

Quasinormal modes of Schwarzschild black holes: Defined and calculated via Laplace transformation

Hans-Peter Nollert

Lehr- und Forschungsbereich Theoretische Astrophysik, Universität Tübingen, 7400 Tübingen, Germany

Bernd G. Schmidt

Max-Planck-Institut für Astrophysik, Karl-Schwarzschild-Strasse 1, 8046 Garching, Germany

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Quasinormal modes play a prominent role in the literature when dealing with the propagation of linearized perturbations of the Schwarzschild geometry. We show that space-time properties of the solutions of the perturbation equation imply the existence of a unique Green's function of the Laplace-transformed wave equation. This Green's function may be constructed from solutions of the homogeneous time-independent equation, which are uniquely characterized by the boundary conditions they satisfy. These boundary conditions are identified as the boundary conditions usually imposed for quasinormal-mode solutions. It turns out that solutions of the homogeneous equation exist which satisfy these boundary conditions at the horizon and at spatial infinity simultaneously, leading to poles of the Green's function. We therefore propose to define quasinormal-mode frequencies as the poles of the Green's function for the Laplace-transformed equation. On the basis of this definition a new technique for the numerical calculation of quasinormal frequencies is developed. The results agree with computations of Leaver, but not with more recent results obtained by Guinn, Will, Kojima, and Schutz.

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I. INTRODUCTION

Compact linear oscillating systems such as finite strings, membranes, or cavities, have preferred time harmonic states of motion if dissipation is neglected. There is always a countable number of such solutions, characterized by normal-mode frequencies ω_n and the corresponding eigenfunctions which describe the spatial pattern of the time harmonic motion. Any motion of the system is a superposition (in general infinite) of harmonic motions, which means that the eigenfunctions form a complete system.

Mathematically this is based on the equation

$$\frac{\partial \Phi}{\partial t} = \mathbf{A} \Phi, \tag{1}$$

where \mathbf{A} is a differential operator acting only on the spatial variables. Because of the boundary conditions dictated by the physical problem, \mathbf{A} is self-adjoint with a pure point spectrum on a Hilbert space. As a prototype, one can consider the motion of a string whose ends are both fixed. The energy in the system stays there forever "traveling forwards and backwards."

Completely different is the behavior, and, consequently, the mathematical properties, of infinite systems which may lose energy to infinity. Examples are waves propagating on an infinite string or electromagnetic waves reflected at a metallic body. Let us discuss and compare for clarity the following two problems: (1) The propagation of waves described by the scalar wave equation in a perfectly reflecting spherical cavity of radius 1. This implies the idealized condition $\Phi=0$ at the boundary. (2)

The propagation of waves outside of a perfectly reflecting sphere of radius 1. For simplicity we consider only the $l=1$ spherical harmonic angular behavior.

The general spherically symmetric solution of the scalar wave equation which is regular (smooth) at the origin is given by

$$\frac{1}{r} [f(t+r) - f(t-r)]. \tag{2}$$

Differentiating with respect to $x, y,$ or z gives the radial part of the general $l=1$ solution which is regular at the origin:

$$\begin{aligned} \Phi(t,r) = & \frac{1}{r} [f'(t+r) + f'(t-r)] \\ & - \frac{1}{r^2} [f(t+r) - f(t-r)], \end{aligned} \tag{3}$$

where $f'(z) = df/dz$. We try to find a solution with harmonic time behavior which vanishes at $r=1$. The ansatz $f(z) = e^{i\omega z}$ leads to the equation

$$i\omega e^{i\omega} + i\omega e^{-i\omega} - (e^{i\omega} - e^{-i\omega}) = 0, \tag{4}$$

which is equivalent to

$$\omega = \tan(\omega). \tag{5}$$

This equation has an infinite number of solutions ω_n . The corresponding time harmonic solutions of the wave equation are

$$\Phi_n = e^{i\omega_n t} \left[\frac{\omega_n}{r} \cos(\omega_n r) - \frac{1}{r^2} \sin(\omega_n r) \right]. \tag{6}$$

Any solution of the problem is a superposition of these solutions; this is a consequence of the completeness of the eigenfunctions.

In the second case the situation is quite different. Because we want a solution of the wave equation *outside* of the sphere, the regularity at the center is not required, and the general solution is

$$\begin{aligned} \Phi(t,r) = & \frac{1}{r} [A'(t-r) + B'(t+r)] \\ & + \frac{1}{r^2} [A(t-r) - B(t+r)]. \end{aligned} \quad (7)$$

The boundary condition that Φ vanishes at $r=1$ implies

$$0 = A'(t-1) + B'(t+1) + A(t-1) - B(t+1). \quad (8)$$

We see that only one of the two functions can be chosen freely. If we choose B , then A is determined as the solution of the following linear differential equation:

$$A'(t) + A(t) = -B'(t+2) + B(t+2) = f(t). \quad (9)$$

Its general solution is

$$A(t) = e^{-t} \left[\int_0^t e^{t'} f(t') dt' + A(0) \right]. \quad (10)$$

The function $B(t)$ describes a wave coming in from “past null infinity” which is modified by the reflection at the sphere and travels outward as a wave described by $A(t)$ to “future null infinity.”

We may, for example, choose $B(t)$ such that it vanishes outside of the interval, say, $2 < t < C$, where C is an arbitrary constant larger than 2. Such a solution represents an incoming pulse of finite duration, it has spatially compact support for all times. We then have to set $A(0)=0$ to obtain the solution of finite energy—otherwise, the solution would grow exponentially in the past. The result is a finite pulse coming in and a pulse of infinite duration going out. Inserting (10) into (7), we observe that, for large t (i.e., $t > r + 2 - C$), the solution is

$$\begin{aligned} \Phi(t,r) = & \alpha \left[-\frac{1}{r} e^{-t+r} + \frac{1}{r^2} e^{-t+r} \right] \\ = & \alpha e^{-t} e^r \left[-\frac{1}{r} + \frac{1}{r^2} \right], \end{aligned} \quad (11)$$

where

$$\alpha = \int_0^\infty e^{t'} f(t') dt'. \quad (12)$$

Note that the form of solution (11) is independent of the particular structure of the incoming wave, which is characterized by the function $B(t)$. Only the constant α depends on $B(t)$.

The spatial part of this solution represents a “quasinormal mode” with frequency $\omega=i$. In this particular example there exists only one quasinormal mode. Furthermore, the frequency is imaginary, making the solution not oscillatory, but just damped. Had we treated the case $l=2$, then there would be a $-\frac{2}{3} \pm i\sqrt{3}$ in the exponent; for higher l there are l quasinormal modes. In other cases (more complicated obstacles), there may be an

infinite number.

The radial part of the function in (11) is, in a way, analogous to the eigenfunctions of the cavity. If we just ask for a behavior like $e^{i\omega t}$, we will find such a solution for arbitrary values of ω . Only for $i\omega = -1$, however, will the radial part be “outgoing at infinity” as in (11). It is this function, which is defined by the usual “quasinormal-mode boundary conditions,” $r\Phi \sim e^{i\omega(t-r)}$ for $r \rightarrow \infty$. The physical meaning of the quasinormal modes is that they describe the asymptotic behavior of the waves as $t \rightarrow \infty$ at fixed r .

