Gravity and the Poincaré group

G. Grignani* and G. Nardelli†
Center for Theoretical Physics, Laboratory for Nuclear Science, and Department of Physics, Massachusetts Institute of Technology, Cambridge, Massachusetts 02139
(Received 17 October 1991)

We discuss gravity as a gauge theory of the Poincaré group in three and four dimensions, i.e., in a metric-independent fashion. The fundamental fields of the theory are the gauge potentials, the matter fields, and the so-called Poincaré coordinates $q^a(x)$: a set of fields that are defined on the space-time manifold, but that transform as Poincaré vectors under gauge transformations. The presence of such coordinates is necessary in order to construct a gauge theory of the Poincaré group. We discuss the procedure needed to connect this theory with the Einsteinian formulation of gravity, and we show that the field equations for the gauge potentials, for pointlike sources, and for scalar and spinor matter fields reproduce the Einstein equations, the geodesics equations, and the Klein-Gordon and the Dirac equations in curved space-time, respectively. In $2+1$ dimensions and in the presence of pointlike sources this gauge-theoretical approach can be further developed: the gauge potentials can be written almost everywhere as pure gauge, and a solution of the field equations provides, at the same time, the space-time metric and the set of coordinates that globally flatten the metric.

PACS number(s): 04.20.Cv, 11.15.$-$q

I. INTRODUCTION

By means of a first-order formalism, pure gravity in $2+1$ dimensions can be shown to be a Chern-Simons (CS) gauge theory of the Poincaré group ISO(2,1) where the components of the gauge potential along the ISO(2,1) generators are the dreibein $e^a{}_\mu$ and the spin connection $\omega^{ab}{}_\mu$ [1,2]. How to couple this CS theory to matter in an ISO(2,1) gauge-invariant fashion is an interesting and open problem. In this paper we shall show that such a coupling can indeed be realized both for pointlike and for extended (fields) matter sources so that the field equations of the theory, upon postulating the invertibility of the *dreibein,* reproduce the corresponding Einstein's equations. In constructing such a theory, however, one easily realizes that the *dreibein* cannot be identified with a component of the gauge potential and consequently the corresponding component of the field strength no longer has the meaning of torsion in space time. Actually the relation between Poincaré gauge potentials and *vielbein* can be established in a framework which does not depend on the space-time dimensionality and which allows one to construct a Poincaré gauge theory for gravity interacting with matter in any dimension. For this reason we shall discuss also the four-dimensional case, but focusing our interest in particular on $(2+1)$-dimensional gravity where this gauge-theoretical approach could be particularly convenient.

One may think that gravity as a gauge theory of the Poincaré group does not really represent a novelty; since the papers by Utiyama [3], Kibble [4], Sciama [5], as well as Chamseddine and West [6], MacDowell and Mansouri [7], there is by now an extensive literature on this subject. (For a review, see for instance, Ref. [8].) However, in these works the translational part of the Poincaré symmetry was parametrized so that it reproduces general coordinate transformations in space time considered actively, i.e., transforming fields but not the coordinates of the space time. Here, instead, we shall follow a different approach, which is more similar to the one first introduced by Stelle and West [9] for the SO(3,2) group spontaneously broken to the Lorentz group, successively reexamined by Pagels [10] for the O(5) group and also used by Kawai [11] following the lines of the standard geometrical formulation of gauge theories. In this framework, we shall consider the Poincaré gauge theory as closely as possible to any ordinary non-Abelian gauge theory, without discarding the translational part of the Poincaré symmetry in favor of general coordinate transformations. This can only be realized by introducing an extra degree of freedom in the theory: the set of Poincaré coordinates $q^a(x)$ that transform as Poincaré vectors under gauge transformation. As we shall see, such a Higgs-type field, whose geometrical meaning has been discussed in Refs. [9,11], naturally arises in our formalism by gauging the action of a free relativistic particle in Minkowski space so that it becomes invariant under *local* (depending on the space-time coordinates) Poincaré transformations. The $q^a(x)$ will then be interpreted as the coordinates of an internal (i.e., gauge) Poincaré space. Any choice of the Poincaré coordinates $q^a(x)$ can be performed just by fixing the translational part of the Poincaré group leaving the theory invariant under residual (local) Lorentz transformations. Among these gauge choices, the so-called "physical" gauge $q^a=0$ plays a particularly important role. The name "physical" derives from the fact that only in this specific gauge choice the components of the

*On leave from University of Perugia, Perugia, Italy.
†On leave from University of Trento, Trento, Italy.

gauge potential along the Poincaré generators become the physical *vielbein* and spin connection. Hence, this interpretation of the gauge potential only holds in the physical gauge choice and in the framework of a *Lorentz* gauge theory of gravity; alternatively, one can maintain the whole gauge invariance under the *Poincaré* group [without fixing a choice for the internal coordinate $q^a(x)$] or choose a gauge different from the physical one, but then the gauge potential can no longer be interpreted as *vielbein* and spin connection. We shall see that in some cases it is even possible to choose a gauge for which the translational components of the gauge field vanish. Actually the possibilities opened by this enlarged gauge invariance are not yet fully explored and the study of the quantum level of such a Poincaré gauge theory, for example, in gauge choices different from the "physical" one, would be extremely interesting.

The organization of the paper is as follows. In Sec. II we formulate the theory in four dimensions. The gauge potentials are introduced by gauging the coordinates q^a of a particle in the Minkowski space and the ISO(3,1) gauge-invariant actions for point particles, scalar and spinor fields, and for the gauge potentials are then presented. We discuss the procedure needed to connect this ISO(3,1) gauge theory with the Einsteinian formulation of gravity, and we show that the equations of motion obtained from the previous actions, upon supposing the *vierbein* invertible, reproduce the geodesic equations, the Klein-Gordon and the Dirac equations in curved space time, and the Einstein equations. We also discuss the geometrical interpretation of the Poincaré coordinates.

In Sec. III we specialize to three dimensions, where the formulation of the theory is particularly appealing for the following reasons.

(a) The Einstein-Hilbert action can be written in this case as a pure Chern-Simons term of the gauge potential and does not involve the Poincaré coordinates $q^a(x)$.

(b) When the matter sources are pointlike, the gauge potentials are almost everywhere pure gauges $A_\mu = U^{-1}\partial_\mu U, U \in \mathrm{ISO}(2,1)$, where U is the (multivalued) gauge group element that maps the *Poincaré* coordinates $q^a(x)$ in the space-time coordinates $\hat{q}^a(x)$ that globally flatten the metric. Consequently, as far as pointlike sources are concerned, a solution of the field equations in terms of U not only provides the space-time metric but, at the same time, also the set of coordinates that makes the metric globally Minkowskian.

In Sec. IV we draw our conclusions and discuss further possible developments.

II. POINCARÉ GROUP GAUGE INVARIANCE

One of the simplest ways to construct a gauge theory is to gauge the corresponding action invariant under global transformations. Let us consider a free relativistic particle in a four-dimensional Minkowskian manifold $\mathcal{M}_q$. The action can be written as

$$S_{\text{free}} = \int d\tau [p_a q^a + \lambda(p^2 - m^2)] \ , \tag{2.1}$$

where the momentum p_a is the variable canonically conjugate to the position variable q^a, τ denotes the proper time of the particle, and the Lagrange multiplier λ has been introduced to enforce the constraint $p^2 = m^2$, m being the mass of the particle. In Eq. (2.1) the indices a are raised and lowered with the Minkowskian metric tensor $\eta_{ab} = \mathrm{diag}(+,-,-,-)$. Since the particle is free, the canonical variables (q^a, p_b) can be identified with the space-time canonical variables (x^μ, π_ν), because in this case the space time is Minkowskian; if instead we wanted to include gravitational interactions, the previous identification would be incorrect and the q^a and p_b variables should be thought of as functions of the space-time trajectory of the particle $x^\mu(\tau)$. Obviously, S_{tree} is invariant under global Poincaré transformations

$$\begin{aligned} q^a &= \Lambda^a{}_b q^b + \rho^a \ , \\ p^a &= \Lambda^a{}_b p^b \ , \end{aligned} \tag{2.2}$$

$\Lambda^a{}_b$ and ρ^a being a constant Lorentz matrix and a constant translation, respectively. The Poincaré group ISO(3,1) is the semidirect product of the SO(3,1) group of Lorentz transformations with angular-momentum-boost generators J_{ab} and the group of translations, with generators P_a. The (anti-Hermitian) generators J_{ab} and P_a satisfy the Poincaré algebra

$$\begin{aligned} [P_a, P_b] &= 0 \ , \\ [P_a, J_{bc}] &= \eta_{ac} P_b - \eta_{ab} P_c \ , \\ [J_{ab}, J_{cd}] &= \eta_{ac} J_{bd} - \eta_{bc} J_{ad} + \eta_{bd} J_{ac} - \eta_{ad} J_{bc} \ . \end{aligned} \tag{2.3}$$

By parametrizing $\Lambda^a{}_b = \exp[-(\kappa^{cd}/2) J_{cd}]^a{}_b$, the infinitesimal form of the transformations (2.2) reads

$$\begin{aligned} \delta q^a &= \kappa^a{}_b q^b + \rho^a \ , \\ \delta p^a &= \kappa^a{}_b p^b \ . \end{aligned} \tag{2.4}$$

Now we gauge the transformations (2.2) by requiring that the action (2.1) becomes invariant under local transformations depending on the space-time coordinates x^μ. To this aim, we have to introduce the trajectory of the particle in the space time $x^\mu(\tau)$, and consider the Poincaré vector q^a as depending on the proper time τ only through the space-time trajectory $x(\tau)$. Clearly, to restore the ISO(3,1) invariance we have now to replace the derivative with respect to the proper time $\dot{q}^a$ with a covariant derivative $\mathcal{D}_\tau q^a = \mathcal{D}_\mu q^a \dot{x}^\mu$.

