the papers by Myers and Periwal, the polynomials are orthogonal with respect to $d\mu_1$.

The recursion relations for the polynomials are $(z_\pm\equiv z+1/z)$

$$\begin{aligned} z_+c_k&=c_{k+1}+\bar{X}_k c_k+\alpha_k c_{k-1} , \\ z_+s_k&=s_{k+1}+X_k s_k+\frac{\gamma_k}{\gamma_{k-1}}\alpha_k s_{k-1} , \\ z_-c_k&=s_{k+1}+\bar{Y}_k s_k+\frac{\alpha_k}{\gamma_{k-1}}s_{k-1} , \\ z_-s_k&=c_{k+1}+\bar{Y}_k\gamma_k c_k+\gamma_k\alpha_k c_{k-1} , \end{aligned} \tag{15}$$

where $\alpha_k\equiv\bar{h}_k/\bar{h}_{k-1}$ and $\gamma_k\equiv/\bar{h}_k$. These equations have the same form for any potential. The following equations are valid for $V(y)=y$ $(zd\equiv z\partial/\partial z)$:

$$\begin{aligned} z\,dc_k&=(k+\tfrac{1}{2})s_k+2\beta\frac{\alpha_k}{\gamma_{k-1}}s_{k-1} , \\ z\,ds_k&=(k+\tfrac{1}{2})c_k+2\beta\gamma_k\alpha_k c_{k-1} . \end{aligned} \tag{16}$$

Meyers and Periwal obtain the exact same recursion relations in their paper on self-dual unitary matrices, *except* for the substitution of β by 2β. With the understanding that $g\equiv(N+\frac{1}{2})/2\beta$ [not $(N+\frac{1}{2})/\beta$] we arrive at the same critical values in the double scaling limit: $\gamma=-1$, $\alpha=+1$, and $g=\pm1$. The scaling solutions are also the same:

$$\begin{aligned} &2f-h^2-h'=0 , \\ &h''-4hf+2hh'+8ht=0 , \end{aligned} \tag{17}$$

and therefore

$$h''-2h^3+8ht=0 . \tag{18}$$

These differential equations were obtained using the *Ansätz*

$$\gamma_N=-1+(2\beta)^{-1/3}h(t_N) ,$$

$$\alpha_N=1-(2\beta)^{-2/3}f(t_N) ,$$

with[2]

$$t\equiv(2\beta)^{-1/3}[(2\beta)-(N+\tfrac{1}{2})] .$$

Equation (18) is the Painlevé II equation. For equivalent potentials, the scaling equations will always be identical to those of the self-dual unitary case because the recursion relations for the polynomials only differ by the change $\beta\to2\beta$.

---

[1]Observe that for $N=2M+1$, similar polynomials of integer powers can be defined. The bottom row of the matrix will be integrated from $z(-\pi)$ to $z(\pi)$ so that all of its elements will be zero except the center term, which will be $\bar{h}_0$ in the new polynomial basis. All of the remaining calculations of this section are the same except that the normalization of the integral is changed.

[2]For the case of $N=2M+1$, the scaling equations above are exactly the same. The only change in the recursion relations is the replacement of $k+\frac{1}{2}$ by k. In addition, all further calculations in the scaling limit are identical up to the previously mentioned normalization adjustment to the partition function.

IV. RECURSION RELATIONS FOR Z_{2M}

Recall that

$$g_{ij}=2\int_{z(-\pi)}^{z(\pi)}d\mu_1(y)\int_{z(-\pi)}^{y}d\mu_1(x)c_i(y)s_j(x)\ , \tag{19}$$

$$Z_{2M}=|C_{2M}\det[g_{ij}]|\ , \tag{20}$$

and $C_{2M}=i^{-M}(2M)!$. We will now use the properties of c_i and s_i to express Z_{2M} as the determinant of a banded matrix.