What we described in this example is true in more general cases. The scattering theory by Lax and Phillips [1] demonstrates the following for the reflection of waves at quite general static obstacles in Minkowski space: There is a finite or infinite number of “quasinormal modes” (this terminology, however, is not used in the mathematical literature) which appear as poles of the scattering matrix and govern the asymptotic behavior for $t \rightarrow \infty$ at fixed r . A theorem by Lax and Phillips [2] demonstrates that these numbers can be calculated as the solutions of the Helmholtz equation which satisfy the quasinormal-mode boundary conditions.

The question of completeness of the quasinormal modes is never an adequate one. In the case we discussed, the $l=1$ scattering at the sphere, there exists just one quasinormal mode. Furthermore, the analogues of the eigenfunctions grow, in general, at infinity and are not elements of the Hilbert spaces of solutions of finite energy. It is only the late time asymptotic expansion which is determined by these solutions. For generic data the mode with the weakest damping dominates this behavior.

For wave propagation on the Schwarzschild background—the subject of this paper—the situation is more complicated. For the Lax-Phillips theory to be applicable it is essential that space is flat outside of the obstacle. We could only apply this theory directly if we would cut off the Schwarzschild metric at some large radius R and take the flat metric near infinity. Certain calculations [3,4] indicate that, due to backscatter at the curvature of the nonflat metric up to infinity, the exponential decay changes to a polynomial one. It is plausible that the Schwarzschild quasinormal modes are the limits of the modes calculated with the cutoff at R , letting R go to infinity. This makes the late-time asymptotics more difficult to determine. Nevertheless, numerical calculations [4] show that the fundamental quasinormal mode (i.e., the one which is damped least) dominates the decay for intermediate times.

In Sec. II of this paper, we will demonstrate how quasinormal modes of Schwarzschild black holes may be defined uniquely, even though the Schwarzschild metric is nowhere flat. In Sec. III, we present a technique for the explicit construction of the quasinormal-mode solutions and for the numerical calculation of the quasinormal frequencies. In Appendix A, we discuss mathematical problems arising from the approach which is commonly used to deal with the perturbation equation. This approach is based on a quasistationary picture, appealing to an analogy with a resonant system [5]. In Appendix B,

some details of our construction of the solutions are outlined. In Appendix C we prove that the solution defined by our approach is indeed the correct one.

II. DEFINITION OF QUASINORMAL MODES

The following master equation has been found [6,7] to describe scalar as well as electromagnetic or gravitational perturbations on the static part of the Schwarzschild space time:

$$\frac{\partial^2}{\partial t^2} r \phi_l(t, x) + \mathbf{A} r \phi_l(t, x) = 0, \quad (13)$$

where $\Phi(t, x, \theta, \phi) = \phi_l(t, x) Y_{lm}(\theta, \phi)$ is the perturbing field, and Y_{lm} indicate the spherical harmonics appropriate to the type of perturbation under consideration. The radial coordinate r has been scaled such that the horizon is at $r=1$. In order to obtain (13), the so-called "tortoise coordinate" $x = r + \ln(r-1)$ has been used, the horizon being pushed to $x = -\infty$.

\mathbf{A} is the following differential operator:

$$\mathbf{A} = -\frac{\partial^2}{\partial x^2} + \left[1 - \frac{1}{r} \right] \left[\frac{l(l+1)}{r^2} + \frac{\sigma}{r^3} \right], \quad (14)$$

where r is to be considered a function of x . σ denotes the type of perturbation:

$$\sigma = 1 - \tau^2 = \begin{cases} 1 & \text{for a scalar perturbation,} \\ 0 & \text{for an electromagnetic perturbation,} \\ -3 & \text{for a gravitational perturbation,} \end{cases} \quad (15)$$

i.e., τ is the spin of the perturbing field.

The general theory of partial differential equations for wave equations of the above type shows that smooth initial data $\Phi(0, x, \theta, \phi)$ and $(\partial/\partial t)\Phi(0, x, \theta, \phi)$ determine a unique, smooth solution $\Phi(t, x, \theta, \phi)$ for all t and (x, θ, ϕ) .

In the context of quasinormal modes, a quasistationary picture, using Fourier analysis in time, is generally employed to obtain a time-independent problem. Specific boundary conditions are then imposed to determine a discrete set of "resonant frequencies." This approach, however, suffers from several problems: The quasinormal-mode solutions do not form a complete set, and they become unbounded both at the horizon and at spatial infinity. Therefore, they cannot be used to deal with a "real" problem which is given, for example, in terms of initial values to (13). Mathematically, the boundary conditions, especially at spatial infinity, do not uniquely define solutions (see Appendix A and [8]). Numerically, it is extremely difficult to find solutions which satisfy the boundary conditions, as this involves tracking a small quantity in an exponentially growing one.

We will show that these problems can be avoided if the initial data at $t=0$ are taken into account explicitly, and a Laplace transformation instead of a Fourier transformation is used on Eq. (13).

Kay and Wald [9] have shown that the solution is bounded if the data have compact support on the

Kruskal extension of the Schwarzschild spacetime. If, in particular, the data have compact support on the Schwarzschild part, then there is a positive constant C such that

$$|\Phi(t, x, \theta, \phi)| \leq C \quad \text{for all } t \text{ and } (x, \theta, \phi). \quad (16)$$

This implies that such a solution has a Laplace transform

$$\hat{\phi}_l(s, x) = \int_0^\infty e^{-st} \phi_l(t, x) dt, \quad (17)$$

which is a holomorphic function of $s = s_0 + is_1$ for $s_0 > 0$ and satisfies the differential equation

$$s^2 r \hat{\phi}_l + \mathbf{A} r \hat{\phi}_l = J_l, \quad (18)$$

where $J_l(x)$ is determined by the data at $t=0$ (in the following we will omit the subscript l). Equation (18) corresponds to an inhomogeneous Schrödinger-like differential equation for the function $f = r\hat{\phi}$:

$$f''(s, x) - s^2 f(s, x) - V(x) f(s, x) = J(s, x), \quad (19)$$

with the potential

$$V(x) = \left[1 - \frac{1}{r} \right] \left[\frac{l(l+1)}{r^2} + \frac{\sigma}{r^3} \right]. \quad (20)$$

Without loss of generality, we consider only Cauchy data with vanishing field at $t=0$. J then depends only on x .

The spectrum of the corresponding self-adjoint operator turns out to be purely continuous, there are no periodic solutions of finite energy. There exist only improper eigenfunctions, similar to plane waves, which can be used to build packets of finite energy. These statements have been proven by Dimock and Kay [10].

We may restrict our treatment to initial data of compact support, since these are dense in the space of data of finite energy. We then know that the corresponding solutions must have a Laplace transform which is analytic in the right complex half-plane. In the following we will use the Laplace transformation technique not to show existence, but to obtain certain representations of the solutions.