We notice that two kinds of covariant derivatives can be constructed. The first one $D_\mu q^a$ is such that it transforms like q^a itself (i.e., inhomogeneously) and is given by

$$D_\mu q^a = \partial_\mu q^a + \omega^{ab}{}_\mu q_b \ . \tag{2.5}$$

The condition

$$\delta D_\mu q^a = \kappa^a{}_b D_\mu q^b + D_\mu \rho^a \tag{2.6}$$

enforces the transformation law for the gauge potential $\omega^{ab}{}_\mu$ to be

$$\delta \omega^{ab}{}_\mu = -\partial_\mu \kappa^{ab} - \omega^{ac}{}_\mu \kappa_c{}^b + \omega^{cb}{}_\mu \kappa^a{}_c \ . \tag{2.7}$$

However, if we want to restore the invariance in the action (2.1) we need a covariant derivative that transforms homogeneously. Hence we look for a covariant derivative of the kind

$$\mathcal{D}_\mu q^a = D_\mu q^a + e^a{}_\mu \qquad (2.8)$$

and the condition

$$\delta \mathcal{D}_\mu q^a = \kappa^a{}_b \mathcal{D}_\mu q^b \qquad (2.9)$$

together with Eq. (2.7) implies the transformation law

$$\delta e^a{}_\mu = -\partial_\mu \rho^a + \kappa^a{}_b e^b{}_\mu - \omega^{ab}{}_\mu \rho_b \ . \qquad (2.10)$$

The finite form of the gauge transformations of the gauge potentials Eqs. (2.7), (2.10) are

$$\omega'^{ab}_\mu(x) = \Lambda^a{}_c(x)\Lambda^b{}_d(x)\omega^{cd}{}_\mu(x) + \Lambda^{ac}(x)\partial_\mu \Lambda^b{}_c(x) \ , \qquad (2.11a)$$

$$e'^a_\mu(x) = -\partial_\mu \rho^a(x) - \Lambda^a{}_b(x)\Lambda_{cd}(x)\omega^{bd}{}_\mu(x)\rho^c(x) + \Lambda^a{}_b(x)e^b{}_\mu(x) + \Lambda_b{}^c(x)\partial_\mu \Lambda^a{}_c(x)\rho^b(x) \ . \qquad (2.11b)$$

By construction, the ISO(3,1) gauge-invariant action for a particle reads

$$S_P = \int d\tau [p_a \mathcal{D}_\mu q^a \dot{x}^\mu + \lambda(p^2 - m^2)] \ , \qquad (2.12)$$

where $\dot{x}^\mu(\tau)$ is the vector tangent to the particle trajectory in space time and $q^a(x(\tau))$ describes an image trajectory in the (internal) Poincaré space. By comparing Eq. (2.12) with the usual action for a particle in the space time written in the first-order formalism (π_μ is the canonically conjugate variable to x^μ)

$$\begin{aligned} S &= \int d\tau [\pi_\mu \dot{x}^\mu + \lambda(\pi^\mu \pi_\mu - m^2)] \\ &= \int dt [p_a V^a{}_\mu \dot{x}^\mu + \lambda(p^2 - m^2)] \end{aligned} \qquad (2.13)$$

we are led to define the *vierbein* $V^a{}_\mu$ as

$$V^a{}_\mu \equiv \mathcal{D}_\mu q^a = \partial_\mu q^a + \omega^{ab}{}_\mu q_b + e^a{}_\mu \qquad (2.14)$$

and $V^a{}_\mu$ has the meaning of the soldering form between the space time and the Poincaré space $\mathcal{M}_q$. The validity of Eq. (2.14) can be further supported by a very simple argument. For any fixed space-time point x^μ we can choose a frame in which the space time is locally Minkowskian; in such a point we can choose $\omega^{ab}{}_\mu = e^a{}_\mu = 0$ and the *vierbein* becomes $V^a{}_\mu = \partial_\mu q^a$, i.e., the q^a can be interpreted as the local orthonormal coordinates at a fixed point. Then, by general covariance, the form of the *vierbein* at any other point can be obtained by the covariant replacement $\partial_\mu q^a \to \mathcal{D}_\mu q^a$, which is precisely Eq. (2.14).

By introducing the Lie-algebra-valued gauge potential

$$A_\mu = P_a e^a{}_\mu - \tfrac{1}{2} J_{ab} \omega^{ab}{}_\mu \ , \qquad (2.15)$$

Eqs. (2.7) and (2.10) become the usual gauge transformations of non-Abelian gauge theories: namely,

$$\begin{aligned} \delta A_\mu &= -\partial_\mu u - [A_\mu, u] = -\Delta_\mu u \\ u &= P_a \rho^a - \tfrac{1}{2} J_{ab} \kappa^{ab} \ . \end{aligned} \qquad (2.16)$$

The Lie-algebra-valued field strength, defined by

$$F_{\mu\nu} = [\Delta_\mu, \Delta_\nu] = P_a T^a{}_{\mu\nu} - \tfrac{1}{2} J_{ab} R^{ab}{}_{\mu\nu} \ , \qquad (2.17)$$

where $T^a{}_{\mu\nu} = \partial_\mu e^a{}_\nu - \partial_\nu e^a{}_\mu + \omega^{ab}{}_\mu e_{b\nu} - \omega^{ab}{}_\nu e_{b\mu}$ and $R^{ab}{}_{\mu\nu} = \partial_\mu \omega^{ab}{}_\nu - \partial_\nu \omega^{ab}{}_\mu + \omega^{ac}{}_\mu \omega_c{}^b{}_\nu - \omega^{ac}{}_\nu \omega_c{}^b{}_\mu$, transforms covariantly under gauge transformations. The Einstein-Hilbert action $\int d^4x \sqrt{-g} R$ where R is the scalar curvature, can be rewritten, within our formalism, as the Poincaré gauge-invariant action

$$S_{\text{EH}} = \tfrac{1}{4} \int d^4x \, \epsilon^{\mu\nu\rho\sigma} \epsilon_{abcd} \mathcal{D}_\mu q^a \mathcal{D}_\nu q^b R^{cd}{}_{\rho\sigma} \ . \qquad (2.18)$$

It should be noticed that S_{EH} does not depend only on the gauge potential, but also involves the Poincaré coordinates $q^a(x)$, and their presence is necessary in order to preserve Poincaré gauge invariance, as was first discussed by Stelle and West [9], in the framework of an SO(3,2) gauge theory spontaneously broken to SO(3,1).

A choice of the map $q^a(x)$ corresponds to a gauge choice that leaves invariant the whole Lorentz group as a residual gauge freedom. This can be realized by noticing that a choice of $q^a(x)$ entails a choice of $\rho^a(x)$ in the gauge transformations (2.2). Now we are ready to clarify an important point that has caused some confusion in the recent literature. It has been said that the gauge potentials of the Poincaré gauge theory are the *vierbein* and the spin connection. Strictly speaking this statement is incorrect because in general the *vierbein* has the more complicated structure given in Eq. (2.14), the translational component of the gauge potential being $e^a{}_\mu$ and not the whole *vierbein*. If we should replace $\mathcal{D}_\mu q^a$ with $e^a{}_\mu$ in Eq. (2.18) the Einstein-Hilbert action would not be gauge invariant. There is a gauge choice in which $e^a{}_\mu$ actually becomes the *vierbein*. It is the so-called "physical gauge" $q^a = 0$. Once the physical gauge is chosen, from Eq. (2.14) it follows that $V^a{}_\mu \equiv e^a{}_\mu$ and the Einstein-Hilbert action takes the more familiar form

$$S_{\text{EH}} = \tfrac{1}{4} \int d^4x \, \epsilon^{\mu\nu\rho\sigma} \epsilon_{abcd} e^a{}_\mu e^b{}_\nu R^{cd}{}_{\rho\sigma} \ . \qquad (2.19)$$

Having chosen a gauge, however, the theory is no more Poincaré but rather Lorentz gauge invariant, as the Lorentz group is the Poincaré subgroup which is left invariant by the gauge condition $q^a = 0$. But, since q^a are gauge degrees of freedom, any other choice is allowed to describe the dynamics of the fields. For example, a particularly convenient gauge choice that has been used in the (2+1)-dimensional case is $q^a=\delta^a{}_\mu x^\mu$, and the *vierbein* becomes $V^a{}_\mu=\delta^a{}_\mu+e^a{}_\mu+\omega^{ab}{}_\mu q_b$ which, again, is not a gauge potential.

For the very same reason, if one does not choose the physical gauge, the component $T^a{}_{\mu\nu}$ of the field strength cannot be interpreted as a space-time torsion $\mathcal{T}^a{}_{\mu\nu}$: in order to establish full contact with the corresponding space-time objects, it is necessary to express $\mathcal{T}^a{}_{\mu\nu}$ in terms of $T^a{}_{\mu\nu}$ by a redefinition, involving the Poincaré coordinates, which is analogous to the one used to obtain the *vierbein*. As a matter of fact, using Eq. (2.14), one can get

$$\mathcal{T}^a{}_{\mu\nu}=T^a{}_{\mu\nu}+R^{ab}{}_{\mu\nu}q_b \qquad (2.20)$$

and, as expected, only in the physical gauge $T^a{}_{\mu\nu}$ coincides with the space-time torsion. Since we are interested in a formulation of gravity as a gauge theory of the Poincaré group, we shall not fix any gauge choice for the Poincaré variables q^a.

As the Einstein-Hilbert action, Eq. (2.18), is invariant under ISO(3,1) gauge transformations, it must be possible to write it in an ISO(3,1) manifest scalar form. To this aim, we have to introduce a Lie-algebra-valued tensor $Q_{\mu\nu}$ which contains the Poincaré coordinate q^a and that transforms covariantly under gauge transformations. A natural choice is

$$Q_{\mu\nu}=D_\mu\mathcal{D}_\nu q^aP_a-\tfrac{1}{2}\mathcal{D}_\mu q^a\mathcal{D}_\nu q^bJ_{ab}\ , \qquad (2.21)$$

where D_μ and $\mathcal{D}_\mu$ have been defined in Eqs. (2.5) and (2.8). Introducing the degenerate, invariant, and associative inner product

$$\langle P_a,P_b\rangle=\langle P_a,J_{bc}\rangle=0\ ,\quad \langle J_{ab},J_{cd}\rangle=\epsilon_{abcd} \qquad (2.22)$$

the action (2.18) can be written in a manifest ISO(3,1) scalar form

$$S_{\mathrm{EH}}=\int d^4x\ \epsilon^{\mu\nu\rho\sigma}\langle Q_{\mu\nu},F_{\rho\sigma}\rangle\ . \qquad (2.23)$$

