Quantum-mechanical scattering of charged black holes

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(Received 30 November 1989)

We describe the quantum-mechanical scattering of slowly moving maximally charged black holes. Our technique is to develop a canonical quantization procedure on the parameter space of possible static classical solutions. With this, we compute the capture cross sections for the scattering of two black holes. Finally, we discuss how quantization on this parameter space relates to quantization of the degrees of freedom of the gravitational field.

PACS number(s): 97.60.Lf, 04.20.Me

I. INTRODUCTION

Here we study quantum-mechanical interactions of charged black holes. In classical general relativity there exist exact static solutions for N maximally charged black holes; the black holes can be placed anywhere, and will remain at rest. This suggests that, for slowly moving maximally charged black holes, the spatial geometry at any time will be well approximated by the static solution for the black-hole configuration at that time. The classical solutions were worked out [2] following this type of adiabatic approach. Now, the parameter or moduli space of possible static solutions is a $3N$-dimensional manifold, consisting of the positions of the N black holes. Therefore, for the slowly moving black holes, a path is traced out in moduli space as the three geometry evolves to the four-dimensional spacetime.

In studying the classical solutions it was found that the motion of the black holes was governed by an effective Hamiltonian for N point particles. The approach here is to use this Hamiltonian to evolve a Schrödinger wave function, for the case of two black holes. The wave function is a function of the positions of the two black holes; that is, the configuration space of the wave function is the parameter space of possible classical static solutions. Hence the degree of freedom being quantized is the proper distance between the two black holes.

One can imagine that if the full theory for quantum gravity were known, one could compute the motion of maximally charged black holes and then consider the slow-motion, low-energy limit. Here, we can hope that we are computing an approximation to the actual motion. Of course, since we do not know the theory of quantum gravity we have no way of knowing how good an approximation this is. On the other hand, we do know of some ways this approximation may fail. In particular, the classical metrics were assumed to have a particular form, corresponding to slow motion. However, in the full quantum-theoretic description of the system, there will surely be excitation of degrees of freedom which are not included in this model; i.e., the configuration space for the true wave function must include other degrees of freedom than those in the moduli space considered here. In our approximation we will never see these. This is similar to minisuperspace models, and also quantized nonlinear σ models (when viewed as models of approximately constrained systems), which suffer the same shortcoming—there is no way excite modes which are not in the approximation [1].

Perhaps the most interesting issue is the meaning of the wave function for the geometry of the spacetime. In the classical case the metric can be found by solving constraint equations, with the black-hole positions and momenta as sources, after the motion of the black holes is known. When the sources are quantized, one can no longer speak of a position and velocity of the source, and instead must speak in terms of probabilities. The question, then, is what does it mean for the metric field equations to have sources which are probabilities? Our preferred interpretation, but by no means the only one, is to consider the probabilities as probabilities for finding different spacetimes. We discuss these issues in the final section.

II. THE SYSTEM FOR SCATTERING OF MAXIMALLY CHARGED BLACK HOLES

We now review the work of Ferrell and Eardley [2]. This section will conclude with the classical Hamiltonian for black holes. In the next section we will discuss the quantization procedure, e.g., the Hilbert space of states, and the quantum-mechanical operator which corresponds to the classical Hamiltonian.

The sources in our study will be maximally charged black holes. A maximally charged black hole is an electrically charged (Reissner-Nordström) black hole with charge $Q=G^{1/2}M$. Such a black hole is maximally charged, since if Q is increased but M is held fixed, the event horizon which is associated with the back hole disappears and a naked singularity arises.

The theory for charged matter in curved space is assumed to be the coupled Einstein-Maxwell field theory.

The action for this theory is [3]

$$S=\frac{1}{16\pi G}\int\sqrt{-g}\,{}^{(4)}R\,d^4x-\frac{1}{8\pi}\int F_{\mu\nu}F^{\mu\nu}\sqrt{-g}\,d^4x+\int A_\mu j^\mu\sqrt{-g}\,d^4x+\sum_a\int m_a d\tau_a\ . \tag{2.1}$$

In this expression, ${}^{(4)}R$ is the scalar curvature of the spacetime, A_μ is the electromagnetic four-potential, j^μ is the electromagnetic current and $F_{\mu\nu}=\partial_\mu A_\nu-\partial_\nu A_\mu$ is the electromagnetic field strength. The last term is the action for a collection of free particles, and does not make sense for black holes. In deriving the classical effective action, Eardley and Ferrell replace the singularities with charged dust, with charge density equal to the mass density. So in performing the manipulations the sources are smooth. At the end of the derivation, the limit is taken in which the charge density goes to a sum of point singularities. This regularization scheme works because the dust is maximally charged, and hence in a static configuration. Dust with a charge less than the mass would tend to collapse. For this paper we will not make the distinction between black holes and the smoothed distribution used to represent them.

G is Newton's gravitational constant. In this paper we will work in units where $G=1$ and $c=1$. In these units $\hbar=m_{\rm Planck}^2$ where $m_{\rm Planck}$ is the Planck mass.

In nonrelativistic theories, it is clear that there exists a static configuration of maximally charged black holes, since the condition for maximal charge is just the condition that the static gravitational attraction exactly cancels the electrostatic repulsion. Remarkably, in the full Einstein-Maxwell theory for gravitation coupled to electricity and magnetism there also can be static configurations of maximally charged black holes.