Leaver [4] and Sun and Price [11,12] never discuss the question whether the solutions actually have a Laplace transform. Rather, they take the existence of the Laplace transform for granted, as well as the existence of quasinormal modes. They assume that the quasinormal modes are defined and calculated by the quasistationary approach, relying on Fourier transformation of the time-dependent perturbation equation. Their emphasis is on treating astrophysical problems using the Laplace transform, they do not attempt to calculate the quasinormal frequencies themselves on this basis.

A solution of the homogeneous, linear ordinary differential equation corresponding to (19) is determined by the initial values $f(a)$ and $f'(a)$ at an arbitrary value a of the argument, it is then defined for all x . Any such solution is an entire function in s [13], there are two linearly independent solutions for each value of s .

The equation above plays a prominent role in the theory of Sturm-Liouville operators. A theorem of Weyl demonstrates that, for nonreal s , there is—depending on

the behavior of $V(x)$ for $x \rightarrow \infty$ —either one solution which is square integrable in $0 < x < \infty$, or all solutions are square integrable. The first case is called the limit-point case, the second the limit-circle case. For the potential in (20), one is in the limit-point case at both ends. Hence, there is a solution f_+ which is square integrable to the right and one, f_- , which is square integrable to the left.

The solution of the inhomogeneous equation is unique up to a solution of the homogenous equation. Among all possible solutions, we have to find the unique solution of the inhomogeneous equation which is the Laplace transform of the original solution in space time. Any two linearly independent solutions of the *homogeneous* equation define a particular Green’s function of the *inhomogeneous* equation. If we build the Green’s function using f_- and f_+ , and if $J(x)$ is of compact support, we obtain a solution of the inhomogeneous equation which is square integrable in x :

$$f(s, x) = \int_{-\infty}^{\infty} G(s, x, x') J(s, x') dx', \tag{21}$$

$$G(s, x, x') = \frac{1}{W(s)} \begin{cases} f_-(s, x') f_+(s, x) & (x' < x), \\ f_-(s, x) f_+(s, x') & (x' > x), \end{cases} \tag{22}$$

where $W(s)$ is the Wronskian of f_- and f_+ .

The boundedness of the solution in space time (16) implies that the Laplace transform (17) is bounded in x as well. The choice (22) for the Green’s function is therefore correct.

The general theory of the limit-point case at both ends tells us that the Green’s function, which yields square-integrable solutions of the inhomogeneous equation, is unique. All other Green’s functions lead to solutions of the inhomogeneous equation which are unbounded in x . Therefore, (22) is the only possible choice for the correct Green’s function.

So far, we have defined the Green’s function only in the right half-plane of s , where the path of integration for the inverse of the Laplace transform is located. We may now continue the Green’s function analytically into the left half-plane of s . This may be done by continuing f_- and f_+ from the right into the left half-plane and using (22) again. It turns out that the Wronskian of f_- and f_+ has isolated zeros there, leading to poles of the Green’s function (22) (see Fig. 1).

The formal replacement $s = i\omega$ turns the Laplace-transformed equation (19) into the well-known Fourier-transformed equation. Note that a real solution of the time-dependent equation will have a real Laplace transform for a real Laplace parameter s . In this case, the inhomogeneity $J(s, x)$ in (18) is also real, as well as f_- and f_+ and the Green’s function constructed from them.

Investigating (19) with standard techniques [15], we find that the solutions which are bounded at either end must behave like

$$f_-(s, x) \sim e^{sx} \text{ as } x \rightarrow -\infty$$

and (23)

$$f_+(s, x) \sim e^{-sx} \text{ as } x \rightarrow \infty$$

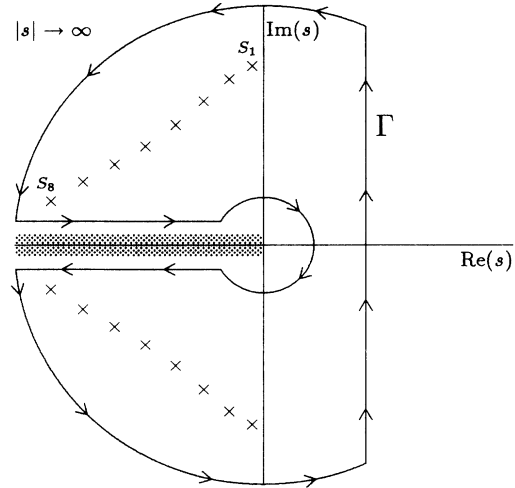


FIG. 1. Path of integration Γ in the complex plane for the inverse Laplace transformation. S_1, \dots, S_8 are the first eight poles of the Green’s function. The shaded region indicates a branch cut of the Green’s function along the negative real axis including $\{0\}$.

in the right half-plane of s . Their analytic continuations into the left half-plane must therefore show the same behavior, even though this will make them unbounded at the boundaries. The replacement $s = i\omega$ then shows that f_- and f_+ satisfy the usual quasinormal-mode boundary conditions. Thus, the solutions of the homogeneous equation used to construct the Green’s function (22) always satisfy “outgoing” boundary conditions at either spatial infinity or at the horizon [usually, waves going into the horizon are called “outgoing” with regard to the static part of the Schwarzschild space time, especially when the tortoise coordinate x of (13) is used].

The vanishing of the Wronskian indicates that we have indeed found values of s (and therefore ω) where there exists a solution of the transformed equation which satisfies the conditions at *both* boundaries simultaneously. We may therefore regard the zeros of the Wronskian of f_- and f_+ as quasinormal modes of the Regge-Wheeler potential (20).

It is now possible to close the path of integration Γ (Fig. 1) in order to compute the contributions to the inverse of the Laplace transform which arise from the corresponding poles of the Green’s function. This has been done by Leaver [4], and it turns out that these contributions indeed dominate the response of the metric some time after the disappearance of the initial perturbation, as quasinormal modes are expected to do.

We note that the mathematical ambiguities of the quasistationary picture, arising from the fact that it has never been shown how the quasinormal-mode boundary conditions should be used to actually pick a *unique* solution (see Appendix A), do not occur here at all. It is also quite clear how actual physics problems may be treated using the Laplace transform. Indeed, Leaver [4] and Sun

and Price [11,12] have already presented such calculations. Therefore, we suggest that the quasistationary picture be dispensed with altogether, and that quasinormal modes be regarded as the poles of the Green's function for the Laplace transformed solutions.

Having defined the quasinormal modes, the problem of their physical meaning has to be solved. The representation of the solution in terms of a Laplace transform allows one to relate decay properties in time to analyticity properties of the Laplace transform. We know that it is analytic in the right complex half-plane. The zeros of the Wronskian lead to poles in the left half-plane. If these were the only singularities, we would obtain exponential decay in time with frequencies determined by the quasinormal modes. The analysis by Leaver [4] indicates that there is an essential singularity at $s=0$, leading to polynomial decay in time like t^{-2l-3} . In addition, Leaver shows that contributions from the part of the integration path which lies at $|s|=\infty$ cause further modifications of the decay behavior.

III. CONSTRUCTION OF THE SOLUTIONS AND NUMERICAL RESULTS

We will now address the question how the Green's function can actually be constructed and how its singularities may be determined. On this basis we will develop a new technique for the numerical calculation of quasinormal frequencies. The following results are presented in more detail in the Ph.D. thesis of Nollert [14].