The next step is to derive the equation of motion, and to verify their equivalence to those obtained in the Einsteinian formulation of the theory. Let us consider the gauge-invariant action

$$S=\int d\tau[p_a\mathcal{D}_\mu q^a\dot{x}^\mu+\lambda(p^2-m^2)] -\frac{1}{4\pi G}\int d^4x\ \epsilon^{\mu\nu\rho\sigma}\epsilon_{abcd}\mathcal{D}_\mu q^a\mathcal{D}_\nu q^bR^{cd}{}_{\rho\sigma} \qquad (2.24)$$

where G is Newton's constant. The equations of motion for the fields obtained by varying S with respect to $q^a(x)$, $e^a{}_\mu(x)$, and $\omega^{ab}{}_\mu(x)$ are, respectively,

$$\epsilon^{\mu\nu\rho\sigma}\epsilon_{abcd}(T^b{}_{\mu\nu}R^{cd}{}_{\rho\sigma}+R^{be}{}_{\mu\nu}R^{cd}{}_{\rho\sigma}q_e) =4\pi G\int d\tau(\partial_\mu p_a+\omega_a{}^b{}_\mu p_b)\dot{x}^\mu\delta^{(4)}(x-x(\tau))\ , \qquad (2.25a)$$

$$\epsilon^{\mu\nu\rho\sigma}\epsilon_{abcd}\mathcal{D}_\nu q^bR^{cd}{}_{\rho\sigma}=2\pi G\int d\tau\, p_a\dot{x}^\mu\delta^{(4)}(x-x(\tau))\ , \qquad (2.25b)$$

$$\epsilon^{\mu\nu\rho\sigma}(\epsilon_{aecd}q_b\mathcal{D}_\nu q^eR^{cd}{}_{\rho\sigma}-\epsilon_{becd}q_a\mathcal{D}_\nu q^eR^{cd}{}_{\rho\sigma} -\epsilon_{abcd}q_e\mathcal{D}_\nu q^cR^{ed}{}_{\rho\sigma}+\epsilon_{abcd}\mathcal{D}_\nu q^cT^d{}_{\rho\sigma}) =2\pi G\int d\tau(p_aq_b-q_ap_b)\dot{x}^\mu\delta^{(4)}(x-x(\tau))\ . \qquad (2.25c)$$

With the previous identifications for the *vierbein* and for the space-time torsion and postulating the invertibility of the *vierbein*, Eqs. (2.25b), (2.25c) reproduce the correct Einstein's equations

$$T^a{}_{\mu\nu}\equiv\partial_\mu V^a{}_\nu-\partial_\nu V^a{}_\mu+\omega^{ab}{}_\mu V_{b\nu}-\omega^{ab}{}_\nu V_{b\mu}=0\ , \qquad (2.26a)$$

$$\epsilon^{\mu\nu\rho\sigma}\epsilon_{abcd}V^b{}_\nu R^{cd}{}_{\rho\sigma}=2\pi G\int d\tau\, p_a\dot{x}^\mu\delta^{(4)}(x-x(\tau)) \qquad (2.26b)$$

whereas Eq. (2.25a) becomes, taking into account Eqs. (2.25b), (2.25c),

$$\dot{x}^\mu(\tau)(\partial_\mu p_a+\omega_a{}^b{}_\mu p_b)=0\ . \qquad (2.27)$$

As expected, as far as the angular momentum of the particle is purely orbital ($M_{ab}=q_ap_b-q_bp_a$), the space-time torsion $\mathcal{T}^a{}_{\mu\nu}$ vanishes. On the contrary, a spin contribution S_{ab} in the angular momentum would entail a spin-interaction term in the action of the type

$$\tfrac{1}{2}\int d\tau\omega^{ab}{}_\mu S_{ab}\dot{x}^\mu(\tau)\ , \qquad (2.28)$$

and Eq. (2.26a) would become

$$\epsilon^{\mu\nu\rho\sigma}\epsilon_{abcd}V^c{}_\nu\mathcal{T}^d{}_{\rho\sigma}=2\pi G\int d\tau\dot{x}^\mu S_{ab}\delta^{(4)}(x-x(\tau))\ , \qquad (2.29)$$

leading to a nonvanishing space-time torsion.

The equations of motion for the particle are obtained by varying the action with respect to $q_a(x(\tau))$, $p_a(x(\tau))$, $x(\tau)$, and λ. The variation with respect to $q_a(x(\tau))$ reproduces Eq. (2.27) whereas the variation with respect to $p_a(x(\tau))$ gives

$$2\lambda p^a+\dot{x}^\mu(\partial_\mu q^a+e^a{}_\mu+\omega^{ab}{}_\mu q_b)=0\ . \qquad (2.30)$$

Substituting Eq. (2.30) in Eq. (2.27) and taking into account Eq. (2.14) we get

$$V^a{}_\rho\ddot{x}^\rho+\dot{x}^\mu\dot{x}^\rho(\partial_\mu V^a{}_\rho+\omega^a{}_{b\mu}V^b{}_\rho)=0\ , \qquad (2.31)$$

namely, the geodesics equation. Finally $\delta S/\delta\lambda=0$ obviously enforces the constraint $p^2=m^2$ whereas $\delta S/\delta x^\mu(\tau)=0$ gives an equation which is identically satisfied, that is

$$p^a\dot{x}^\nu(T^a{}_{\mu\nu}+R^{ab}{}_{\mu\nu}q_b)=p_a\dot{x}^\nu\mathcal{T}^a{}_{\mu\nu}=0\ . \qquad (2.32)$$

Hence, all the variations of the action reproduce the corresponding equations in the Einsteinian approach; moreover, the extra degrees of freedom that we introduced in order to construct a gauge theory of the Poincaré group turn out to be harmless. The corresponding (extra) equations either collapse with other equations [see Eq. (2.27)] or provide identically satisfied conditions [see Eq. (2.32)].

Next we consider the coupling with matter fields (spinors and scalars). It is important to stress that in our approach the space-time coordinates do not change under a

Poincaré transformation and consequently the matter fields transform only because of their internal structure and not because of their x dependence. The fact that the x^μ do not transform is compensated by the presence of the Poincaré coordinates $q^a(x)$ so that we will obtain gauge-invariant actions that not only reproduce the correct gravitational equations of motion, but are also invariant under diffeomorphisms. We intend to provide actions that are ISO(3,1) gauge invariant and independent of the metric tensor $g_{\mu\nu}$. We naturally require that the matter fields carry a representation of the whole Poincaré group. Let us first consider a scalar field. In order to formulate a metric-independent theory we have to work in a first-order formalism. To this purpose we introduce a field φ^A, $A=0,\ldots,4$ carrying a vectorial representation of the Poincaré group; in components φ^A reads $\varphi^A\equiv(\varphi^a,m^2\varphi)$, where φ^a is a Lorentz vector, $\varphi^4\equiv\varphi$ a scalar field and m a mass parameter. The transformation law for such a field, along the Lorentz and the translational (fourth) component can be written, in terms of the vectorial (5×5) representation of the Poincaré group generators, as

$$\delta\varphi^a=-\tfrac{1}{2}(J_{cd})^a{}_b\kappa^{cd}\varphi^b+(P_b)^a{}_4\rho^b m^2\varphi \equiv\kappa^a{}_b\varphi^b+\rho^a m^2\varphi\ , \tag{2.33a}$$

$$\delta\varphi=0\ , \tag{2.33b}$$

where

$$(J_{cd})^a{}_b=\eta_{cb}\delta_d{}^a-\eta_{db}\delta_c{}^a\ ,\quad (J_{cd})^4{}_a=(J_{cd})^a{}_4=0\ ,\qquad (P_b)^a{}_4=\delta^a{}_b\ ,\quad (P_b)^a{}_c=(P_b)^4{}_a=0\ . \tag{2.34}$$

Accordingly we can construct the covariant derivatives

$$\mathcal{D}_\mu\varphi^a=\partial_\mu\varphi^a+\omega^a{}_{b\mu}\varphi^b+m^2e^a{}_\mu\varphi\ , \tag{2.35a}$$

$$\mathcal{D}_\mu\varphi=\partial_\mu\varphi\ , \tag{2.35b}$$

that under the gauge-group transform as the fields φ^a and φ, respectively. An ISO(3,1) gauge-invariant action is then

$$S_S=-\tfrac{1}{3}\int d^4x\ \epsilon^{\mu\nu\rho\lambda}\epsilon_{abcd}\mathcal{D}_\mu q^a\mathcal{D}_\nu q^b\mathcal{D}_\rho q^c \times[\varphi\mathcal{D}_\lambda\varphi^d-\partial_\lambda\varphi\varphi^d+\mathcal{D}_\lambda q^d(\varphi_e\varphi^e-2q_e\varphi^e\varphi m^2 +q_eq^e\varphi^2m^4)]\ . \tag{2.36}$$

S_S in the physical gauge and in the flat-space-time limit becomes

$$S_S=-2\int d^4x(\varphi\partial_a\varphi^a-\partial_a\varphi\varphi^a+4m^2\varphi^2+4\varphi_a\varphi^a)\ , \tag{2.37}$$

that, after eliminating the field φ_a, reproduces the flat-space-time action for a scalar field of mass $4m$. The variation of Eq. (2.36) with respect to φ^a [supposing $\det(\mathcal{D}_\mu q^a)\neq0$] gives a relation between φ and φ^a:

$$\mathcal{D}_\mu q^a\varphi_a=\tfrac{1}{4}\partial_\mu\varphi\ . \tag{2.38}$$

By means of Eq. (2.38) the variation with respect to φ provides the Klein-Gordon equation in curved space time for a scalar particle with mass $4m$.

It has to be stressed that the requirement of gauge invariance of the theory under the whole Poincaré group and the consequent requirement that the fields carry a representation of such a group, entail the presence of a mass term in the action. Had we considered massless fields, the field φ^A would have been a Lorentz-vector field and would have transformed only through $\kappa^a{}_b$.

As one should expect, the variation of S_S with respect to the gauge fields reproduces the correct energy-momentum tensor for a massive scalar field in the Einstein equations and in particular does lead to a vanishing torsion.