The metric that describes a system of N static maximally charged black holes (first discovered by Majumdar [4] and Papapetrou [5] is

$$ds^2=-\Psi^{-2}dt^2+\Psi^2\delta_{ij}dx^i dx^j\ , \tag{2.2}$$

where $\nabla^2\Psi=-4\pi\sum_a m_a\delta(\mathbf{x}-\mathbf{x}_a)$ with the black holes at the points $\mathbf{x}_a$ and the boundary condition $\Psi=1$ at infinity. Here ∇^2 is the Laplacian on the $\{\mathbf{x}\}$ regarded as coordinates on flat $\mathbf{R}^3$. In this coordinate system the hypersurfaces $t=$const intersect the singularities at the points $\mathbf{x}=\mathbf{x}_a$, and the event horizons are the same points (i.e., the event horizons are represented as points, but have surface area $4\pi m_a$). The Maxwell four-potential is

$$A=-(1-1/\Psi)dt\ . \tag{2.3}$$

The space of possible Majumdar-Papapetrou metrics is the $3N$-dimensional configuration space of the positions of the black holes—once the $\mathbf{x}_a$ are known, Ψ is known and therefore the metric and scalar potential are also known. If the black holes are moving slowly then the solution will trace out a trajectory in the space of static solutions [2,6]. These quasistatic solutions exist because in the slow motion approximation radiation can be neglected [3]: The static forces cancel exactly, and the only relevant forces are the velocity-dependent magnetic force and the gravitational analogue, the gravitomagnetic force. As long as the velocities remain small, there will be very little radiation. In this approximation then, the fields are fully fixed by the positions and velocities of the matter sources. [A similar example of this approximation comes from electrodynamics in flat space. If the position of a charge q at $\mathbf{r}_0$ is changing slowly, with $\mathbf{v}=d\mathbf{r}_0/dt$, then the scalar potential is approximately $\phi(\mathbf{r},t)\approx q/|\mathbf{r}-\mathbf{r}_0(t)|$, and the vector potential is approximately $A\approx q\mathbf{v}/|\mathbf{r}-\mathbf{r}_0(t)|$. The space of field configurations is just the configuration space for the particle, since a field configuration is determined by the position and velocity of the particle.]

To describe slowly moving black holes, one looks for a metric and four-potential of the form

$$ds^2=-\left[\frac{1}{\Psi^2}\right]dt^2+2N_i dx^i dt+\Psi^2\delta_{ij}dx^i dx^j\ , \tag{2.4}$$

$$A=-(1-1/\Psi)dt+A_i dx^i \tag{2.5}$$

with Ψ as before, but now the $\mathbf{x}_a$ will be functions of time. Both N_i and A_i will depend on the velocities of the black holes. However, since we are assuming velocities are small we can truncate the field equations for N_i and A_i to first order in velocity.

Using the Einstein constraint equations, the fields N_i and A_i can be solved for in terms of the source positions and velocities. These expressions are substituted into the Einstein-Maxwell action (2.1). The resulting effective Lagrangian depends only on the positions and velocities of the sources. The black-hole limit is shown to be well defined. This is the Lagrangian for the interaction of the black holes. For details, see [2]. Finally, one finds that the Hamiltonian for two black holes, with masses m_1 and m_2, is

$$H_{2\ \rm body}=\frac{1}{2M}\mathbf{P}\cdot\mathbf{P}+\frac{1}{2\mu}g^{ab}p_a p_b\ . \tag{2.6}$$

where $\mathbf{P}$ is the momentum of the center of mass, $M=m_1+m_2$, $\mathbf{r}=\mathbf{x}_1-\mathbf{x}_2$ is the relative coordinate, $\mathbf{p}$ is the momentum conjugate to that coordinate, and $\mu=m_1m_2/M$ is the reduced mass of the system. In the first term, a flat metric is implied. The metric in the second term is

$$g_{ab}=\gamma(r)\delta_{ab}\ , \tag{2.7}$$

$$\gamma(r)=1+\frac{3M}{r}+\frac{3M^2}{r^2}+\frac{3\alpha M^3}{r^3}\ , \tag{2.8}$$

where $\alpha=1-2\mu/M$. (Recall that $\mathbf{R}$ is the position of the center of mass, and $\mathbf{r}$ is the relative position; hence, these are coordinates on the moduli space of solutions, not on spacetime.)

The evolution of the center-of-mass coordinate is just free particle motion. The evolution for the relative coordinate can have one of two behaviors. If the two black holes start at infinite separation, then when they interact they can either scatter back out to infinity, or they can evolve toward zero separation, depending on their angular momentum. Although in the $r\to 0$ limit the slow motion approximation probably breaks down, it is reasonable to guess that in this case the two black holes coalesce into one black hole [2,3]. We will call the two possible classes of orbits scattering orbits or coalescence orbits, respectively. This completes the review of the classical behavior.