In the following we will use the Schwarzschild coordinate r as radial coordinate, since all coefficients in the differential equation corresponding to (19) are known explicitly as functions of r , but not as functions of x . The homogeneous part of (19) then becomes

$$r^2(r-1)^2 f''(r) + r(r-1) f'(r) + [-s^2 r^4 - l(l+1)r(r-1) - \sigma(r-1)] f(r) = 0. \quad (24)$$

Using standard techniques [15], we obtain two series expansions for two linearly independent solutions of (24):

$$\begin{aligned} f_-^{(1)}(r) &= (r-1)^s \sum_{n=0}^{\infty} a_n (r-1)^n, \\ f_-^{(2)}(r) &= (r-1)^{-s} \sum_{n=0}^{\infty} b_n (r-1)^n. \end{aligned} \quad (25)$$

The coefficients a_n and b_n may be obtained by substituting these expansions into (24) and choosing, e.g., $a_0 = b_0 = 1$.

If $\text{Re}(s) > 0$, then $f_-^{(2)}(r)$ is unbounded as $r \rightarrow 1$, while $f_-^{(1)}(r)$ is bounded. Since there is only one bounded solution, $f_-^{(1)}(r)$ must be identical to the desired solution $f_-(r)$. Note that the series in (25) converge absolutely for $|r-1| < 1$. $f_-(r)$ is therefore well defined for $0 < r < 2$ and may be continued analytically to $0 < r < \infty$. In fact, it is possible to obtain a series expansion for $f_-(r)$ which covers $\frac{1}{2} < r < \infty$ in one piece (see, e.g., Leaver [4]).

So far, the identification of $f_-(r)$ with $f_-^{(1)}(r)$ is valid

only in the right half-plane of s . However, the expansion of $f_-(r)$ converges for all s and is analytic in s . Therefore, it also represents the analytic continuation of $f_-(r)$ into the left half-plane of s .

We also know that there are two linearly independent solutions $f_+^{(1)}(r)$ and $f_+^{(2)}(r)$ which have the following asymptotic expansions for $r \rightarrow \infty$:

$$\begin{aligned} f_+^{(1)}(r) &= e^{-sr} r^{-s} \left[\sum_{n=0}^N \alpha_n r^{-n} + O(r^{-N-1}) \right], \\ f_+^{(2)}(r) &= e^{+sr} r^{+s} \left[\sum_{n=0}^N \beta_n r^{-n} + O(r^{-N-1}) \right]. \end{aligned} \quad (26)$$

Again, if $\text{Re}(s) > 0$, then $f_+^{(2)}(r)$ is unbounded as $r \rightarrow \infty$ and $f_+^{(1)}(r)$ is bounded. Therefore, the required solution $f_+(r)$ must be identical to $f_+^{(1)}(r)$. However, since the expansions (26) are only asymptotic and do not converge for any value of r , the problem of actually constructing $f_+(r)$ remains. In principle, it is possible to start with some initial condition [which may be obtained from (26)], to integrate to large values of r , and to check if the solution stays bounded. If not, change the initial conditions and try again, until $f_+(r)$ has been obtained with sufficient accuracy.

This method works for values of s in the right half-plane, but not in the left half-plane: There we would have to look for a solution which becomes unbounded as we integrate towards large values of r . However, almost all solutions of (24) are unbounded as r approaches infinity. Thus, it will be impossible to distinguish the solution $f_+(r)$ from others in this way.

If we need $f_+(r)$ in the left half-plane (to compute quasinormal modes, for example), the "trial-and-error" method would have to be repeated for many values of s in the right half-plane and the resulting solutions be continued analytically (as functions of s) into the left half-plane. While this procedure is well-defined mathematically, it is impossible to realize numerically, especially for large negative real parts of s . We must therefore look for some representation which allows us to obtain $f_+(r)$ directly for all values of s .

The construction of such a representation will be based on the asymptotic expansion (26) of $f_+^{(1)}(r)$. We will define a sequence of solutions which will be coupled to the values of the asymptotic expansions at growing values of r . We will then show that this sequence converges to a solution which we will identify as the desired solution $f_+(r)$.

Define a sequence $A_n(r)$ of functions by

$$A_n(r) = e^{-sr} r^{-s} \sum_{m=0}^n \alpha_m r^{-m}, \quad (27)$$

i.e., the A_n are just the asymptotic expansion of $f_+^{(1)}(r)$ truncated after $n+1$ terms. Define a sequence of solutions of (24) by the initial conditions

$$\begin{aligned} f_n^+(r_n) &= A_n(r_n), \\ \frac{df_n^+}{dr}(r_n) &= \frac{dA_n}{dr}(r_n) \quad \text{for all } n \geq N_0, \end{aligned} \quad (28)$$

where

$$r_n = \frac{n}{2|s|} \tag{29}$$

and N_0 has to be chosen such that $r_{N_0} > 1$. This coupling between r and n will be necessary to guarantee convergence of the sequence $f_n^+(r)$.

Let $g(r)$ be an arbitrary solution of (24). Consider the sequence $W_n(r) = W[g, f_n^+](r)$ of Wronskians of $g(r)$ with $f_n^+(r)$. The Wronskian of any two solutions of (24) has the form [16]

$$W[f_1, f_2](r) = K_{12} w(r) = K_{12} \frac{r}{r-1}, \tag{30}$$

where K_{12} is a complex constant.

We therefore have a sequence K_n of numbers corresponding to the sequence of Wronskians $W_n(r)$ such that

$$W_n(r) \equiv W[g, f_n^+](r) = K_n w(r). \tag{31}$$

This sequence K_n is given by the following expression (see Appendix B):

$$K_n = K_{N_0} + \sum_{m=N_0+1}^n k_m + \sum_{m=N_0+1}^n l_m, \tag{32}$$

where

$$k_m = w^{-1}(r_m) W[g, A_m - A_{m-1}](r_m), \tag{33}$$

$$l_m = w^{-1}(r_m) W[g, \Delta_{m-1}](r_m) = \int_{r_{m-1}}^{r_m} \frac{g(r') I_{m-1}(r')}{r'^2 (r'-1)^2 w(r')} dr', \tag{34}$$

and

$$\Delta_m(r) = A_m(r) - f_m^+(r).$$

$I_m(r)$ is obtained by applying the differential operator corresponding to (24) to $A_m(r)$ [see (B14)]. k_n and l_n represent the difference between K_{n-1} and K_n . We may regard k_n as the contribution which arises from using one more term in the asymptotic expansion (26) and l_n as the contribution due to changing from r_{n-1} to r_n in the initial condition (28) for $f_n^+(r)$. K_{N_0} may be determined by using (28) to evaluate $W_{N_0}(r_{N_0})$.