The potential term for scalar fields can also be constructed and reads

$$S_S^P=\int d^4x\ \epsilon^{\mu\nu\rho\lambda}\epsilon_{abcd}\mathcal{D}_\mu q^a\mathcal{D}_\nu q^b\mathcal{D}_\rho q^c\mathcal{D}_\lambda q^d \times V(\varphi_e\varphi^e-2q_e\varphi^e\varphi m^2+q_eq^e\varphi^2m^4)\ , \tag{2.39}$$

where $V(\varphi_e\varphi^e-2q_e\varphi^e\varphi m^2+q_eq^e\varphi^2m^4)$ is a second-order polynomial of the argument. To include fermions we have to introduce 4×4 Dirac matrices γ^a normalized by $\{\gamma^a,\gamma^b\}=2\eta^{ab}$ (we use the conventions of Ref. [12]). Spinors belong to the fundamental representation of ISO(3,1) transforming as

$$\delta\psi=u\psi=(-\tfrac{1}{2}J_{ab}\kappa^{ab}+P_a\rho^a)\psi \equiv\frac{i}{4}\sigma^{ab}\kappa_{ab}\psi-im\gamma_a\rho^a(1+s\gamma_5)\psi\ , \tag{2.40}$$

where $\sigma^{ab}=i[\gamma^a,\gamma^b]/2$ are the ten matrices representing the generators of a Lorentz transformation on a spinor, $P_a=im\gamma_a(1+s\gamma_5)$ is the generator of the translations on a spinor, m is a mass parameter, and s is a real number such that $s^2=1$. From Eq. (2.40) the covariant derivative that transforms as the field itself is

$$\mathcal{D}_\mu\psi=\partial_\mu\psi-\frac{i}{4}\omega^{ab}{}_\mu\sigma_{ab}\psi+im\gamma_ae^a{}_\mu(1+s\gamma_5)\psi\ . \tag{2.41}$$

The gauge-invariant action for a massive Dirac spinor is

$$S_F=\tfrac{1}{12}\int d^4x\ \epsilon^{\mu\nu\rho\lambda}\epsilon_{abcd}\mathcal{D}_\mu q^a\mathcal{D}_\nu q^b\mathcal{D}_\rho q^c(\bar\psi\{\gamma^d-imq^e[\gamma^d\gamma_e(1+s\gamma_5) -\gamma_e\gamma^d(1-s\gamma_5)]+4m^2(q^d\gamma_eq^e-\tfrac{1}{2}|q|^2\gamma^d)(1+s\gamma_5)\}\mathcal{D}_\lambda\psi-\text{H.c.}) \tag{2.42}$$

where H.c. denotes the Hermitian conjugate and $|q|^2=q^aq_a$. Again, in the physical gauge and assumed the invertibility of the *vierbein* $V^a{}_\mu=\mathcal{D}_\mu q^a$, this action reproduces the correct Dirac equation in curved space time for a spinor of mass $4m$, and provides a nonvanishing torsion term in the gauge-field equations.

For the spinors, as well as for the scalars, the mass term is related to the translational part of the gauge group so that a massless fermion would carry a representation only of the Lorentz group.

We conclude that all the renormalizable field theories can be included into this metric-independent, Poincaré gauge-invariant formalism.

Some remarks concerning the geometrical interpretation of the Poincaré coordinates are now in order. A purely geometrical description of the problem goes beyond the aim of the present paper, and for a deeper analysis of this subject we shall refer to the works by Stelle and West [9] and by Kawai [11]. We shall limit ourselves to some comments which can be useful for a better understanding of the meaning of the fields introduced so far.

The fields $q^a(x)$ were obtained by gauging the action of a free particle defined in the (internal) Minkowski space. However, the presence of such degrees of freedom is fundamental, in order to construct an ISO(3,1) gauge theory, not only for describing gravitational point-particle interactions, but also matter-field interactions and even pure gravity with no matter degrees of freedom. Consequently, it turns out that the Poincaré coordinates $q^a(x)$ must have a more fundamental and intrinsic role in a formulation of gravity as a Poincaré gauge theory.

Following the usual geometrical interpretation of gauge theories, the gauge potentials $e^a{}_\mu$ and $\omega^{ab}{}_\mu$ are the connections defined on the principal ISO(3,1) bundle. From the principal bundle, a fiber bundle admitting a global cross section Σ can be constructed, whose fibers are the coset spaces $\{\mathrm{ISO}(3,1)/\mathrm{SO}(3,1)\}$. Moreover, the tangent spaces to the fibers $\{\mathrm{ISO}(3,1)/\mathrm{SO}(3,1)\}$ at the points selected by the cross section can be smoothly mapped onto the tangent space at the corresponding space-time point; i.e., there must exist a form soldering the two tangent spaces. Such soldering form is just the *vierbein* $V^a{}_\mu=\mathcal{D}_\mu q^a$, the coset spaces $\{\mathrm{ISO}(3,1)/\mathrm{SO}(3,1)\}$ are the Poincaré spaces $\mathcal{M}_q$, and a point on the cross section Σ is represented by the Poincaré coordinates $q^a(x)$. For a fixed space-time point $P_1\equiv x_1^\mu$, one can always choose a cross section Σ_1 which coincides with the tangent space at P_1 and the *vierbein* at that point is simply given by $\partial_\mu q_1^a(x_1)$. Had we chosen a different cross section Σ_2, the coordinates $q_2^a(x_1)$ on Σ_2 would not represent the locally orthonormal coordinates anymore. However, $q_1^a(x_1)$ and $q_2^a(x_1)$ are related by a Poincaré transformation $q_1^a(x_1)=\Lambda^a{}_b(x_1)q_2^b(x_1)+\rho^a(x_1)$ so that, for any cross section other than Σ_1, the *vierbein* has the more general structure exhibited in Eq. (2.14). Once a cross section has been picked, the residual structure group is SO(3,1) and two *vierbein* which differ by a Lorentz transformation must be identified. Now we are able to explain what the field strength $F_{\mu\nu}$ represents: given an infinitesimal closed curve $x(\tau)$ in the space time, $\tau\in[\tau_0,\tau_1]$ and $x(\tau_0)=x(\tau_1)$, the Poincaré torsion $T^a{}_{\mu\nu}$ gives the gap by which the image curve $q^a(x(\tau))=q^a(\tau)$ in the internal space fails to close and the Poincaré curvature $R^{ab}{}_{\mu\nu}$ (which coincides with the space-time curvature) gives the relative rotation of the image vector $q^a(x(\tau_1))$ developed around such an infinitesimal curve with respect to the original vector $q^a(x(\tau_0))$. In fact, by parallel transporting $q^a(x)$ when $x(\tau)$ describes an infinitesimal closed curve one can check that the quantity $q^a(x(\tau_1))-q^a(x(\tau_0))$ has precisely the form of an infinitesimal gauge transformation, with translation ρ^a proportional to $T^a{}_{\mu\nu}$ and rotation κ^{ab} proportional to $R^{ab}{}_{\mu\nu}$, namely,

$$\begin{aligned}\delta q^a&=q^a(\tau_1)-q^a(\tau_0)=\kappa^{ab}q_b(\tau_0)+\rho^a\\&=\epsilon^{\mu\nu}[R^{ab}{}_{\mu\nu}(\tau_0)q_b(\tau_0)+T^a{}_{\mu\nu}(\tau_0)]\ ,\end{aligned}\qquad(2.43)$$

where the antisymmetric matrix $\epsilon^{\mu\nu}$ is given by

$$\epsilon^{\mu\nu}=\frac{1}{2}\int_{\tau_0}^{\tau_1}d\tau\frac{dx^\mu}{d\tau}x^\nu(\tau)=\frac{1}{2}\oint dx^\mu x^\nu\ .\qquad(2.44)$$

Incidentally, the square brackets on the right-hand side of Eq. (2.43) give precisely the space-time torsion $\mathcal{T}^a{}_{\mu\nu}$ evaluated at the point $x(\tau_0)$ [see Eq. (2.20)] so that, for any physical system without spin, $\delta q^a=0$ on shell: for any closed curve in the space time, the image curve is closed and the vector $q^a(\tau_1)$ returns to its original position $q^a(\tau_0)$ and with the same orientation.

III. (2+1)-DIMENSIONAL CASE

A. General framework

From the gauge-theoretical point of view, gravity in 2+1 dimensions [13] is even more interesting than the (3+1)-dimensional case. This is due to the fact that in 2+1 dimensions the Einstein-Hilbert actions can be written as a Chern-Simons term of the ISO(2,1) gauge group. After the original works by Achucarro and Townsend [1] and Witten [2], there have been several attempts to couple this gauge theory to pointlike sources, both for the study of classical [14,15] and quantum graviational effects [16–22]. In this section after a brief review of Witten's approach, we introduce the actions coupling this theory to matter fields and to pointlike sources. In the latter case, being the gauge potentials almost everywhere pure gauges, an ISO(2,1) gauge-group element not only specifies the gauge potentials but also provides the gauge transformation that, starting from the *Poincaré* coordinates q^a, gives the space-time coordinates $\hat{q}^a$ that globally flatten the metric.

Following the general method developed in the previous section, it is immediate to write an ISO(2,1) gauge-invariant form for the Einstein-Hilbert action:

$$S_{\mathrm{EH}}=\tfrac{1}{2}\int d^3x\ \epsilon^{\mu\nu\rho}\epsilon_{abc}\mathcal{D}_\mu q^a R^{bc}{}_{\nu\rho}\ .\qquad(3.1)$$

However, in 2+1 dimensions the action (3.1) can be written only in terms of gauge potentials $e^a{}_\mu$ and $\omega^{ab}{}_\mu$, yet remaining Poincaré gauge invariant. In fact, all the terms containing q^a in Eq. (3.1) give, taking into account the Bianchi identity, only a surface term and S_{EH} becomes

$$S_{\mathrm{EH}}=\tfrac{1}{2}\int d^3x\ \epsilon^{\mu\nu\rho}\epsilon_{abc}e^a{}_\mu R^{bc}{}_{\nu\rho}\ .\qquad(3.2)$$

In order to write S_{EH} as a Chern-Simons action, it is convenient to redefine the SO(2,1) generators and gauge potential

$$J_a=\tfrac{1}{2}\epsilon_a{}^{bc}J_{bc}\ , \tag{3.3a}$$

$$\omega^a{}_\mu=-\tfrac{1}{2}\epsilon^a{}_{bc}\omega^{bc}{}_\mu\ , \tag{3.3b}$$

in such a way that the Poincaré algebra takes the simpler form

$$[J_a,J_b]=\epsilon_{abc}J^c\ ,\quad [J_a,P_b]=\epsilon_{abc}P^c\ ,\quad [P_a,P_b]=0\ . \tag{3.4}$$

The Lie-algebra-valued gauge potential can then be written as

$$A_\mu=P_a e^a{}_\mu+J_a\omega^a{}_\mu\ , \tag{3.5}$$

and the ISO(2,1) gauge transformations that leave the action (3.2) invariant take the form

$$\delta A_\mu=-\Delta_\mu u=-\partial_\mu u-[A_\mu,u]\ ,\quad u=P_a\rho^a+J_a\kappa^a\ . \tag{3.6}$$

Finally, introducing the invariant, nondegenerate, associative inner product

$$\langle P_a,P_b\rangle=\langle J_a,J_b\rangle=0\ ,\quad \langle P_a,J_b\rangle=\eta_{ab}\ , \tag{3.7}$$

$S_{\rm EH}$ can be written as a pure Chern-Simons term

$$S_{\rm EH}=-\tfrac{1}{2}\int d^3x\ \epsilon^{\mu\nu\rho}\langle A_\mu,\partial_\nu A_\rho+\tfrac{1}{3}[A_\nu,A_\rho]\rangle\ . \tag{3.8}$$

It is worth stressing that in $S_{\rm EH}$ the $q^a(x)$ fields do not appear; consequently, as far as the vacuum solutions are concerned, $e^a{}_\mu$ can indeed be interpreted as the space-time *dreibein* and yet the theory is fully Poincaré (and not only Lorentz) gauge invariant, in contrast with what happens in 3+1 dimensions. In this particular case, the Poincaré and space-time torsions coincide and the equations of motion obtained from $S_{\rm EH}$ reproduce, upon postulating the invertibility of the *dreibein,* Einstein's equations of vanishing curvature.