III. THE QUANTUM-MECHANICAL SYSTEM

In this section we will discuss the quantum-mechanical system induced by the classical Hamiltonian $H_{2\text{ body}}$. We shall quantize the two-black-hole system using canonical quantization. The configuration space is the six-dimensional moduli space of possible static solutions. The state of the system is described by a wave function on moduli space; we will describe the Hilbert space more precisely shortly. We must also construct the quantum mechanical Hamiltonian operator. The center-of-mass coordinate represents free particle motion on a flat $\mathbf{R}^3$ so we can use the usual correspondence $P^2 = -\hbar^2\partial^2/\partial X^2$. The classical motion of the relative coordinate is like that of a particle of mass μ on a curved manifold with metric $g_{ab}(r)$. There is some ambiguity in defining the quantum mechanical operator for the kinetic energy on a curved manifold since the metric does not commute with the usual momentum operator $-i\hbar\partial/\partial x$. A choice which ensures that $H_{2\text{ body}}$ is Hermitian, is to replace the classical kinetic term $g^{ab}p_a p_b$ by the operator $-\hbar^2 g^{ab}\nabla_a\nabla_b$, where ∇_a is the derivative operator compatible with the metric g_{ab} [7]. In addition, in the Hamiltonian we can add a term which is proportional to the scalar curvature, $\hbar^2\mathcal{R}$, since this is Hermitian, and vanishes in the classical limit [8].

A. Derivation of the Schrödinger equation for maximally charged black holes

Now we view the Hamiltonian for the system of two maximally charged slowly moving black holes, $H_{2\text{ body}}$, as defining the evolution of a quantum mechanical system. That is, the system is described by a wave function Ψ_{total} with the evolution operator $H_{2\text{ body}}$:

$$-\frac{\hbar}{i}\dot{\Psi}_{\text{total}} = H_{2\text{ body}}\Psi_{\text{total}} \ .$$

As in the classical case, the center-of-mass degrees of freedom $\mathbf{X}$ separate, so that the total wave function can be taken to be of the form $\Psi_{\text{total}}(\mathbf{x},\mathbf{X},t) = \Psi(\mathbf{x},t)\exp[i(-E_c t/\hbar + \mathbf{P}\cdot\mathbf{X}/\hbar)]$, where E_c and $\mathbf{P}$ are the center-of-mass energy and momentum, and $\mathbf{x}$ are the relative coordinates.

Now consider the evolution of a wave packet. Suppose that at early times a wave packet is given which has support at large relative coordinate $|\mathbf{r}|$, so the two black holes are far apart. As time evolves the black holes approach each other; that is, the wave packet evolves towards smaller r. Part of the packet will be scattered and part will be absorbed. This is in contradistinction to the classical case, in which the black holes either scatter or coalesce. Here we will compute the absorption coefficient as a function of the angular momentum and energy of the wave.

The Schrödinger equation for the wave function of the relative coordinates is

$$-\frac{\hbar}{i}\dot{\Psi}(\mathbf{x},t) = H\Psi = \left[-\frac{\hbar^2}{2\mu}g^{ab}\nabla_a\nabla_b + \hbar^2\xi\mathcal{R}\right]\Psi \ ,$$

where g_{ab} is given in (2.7). $\mathcal{R}$ is the scalar curvature of the three-metric g_{ab}. The Hilbert space of states can be taken to be square-integrable functions. We will see that the energy eigenfunctions are square integrable as $r \to 0$. As $r \to \infty$, the eigenfunctions become plane waves, so just as usual, one would need to form wave packets in the free particle regime. When checking that H is Hermitian, one finds that boundary term contributions of probability flux cancel, between large r and the horizon. This is similar to the cancellation for plane waves in a box.

The eigenstates of H will have three quantum numbers, and we will denote the eigenstates by $\Psi_{qlm}(\mathbf{x},t)$. Because the potential is spherically symmetric, we take them to be of the form

$$\Psi_{qlm}(\mathbf{x},t) = \psi_{ql}(r)Y_{lm}(\theta,\phi)\exp(-iEt/\hbar) \ ,$$

so that $H\Psi = E\Psi$ implies that the radial wave functions satisfy

$$\psi''(r) + \frac{2}{r}\psi' + \frac{1}{2}\frac{\gamma'}{\gamma}\psi'(r) - \frac{l(l+1)}{r^2}\psi - 2\mu\xi R\gamma(r)\psi = -q^2\gamma(r)\psi(r) \ , \qquad (3.1)$$

where

$$q^2 = \frac{2\mu E}{\hbar^2} \ . \qquad (3.2)$$

It is useful to change to a new "tortoise" coordinate R, which measures radial length along a path. Let $R = \int\sqrt{\gamma}\,dr$, with γ as defined in (2.8), and let $\chi = r\gamma^{1/2}\psi$. This casts the problem into the form of a standard one-dimensional quantum-mechanical scattering problem, and the resulting effective potential is better behaved. Then the Schrödinger equation (3.1) becomes

$$\chi_{,RR} + (q^2 - V)\chi = 0 \qquad (3.3)$$

where

$$V = \frac{1}{2r\gamma^3}[\gamma(r\gamma_{,r})_{,r} - r\gamma^2] + \frac{l(l+1)}{r^2\gamma} + 2\mu\xi\mathcal{R} \ .$$

For $\mu = 0$ we have

$$V(R) = \frac{3}{2}\frac{1}{M^2}\frac{(r/M)^2}{(1+r/M)^5} + \frac{l(l+1)}{M^2}\frac{r/M}{(1+r/M)^3} \ . \qquad (3.4)$$

Here r is an implicit function of R.