Define new functions $f_k(z)=c_k(z)e^{-\beta V(z_+)}/z$ and $g_k(z)=s_k(z)e^{-\beta V(z_+)}/z$. Thus g_{ij} is written as

$$g_{ij}=2\int_{z(-\pi)}^{z(\pi)}dy\int_{z(-\pi)}^{y}dx\,f_i(y)g_j(x)\ . \tag{21}$$

Using Appendix A, we find recursion relations for the new functions:

$$\begin{aligned}d(zf_k)&=\sum_{i=1}^{p}(-a_{k-i,k}g_{k-i}+a_{k+i,k}g_{k+i})+\chi_k g_k\ ,\\ d(zg_k)&=\sum_{i=1}^{p}(-\bar{a}_{k-i,k}f_{k-i}+\bar{a}_{k+i,k}f_{k+i})\\ &\quad+(-\gamma_k)\chi_k f_k\ ,\end{aligned} \tag{22}$$

where
$\int d\mu_2 s_l[-\beta V'(z_+)(z_-)]c_m=a_{l,m}h_l=\bar{a}_{m,l}\bar{h}_m$
for $l\neq m$ and $\chi_k=\frac{1}{2}(k+\frac{1}{2})(1-1/\gamma_k)$.

In our case, $V(z_+)=z_+$ so that $p=1$ and

$$\begin{aligned}a_{j-1,j}&=-\beta\frac{\bar{h}_j}{h_{j-1}}\ ,\quad \bar{a}_{j-1,j}=-\beta\frac{h_j}{\bar{h}_{j-1}}\ ,\\ a_{j+1,j}&=\bar{a}_{j+1,j}=-\beta\ .\end{aligned} \tag{23}$$

If we proceed through the matrix adding two consecutive columns with the correct weights to the following column, we change the $g_i(x)$'s into $[1/(-\beta)]\,[d(xf_{j-1}(x))]$'s. Integrating by parts, we obtain new values for g_{ij}:

$$g_{ij}=2\int_{z(-\pi)}^{z(\pi)}\frac{dy}{2\pi i}\int_{z(-\pi)}^{y}\frac{dx}{2\pi i}f_i(y)[d(xf_{j-1}(x))]\left[\frac{-1}{\beta}\right]=-\delta_{i,j-1}\frac{\bar{h}_i}{i\pi\beta}\ . \tag{24}$$

Notice that we can add the first and second column to the third, the first column to the second, but we cannot add anything to the first column. We are left with

$$Z_{2M}=C_{2M}(-1)^M\det\begin{bmatrix} g_{0,0} & -\frac{\bar{h}_0}{i\pi\beta} & 0 & \cdots & 0\\ g_{1,0} & 0 & -\frac{\bar{h}_1}{i\pi\beta} & \cdots & 0\\ g_{2,0} & 0 & 0 & \cdots & 0\\ \vdots & \vdots & \vdots & \cdots & \vdots\\ g_{M-2,0} & 0 & 0 & \cdots & -\frac{\bar{h}_{M-2}}{i\pi\beta}\\ g_{M-1,0} & 0 & 0 & \cdots & 0\end{bmatrix}\ . \tag{25}$$

[The factor of $(-1)^M$ assures positivity of Z_{2M}, since $\bar{h}_i<0$, $g_{i,1}$ is negative imaginary, and $C_{2M}\propto i^{-M}$.]

In order to similarly set terms in column 1 to zero, we first notice that

$$g_{i0}=-2\int_{z(-\pi)}^{z(\pi)}dy\int_{z(-\pi)}^{y}dx\,g_0(y)f_i(x)\ . \tag{26}$$

Then we add rows in the same way as we added columns previously. The result is

$$Z_{2M}=\frac{(2M)!}{-i}\det\begin{bmatrix} g_{0,0} & -\frac{\bar{h}_0}{\pi\beta} & & & 0\\ \frac{\bar{h}_0\gamma_0}{\pi\beta} & \frac{(1-\gamma_0)\bar{h}_0}{4\pi\beta^2} & -\frac{\bar{h}_1}{\pi\beta} & & \\ & \frac{\bar{h}_1\gamma_1}{\pi\beta} & \frac{3(1-\gamma_1)\bar{h}_1}{4\pi\beta^2} & \ddots & \\ & & & \ddots & \\ & \ddots & & & -\frac{\bar{h}_{M-2}}{\pi\beta}\\ 0 & & & \frac{\bar{h}_{M-2}\gamma_{M-2}}{\pi\beta} & \frac{(M-\frac{3}{2})(1-\gamma_{M-2})\bar{h}_{M-2}}{2\pi\beta^2}\end{bmatrix}\ . \tag{27}$$

Now we write the recursion relation, ignoring the first 2×2 matrix,

$$\frac{Z_{2M+4}}{(2M+4)!}-\frac{(M+\frac{1}{2})(1-\gamma_M)\bar{h}_M}{2\pi\beta^2}\frac{Z_{2M+2}}{(2M+2)!}-\frac{\bar{h}_M^2\gamma_M}{\pi^2\beta^2}\frac{Z_{2M}}{(2M)!}=0\ . \quad (28)$$