It has to be noticed, however, that this is a peculiarity of the vacuum solutions. In fact, if we try to couple this gauge theory with matter sources, then $e^a{}_\mu$ can no longer be interpreted as a *dreibein.* As an example we can consider the ISO(2,1) gauge-invariant action for a relativistic spinless particle with mass m:

$$S_P=\int d\tau[p_a(\partial_\mu q^a+e^a{}_\mu+\epsilon^a{}_{bc}\omega^b{}_\mu q^c)\dot{x}^\mu(\tau)+\lambda(p^2-m^2)]$$
$$=\int d\tau[p_a\mathcal{D}_\mu q^a\dot{x}^\mu+\lambda(p^2-m^2)]\ , \tag{3.9}$$

where the gauge transformations for the variables q^a and p^a are now parametrized as

$$\delta q^a=\epsilon^a{}_{bc}\kappa^b q^c+\rho^a\ ,\quad \delta p^a=\epsilon^a{}_{bc}\kappa^b p^c\ . \tag{3.10}$$

As in the (3+1)-dimensional case, a comparison between Eq. (3.9) and the corresponding action in the Einsteinian formulation gives the expression for the *dreibein:*

$$V^a{}_\mu=\partial_\mu q^a+e^a{}_\mu+\epsilon^a{}_{bc}\omega^b{}_\mu q^c\ . \tag{3.11}$$

The only case in which $e^a{}_\mu$ can be identified with the space-time *dreibein* is when the physical gauge $q^a=0$ is picked; however, in this case one is not dealing with a Poincaré gauge theory but rather with a Lorentz gauge theory, as the action (3.9) with $q^a=0$ is invariant only under Lorentz transformation. This point, if not carefully taken into account, can give rise to meaningless results as, for instance, a nonvanishing torsion associated with spinless particle couplings. To overcome the problem we notice that, again, the space-time torsion $\mathcal{T}^a{}_{\mu\nu}$ is different from the Poincaré torsion $T^a{}_{\mu\nu}$ (i.e., the coefficient of the field strength $F_{\mu\nu}$ along the P_a generators). In fact, from Eq. (3.11) one can easily obtain the (2+1)-dimensional analogue of Eq. (2.20):

$$\mathcal{T}^a{}_{\mu\nu}=T^a{}_{\mu\nu}+\epsilon^a{}_{bc}R^b{}_{\mu\nu}q^c\ , \tag{3.12}$$

$R^a{}_{\mu\nu}=-\tfrac{1}{2}\epsilon^a{}_{bc}R^{bc}{}_{\mu\nu}$ being the component of the field strength along the J^a generators.

The coupling with matter (spinor and scalar) fields can be performed along the same lines of the (3+1)-dimensional case. The gauge invariant action that reproduces the correct field equations for a scalar field φ in curved space time can be obtained, as above, by introducing a Poincaré vector field $\varphi^A=(\varphi^a,m^2\varphi)\,A=0,\dots,3$, transforming under the Poincaré group as

$$\delta\varphi^a=(J_b)^a{}_c\kappa^b\varphi^c+(P_b)^a{}_3\rho^b m^2\varphi$$
$$\equiv\epsilon^a{}_{bc}\kappa^b\varphi^c+\rho^a m^2\varphi\ , \tag{3.13a}$$

$$\delta\varphi^4\equiv\delta\varphi=0\ , \tag{3.13b}$$

where we have used the (4×4) representation of the Poincaré group generators:

$$(J_c)^a{}_b=-\epsilon_c{}^a{}_b\ ,\quad (J_c)^3{}_a=(J_c)^a{}_3=0\ , \tag{3.14a}$$

$$(P_b)^a{}_3=\delta^a{}_b\ ,\quad (P_b)^a{}_c=(P_b)^3{}_a=0\ . \tag{3.14b}$$

From Eqs. (3.13)–(3.14) one can define covariant derivatives transforming as the fields φ^a and φ,

$$\mathcal{D}_\mu\varphi^a=\partial_\mu\varphi^a+\epsilon^a{}_{bc}\omega^b{}_\mu\varphi^c+m^2e^a{}_\mu\varphi\ , \tag{3.15a}$$

$$\mathcal{D}_\mu\varphi=\partial_\mu\varphi\ , \tag{3.15b}$$

and, in terms of these, the action reads

$$S_S=-\tfrac{4}{3}\int d^3x\ \epsilon^{\mu\nu\lambda}\epsilon_{abc}\mathcal{D}_\mu q^a\mathcal{D}_\nu q^b[\varphi\mathcal{D}_\lambda\varphi^c-\partial_\lambda\varphi\varphi^c+\mathcal{D}_\lambda q^c(\varphi_d\varphi^d-2q_d\varphi^d\varphi m^2+q_dq^d\varphi^2m^4)]\ . \tag{3.16}$$

The coefficient in Eq. (3.14) has been chosen in order to reproduce, once the physical gauge is selected and the vector field φ_d is eliminated through its equations of motion, the action for a scalar field in curved space time.

Like in the (3+1)-dimensional case, a consequence of the fact that the field φ^A carries a vectorial representation of the Poincaré group is the presence of a mass term for the scalar field φ, the mass being related to the translational part of the Poincaré group.

The (2+1)-dimensional action that describes fermions interacting with Poincaré gauge fields cannot be straightforwardly obtained from the analogous (3+1)-dimensional action [Eq. (2.42)], because of the absence in 2+1 dimension of an operator playing the role of γ^5 in 3+1 dimensions. Moreover, the Dirac matrices in 2+1 dimensions being two-dimensional [we will choose the $\gamma^a \equiv (\sigma^3, i\sigma^2, -i\sigma^1)$ representation, σ^i = Pauli matrices], a spinorial representation of the ISO(2,1) algebra, Eq. (3.4), cannot be achieved only in terms of 2×2 matrices γ^a. To represent the operator J_a and P_a we shall choose in fact the 4×4 representation

$$J_a = \frac{i}{2}\begin{bmatrix} \gamma_a & 0 \\ 0 & \gamma_a \end{bmatrix} \equiv \frac{i}{2}\Gamma_a \, , \qquad P_a = \frac{2}{3} im \begin{bmatrix} 0 & 0 \\ \gamma_a & 0 \end{bmatrix} \equiv \frac{2}{3} im I_- \Gamma_a \, , \tag{3.17}$$

where we have defined Γ^a, the step operator $I_- = \left(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right)$, and the coefficient in P_a has been introduced for later convenience. In order to maintain a covariant notation we then have to work with four-dimensional spinors, $\Psi = \left(\begin{smallmatrix} \psi \\ \psi \end{smallmatrix}\right)$, where ψ is two-dimensional. The action of the Poincaré group on Ψ is given by

$$\delta\Psi = u\Psi = (J_a \kappa^a + P_a \rho^a)\Psi \; ; \tag{3.18}$$

i.e., Ψ transforms in the fundamental representation of ISO(2,1). As usual, the covariant derivative $\mathcal{D}_\mu \Psi$, transforming as $\delta \mathcal{D}_\mu \Psi = u \mathcal{D}_\mu \Psi$, is defined by

$$\mathcal{D}_\mu \Psi = (\partial_\mu + A_\mu)\Psi = \partial_\mu \Psi + \omega^a{}_\mu J_a \Psi + e^a{}_\mu P_a \Psi \; . \tag{3.19}$$

Introducing $\overline{\Psi} = \Psi^\dagger \Gamma^0$, the gauge-invariant action that reproduces the Dirac action for a spinor of mass m in the gravitational field reads

$$S_F = \int d^3x \, \epsilon^{\mu\nu\rho} \epsilon_{abc} V^a{}_\mu V^b{}_\nu \left[\overline{\Psi} \left[\frac{\Gamma^c}{8} + \frac{im}{12}(I_+ \not{q} \Gamma^c - \Gamma^c \not{q} I_-) + \frac{m^2}{9} I_+ I_- (2\not{q} q^c - |q|^2 \Gamma^c) \right] \mathcal{D}_\rho \Psi - \text{H.c.} \right] , \tag{3.20}$$

with the obvious definition for I_+ and $\not{q} = q^a \Gamma_a$. Again, the fact that the spinor Ψ carries also a representation of the translational part of the Poincaré group necessarily yields to a mass term in the action (3.20). In two-dimensional notation Eq. (3.20) becomes

$$S_F = \frac{1}{4} d^3x \, \epsilon^{\mu\nu\rho} \epsilon_{abc} V^a{}_\mu V^b{}_\nu \left[\bar{\psi} \gamma^c D_\rho \psi + \frac{im}{3} e^f{}_\rho \left[\delta^c{}_f + \frac{2}{3} m \epsilon^{dc}{}_f q_d \right] \bar{\psi}\psi + \frac{m}{3} \epsilon^c{}_{df} \bar{\psi} \gamma^f (e^d{}_\rho \psi - 2q^d D_\rho \psi) \right.$$
$$\left. + \frac{2}{9} m^2 \bar{\psi} (\delta^c{}_d \delta^g{}_f - \eta_{df} \eta^{gc} + \delta^c{}_f \delta^g{}_d) q^d (2q^f \gamma_g D_\rho \psi - \gamma_g e^f{}_\rho \psi) - \text{H.c.} \right] , \tag{3.21}$$

where $D_\rho \psi = \partial_\rho \psi + (i/2)\gamma_a \omega^a{}_\rho \psi$. The gauge-invariant actions, Eqs. (2.39), (2.42), (3.16), and (3.20), for the interaction of matter fields with Poincaré gravity deserve further investigations either in the direction of the research of new classical or semiclassical solutions (in particular the existence of the soliton solutions presumably can be more easily discovered in this gauge-invariant framework because of the resemblance it bears with a usual non-Abelian gauge theory) or for quantization purposes. In particular the enlarged gauge freedom introduced by the presence of the q^a variables opens unexplored possibilities. Suppose for example that the coordinate $q^a(x)$ are chosen so that $q^a(x) = \delta^a{}_\mu x^\mu$, then the theory would be defined on the flat manifold $\mathcal{M}_q$ even if the space-time manifold is nowhere flat in the presence of matter field. The connection between the manifold $\mathcal{M}_q$ and the space-time manifold would be realized by the soldering from $V^a_\mu = \mathcal{D}_\mu q^a$ that in this gauge choice would become simply $V^a{}_b = \delta^a{}_b + e^a{}_b + \omega^{ac}{}_b q_c$, since there would not be any more distinction among greek and latin indices. The definition of the theory on the manifold $\mathcal{M}_q$ instead of on the space-time manifold would then give us an intriguing "flat theory of gravity." But let us turn to a more prosaic interaction with point particles.