The new coordinate R ranges from $-\infty$ as $r \to 0$, to $+\infty$ at $r \to +\infty$; explicitly $R \cong r + \frac{3}{2}M\ln(r/M)$ as $r \to \infty$ and $R \cong -2\sqrt{\alpha}M(r/M)^{-1/2}$ as $r \to 0$

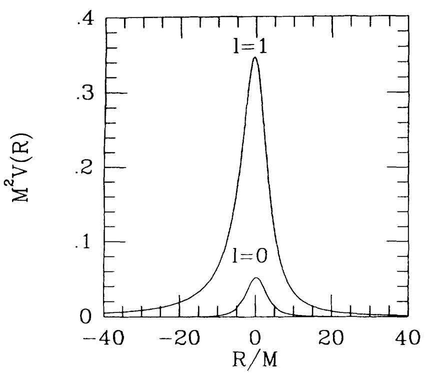

FIG. 1. We plot the potential $V(R)$ as a function of the "tortoise" coordinate, R, for two values of angular momentum. Infinite black-hole separation corresponds to $R\rightarrow\infty$. $R\rightarrow-\infty$ corresponds to zero separation of the two black holes. From the $l=0$ plot, one sees that there is a small barrier to coalescence, independent of the angular momentum. However, already for $l=1$ the angular momentum barrier is much larger than this. As discussed in the text, for low energies there is some tunneling through the $l=0$ barrier, but almost none through the $l>0$ angular momentum barrier.

$(\alpha=1-2\mu/M)$. For the remainder of the paper we will treat the case $\xi=0$.[1] We find, value for all μ,

$$V\cong\frac{3M}{2R^3}+\frac{l(l+1)}{R^2}\quad\text{for } R\gg M\ ,$$
$$V\cong\frac{24M^2}{R^4}+\frac{4l(l+1)}{R^2}\quad\text{for } R\ll -M\ . \tag{3.5}$$

Hence the potential falls off rapidly both in the asymptotic-free region and in the coalescence regime. The potential with $\mu=0$, and for a couple of l values, is plotted in Fig. 1.

If we think of the problem as the quantum mechanics of a particle on a curved surface, the three-geometry on which the particle moves becomes, as the horizon is approached, flat space minus a three-dimensional wedge of solid angle $4\pi/4=\pi$. A two-dimensional cross section is shown in Fig. 2. Because the potential is asymptotically flat, the eigenfunctions behave like free particles on either side of the potential barrier, $\chi\cong A_+e^{iqR}+B_-e^{-iqR}$ as $R\rightarrow\pm\infty$.

What are the correct boundary conditions for the eigenfunctions? If the two holes coalesce, there should be no flux out of their joint horizon. We will abbreviate this as "the horizon." (Indeed, in the classical case one can check that in the limit $m_1\ll m_2$, the solution does reduce to that of a test particle propagating freely on the fixed spacetime determined by the more massive black hole.) Therefore we want the solutions to (3.3) with the

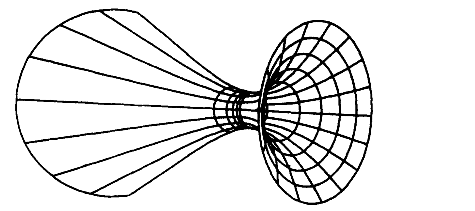

FIG. 2. The geometry of moduli space for two equal-mass black holes $(m_1=m_2)$. A geodesic on this surface represents a classical trajectory for the two-black-hole system. Circles represent lines of constant separation of the black holes. Circles above the throat are separated by $\Delta r=1.46M$, below the throat by $\Delta r=0.0731M$. The point $r=0$ lies infinitely far down the cone. In "tortoise" coordinates, the throat is at $R=0$.

boundary condition that χ is a purely captured, left-moving wave, as $R\rightarrow-\infty$. This implies that as $R\rightarrow+\infty$, the solution will be the sum of an incident wave e^{-iqR}, normalized to unit amplitude, and a scattered wave Se^{iqR}. Using the asymptotic form of the potential (3.5), we find

$$\chi_{ql}\cong qR[(-i)^{(l+1)}h_l^{(-)}(qR)+S_{ql}h_l^{(+)}(qR)],\quad R\gg M\ , \tag{3.6a}$$

$$\chi_{ql}\cong C_{ql}qRh_\nu^{(-)}(qR),\quad R\ll -M\ , \tag{3.6b}$$

where $\nu=\sqrt{4l(l+1)+1/4}-\frac{1}{2}$ and h_ν is a spherical Bessel function of order ν. For a given eigenfunction, the fraction of captured flux is $|C_{ql}|^2$ and the scattered flux is $|S_{ql}|^2=1-|C_{ql}|^2$.

B. General properties of the solutions

Before we estimate C_{ql} and S_{ql} let us consider the motion of wave packets, in which each eigenmode evolves like $\exp(-iEt/\hbar)=\exp(-i\hbar q^2t/2\mu)$ [from the definition of q in (3.2)]. At early times, let a wave packet start in the asymptotic-free region $R\rightarrow\infty$, $t\rightarrow-\infty$. Then, by looking at the point of stationary phase, one sees that initially the center of the packet moves inward according to $R\simeq-v_\infty t$, where $v_\infty=\hbar q/\mu$ is the nonrelativistic (relative coordinate) velocity at infinity, associated with the momentum $\hbar q$. At large times the wave packet has split into two pieces. The center of one piece continues toward $R\rightarrow-\infty$ as $R\simeq-2v_\infty t$; this is the captured flux, corresponding to classical coalescence of the black holes. Apparently, it takes an infinite amount of t-coordinate time for coalescence (this is also true in the classical analysis.) There is also a scattered part of the packet, propagating as $R\simeq v_\infty t$. Hence the motion of the center of the packet is like the classical solution. The motion along the classical path depends only on a rescaled time $v_\infty t$; the parameter v_∞ scales out of the problem, and the classical solutions depend only on the angular momentum, or equivalently, the impact parameter b.