Substituting

$$W_{2M+2}=\frac{Z_{2M+2}}{Z_{2M}}\frac{\pi\beta}{(2M+2)(2M+1)(1-\gamma_{M-1})(\bar{h}_{M-1})} \quad (29)$$

leaves the equation

$$W_{2M+2}W_{2M}-\left[\frac{M+\frac{1}{2}}{2\beta}\right]W_{2M}-\frac{\alpha_M\gamma_M}{(1-\gamma_M)(1-\gamma_{M-1})}=0\ . \quad (30)$$

This is the same equation obtained by Myers and Periwal for the symplectic circular ensemble. In fact, the equations for W_M for the two ensembles are found to be identical for all potentials (see Appendixes A–C). We use the critical values of α, γ, and g to find the limiting value of $W_N, W=\frac{1}{2}$. Using the *Ansatz* $W_N=\frac{1}{2}[1+\beta^{-1/3}w(t_N)]$, we find [14]

$$w'=w^2+2t-\frac{3h^2}{4}\ . \quad (31)$$

Finally, expressing $Z_{2M+2}Z_{2M-2}/Z_{2M}^2$ in terms of α_M, γ_M, and W_M, we find that the susceptibility is $h^2/2+w'$ in the scaling limit. In the unitary matrix model, the susceptibility is h^2. Since there are only orientable diagrams in the unitary matrix model, we conclude that the nonorientable contributions in the other two models come from w'.

V. WEAK-COUPLING EXPANSION

Knowing the differential equations for h and w, Eqs. (18) and (31), we expand the susceptibility as $\beta\rightarrow\infty$. First we expand

$$h(t)=t^{1/2}\sum_{k=0}^{\infty}r_k t^{-3k/2}\ ,\quad w(t)=t^{1/2}\sum_{k=0}^{\infty}s_k t^{-3k/2} \quad (32)$$

in the differential equations. From the Painléve II equation, we see that $r_0=\pm 2$. For $k\geq 0$ and $l,m,n<k+2$ we have

$$r_{k+2}=-\frac{9k^2-1}{64}r_k+\frac{1}{8}\sum_{l+m+n=k+2}r_l r_m r_n\ . \quad (33)$$

Observe that $r_{2k+1}=0$. From Eq. (31), we have for $k\geq 0$ and $l,m<k+1$,

$$w_{k+1}=\frac{1}{w_0}\left[\frac{(3k-1)}{2}w_k-\sum_{l+m=k+1}w_l w_m+\frac{3}{4}\sum_{i+j=k+1}r_i r_j\right]\ . \quad (34)$$

The first coefficient is given by $w_0^{\pm}=\pm 1$ (the superscript is included here because, unlike the double-valued r_{2i}'s, the two cases result in two branches in the susceptibility expansion with the correct leading asymptotic behavior). In fact, the final expansion for the susceptibility is

$$f\sim 2t\pm\frac{1}{2}t^{-1/2}-\frac{5}{16}t^{-2}\pm\frac{65}{128}t^{-7/2}-\frac{321}{256}t^{-5}\pm\cdots\ . \quad (35)$$

Since this expansion is identical to the one obtained for the self-dual unitary matrices, one would suspect that each branch corresponds to one of the two matrix models. The difference between the two solutions is a sign reversal for terms of odd order in N. As explained in the Introduction, however, the finite-order topological expansion predicts a replacement of N by $-N$ from the symmetric unitary matrix model to the self-dual unitary matrix model. Thus we claim the unitary branch, that which has only negative terms following the first positive term, to be the correct expansion for the symmetric unitary matrix model, and the other branch to be the correct one for the self-dual unitary matrix model.

Since the differential equations are nonlinear, we expect to find instanton contributions. By substituting $t^{1/2}+\epsilon_h$ for h in Eq. (18), we find a one-parameter family of solutions [10]

$$\epsilon_h=t^{-1/4}\exp(-\tfrac{8}{3}t^{3/2})\ . \quad (36)$$

The above effect further induces an instanton $\epsilon_2=\pm\frac{3}{2}\epsilon_h$, where the sign ambiguity comes from that present in r_0. There is an additional instanton for $w(t)$,

$$\epsilon_w=\exp(w_0^{\pm}\tfrac{4}{3}t^{3/2})\ . \quad (37)$$

Because of the asymptotic behavior of the susceptibility, this instanton only exists for w_0^-, and thus does not appear in the self-dual unitary matrix model.