It may be shown that the sequence K_n converges for $n \rightarrow \infty$ (see Appendix B). The sequence $W_n(r)$ therefore converges point by point to a function

$$W(r) = \lim_{n \rightarrow \infty} W_n(r) = K w(r) = \left(\lim_{n \rightarrow \infty} k_n \right) w(r).$$

Let $f_1(r)$ and $f_2(r)$ be any two linearly independent solutions of (24). We may then express each $f_n^+(r)$ as a linear combination of $f_1(r)$ and $f_2(r)$:

$$f_n^+(r) = C_n^{(1)} f_1(r) + C_n^{(2)} f_2(r).$$

Due to the identity

$$K_{12} f_3(r) + K_{23} f_1(r) + K_{31} f_2(r) = 0 \tag{35}$$

for any three solutions $f_i(r)$ of (24) [where K_{ij} is obtained

via (30) from the Wronskian of $f_i(r)$ with $f_j(r)$], we have

$$C_n^{(1)} = -\frac{1}{K_{12}} K_{2n}$$

and

$$C_n^{(2)} = \frac{1}{K_{12}} K_{1n}, \tag{36}$$

where K_{1n} and K_{2n} are determined by letting $g(r) = f_1(r)$ and $g(r) = f_2(r)$, respectively, in (32)–(34). Since $K_1 = \lim_{n \rightarrow \infty} K_{1n}$ and $K_2 = \lim_{n \rightarrow \infty} K_{2n}$ exist, so must $C^{(1)} = \lim_{n \rightarrow \infty} C_n^{(1)}$ and $C^{(2)} = \lim_{n \rightarrow \infty} C_n^{(2)}$. Therefore, for any given r , the sequence $f_n^+(r)$ converges to

$$f(r) \equiv C^{(1)} f_1(r) + C^{(2)} f_2(r). \tag{37}$$

On the other hand, any linear combination of $f_1(r)$ and $f_2(r)$ is itself a solution of (24). Thus, the function $f(r)$, which has been defined only point by point so far, must also be a solution of (24).

Note that the solutions $f_n^+(r)$ in the sequence defined by (28) are usually unbounded as $r \rightarrow \infty$. It is therefore necessary to show that the limit function $f(r)$ stays bounded as $r \rightarrow \infty$ (see Appendix C). Since there is only one solution which has this property, we may identify $f(r)$ as the desired solution $f_+(r)$.

The sequences K_{1n} and K_{2n} converge absolutely for all values of s except those where $\text{Re}(s) = -|s|$, i.e., the negative real axis including $\{0\}$. They contain only functions of s which are analytic in s , except at $s=0$ and at isolated points along the negative real axis. Therefore, $f(r)$ as given by (32)–(34), (36), and (37) may again be regarded as a representation of $f_+(r)$ which is valid for (almost) all values of s .

We note that in the derivation above, we have encountered three possible procedures for the definition of $f_+(r)$: (i) Pick out the solution which stays bounded as $r \rightarrow \infty$ for values of s in the right half-plane, continue this solution analytically into the left half-plane, (ii) construct the sequence $f_n^+(r)$ according to (28) and use its limit, or (iii) pick two suitable solutions $f_1(r)$ and $f_2(r)$ and use expression (37). However, while these possibilities are mathematically equivalent, only the third is actually practical for numerical purposes.

For the computation of quasinormal modes, we do not have to construct $f_+(r)$ explicitly. It is sufficient to use $g(r) = f_-(r)$ in (32)–(34) to determine the value of K for f_+ and f_- . If K is zero, the Wronskian of f_+ with f_- vanishes and the Green's function (22) has a singularity.

For the numerical evaluation of $f_-(r)$, a series representation which converges for all values $r > \frac{1}{2}$ may be used instead of (25) [4]. However, the convergence of this series becomes very slow as r grows. It is therefore better to switch to power series in $(r - r_i)$ with growing r_i as $r \rightarrow \infty$.

l_n in (34) may be computed by numerical integration. However, it is much faster to obtain a differential equation for the integrand

$$i_n(r) = \frac{g(r) I_{n-1}(r)}{r^2 (r-1)^2 w(r)}$$

by substituting

$$g(r) = \frac{r^2(r-1)^2 w(r)}{I_{n-1}(r)} i_n(r)$$

in (24). This differential equation may again be solved by a power-series ansatz, this power series may then be integrated trivially.

In this way, all expressions involved in the computation of K are given in terms of convergent series. The convergence properties as well as the remainder terms may be estimated *a priori*, allowing an efficient numerical evaluation of K . However, it turns out that the series K_n may first grow very large before approaching its limiting value. This effect becomes stronger as the real part of s grows more negative. While this obstacle may, in principle, be overcome, it has kept us so far from calculating $K(s)$ for $\text{Re}(s) < -6$.

We have searched for zeros of K as a function of s , for the case of gravitational perturbations ($\sigma = -3$) and different values of l . The results are presented in Table I and in Fig. 2. The values obtained by Leaver [4] via a continued fraction expression and by Guinn, Will, Kojima, and Schutz [17] (GWKS) through a Wentzel-Kramers-Brillouin (WKB) approach are shown for comparison.

It turns out that our results agree completely with Leaver's, with the exception of Leaver's purely imaginary

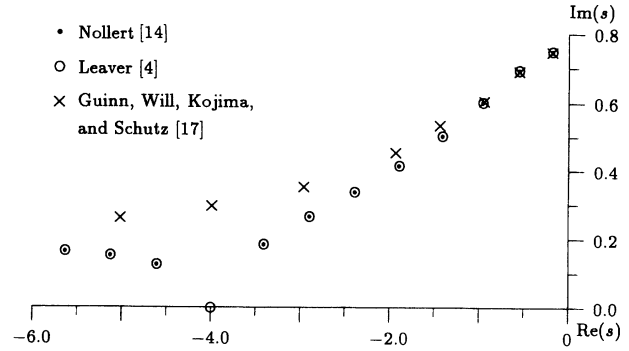


FIG. 2. The first 12 poles of the Green's function for the gravitational case and $l=2$. The results of Leaver and of GWKS have been multiplied by i . Note that the values of GWKS for $n=6, 8, 10$, and 12 are not shown.

frequency, which corresponds to $s = -4$ in the case of $l=2$. We do not find a zero of $K(s)$ at this value of s . However, the question whether there is a quasinormal mode near $s = -4$ has many aspects which go beyond this simple statement [14]. We will discuss these in a future paper.

We would like to stress that our technique, as well as its mathematical foundation, is totally different from the

TABLE I. The first 12 singularities of the Green's function for the gravitational case and $l=2,3$. The results of Leaver and of GWKS have been multiplied by i to compare them with ours.