While this is also true of the center of the wave packet, the quantum mechanical quantities of physical interest, such as the scattering coefficient, *will* depend on v_∞ (or equivalently, q) as well as the analogue of b. Indeed, from

[1]For $\xi\neq0$ the most important effect is that as $R\rightarrow-\infty$, there is an additional term of $-96\mu\xi/R^2$, which can alter the behavior of the eigenfunctions at the horizon.

the eigenfunction equation (3.3) we see that the eigenfunctions depend on the two dimensionless parameters l and qM, or equivalently on qM and b/M, where

$$\frac{b}{M}=\frac{\sqrt{l(l+1)}}{qM} . \tag{3.7}$$

[This definition of b/M comes from equating the quantum-mechanical angular momentum $\hbar\sqrt{l(l+1)}$ with the classical angular momentum $\mu v_\infty b$.] Finally, note that the slow-motion approximation $v_\infty \ll 1$ is equivalent to

$$\hbar q \ll \mu ; \tag{3.8}$$

that is, the (quantum-mechanical) particle is nonrelativistic.

Although we do not have sufficient understanding of the eigenfunction solutions to actually match the solutions from the two asymptotic regions (and thereby derive algebraic relations between C_{ql} and S_{ql}), we can determine how the coefficients C_{ql} and S_{ql} scale with qM in the low-energy limit. In the next section we will use different techniques to get approximate forms for C_{ql} and S_{ql} in essentially all qM and l regimes.

Consider first the low-energy limit $qM \ll 1$. Then for $|R|<1/q$ the solutions to (3.3) are (approximately) independent of q. The solution in the region $M<R<1/q$ inherits an overall scaling dependence on qM from the large-R solution (3.6a). Hence this same scaling with qM is passed onto the solution in the region $-1/q<R<-M$, which can then be matched onto the large negative R region [9].[2]

For details see Appendix A. One finds, for $qM \ll 1$ and all l,

$$S_{ql} \simeq (-i)^{l+1}+O((qM)^{2l+2}) ,$$

$$C_{ql} \simeq \frac{(-i)^{l+1}\pi\kappa_l}{2^{\nu+l}\Gamma(\nu+\frac{1}{2})\Gamma(l+\frac{3}{2})}(qM)^{\nu+l+1} ,$$

where κ_l is an unknown coefficient which is independent of q. For large l then, the capture coefficient is

$$|C_{ql}|^2 \simeq \frac{\pi^2 e^{6l}}{l^2 2^{10l-1}}|\kappa_l|^2 \left[\frac{qM}{l}\right]^{6l+3} , \quad l \gg 1, qM \to 0 \tag{3.9}$$

and

$$|C_{q0}|^2 \propto (qM)^2 \quad \text{for } l=0 . \tag{3.10}$$

[A similar analysis in the case of a Klein-Gordon field scattering off a Schwarzschild background yields $|C_{ql}|^2 \propto (qM)^{2l+2}$.] Therefore at low energies the only significant capture occurs for $l=0$ waves.

[2]We would like to thank Jonathan Simon for explaining this point.

C. Approximate methods for C_{ql}

We now turn to the calculation of the capture coefficient. The behavior of a wave packet will be determined by whether q^2 is larger or smaller than the potential barrier, as in a classical scattering problem. We will use the classical terminology, and say in these two cases that "the particle is over or under the barrier," respectively. Recall that the barrier has a contribution from the angular momentum, as well as a piece independent of l. When the energy of the particle is large, so that q^2 is much greater than the height of the potential barrier, one can use the Born approximation. When q^2 is small so that the particle is well under the barrier there are two cases. If the angular momentum barrier is large, $l \gg 1$, one can use the WKB approximation. If $l=0$, one can use another approximation where the potential is replaced by a δ function.

In general for a given l and qM the roots of a cubic equation determine if the particle is over or under the barrier. In the case $\mu=0$ this simplifies: If $l=0$, then the particle is over the barrier if $qM>\sqrt{2\times 3^4/5^5}\cong 0.23$. If $l \geq 1$, the particle is over the barrier, and hence primarily captured, if $qM>\frac{2}{5}\sqrt{l(l+1)}$, which is equivalent to $b/M<5/2$. (In the classical case, coalescence always occurs if $b/M<5/2$.)

1. Low energies

In the small-qM regime one looks for wave functions of the form $|S'|^{-1/2}e^{iS}$. In the first WKB, or adiabatic, approximation, $S=-\int^R(q^2-V)^{1/2}dR'$, for a wave propagating in from $R=+\infty$. This neglects terms of order $|\partial V/\partial R||q^2-V|^{-3/2}$, which are small except near the classical turning points R_a and R_b, defined by $V(R_a,R_b)=q^2$. Further, for the WKB approximation to be valid, the width of the potential, R_b-R_a, must be large compared to the wavelength of the incident particle. For $l\neq 0$ and small incident energies, $qM \ll 2/5\sqrt{l(l+1)}$, we see from (3.5) that $R_a \cong -2l/q$ and $R_b \cong l/q$. So the WKB approximation is valid if $\sqrt{l(l+1)} \gg 2$. For $l=0$ WKB is never valid; one cannot fit several wavelengths in under the barrier.