B. Interaction with pointlike sources

The action we shall consider for N relativistic spinless particles interacting with Poincaré gauge fields is

$$S = S_P + S_{\text{EH}}$$
$$= \sum_{\alpha=1}^{N} \int d\tau [\eta_{ab} p^a_{(\alpha)} \mathcal{D}_\mu q^b_{(\alpha)} \dot{x}^\mu_{(\alpha)}(\tau) + \lambda_{(\alpha)}(p^2_{(\alpha)} - M^2_{(\alpha)})] - \frac{1}{4\pi G} \int d^3x \, \epsilon^{\mu\nu\rho} e_{a\mu} (\partial_\nu \omega^a{}_\rho - \partial_\rho \omega^a{}_\nu + \epsilon^a{}_{bc} \omega^b{}_\nu \omega^c{}_\rho) \, , \tag{3.22}$$

where G is the Newton's constant, $M_{(\alpha)}$ is the mass of the αth particle, $\dot{x}^\mu_{(\alpha)}(\tau)$ is the vector tangent to the αth trajectory in space-time and $q^a_{(\alpha)}(x(\tau))$ is the image trajectory in $\mathcal{M}_q$. By varying S with respect to $e^a{}_\mu$ and $\omega^a{}_\mu$ we get the field equations

$$\frac{1}{4\pi G}\epsilon^{\mu\nu\rho}R^a{}_{\nu\rho}\equiv\frac{1}{4\pi G}\epsilon^{\mu\nu\rho}(\partial_\nu\omega^a{}_\rho-\partial_\rho\omega^a{}_\nu+\epsilon^a{}_{bc}\omega^b{}_\nu\omega^c{}_\rho)$$
$$=\sum_{\alpha=1}^N\int d\tau p^a_{(\alpha)}\dot{x}^\mu_{(\alpha)}\delta^{(3)}(x-x_{(\alpha)}(\tau))=\sum_{\alpha=1}^N p^a_{(\alpha)}(x^0)\frac{dx^\mu_{(\alpha)}}{dx^0}\delta^{(2)}(\mathbf{x}-\mathbf{x}_{(\alpha)}(x^0))\ , \tag{3.23a}$$

$$\frac{1}{4\pi G}\epsilon^{\mu\nu\rho}T^a{}_{\nu\rho}\equiv\frac{1}{4\pi G}\epsilon^{\mu\nu\rho}[\partial_\nu e^a{}_\rho-\partial_\rho e^a{}_\nu+\epsilon^a{}_{bc}(\omega^b{}_\nu e^c{}_\rho+\omega^c{}_\rho e^b{}_\nu)]$$
$$=\sum_{\alpha=1}^N\int d\tau j^a_{(\alpha)}\dot{x}^\mu_{(\alpha)}\delta^{(3)}(x-x(\tau))=\sum_{\alpha=1}^N j^a_{(\alpha)}(x^0)\frac{dx^\mu_{(\alpha)}}{dx^0}\delta^{(2)}(\mathbf{x}-\mathbf{x}_{(\alpha)}(x^0))\ , \tag{3.23b}$$

where $j^a_{(\alpha)}=\epsilon^a{}_{bc}q^b_{(\alpha)}p^c_{(\alpha)}$ is the orbital angular momentum of the αth particle in $\mathcal{M}_q$ and, after the integration over τ is performed, the trajectory three-vectors are $x^\mu_{(\alpha)}\equiv(x^0,\mathbf{x}_{(\alpha)}(x^0))$. A variation of S with respect to $q^a_{(\alpha)}$ and $p^a_{(\alpha)}$ gives the equations of motion for the αth particle:

$$\mathcal{D}_\nu q^a_{(\alpha)}\dot{x}^\mu_{(\alpha)}+2\lambda_{(\alpha)}p^a_{(\alpha)}=0\ ,$$
$$D_\mu p^a_{(\alpha)}\dot{x}^\mu_{(\alpha)}=(\partial_\mu p^a_{(\alpha)}+\epsilon^a{}_{bc}\omega^b{}_\mu p^c_{(\alpha)})\dot{x}^\mu_{(\alpha)}=0\ . \tag{3.24}$$

The equations obtained by varying S with respect to $x^\mu_{(\alpha)}(\tau)$ provide relations which are identically satisfied whereas Eqs. (3.23) and (3.24) give, with the proper identifications for the space-time *dreibein* and torsion, the correct Einstein's equations and the geodesic equations, respectively. In fact, by means of Eqs. (2.11) and (3.23a) one can show that Eq. (3.23b) becomes

$$\mathcal{T}^a{}_{\mu\nu}\equiv\partial_\mu V^a{}_\nu-\partial_\nu V^a{}_\mu+\epsilon^a{}_{bc}(\omega^b{}_\mu V^c{}_\nu+\omega^c{}_\nu V^b{}_\nu)=0\ . \tag{3.25}$$

Spin interactions can be included in the system by adding to the action (3.22) a gauge-invariant spin term of the form described in Ref. [18], and the field equations of motion have the same form of Eqs. (3.23) where now $j^a_{(\alpha)}=\epsilon^a{}_{bc}q^b_{(\alpha)}p^c_{(\alpha)}+s^a_{(\alpha)},s^a_{(\alpha)}$ being the spin of the αth particle. The equations of motion for the particles Eqs. (3.24), instead, become more complicated if the spin is considered; however, since in 2+1 dimensions the spin of a particle is proportional to its momentum, the equation for the spin precession and the Mathisson-Papapetru equation for the motion of a spinning particle would collapse into a single equation.

Let us now discuss the structure of the solutions of Eqs. (3.23) and (3.25). By means of Eq. (3.23a) one can show that a solution of Eq. (3.25) is given by

$$V^a{}_\mu=\partial_\mu\left[x^\nu-\sum_{\alpha=1}^N x^\nu_{(\alpha)}f_{(\alpha)}(\mathbf{x};\mathbf{x}_{(1)}(x^0)\cdots\mathbf{x}_{(N)}(x^0))\right]\delta^a{}_\nu+\epsilon^a{}_{bc}\omega^c{}_\mu\left[x^\nu-\sum_{\alpha=1}^N x^\nu_{(\alpha)}f_{(\alpha)}(\mathbf{x};\mathbf{x}_{(1)}(x^0)\cdots\mathbf{x}_{(N)}(x^0))\right]\delta^c{}_\nu\ , \tag{3.26}$$

where $f_{(\alpha)}(\mathbf{x};\mathbf{x}_{(1)}(x^0)\cdots\mathbf{x}_{(N)}(x^0))$ is an arbitrary function with the property

$$f_{(\alpha)}(\mathbf{x}_{(\beta)}(x^0);\mathbf{x}_{(1)}(x^0)\cdots\mathbf{x}_{(N)}(x^0))=\delta_{\alpha\beta}\ . \tag{3.27}$$

Moreover, the $f_{(\alpha)}$ must vanish when $GM_{(\alpha)}\to0$, in order to recover the correct flat-space-time limit in Eq. (3.26), the $f_{(\alpha)}$ are nonsingular at finite distance from the origin of the coordinates, they decrease rapidly at large distances, they are single valued at the origin, in summary they have to be well-defined test functions for the δ distributions in Eq. (3.23a). Of course functions with these properties can be constructed explicitly, for example,

$$f_{(\alpha)}=\exp\left[-\frac{1}{G^4M^2_{(\alpha)}}|\mathbf{x}-\mathbf{x}_{(\alpha)}(x^0)|^{2N}\prod_{\beta\neq\alpha}\frac{1}{|\mathbf{x}-\mathbf{x}_{(\beta)}(x^0)|^2}\right]\ . \tag{3.28}$$

A solution of Eq. (3.23a) is known in the case of static particles [14] when

$$p^a_{(\alpha)}=(M_{(\alpha)},0,0)\ , \tag{3.29}$$

and reads

$$\omega^i{}_\mu=\omega^0{}_0=0\ ,\quad \omega^0{}_i=G\partial_i\Psi(\mathbf{x};\mathbf{x}_{(1)},\ldots,\mathbf{x}_{(N)})\ , \tag{3.30}$$

where

$$\Psi(\mathbf{x};\mathbf{x}_{(1)},\ldots,\mathbf{x}_{(N)})=\sum_{\alpha=1}^{N}p^0_{(\alpha)}\arctan\left[\frac{x^2-x^2_{(\alpha)}}{x^1-x^1_{(\alpha)}}\right]. \tag{3.31}$$

In the case of one particle at the origin the metric defined by Eq. (3.26) has the usual form of the one-particle solution discussed for example in Ref. [13]. In the case $N>1$ Eq. (3.26) describes a nice N-particle generalization of the one-particle *dreibein*. Although Eq. (3.26) in the static case has a very different form if compared to the N-particle solution found in Ref. [23], it corresponds to the same metric. In fact the scalar curvature is the same and has the same pointlike singularities.

Now we want to discuss an important feature which is specific of (2+1)-dimensional gravity coupled to pointlike sources, and shows the efficiency of this gauge-theoretical approach in solving Einstein's equations. In the presence of point particles the space time is flat outside the sources and it is always possible to find a set of coordinates x^μ_M which are globally Minkowskian.