$l=2$						
N	Zeros of $K(s)$		Leaver [4]		GWKS [17]	
	$\text{Re}(s)$	$\text{Im}(s)$	$\text{Re}(s)$	$\text{Im}(s)$	$\text{Re}(s)$	$\text{Im}(s)$
1	-0.177 925	0.747 343	-0.177 925	0.747 343	-0.178 40.746 4	
2	-0.547 830	0.693 422	-0.547 830	0.693 422	-0.549 80.692 0	
3	-0.956 554	0.602 107	-0.956 554	0.602 107	-0.942 20.605 8	
4	-1.410 296	0.503 010	-1.410 296	0.503 010	-1.437 60.535 2	
5	-1.893 690	0.415 029	-1.893 690	0.415 029	-1.934 20.454 8	
6	-2.391 216	0.338 599	-2.391 216	0.338 599	Not available	
7	-2.895 821	0.266 505	-2.895 822	0.266 505	-2.961 80.354 0	
8	-3.407 682	0.185 645	-3.407 676	0.185 617	Not available	
9	No zero of $K(s)$		-3.998 000	0	-3.990 40.298 0	
10	-4.605 289	0.126 527	-4.605 289	0.126 527	Not available	
11	-5.121 653	0.153 107	-5.121 653	0.153 107	-5.013 40.262 4	
12	-5.630 885	0.165 196	-5.630 885	0.165 196	Not available	
$l=3$						
1	-0.185 406	1.198 887	-0.185 406	1.198 887	Not available	
2	-0.562 596	1.165 288	-0.562 596	1.165 288	Not available	
3	-0.958 185	1.103 370	-0.958 186	1.103 370	Not available	
4	-1.380 674	1.023 924	-1.380 674	1.023 924	Not available	
5	-1.831 299	0.940 348	-1.831 299	0.940 348	Not available	
6	-2.304 303	0.862 773	-2.304 303	0.862 773	Not available	
7	-2.791 824	0.795 319	-2.791 824	0.795 319	Not available	
8	-3.287 689	0.737 985	-3.287 689	0.737 985	Not available	
9	-3.788 066	0.689 237	-3.788 066	0.689 237	Not available	
10	-4.290 798	0.647 366	-4.290 798	0.647 366	Not available	
11	-4.794 709	0.610 922	-4.794 709	0.610 922	Not available	
12	-5.299 159	0.578 768	-5.299 159	0.578 768	Not available	

one used by Leaver. In fact, these two are the only known methods to compute, without approximation, quasinormal frequencies well beyond the fundamental ones. As we have shown, the basis for our computations is mathematically rigorous and well understood. On the other hand, the continued fraction relation used by Leaver is derived by an analogy to the H_2^+ ion which is not unambiguous mathematically.

The imaginary parts of GWKS's numbers (which have to be multiplied by i) agree well with the real parts of s , the remaining differences are probably due to the approximations contained in their WKB method. However, for $N \gtrsim 4$, there are considerable differences between the real parts of their quasinormal frequencies and the imaginary parts of s , which can probably not be attributed to approximation inaccuracies.

According to Guinn [18], in the case of the Pöschl-Teller potential, the quasinormal frequencies computed on the basis of the boundary conditions indeed agree with those obtained by the WKB technique. The Pöschl-Teller potential, which has been used as an approximation to the Regge-Wheeler potential by Ferrari and Mashhoon [19], never becomes exactly 0 either, but it decays exponentially at infinity, while the Regge-Wheeler potential shows a polynomial behavior. We have already pointed out [20] that the real part of the quasinormal frequencies depends strongly on the decay behavior of the potential at spatial infinity, while the imaginary part is rather insensitive to it. It is therefore possible that the differences between our calculations and those by GWKS show up only if the potential decays slowly at spatial infinity.

IV. CONCLUSIONS

We have first formulated the perturbation problem for the Schwarzschild metric in space time. The approach usually employed to obtain a time-independent problem is Fourier transformation of the time-dependent equation, followed by imposing boundary conditions on the solutions derived from an analogy to quantum mechanics. However, the spectrum of the relevant self-adjoint operator is purely continuous, there are no periodic solutions of finite energy. We have demonstrated that this leads to problems with the mathematics of this definition of quasinormal modes, as well as with the physical interpretation of the resulting "eigenfunctions."

Alternatively, we have first shown that finite energy solutions of the time-dependent equation must have a Laplace transform. We have then proven that the properties of the solution in space time necessarily imply a unique choice of the Green's function for the Laplace-transformed (inhomogeneous) equation. There is neither the necessity nor the freedom to choose *ad hoc* boundary conditions or to appeal to an analogy with quantum mechanics. It turns out that the correct Green's function must be constructed from solutions of the homogeneous equation which may be regarded as "outgoing" at either spatial infinity or the horizon.

The Green's function for the Laplace transform of a real solution of the time-dependent equation is real, if the

Laplace parameter s is real. It is constructed from real solutions of the homogeneous Laplace-transformed equation. This real Green's function has a unique analytic extension onto the complex right half-plane, where the real part of s is positive. This follows from the original solution being bounded in space time. This analytic extension of the real Green's function may be used to compute the inverse of the Laplace transform.

We then construct the *maximal* analytic extension of the Green's function. There we find isolated singularities due to the vanishing Wronskian in the denominator of the Green's function. This implies that the corresponding two solutions of the homogeneous equation become linearly dependent, the resulting single solution satisfying both boundary conditions *simultaneously*.

Since these boundary conditions are identical to those usually imposed on quasinormal-mode solutions, we propose to define quasinormal frequencies as those values of the Laplace parameter s where the poles of the Green's function occur. This definition suffers from none of the difficulties which are characteristic for the (quasistationary) Fourier-transform approach.

We have given analytic expressions for the Green's function in terms of absolutely convergent series expansions. These expressions may also be used to determine quasinormal frequencies numerically, without the difficulties which have stifled such attempts in the past. It turns out that our results are identical with values obtained by Leaver, but they disagree with more recent calculations by Guinn, Will, Kojima, and Schutz.

For the Schwarzschild background, the meaning of the quasinormal frequencies and the corresponding solutions of the perturbation equation is not completely clear. In cases where the Lax-Phillips scattering theory is applicable, the quasinormal frequency with the biggest real part determines the final decay in time of any solution at a fixed point in space. The corresponding "quasinormal-mode eigenfunction" describes the final form of the damped waves in a fixed finite spatial interval. This form is independent of the particular initial data. However, these results cannot be applied directly to the Schwarzschild case. Nevertheless, numerical calculations, e.g., by Leaver and Sun and Price, indicate that quasinormal modes describe a universal exponential decay for some intermediate time, before a polynomial decay behavior takes over to dominate the final evolution for very late times.

APPENDIX A: ASYMPTOTIC BEHAVIOR BECOMES INSUFFICIENT FOR SPECIFYING A UNIQUE SOLUTION IF $\text{Im}(\omega) > 0$

In the literature on quasinormal modes of black holes, a quasistationary approach is commonly used to deal with the time-dependent perturbation equation. Equation (13) is Fourier transformed, expressing $\phi(t, x)$ as

$$\phi_I(t, x) = e^{i\omega t} \hat{\phi}_I(\omega, x). \quad (\text{A1})$$

Initial conditions at some fixed time t_0 cannot be taken into account in this way. Rather, the boundary conditions

$$\widehat{\phi}_l(\omega, x) \xrightarrow{x \rightarrow \pm \infty} e^{\mp i\omega x} \quad (\text{A2})$$

are imposed on the solutions of (A1). With respect to the time dependence chosen in (A1), these boundary conditions specify waves which are outgoing at either spatial infinity or at the horizon. In this context, “outgoing at the horizon” means traveling towards smaller values of x (and therefore of r), i.e., going into the black hole.