Using the standard WBK matching formulas (see, e.g., [10]) one finds the capture coefficient for the case when the particle is well under the barrier, and $\sqrt{l(l+1)} \gg 2$,

$$|C_{ql}|^2=4\left[2\theta+\frac{1}{2\theta}\right]^{-2} , \tag{3.11}$$

where

$$\ln(\theta)=\int_{R_a}^{R_b} dR\,(V-q^2)^{1/2} .$$

In the intervals $R_a<R<-M$ and $M<R<R_b$, V goes like $1/R^2$, and the integration can be done exactly. In the remaining region of integration, q^2 can be neglected compared to V, and the value is approximately $\int_{-M}^{M} dR\sqrt{V} \cong l$ (which actually turns out to be a fair estimate from doing the integral numerically). Using the values for R_a and R_b given above, and keeping terms of

leading order in qM/l we find

$$|C_{ql}|^2 \approx \frac{1}{\theta^2} \approx \left[\frac{qM}{l}\right]^{6l+3} \frac{e^{4l}}{2^{10l+5}} , \quad l \gg 1, \quad \frac{qM}{l} \ll 1 . \tag{3.12}$$

The dependence on qM in (3.12) agree with the expression that was derived from scaling arguments (3.9) (Indeed, one can now approximate the unknown coefficient $|\kappa_l|^2 \approx l^2 e^{-2l}/\pi^2 2^6$.) The important point is that the capture coefficient is very small; performing the integration numerically one finds the capture coefficient equals 0, to five significant figures, in ranges where the WKB approximation is reliable.

For $l=0$ and energies below the potential barrier we must use a different technique. In this case the wavelength of the incident wave is much greater than the width of the potential, as long as $qM \ll 1$; that is, M is small compared to the de Broglie wavelength $\sqrt{2\mu E}/\hbar$. (In this regime there is no particular additional constraint for consistency with the slow velocity requirement (3.8). It is sufficient that $\mu M/\hbar = \mu M/m^2_{\text{Planck}}$ is less than or of order 1.) This long-wavelength wave is not sensitive to the details of the potential, and the potential in (3.3) can be replaced by a δ function, $V \to LV_0 \delta(R)$, with strength fixed by

$$LV_0 = \int dR \; V(R) .$$

This problem can be solved, and the capture coefficient is

$$|C_{q0}|^2 = \frac{1}{1+\left[\frac{LV_0}{2q}\right]^2} \approx \frac{1}{1+\left[\frac{10^{-2}}{qM}\right]^2} , \quad l=0, \quad qM \ll 1 . \tag{3.13}$$

It is simple to estimate LV_0 to be a few times $10^{-2}/M^2$, and numerical integration gives $LV_0 = 2\times 10^{-2}/M^2$. As $qM \to 0$ the capture coefficient goes like $(qM)^2$, which agrees with the previous result (3.10) derived from scaling arguments.

Classically, if the two black holes approach each other with zero angular momentum, they always coalesce. In the quantum mechanical system, for incident energies of $qM = 0.1, 0.01$ and 0.001, the $l=0$ mode has a capture coefficient of 0.99, 0.50, and 0.001, respectively. The behavior interpolates between almost complete capture, as one expects for a particle, to complete scattering to the asymptotic free region, as is characteristic of a wave. (See Fig. 3.) At very long wavelengths the wave barely "sees" the black hole.

In summary, for incident energies under the barrier we can easily compute the captured flux from (3.13), and the higher angular momentum waves are almost completely scattered. Only the $l=0$ mode has a non-negligible capture coefficient. This is not too surprising since an angular momentum barrier is hard to tunnel through. [(3.13) can also be used for incident energies over the barrier, $0.23 < qM < 1$, where it gives a capture coefficient of essentially unity. This agrees with the results of the high-energy approach discussed next.]

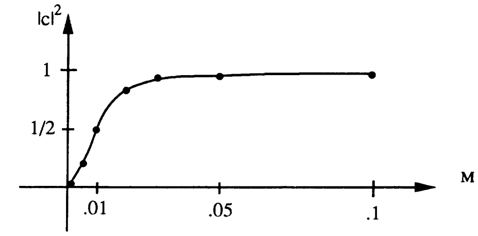

FIG. 3. Capture coefficient vs qM for $l=0$, calculated using (3.13). For very low energies, the behavior is different than for a classical particle with zero angular momentum.

2. High energies

For energies high compared to the height of the potential, the simplest approximation is the Born approximation, which appears to be sufficient here. To this end, we rewrite the eigenfunction equation as an integral equation:

$$\chi(R) = \chi^0(R) + \int dR' G(R,R') V(R') \chi(R') , \tag{3.14}$$

where $\chi^0(R)$ is any solution to the equation with the potential set to zero, and $G(R,R')$ is the Green's function for that equation. For the no-flux-out-of-the-horizon boundary conditions, as before one takes $\chi^0 = e^{-iqR}$ and

$$G(R,R') = \frac{e^{iq(R-R')}}{2iq} \quad \text{for } R > R' ,$$

$$G(R,R') = \frac{e^{iq(R-R')}}{2iq} \quad \text{for } R < R' .$$

Substituting into (3.14), we have χ as the sum of an ingoing and an outgoing wave, and hence can deduce the scattering and capture coefficients S and C. It is simple to check that this approximation does not conserve probability. However, if one computes S and C to second order, probability is conserved. Furthermore, only C gets a correction at this order. S is already correct to second order, so it suffices to compute S to first order and derive $|C|^2$ from $|C|^2 = 1 - |S|^2$.