Moreover, for the very same reason, the gauge potential A_μ is almost everywhere a pure gauge, i.e.,

$$A_\mu=U^{-1}\partial_\mu U\ , \tag{3.32}$$

the gauge-group element $U\in \mathrm{ISO}(2,1)$ being, in general, singular and multivalued at the source location. It is convenient to work in the 4×4 representation of the Poincaré group, so that U can be written as

$$U=\begin{bmatrix}\Lambda^a{}_b & \rho^a\\ 0 & 1\end{bmatrix}, \tag{3.33}$$

where $\Lambda^a{}_b$ is a 3×3 Lorentz matrix and ρ^a is a translation. Then, from Eqs. (3.32) and (3.33) it follows that

$$\begin{aligned}A_\mu=U^{-1}\partial_\mu U&=\begin{bmatrix}(\Lambda^{-1})^a{}_c\partial_\mu\Lambda^c{}_b & (\Lambda^{-1})^a{}_b\partial_\mu\rho^b\\ 0 & 0\end{bmatrix}\\ &=\Lambda_b{}^a\partial_\mu\rho^b P_a-\tfrac{1}{2}\epsilon^a{}_{bc}\Lambda_d{}^b\partial_\mu\Lambda^{dc}J_a\ .\end{aligned} \tag{3.34}$$

Consequently, taking Eq. (3.5) into account, the general form of the gauge potentials reads

$$\omega^a{}_\mu=-\tfrac{1}{2}\epsilon^a{}_{bc}\Lambda_d{}^b\partial_\mu\Lambda^{cd}\ , \tag{3.35a}$$

$$e^a{}_\mu=\Lambda_b{}^a\partial_\mu\rho^b\ . \tag{3.35b}$$

From Eqs. (3.35) the space-time *dreibein* (3.11) can be written in terms of Λ and ρ as

$$V^a{}_\mu=\Lambda_b{}^a\partial_\mu(\Lambda^b{}_c q^c+\rho^b)=\Lambda_b{}^a\partial_\mu\hat{q}^{\,b}\ , \tag{3.36}$$

where we have introduced the coordinates

$$\hat{q}^{\,a}=\Lambda^a{}_b q^b+\rho^a\ . \tag{3.37}$$

In terms of the $\hat{q}^{\,a}$ the space-time line element becomes

$$ds^2=\eta_{ab}V^a{}_\mu V^b{}_\nu dx^\mu dx^\nu=\eta_{ab}\partial_\mu\hat{q}^{\,a}\partial_\nu\hat{q}^{\,b}dx^\mu dx^\nu=\eta_{ab}d\hat{q}^{\,a}d\hat{q}^{\,b}\ . \tag{3.38}$$

Therefore the gauge transformation that parametrizes the gauge potentials as pure gauges (Λ,ρ) is precisely the one that, from the *Poincaré* coordinates q^a, gives the *space-time* coordinates $\hat{q}^{\,a}$ that globally flatten the metric, i.e., $\hat{q}^{\,a}\equiv x^\mu_M$, as Eq. (3.38) exhibits. For this reason, to solve the field equations in terms of (Λ,ρ) instead of (e,ω) turns out to be much more convenient, as the solution automatically leads not only to the space-time metric, but also to the globally Minkowskian space-time coordinates.

In order to write Eqs. (3.23) in terms of (Λ,ρ), we first parametrize the Lorentz matrix Λ as $\Lambda=\exp(\kappa^a J_a)$; since the 3×3 representation of the Lorentz generators is $(J_a)^b{}_c=-\epsilon_a{}^b{}_c$, we have a Lorentz matrix in terms of κ^a:

$$\Lambda^a{}_b=(e^{\kappa^c J_c})^a_b=-\frac{\sin|\kappa|}{|\kappa|}\epsilon^a{}_{bc}\kappa^c-\cos|\kappa|\left[\frac{\kappa^a\kappa_b}{|\kappa|^2}-\delta^a{}_b\right]+\frac{\kappa^a\kappa_b}{|\kappa|^2}\ , \tag{3.39}$$

where $|\kappa|=\sqrt{(\kappa^0)^2-(\kappa^1)^2-(\kappa^2)^2}$. Substituting Eq. (3.39) in Eqs. (3.35) we can write the gauge potentials in terms of κ^a and ρ^a,

$$\omega^a{}_\mu=-\left[(1-\cos|\kappa|)\frac{\epsilon^{abc}\kappa_b}{|\kappa|^2}+\frac{\sin|\kappa|}{|\kappa|}\left[\frac{\kappa^a\kappa^c}{|\kappa|^2}-\eta^{ac}\right]-\frac{\kappa^a\kappa^c}{|\kappa|^2}\right]\partial_\mu\kappa_c\ , \tag{3.40a}$$

$$e^a{}_\mu = -\left[\frac{\sin|\kappa|}{|\kappa|}\epsilon_b{}^a{}_c\kappa^c + \cos|\kappa|\left[\frac{\kappa_b\kappa^a}{|\kappa|^2} - \delta_b{}^a\right] - \frac{\kappa_b\kappa^a}{|\kappa|^2}\right]\partial_\mu\rho^b\ , \tag{3.40b}$$

so that the field equations (3.22) can be finally rewritten as

$$\left[(1-\cos|\kappa|)\frac{\epsilon^{abc}\kappa_c}{|\kappa|^2} + \frac{\sin|\kappa|}{|\kappa|}\left[\frac{\kappa^a\kappa^b}{|\kappa|^2} - \eta^{ab}\right] - \frac{\kappa^a\kappa^b}{|\kappa|^2}\right]\epsilon^{\mu\nu\rho}\partial_\nu\partial_\rho\kappa_b = -2\pi G\sum_{\alpha=1}^{N} p^a_{(\alpha)}(x^0)\frac{dx^\mu_{(\alpha)}}{dx^0}\delta^{(2)}(\mathbf{x}-\mathbf{x}_{(\alpha)}(x^0))\ , \tag{3.41a}$$

$$\left[\frac{\sin|\kappa|}{|\kappa|}\epsilon^{abc}\kappa_c + \cos|\kappa|\left[\frac{\kappa^a\kappa^b}{|\kappa|^2} - \eta^{ab}\right] - \frac{\kappa^a\kappa^b}{|\kappa|^2}\right]\epsilon^{\mu\nu\rho}\partial_\nu\partial_\rho\rho_b = -2\pi G\sum_{\alpha=1}^{N} j^a_{(\alpha)}(x^0)\frac{dx^\mu_{(\alpha)}}{dx^0}\delta^{(2)}(\mathbf{x}-\mathbf{x}_{(\alpha)}(x^0))\ . \tag{3.41b}$$

Equations (3.41) clearly show that κ^a and ρ^a have to be, in general, singular and multivalued, otherwise, the left-hand sides would vanish by symmetry and would never reproduce the δ distributions in the right-hand sides.

A general feature of gauge theories is that an appropriate gauge choice is able to simplify dramatically the solutions of the field equations. In the physical gauge $q^a=0$, $e^a{}_\mu$ can be interpreted as *dreibein* and Eqs. (3.23) essentially collapse to the Einstein's equations in the first-order formalism. However, even if this gauge choice can be useful to establish full contact with the corresponding Einsteinian theory, $q^a=0$ is not always the most suitable choice to solve the field equations. For example suppose that the physical system we are considering is such that each particle has a vanishing angular momentum. Then the most convenient gauge choice will be $q^a=\delta^a{}_\mu x^\mu$, so that the right-hand side of Eq. (3.23b) vanishes and we can consistently choose the trivial solution $\rho^a=0$ for this equation. Hence, half of the field equations are immediately solved. Notice that from Eq. (3.40b) $\rho^a=0$ implies $e^a{}_\mu=0$ and, whereas the vanishing solution is acceptable in our approach, it would have been meaningless if the $e^a{}_\mu$ were interpreted as space-time *dreibein.*

Once a solution of Eq. (3.41a) has been obtained, the solution for the *dreibein* of the corresponding metric is given by Eq. (3.26). It has to be stressed, however, that to find a consistent solution of Eq. (3.41a) is a formidable task, not because of the difficulties in solving the equation itself but because one has to find solutions for κ^a that give a single-valued spin connection in order to have a single-valued and smooth metric. Related problems were addressed in Ref. [15], where singular solutions of Eqs. (3.23) for N particles were provided.

It is worth pointing out that different solutions of Eqs. (3.41) correspond to different gauge choices for the gauge potentials in Eqs. (3.23), and that solutions in different gauges lead to the same space-time metric tensor, but in a different coordinate frame. In this regard, a simple example can be instructive. Let us consider a particle of mass M moving with constant velocity v along the positive direction of the x^1 axis. If we choose the $q^a=\delta^a{}_\mu x^\mu$ gauge, we have to reproduce the space-time matter distribution and motion laws in $\mathcal{M}_q$, namely,

$$q^a(\tau)=(\gamma\tau,\gamma v\tau,0)\ ,\quad p^a(\tau)=(M\gamma,M\gamma v,0) \tag{3.42}$$

where $\gamma=(1-v^2)^{-1/2}$. Since $j^a(\tau)=0$, we can choose $\rho^a=0$ (and then $e^a{}_\mu=0$) as a solution of Eq. (3.41b) and we are still free to perform a Lorentz gauge choice. The most appropriate is the radial gauge $x^\mu\omega^a{}_\mu=0$ and a solution of Eq. (3.41a) is this gauge is simply

$$\kappa^a = Gp^a\arctan\left[\frac{\beta y}{x-vt}\right] \equiv Gp^a\Psi(\beta) \tag{3.43}$$

where β is any arbitrary positive constant, whose presence is related to the fact that the radial gauge condition does not fix completely the Lorentz gauge invariance.

The spin connection corresponding to Eq. (3.43) is given by

$$\omega^a{}_\mu = Gp^a\partial_\mu\Psi(\beta)\ , \tag{3.44}$$

leading to the space-time metric [see Eq. (3.11)]

$$ds^2 = \{\eta_{\mu\nu} - G^2M^2[(x^1-vt)^2\gamma^2+(x^2)^2]\partial_\mu\Psi(\beta)\partial_\nu\Psi(\beta) - 2\epsilon_{\mu\rho\sigma}x^\rho\delta^\sigma{}_a p^a\partial_\nu\Psi(\beta)\}dx^\mu dx^\nu\ . \tag{3.45}$$

This metric has to be equivalent to the one obtained by boosting in the x^1 direction the metric for a static particle at the origin, namely

$$ds^2 = \gamma^2(dt-vdx)^2 - \gamma^2(dx-vdt)^2\frac{y^2(1-GM)^2+\gamma^2(x-vt)^2}{y^2+\gamma^2(x-vt)^2} - (dy)^2\frac{\gamma^2(x-vt)^2(1-GM)^2+y^2}{y^2+\gamma^2(x-vt)^2} - \frac{1}{2}\gamma^2GM(2-GM)\frac{dy\,d(x-vt)}{y^2+\gamma^2(x-vt)^2}\ . \tag{3.46}$$

If we naively choose $\beta=1$ in Eq. (3.45), the metric we get has a very different structure from the one above. Instead, by choosing $\beta=1/\gamma$, the two metrics, coincide. Since a different gauge choice must lead to equivalent metrics, there must be a coordinate transformation relating the two metrics. It does exist and is given by

$$\Psi(1)=\arctan\left[\frac{y}{x-vt}\right]\to\Psi'$$
$$=\frac{1}{1-GM}\left[\Psi(1)-GM\arctan\left[\frac{1}{\gamma}\tan\Psi(1)\right]\right]. \tag{3.47}$$

Notice that $\beta=1/\gamma$ is a very natural gauge choice, as $\Psi(1/\gamma)$ gives the azimuthal angle in the reference frame of the moving particle.