VI. DISCUSSION

In this paper, we have derived the scaling equation for the $m=1$ critical potential [14]. However, we follow Sec. III to state that the same modified Korteweg–de Vries (MKdV) hierarchy of differential equations that result from the multicritical potentials for the self-dual unitary matrices will appear here for the symmetric unitary matrices. These potentials are given by [14]

$$V(z)=\int_0^1\frac{dt}{t}[(1-t)(1-tz)]^k\ . \quad (38)$$

Although the integral is divergent, the divergence is z independent and does not cause problems in the integral's role as a definition of the potential as a function of z.

We also conjecture the equivalence of the $m=1$ symmetric unitary model with the real symmetric model for an inverted potential $V(x)=-x^2+gx^4$ and separate scaling limits for R_{2n} and R_{2n+1}. This guess extends the work of Douglas, Seiberg, and Sheuker [16], who recently demonstrated the equivalence of the $m=1$ unitary model and the Hermitian model with both the above potential and mentioned scaling limit. In both cases, the $f\propto h^2$, where h is defined by the Painléve II equation. They attribute the similarity to existence of a "hump" that separates two groups of eigenvalues, and which disappears at the phase transition (due to a quadratic singularity). More generally, we suggest that higher multicritical behavior in the symmetric unitary and the real symmetric models with the inverted potentials and scaling limits will be identical.

ACKNOWLEDGMENTS

I am especially grateful to G. Harris for many helpful discussions and valuable guidance. I would like to thank E. Martinec for useful insights, and for suggesting the problem. This research was supported in part by DOE Contract No. DE-FG02-90ER-40560 and the Fairchild Foundation.

APPENDIX A

In this appendix, we mix c,s and f,g notations when convenient. Remember that $f[,g]=c[,s]e^{-\beta V(z_+)}/z$ and $d\mu_2=(dz/2\pi iz)e^{-2\beta V(z_+)}$, where $z_\pm \equiv z\pm 1/z$. Let us calculate the quantities $z\,df, z\,dg$. For simplicity we only discuss $z\,df$ and later generalize. First we note that

$$z\,df_k=-\beta V'(z_+)(z_-)f_k+z\,dc_k e^{-\beta V(z_+)}(1/z)-f_k \ . \tag{A1}$$

For V as a polynomial of order p, we can rewrite this equation in the suggestive form

$$d(zf_k)=\sum_{i=-p}^{p} a_{k+i,k}g_{k+i}+\sum_{i=-p}^{k} b_{k+i,k}g_{k+i} \ , \tag{A2}$$

where

$$\int d\mu_2 s_l[-\beta V'(z_+)(z_-)]c_m=a_{l,m}h_l$$

and

$$\int d\mu_2 s_l z\,dc_m=b_{l,m}h_l \ .$$

We can simplify things if we integrate the second integral by parts, and compare it to the first integral. We note that for $l<m$, $-2a_{l,m}=b_{l,m}$. For $l=m$, $-2a_{m,m}=b_{m,m}+b_{m,m}(\bar{h}_m/h_m)=b_{m,m}(1+1/\gamma_m)=(k+\frac{1}{2})(1+1/\gamma_m)$. For $l>m$, $b_{l,m}=0$ and $a_{l,m}h_l=\bar{a}_{m,l}\bar{h}_m$. Now,

$$d(zf_k)=\sum_{i=1}^{p}(-a_{k-i,k}g_{k-i}+a_{k+i,k}g_{k+i})+\chi_k g_k \ . \tag{A3}$$

For convenience, we have defined $\chi_k\equiv\frac{1}{2}(k+\frac{1}{2})(1-1/\gamma_k)$. Similarly,

$$d(zg_k)=\sum_{i=1}^{p}(-\bar{a}_{k-i,k}f_{k-i}+\bar{a}_{k+i,k}f_{k+i})+(-\gamma_k)\chi_k f_k \ . \tag{A4}$$

Observe that the coefficient of the $(k+p)$th term in the above equations has the same coefficient $-\beta v_p$, where v_p is the coefficient of the pth-order term in the polynomial V.

APPENDIX B

In this appendix we continue the calculations of Appendix A to prove that the scaling equation that follows from the recursion relations for the partition function [Eq. (31) in the text] is the same for the symmetric unitary and self-dual unitary matrix models for a general potential.