The range of validity of the Born approximation is

$$qM \gg 0.23, \quad \text{if } l=0 ,$$

$$qM \gg \tfrac{2}{5}\sqrt{l(l+1)}, \quad \text{if } l>0 .$$

In these ranges one has

$$|S_{ql}|^2 = \frac{1}{4q^2} \left| \int dR \; V(R) e^{-2iqR} \right|^2 . \tag{3.15}$$

The integral is approximately zero where the Born approximation is valid: the width of the potential at half-maximum is about $3M$, which means that for $qM \gg 1$ the exponential in the integrand is rapidly oscillating and different contributions cancel. We computed the integral numerically and indeed one finds that, for $qM \geq 10$, once the particle is over the barrier, the scattering coefficient is 0, to five significant figures.

3. *Summary of scattering behavior*

For $qM \geq 10$ we see particlelike behavior. Recall that the condition for the wave to be over the barrier, and hence captured, is

$$\frac{b}{M} = \frac{\sqrt{l(l+1)}}{qM} \leq 2.5 \ ,$$

which is the same as the classical condition for coalescence. For high enough energies the discreteness of l is no longer important; a sketch of the capture coefficient vs l looks very much like the plot of capture vs a continuous b/M. (See Fig. 4.) Thus, for high energies, $|C|^2 \approx 1$ for $b/M < 2.5$. For $b/M \geq 3$, where we know the WKB approximation is valid, $|C|^2 \approx 0$. Indeed, $|C|^2$ is probably close to zero at an impact parameter closer to 2.5 than 3.

At intermediate energies, $0.23 \leq qM \leq 10$, q^2 is still above the potential barrier for some l values. As l, or b, is increased, $|C^2|$ decreases from (nearly) unity to (nearly) zero. The transition from the Born regime, where $|C|^2 \approx 1$, to the WKB, where $|C|^2 \approx 0$, is not as sharp as in the high-energy case. (See Fig. 4.) For $qM = 0.5$ this transition occurs over a change in b of about $14M$, compared to $M/2$ above.

For $qM < 0.23$, we see transition to wavelike behavior, as previously discussed. The wave is almost completely scattered, the captured flux going to zero like $(qM)^{6l+3}$ for large l and as $(qM)^2$ for $l = 0$.

A priori, one might expect more back scattering in the over-the-barrier case, and more capture in the under-the-barrier case, than we have found; except for the lowest angular momentum modes, the transition from a capture coefficient of 1 to 0 is quite abrupt in b/M. This is due to the "featurelessness" of the black-hole potential. There is only one length scale M which determines both the height and the width of the potential. Typically in scattering problems there are two independent parameters to vary.

One can also think of the over-the-barrier condition as fixing q and letting M increase, since the analysis is valid in a strong-gravity regime. Then, as one expects for a black hole, once the particle flux is in, it never gets out. Note that because of the slow-motion approximation (3.8), in the regime $qM > 1$ one can only consider black holes with masses such that

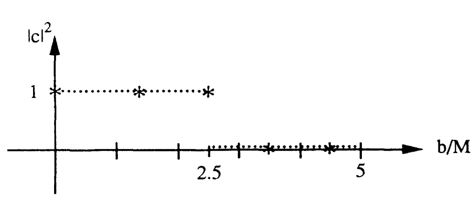

FIG. 4. Comparison of capture coefficient for $qM = 1$ ($*$) and $qM = 10$ ($\bullet$), vs $b/M = \sqrt{l(l+1)}/9M$. For $qM = 10$, the graph is very much like the classical case.

$$\frac{\mu M}{h} = \frac{\mu M}{m^2_{\text{Planck}}} > 1 \ , \tag{3.16}$$

in the case when the particle is over the barrier.

IV. DISCUSSION

At this point the most interesting question is what has been quantized? When solving the classical problem one first solves for the trajectories of the black holes—which immediately gives the diagonal metric components and the time component of the Maxwell potential. Then one finds (at least in principle) the other field components by solving given constraint equations, in which the black-hole positions and velocities appear as source terms [2]. The complete solution can be found in this sequence of steps because of the slow-motion approximation.

In the present calculation we have quantized the distance between the two black holes. Instead of having a solution $\mathbf{r}(t)$ which is this distance as a function of time, one now has an amplitude $\Psi(\mathbf{r},t)$.

This is fundamentally different from calculations in which "gravitons" are quantized as linear perturbations off some fixed classical background (typically Minkowski or Robertson-Walker type). In the charged black hole problem, one of the degrees of freedom of the full metric has been quantized, not just a fluctuation. To see this, just write the spacetime metric (2.4) in coordinates such that one of the black holes is at the origin and the other is at comoving coordinate z, so that one of the metric components is $r(t) = a(t)z$, the distance between the two black holes.