Point-like matter distributions in (2+1)-dimensional gravity have a relevant physical application as they give the cross sections of the space-time geometries created by parallel moving infinite cosmic strings in four dimensions. Very recently it has been claimed that in some particular cases two cosmic strings in motion can support closed timelike curves [24]. However, by using the method based on the matching conditions of the coordinates that flatten the metric [23], the configuration analyzed in Ref. [24] has been proved to correspond to an unphysical (exotic) space time [25]. With our gauge-theoretical approach, one could obtain not only the matching conditions obeyed by the globally Minkowskian coordinates, but also the analytic expression of the space-time metric, thus providing useful tools for a deeper understanding of the geometries created by moving cosmic strings.

IV. CONCLUSIONS AND POSSIBLE DEVELOPMENTS

Our main results can be summarized as follows.

(1) In order to construct a gauge theory of the Poincaré group, it is necessary to introduce, in addition to the gauge potentials, nondynamical degrees of freedom $q^a(x)$ (Poincaré coordinates), whose equations are automatically satisfied if those of the gauge and matter fields are satisfied. The necessity to introduce the Poincaré coordinates q^a is discussed in full detail, as well as their geometrical interpretation.

(2) The correspondence with the usual Einsteinian formulation of the theory has been established by giving the expressions, in terms of the gauge fields, of the spin connection and the *vierbein*. The former trivially coincides with the gauge potentials $\omega^{ab}{}_\mu$ associated to the Lorentz transformation whereas the physical *vierbein* is given by a more complicated expression involving q^a, $\omega^{ab}{}_\mu$ and the gauge potentials $e^a{}_\mu$ associated with the translation generators. In particular, it is *not* true that the field $e^a{}_\mu$ can be identified with the space-time *vierbein* $V^a{}_\mu$. Only in the "physical" gauge $q^a=0$ the identification $V^a{}_\mu\equiv e^a{}_\mu$ holds but, in this case we are not dealing anymore with a Poincaré gauge theory of gravity but rather with a Lorentz gauge theory, as the Lorentz is the residual gauge group which is left invariant by a choice of the gauge coordinates $q^a(x)$.

(3) We have provided Poincaré gauge-invariant actions for the gauge fields as well as for the couplings with matter, i.e., point particles, scalar and spinor fields, and we have shown that, with the identifications discussed in point (2) and upon postulating the invertibility of the *vierbein,* the corresponding equations of motion reproduce the Einstein equations, the geodesics equations, the Klein-Gordon and Dirac equations in curved space time, respectively.

(4) The procedure we have presented is independent on the dimension of the space time. A particularly interesting case is the (2+1)-dimensional one, where the q^a dependence in the action for the gauge potentials can be ruled out, and we recover the well-known property that the Einstein-Hilbert action can be written as a pure Chern-Simons term of the Poincaré group. But what is even more appealing is that in 2+1 dimensions, as far as point-particle interactions are concerned, the gauge potentials can always be chosen almost everywhere as pure gauges. This is a consequence of the fact that in 2+1 dimensions the space time is always flat outside pointlike sources. Thus our procedure can be further developed by writing all the field equations as well as the expressions for any relevant physical quantity in terms of a gauge group element $U=(\Lambda,\rho)\in \mathrm{ISO}(2,1)$. In this case, a solution of the field equations simultaneously provide both the metric and the set of globally Minkowskian space-time coordinates. In fact, the physical meaning of the solution $U=(\Lambda,\rho)$ is the gauge transformation that, out of the *Poincaré* coordinates $q^a(x)$, gives the *space-time* coordinates $\hat{q}^a=\Lambda^a{}_b q^b+\rho^a$ that globally flatten the metric.

Several aspects deserve consideration and many possible developments can be worked out. First of all, an old and still unsolved problem is whether or not a metric-independent quantization procedure for gravity exits. If it does, the Poincaré gauge theory we have presented seems to be the most reasonable choice to being with. Maybe the standard and well-established techniques for the quantization of non-Abelian gauge theories could prove useful in such an attempt.

Another interesting issue is the connection between the present paper and the procedure developed in Ref. [14] in the context of a formulation of gravity as a Poincaré gauge theory defined on a flat space. To this regard, we notice that all the actions given in this work are separately invariant under gauge transformations and diffeomorphisms in space time. In 2+1 dimensions and with pointlike sources, since space time is always flat outside the sources, it is always possible to find a particular set of Poincaré coordinates $\hat{q}^a(x)$ which coincides with the globally Minkowskian space-time coordinate. Consequently it is natural to define the whole theory on the Minkowskian manifold $\mathcal{M}_q$. In this case, a general set of Poincaré coordinates $q^a\in\mathcal{M}_q$ will not represent space-time coordinates, but there will always exist a (singular and multivalued) gauge transformation relating q^a to $\hat{q}^a$. This is precisely what we did in Ref. [14]. However, since in Ref. [14] the theory was completely defined on the Poincaré space $\mathcal{M}_q$, a gauge transformation on the gauge potentials $(e(q),\omega(q))$ also entailed a transformation of the arguments q^a, which automatically took into account the invariance under diffeomorphisms.

An intriguing possibility would be to try to extend this picture also to four dimensions or to the case of matter fields, by considering a Poincaré group theory defined in the flat manifold $\mathcal{M}_q$ and interpreting a gauge transformation as acting both on the fields and on their arguments, the Poincaré vectors $q^a \in \mathcal{M}_q$. In this way, the diffeomorphisms invariance would be parametrized with the same functions which define the gauge transformations ($\delta q^a = \kappa^a{}_b + \rho^a$), which is precisely the procedure followed in Ref. [14]. Obviously, in four dimensions or in the presence of matter fields, there would never exist a set of Poincaré coordinates which could be interpreted as space-time coordinates, and thus the relation between this flat Poincaré gauge theory and gravity would not be so direct as in the $(2+1)$-dimensional case with pointlike sources. Nevertheless, for the Poincaré gauge theory defined on the flat space $\mathcal{M}_q$ in four dimensions, the connection with the space time could be given by the expression of the *vierbein* in terms of q^a that, in this case, should have to be considered as a definition. The possibilities opened by this "flat theory of gravity" are still to be found out. Other interesting subjects that might be successfully treated without our formalism, concern lower-dimensional theories. In $2+1$ dimensions, since it simultaneously gives both the metric and the coordinates that flatten the metric, the theory we have presented could give insights on the space-time geometry generated by moving point particles and, consequently, on the structure of the space time generated by moving four-dimensional infinite cosmic strings [23–25].

Moreover, it should not be difficult to generalize the procedure developed in Sec. III B to other types of null-measure matter distributions such as, for instance, string sources for which the gauge potentials can still be written as pure gauges.

ACKNOWLEDGMENTS

We would like to thank Roman Jackiw for stimulating discussions and constant encouragements. We are also grateful to Stanley Deser and Kellog Stelle for a useful discussion. We acknowledge the Istituto Nazionale di Fisica Nucleare for financial support during the preparation of this work. This work was supported in part by funds provided by the U.S. Department of Energy (DOE) under Contract No. DE-AC02-76ER03069.

[1] A. Achucarro and P. Townsend, Phys. Lett. B **180**, 85 (1986).
[2] E. Witten, Nucl. Phys. **B311**, 46 (1988).
[3] R. Utiyama, Phys. Rev. **101**, 1597 (1956).
[4] T. W. Kibble, J. Math. Phys. **2**, 212 (1961).
[5] D. W. Sciama, in *Recent Developments in General Relativity* (Pergamon, Oxford, England, 1962).
[6] A. Chamseddine and P. West, Nucl. Phys. **B129**, 39 (1977).
[7] S. W. MacDowell and F. Mansouri, Phys. Rev. Lett. **38**, 739 (1977).
[8] F. W. Hehl, P. von der Heyde, G. D. Kerlick, and J. M. Nester, Rev. Mod. Phys. **48**, 393 (1976).
[9] K. S. Stelle and P. C. West, Phys. Rev. D **21**, 1466 (1980).
[10] H. R. Pagels, Phys. Rev. D **29**, 1690 (1984).
[11] T. Kawai, Gen. Relativ. Gravit. **18**, 995 (1986); and in *Sixth Marcel Grossman Meeting on General Relativity,* Proceedings, Kyoto, Japan, 1991, edited by T. Nakamura (World Scientific, Singapore, 1992); Osaka City University report (see also references therein).
[12] C. Itzykson and J. B. Zuber, *Quantum Field Theory* (McGraw-Hill, Singapore, 1980).
[13] For recent reviews on the subject, see for instance R. Jackiw, in Proceedings of SILARG VII, Cocoyoc, Mexico, 1990 (unpublished); or in *Sixth Marcel Grossman Meeting on General Relativity* [11].
[14] G. Grignani and G. Nardelli, Phys. Lett. B **264**, 45 (1991); Nucl. Phys. **B370**, 491 (1992).
[15] A. Cappelli, M. Ciafaloni, and P. Valtancoli, Nucl. Phys. **B369**, 669 (1992); Phys. Lett. B **273**, 431 (1991).
[16] S. Carlip, Nucl. Phys. **B324**, 106 (1989).
[17] S. Martin, Nucl. Phys. **B327**, 178 (1989).
[18] P. de Sousa Gerbert, Nucl. Phys. **B346**, 440 (1990).
[19] K. Koheler, F. Mansouri, Cenalo Vaz, and L. Witten, Nucl. Phys. **B348**, 373 (1990).
[20] B. S. Skagerstam and A. Stern, Int. J. Mod. Phys. **5**, 1575 (1990).
[21] K. Gupta and A. Stern, Phys. Rev. D **44**, 2432 (1991).
[22] Cenalo Vaz and L. Witten, University of Cincinnati Report No. UCTP-103/91, 1991 (unpublished).
[23] S. Deser, R. Jackiw, and G. 't Hooft, Ann. Phys. (N.Y.) **152**, 220 (1984).
[24] J. Richard Gott III, Phys. Rev. Lett. **66**, 1126 (1991).
[25] S. M. Carroll, E. Farhi, and A. H. Guth, Phys. Rev. Lett. **68**, 263 (1992); S. Deser, R. Jackiw, and G. 't Hooft, *ibid.* **68**, 267 (1992).