For the purpose of studying quasinormal modes, it is commonly assumed that (A1), applied to either $x \rightarrow -\infty$ or $x \rightarrow +\infty$, may be used to uniquely determine a solution of

$$\mathbf{A}\phi(x) - \omega^2\phi(x) = 0, \quad (\text{A3})$$

where \mathbf{A} is the differential operator defined in (14) (we have dropped the subscript l , the argument ω , and the caret on ϕ). However, this is no longer true if ω is complex and has a positive imaginary part. Note that quasinormal frequencies can occur only at such values of ω where this is the case.

We will look especially at the situation when $x \rightarrow +\infty$. Stated more accurately, the boundary condition (A2) specifies

$$\phi(x) = e^{-i\omega x} [1 + O(x^{-1})] \text{ as } x \rightarrow \infty. \quad (\text{A4})$$

Suppose now we had singled out a specific solution $\phi_1(x)$ whose asymptotic behavior is given by (A4). There is another solution $\phi_2(x)$ of (A3) which behaves as

$$\phi_2(x) = e^{+i\omega x} [1 + O(x^{-1})] \text{ as } x \rightarrow \infty. \quad (\text{A5})$$

We may now define $\phi_0(x) = \phi_1(x) + \phi_2(x)$. Then $\phi_0(x)$ is a solution of (A3) and it has the asymptotic behavior

$$\begin{aligned} \phi_0(x) &= e^{-i\omega x} \{1 + O(x^{-1}) + e^{2i\omega x} [1 + O(x^{-1})]\} \\ &= e^{-i\omega x} [1 + O(x^{-1})], \end{aligned} \quad (\text{A6})$$

since, with a positive imaginary part of ω , the term $e^{2i\omega x}$ vanishes faster than any power of $1/x$.

Therefore, $\phi_0(x)$ [and almost every other solution of (A3)] has identically the same asymptotic behavior as $\phi_1(x)$. Thus, it is clear that (A4) is not sufficient to single out a *unique* solution of (A3) if ω is complex and has a positive imaginary part.

This is not surprising since an asymptotic expression never defines a single function, but always a class of functions. In the case of real ω (or complex ω with negative imaginary part), the class defined by (A4) happens to contain only one solution of (A2), while in the case of a positive imaginary part of ω it contains almost all solutions of (A3).

A similar problem exists at the horizon, i.e., for $x \rightarrow -\infty$. However, we may derive a convergent series for solutions of (A3), which may formally be compared with the boundary condition. In this way we can identify a unique solution which we choose to regard as “outgoing at the horizon.” This procedure does not work for $x \rightarrow +\infty$: The differential equation (A3) has a strong singularity here. Therefore, a convergent series expansion for $x \rightarrow +\infty$ does not exist, asymptotic expansions like (A4) are all we have to work with.

APPENDIX B: COMPUTATION OF $K[g, f_+]$ AND PROOF OF CONVERGENCE

Since the Wronski determinant is linear in any one of both functions, we obtain

$$\begin{aligned} W_n(r_n) &= W_{n-1}(r_n) + W[g, f_n^+ - f_{n-1}^+](r_n) \\ &= W_{n-1}(r_n) + W[g, A_n - A_{n-1}](r_n) \\ &\quad + W[g, A_{n-1} - f_{n-1}^+](r_n), \end{aligned} \quad (\text{B1})$$

where we have used that $f_n^+(r)$ and $A_n(r)$ are identical at $r = r_n$.

Dividing by $w(r_n)$, we find

$$\begin{aligned} K_n &= K_{n-1} + w^{-1}(r_n) W[g, A_n - A_{n-1}](r_n) \\ &\quad + w^{-1}(r_n) W[g, \Delta_{n-1}](r_n). \end{aligned} \quad (\text{B2})$$

The term $k_n = w^{-1}(r_n) W[g, A_n - A_{n-1}](r_n)$ may be evaluated in a straight-forward way. For the computation of $l_n = w^{-1}(r_n) W[g, \Delta_{n-1}](r_n)$ we apply the differential operator corresponding to (24) to $A_n(r)$ and obtain

$$\text{DE}(A_n)(r) = I_n(r), \quad (\text{B3})$$

since $A_n(r)$ is not a solution of (24).

However, we may as well regard (B3) as an inhomogeneous differential equation which has $A_n(r)$ as a solution. Since $f_n^+(r)$ is a solution of the homogeneous part of (B3), the function $\Delta_n(r) = A_n(r) - f_n^+(r)$ must also be a solution of (B3).

The Wronski determinant $W[g, \Delta_n](r)$ satisfies the differential equation

$$\begin{aligned} r^2(r-1)^2 \frac{dW[g, \Delta_n]}{dr}(r) + r(r-1)W[g, \Delta_n](r) \\ = g(r)I_n(r). \end{aligned} \quad (\text{B4})$$

Substituting

$$W[g, \Delta_n](r) = V_n(r)W_n(r) \quad (\text{B5})$$

into (B4) yields the differential equation

$$r^2(r-1)^2 W_n(r) V_n'(r) = g(r)I_n(r) \quad (\text{B6})$$

for $V_n(r)$. This may readily be integrated, leading to the solution

$$V_n(r) = \int_{r_n}^r \frac{g(r')I_n(r')}{r'^2(r'-1)^2 w(r')} dr'. \quad (\text{B7})$$

This solution satisfies the initial condition $V_n(r_n) = 0$, which follows from (28) via $\Delta_n(r_n) = 0 = \Delta_n'(r_n)$ and $W_n(r_n) = 0$.

Using the definition of l_n , (B5), and (30), we obtain

$$l_n = \int_{r_{n-1}}^{r_n} \frac{g(r')I_{n-1}(r')}{r'^2(r'-1)^2 w(r')} dr'. \quad (\text{B8})$$

We will prove the convergence of the series $\sum(k_n + l_n)$ by treating $\sum k_n$ and $\sum l_n$ separately. In order to examine the behavior of k_n for large n , we will approximate $g(r)$

by the first two terms of its asymptotic expansion:

$$g(r) \sim e^{\pm sr} r^{\pm s} (g_0 + g_1 r^{-1}). \tag{B9}$$

We will only demonstrate the case where the asymptotic expansion of $g(r)$ has minus signs in both exponentials. The case corresponding to two plus signs may be treated in a completely analogous way and, in fact, leads to a faster convergence of $\sum k_n$.