On the other hand, we have not allowed for quantum fluctuations in "directions in the space of metrics" other than this single function. In a fully quantum mechanical theory one must allow all the field components to vary from the classical values. The situation is similar to the quantization of a nonlinear sigma model. Suppose we have a classical theory in D dimensions (or with D fields) with a strong potential that approximately constrains the dynamics to a (D-1)-dimensional submanifold with metric g_{ab}. This leads us to model the system by the σ model $L = g^{ab}\nabla_a\phi\nabla_b\phi$, where ϕ is constrained to take values on the (D-1)-dimensional submanifold. Quantizing the σ model also misses out on quantum fluctuations in directions off the submanifold. One expects that quantizing the reduced system is a good approximation to the full system, if the constraining potential is sufficiently strong. In the black-hole problem this corresponds to the slow-motion approximation remaining good.

An alternative way to say this is that the approximation made here is the same as in "minisuperspace" models. In these quantum cosmology models, the wave function is taken to depend on only one metric component, namely, the scale factor.

We also note that in deriving our black-hole Hamiltonian H_{eff}, the classical equations of motion have been inserted into the action. Now, it was checked by Eardly that to derive the classical effective Hamiltonian one could either work exclusively with the equations of motion or with the action and equations of motion for some of the fields, as described earlier. However, since all field configurations contribute to a functional integral, one expects that use of the classical equations for N_i and A_i also leads to differences from a full quantum theory. Again, this is similar to the σ model approximation discussed above.

Next, what about the other fields in the problem? We suggest that the prescription for recovering the metric and electromagnetic fields from the wave function for the black hole coordinates is similar to the prescription for recovering the states of Schrödinger's cat in that famous demonstration of quantum mechanics [11]. That is, observed macroscopic states occur with probabilities predicted by quantum mechanics, but are not superpositions of multiple quantum-mechanical states. Suppose that for particular initial conditions of the wave function there is a scattering coefficient p. Then at late times, according to this prescription, one observes fields due to two widely separated black holes with masses m_1 and m_2 with probability p, and the set of field corresponding to one large black hole with mass M, with probability $1-p$. That is, one predicts that certain field configurations occur with certain probabilities.

An alternative prescription would be to couple the classical fields to the expectation value of the charge operator. Then, e.g., x would be the sum of two Coulombic pieces, with widely separated poles, one with strength $m_1+m_2(1-p)$ and the other with strength m_2p. That is, the resulting fields are prescribed deterministically, and correspond to sources where "part of the black holes coalesced and part scattered," rather than different classical configurations occurring with different probabilities.

Clearly in the gravitational problem one can only argue by analogy and according to what we observe at accessible energies, to choose the theory for specifying the classical fields. At this point in our understanding of quantum gravity, we leave the choice to the sense and sensibility of the reader.

ACKNOWLEDGMENTS

We would like to thank Doug Eardley and Gary Horowitz for frequent thoughtful and helpful discussions. This research was supported in part by the National Science Foundation under Grants Nos. PHY85-06686 and PHY82-17853, supplemented by funds from the National Aeronautics and Space Administration, at the University of California at Santa Barbara.

APPENDIX A: DEPENDENCE OF CAPTURE COEFFICIENT ON q AS $q \to 0$

As described in Sec. III, one can only solve for the eigenfunctions in the two asymptotic regions, and so cannot actually do the matching to solve for the unknown coefficients C_{ql} and S_{ql}. However, in the limit $qM \to 0$, the qM dependence can be determined, following [9]. From the asymptotic expressions for the potential (3.5) we find

$$\chi \simeq C_{ql}\left[\frac{(qM)^{\nu+1}(R/M)^{\nu+1}\sqrt{\pi}}{2^{\nu+1}\Gamma(\nu+\frac{3}{2})} + i\frac{2^{\nu}\Gamma(\nu+\frac{1}{2})}{(qM)^{\nu}(R/M)^{\nu}\sqrt{\pi}}\right], \quad -\frac{1}{q} \ll R \ll -M , \tag{A1}$$

$$\chi \simeq \frac{[S_{ql}+(-i)^{l+1}](qM)^{l+1}\sqrt{\pi}}{2^{l+1}\Gamma(l+\frac{3}{2})(R/M)^{l+1}} - i\frac{[S_{ql}-(-i)^{l+1}]2^{l}\Gamma(l+1/2)}{(qM)^{l}(R/M)^{l}\sqrt{\pi}}, \quad \frac{1}{q} \ll R \ll M . \tag{A2}$$

As $|qM| \to 0$, the second term in (A1) dominates. In (A2), we guess that as $qM \to 0$, $S_{ql} \simeq (-i)^{l+1} + (qM)^{2l+2}s_{ql}$, where $|s_{ql}| \lesssim 1$. (Later we will show that this is self-consistent.) Then, the first term in (A2) dominates. Since in the regime $-1/q \ll R \ll 1/q$ the differential equation is roughly independent of q, the dependence on q for each of these regions must be the same. Matching the dominant pieces of (A1) and (A2) gives the scaling as seen in (3.9) and (3.10). To check the self-consistency, suppose we tried $S_{ql} \simeq -(-i)^{l+1}+$ (lower order), so that the second term in (A2) dominates. Doing the matching one finds that the capture coefficient for $l=0$ is of order 1. However, we know that in the long-wavelength limit, the capture coefficient for $l=0$ goes to 0.

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