Let us first calculate this recursion relation for the symmetric unitary matrix model. We start the Z_{2M} as a determinant

$$Z_{2M}=|C_{2M}\det[g_{ij}]| \ . \tag{B1}$$

Knowing

$$g_{ij}=2\int_{z(-\pi)}^{z(\pi)}\frac{dy}{2\pi i}\int_{z(-\pi)}^{y}\frac{dx}{2\pi i}f_i(y)g_j(x) \ , \tag{B2}$$

we then add the $2p$ previous columns to the following column with the correct weights as ordained by Eq. (A3). As in the calculation in the main text, this leaves

$$Z_{2M}=\frac{(2M)!}{(-i)^p}\det\begin{bmatrix} g_{0,0} & \cdots & g_{0,p-1} & \frac{\bar{h}_0}{\pi\beta v_p} & 0 & \cdots & 0 \\ g_{1,0} & \cdots & g_{1,p-1} & 0 & \frac{\bar{h}_1}{\pi\beta v_p} & \cdots & 0 \\ \vdots & \cdots & \vdots & \vdots & \vdots & \cdots & \vdots \\ g_{M-1,0} & \cdots & g_{M-1,p-1} & 0 & 0 & \cdots & \frac{\bar{h}_{M-1}}{\pi\beta v_p} \end{bmatrix} . \tag{B3}$$

To turn most of the terms in the first p columns to zero we first write (for $j=0,\ldots,p-1$)

$$g_{ij}=-2\int_{z(-\pi)}^{z(\pi)}\frac{dy}{2\pi i}\int_{z(-\pi)}^{y}\frac{dx}{2\pi i}g_j(y)f_i(x) \tag{B4}$$

and then add the $2p$ previous rows to the following row with the correct weights according to Eq. (A4). This leaves the partition function as

$$Z_{2M}=\frac{(2M)!}{(-i)^p}\det\begin{bmatrix} & & & & 0 \\ & 2p\times 2p & & \bar{a}_{p,0}\frac{\bar{h}_p}{\pi\beta^2 v_p^2} & \\ & \text{matrix} & & \bar{a}_{p,1}\frac{\bar{h}_p}{\pi\beta^2 v_p^2} & \ddots \\ & & & \vdots & \ddots \\ -\bar{a}_{0,p}\frac{\bar{h}_0}{\pi\beta^2 v_p^2} & -\bar{a}_{1,p}\frac{\bar{h}_1}{\pi\beta^2 v_p^2} & \cdots & -\gamma_p\chi_p\frac{\bar{h}_p}{\pi\beta^2 v_p^2} & \ddots \\ 0 & \ddots & \ddots & \ddots & \ddots \end{bmatrix} . \tag{B5}$$

Since the positions of the matrix elements may be confusing, we note the following: terms containing χ are along the diagonal, and terms containing $\bar{a}_{m,n}$ appear at the $(n+p+1)$th row and the $(m+p+1)$th column.

APPENDIX C

Now we are ready to look at the partition function for the self-dual unitary matrix model. After diagonalizing the $2N\times 2N$ matrices, the integral can be written in terms of the eigenvalues z_l,

$$Z_N=\int\prod_{l=1}^{N}\frac{dz_l}{2\pi i z_l}\prod_{k<l}|z_k-z_l|^4\exp\left[-\sum_{j=1}^{N}\beta V(z_j+z_j^{-1})\right] . \tag{C1}$$

Removing the absolute magnitude and then adding and subtracting the resulting columns leaves us with [14]

$$\prod_{k<l}|z_k-z_l|^4=\frac{1}{2^N}\det\begin{bmatrix} s_{N-1}(z_1) & \cdots & s_0(z_1) & c_0(z_1) & \cdots & c_{N-1}(z_1) \\ z\,ds_{N-1}(z_1) & \cdots & z\,ds_0(z_1) & z\,dc_0(z_1) & \cdots & z\,dc_{N-1}(z_1) \\ s_{N-1}(z_2) & \cdots & s_0(z_2) & c_0(z_2) & \cdots & c_{N-1}(z_2) \\ \vdots & \cdots & \vdots & \vdots & \cdots & \vdots \\ z\,ds_{N-1}(z_N) & \cdots & z\,ds_0(z_N) & z\,dc_0(z_N) & \cdots & z\,dc_{N-1}(z_N) \end{bmatrix} . \tag{C2}$$

In this case, the c_i's and s_i's are defined to be orthogonal with respect to $d\mu_1$, not $d\mu_2$.

From here, we can follow our earlier procedure carried out from Eqs. (11)–(14). This will leave us with

$$Z_N=\frac{N!}{2^N}|\det[g'_{ij}]| , \tag{C3}$$

where

$$g'_{ij}=\int_{z(-\pi)}^{z(\pi)}d\mu_1(z)[c_i(z)z\,ds_j(z)-s_j(z)z\,dc_i(z)] . \tag{C4}$$

For the symplectic circular ensemble,

$$z\,dc_k=\sum_{i=1}^{p}-a_{k-i,k}s_{k-i}+(k+\tfrac{1}{2})s_k ,$$
$$z\,ds_k=\sum_{i=1}^{p}-\bar{a}_{k-i,k}c_{k-i}+(k+\tfrac{1}{2})c_k , \tag{C5}$$

so that the partition function is

$$Z_{2M}=\frac{N!}{2^N}\det\begin{bmatrix} & & & -\bar{a}_{0,p}\bar{h}_0 & 0 \\ & p\times p\ \text{matrix} & & -\bar{a}_{1,p}\bar{h}_1 & \ddots \\ & & & \vdots & \ddots \\ a_{0,p}h_0 & a_{1,p}h_1 & \cdots & 2\gamma_p\chi_p\bar{h}_p & \ddots \\ 0\ \ddots & \ddots & \ddots & \ddots & \end{bmatrix} . \tag{C6}$$

Here, terms containing χ appear along the diagonal, terms containing $a_{m,n}$ appear at the $(n+1)$th row and the $(m+1)$th column, and terms containing $\bar{a}_{m,n}$ appear at the $(m+1)$th row and the $(n+1)$th column.

Taking the transpose of the above matrix and recalling that $a_{l,m}h_l=\bar{a}_{m,l}\bar{h}_m$, it is apparent that the recursion relations for W_N will be identical to those produced by Eq. (B5) provided that

$$W_N^{\rm symp}=\frac{Z_N}{Z_{N-1}N}\frac{1}{\beta}G_N \ ,$$

$$W_N^{\rm orth}=\frac{Z_{N+2}}{Z_N(N+2)(N+1)}\pi\beta v_p^2 G_N \ , \tag{C7}$$

where the β dependence of G_N is removed through the relations $(N+\frac{1}{2})/g=\beta$ and $(N+\frac{1}{2})/g=2\beta$, respectively. A factor of 2 discrepancy in these relations causes the doubling of the diagonal elements from Eqs. (B5)–(C6). From the identical recursion relations, we conclude that the models have the same scaling-limit equations and critical values. In terms of the scaling variables, the susceptibility will be identical for any potential V.

---

[1] E. Brézin, C. Itzykson, G. Parisi, and J.-B. Zuber, Commun. Math. Phys. **59**, 35 (1978).

[2] D. Bessis, Commun. Math. Phys. **69**, 147 (1979); C. Itzykson and J.-B. Zuber, J. Math. Phys. **21**, 411 (1980).

[3] E. Brézin and V. A. Kazakov, Phys. Lett. B **236**, 144 (1990).

[4] M. R. Douglas and S. H. Shenker, Nucl. Phys. **B335**, 635 (1990).

[5] D. J. Gross and A. A. Migdal, Phys. Rev. Lett. **64**, 127 (1990).

[6] G. Harris and E. J. Martinec, Phys. Lett. B **245**, 384 (1990).

[7] E. Brézin and H. Neuberger, Phys. Rev. Lett. **65**, 2098 (1990).

[8] E. Brézin and H. Neuberger, Nucl. Phys. **B350**, 513 (1991).

[9] R. C. Meyers and V. Periwal, Phys. Rev. Lett. **64**, 3111 (1990).

[10] V. Periwal and D. Shevitz, Phys. Rev. Lett. **64**, 1326 (1990).

[11] V. Periwal and D. Shevitz, Nucl. Phys. **B344**, 731 (1990).

[12] H. Neuberger, Nucl. Phys. **B340**, 703 (1990).

[13] K. Demeterfi and C. Tan, Mod. Phys. Lett. A **5**, 1563 (1990).

[14] R. C. Meyers and V. Periwal, Phys. Rev. Lett. **65**, 1088 (1990).

[15] M. L. Mehta, *Random Matrices* (Academic, New York, 1967).

[16] M. R. Douglas, N. Seiberg, and S. H. Shenker, Phys. Lett. B **244**, 381 (1990).