Evaluating the expressions on the right-hand side of (33) to first order in r^{-1} , we find

$$k_n \sim -g_0 \frac{n}{r_n} \alpha_n e^{-2sr_n} r_n^{-2s} r_n^{-n}. \tag{B10}$$

Using $r_n = n/2|s|$, we obtain

$$\begin{aligned} \frac{k_{n+1}}{k_n} &\sim e^{-s/|s|} \frac{\alpha_{n+1}}{\alpha_n} \frac{r_{n+1}^{-n-1}}{r_n^{-n}} \\ &= e^{-s/|s|} \frac{\alpha_{n+1}}{\alpha_n} \frac{2|s|}{n+1} \left(\frac{n+1}{n} \right)^{-n}. \end{aligned} \tag{B11}$$

In order to determine α_{n+1}/α_n , we need to know the recurrence relation for the α_n explicitly:

$$c_0(n)\alpha_n + c_1(n)\alpha_{n-1} + c_2(n)\alpha_{n-2} + c_3(n)\alpha_{n-3} = 0, \tag{B12}$$

where

$$\begin{aligned} c_0(n) &= 2sn, \\ c_1(n) &= n(n-1) - 2s(n-1) - 2s^2 - l(l-1), \\ c_2(n) &= -n(2n-5) - 2s(n-1) + l(l-1) - \sigma - 2, \\ c_3(n) &= n(n-4) + 2s(n-2) + s^2 + \sigma + 3. \end{aligned}$$

Usually, for large n , the behavior of α_n will be characterized by $\alpha_n/\alpha_{n-1} \sim n/2s$. We therefore obtain

$$\left| \frac{k_{n+1}}{k_n} \right| \xrightarrow{n \rightarrow \infty} |e^{-1-s/|s|}| = e^{-[1+\text{Re}(s)/|s|]} \begin{cases} = 1 & \text{if } \text{Re}(s) = -|s|, \\ < 1 & \text{if } \text{Re}(s) \neq -|s|, \end{cases} \tag{B13}$$

where we have used $\lim_{n \rightarrow \infty} (1 + 1/n)^n = e$. $\sum k_n$ therefore converges absolutely as long as s does not lie on the negative real axis.

In order to prove the convergence of $\sum l_n$, we will use $|\sum l_n| \leq \sum |l_n|$ and

$$|l_n| = \left| \int_{r_{n-1}}^{r_n} \frac{g(r') I_{n-1}(r')}{r'^2 (r'-1)^2 w(r')} dr' \right| \leq \int_{r_{n-1}}^{r_n} |\dots| dr' \leq \frac{1}{2|s|} \max |\dots|.$$

For simplicity, we will assume that the maximum is attained at $r = r_n$.

At this point we need to know $I_n(r)$ explicitly:

$$\begin{aligned} I_n(r) &= e^{-sr} r^{-s} r^{-n} \alpha_n c_3(n+3) + e^{-sr} r^{-s} r^{-n+1} [\alpha_{n-1} c_3(n+2) + \alpha_n c_2(n+2)] \\ &\quad + e^{-sr} r^{-s} r^{-n+2} [\alpha_{n-2} c_3(n+1) + \alpha_{n-1} c_2(n+1) + \alpha_n c_1(n+1)], \end{aligned} \tag{B14}$$

where the $c_i(n)$ are the same as in (B12).

We will split up l_n into a sum of six contributions, each corresponding to a term of the form $e^{-sr} r^{-s} r^{-(n-j)} \alpha_{n-i} c_k(n+3-j)$, $i, j, k \in \{0, 1, 2\}$, in $I_n(r)$. The convergence of each of these contributions may be examined separately. It is sufficient to consider, as an example, the contribution arising from $e^{-sr} r^{-s} r^{-n} \alpha_n c_3(n+3)$. For large n , its dominant behavior is

$$\begin{aligned} l_n^{(1)} &= \frac{1}{2|s|} \left| \frac{g(r_n) e^{-sr_n} r_n^{-s} r_n^{-n+1} \alpha_{n-1} c_3(n+2)}{r_n^2 (r_n-1)^2 w(r_n)} \right| \\ &\sim \frac{1}{2|s|} \left| \frac{g_0 c_3(n+2) \alpha_{n-1} e^{-2sr_n} r_n^{-2s} r_n^{-n}}{r_n^2 (r_n-1)} \right|. \end{aligned} \tag{B15}$$

Again, we are only presenting the case where $g(r) \sim g_0 e^{-sr} r^{-s}$ for large r . Obviously, $l_n^{(1)}$ is dominated by the same terms which are characteristic for k_n . The

convergence of $\sum l_n^{(1)}$ may therefore be shown in the same way as for $\sum k_n$.

**APPENDIX C: PROOF THAT
THE LIMIT SOLUTION $f(r)$ IS BOUNDED
FOR $\text{Re}(s) > 0$**

Every function $f_n^+(r)$ in the sequence of solutions defined by (28) may be expressed as a linear combination of $f_+^{(1)}(r)$ and $f_+^{(2)}(r)$ [see (26)]:

$$f_n^+(r) = C_n^{(1)} f_+^{(1)}(r) + C_n^{(2)} f_+^{(2)}(r) \text{ for all } n > N_0. \tag{C1}$$

Since the limits $C^{(1)} = \lim_{n \rightarrow \infty} C_n^{(1)}$ and $C^{(2)} = \lim_{n \rightarrow \infty} C_n^{(2)}$ exist, there must be some index N_1 and real, positive numbers m_1 and m_2 such that

$$|C_n^{(1)}| < m_1$$

and

$$(C2)$$

$$|C_n^{(2)}| < m_2 \text{ for all } n > N_1.$$

We want to show that $C^{(2)}$ must be zero. Suppose $C^{(2)} \neq 0$: In this case there is also an index N_2 and some real, positive number M_2 such that

$$|C_n^{(2)}| > M_2 \quad \text{for all } n > N_2. \quad (\text{C3})$$

Since $f_+^{(1)}(r) \sim e^{-sr} r^{-s} [\alpha_0 + O(r^{-1})]$ for large r , we can find an index N_3 for any positive number m_+ such that

$$|f_+^{(1)}(r_n)| < m_+ \quad \text{for all } n > N_3 \quad (\text{C4})$$

as long as $\text{Re}(s) > 0$. For the same reason, there is an index N_4 for any positive number m_A such that

$$|A_n(r_n)| < m_A \quad \text{for all } n > N_4. \quad (\text{C5})$$

On the other hand, $f_+^{(2)}(r) \sim e^{sr} r^s [\beta_0 + O(r^{-1})]$, so there is an index N_5 for any positive number M_+ such that

$$|f_+^{(2)}(r_n)| > M_+ \quad \text{for all } n > N_5. \quad (\text{C6})$$

Choose suitable values for m_1 and M_2 with corresponding indices N_1 and N_2 . Let $m_+ = 1/(2m_1)$,

$m_A = \frac{1}{2}$, and $M_+ = 1/M_2$, with indices N_3 , N_4 , and N_5 determined accordingly. Let $N = \max(N_1, \dots, N_5)$. We then have

$$\begin{aligned} A_N(r_N) &= f_N^+(r_N) \\ &= C_N^{(1)} f_+^{(1)}(r_N) + C_N^{(2)} f_+^{(2)}(r_N) \\ &\Leftrightarrow |A_N(r_N) - C_N^{(1)} f_+^{(1)}(r_N)| \\ &= |C_N^{(2)} f_+^{(2)}(r_N)|. \end{aligned} \quad (\text{C7})$$

However, due to (C2), (C4), and (C5), we find

$$|A_N(r_N) - C_N^{(1)} f_+^{(1)}(r_N)| < m_A + m_1 m_+ = 1. \quad (\text{C8})$$

On the other hand, due to (C3) and (C6),

$$|C_N^{(2)} f_+^{(2)}(r_N)| > M_2 M_+ = 1. \quad (\text{C9})$$

Obviously, (C8) and (C9) together are in contradiction to (C7). Therefore, $C^{(2)}$ cannot be different from 0, and the limit function $f(r)$ must be proportional to $f_+^{(1)}(r) = f_+(r)$.

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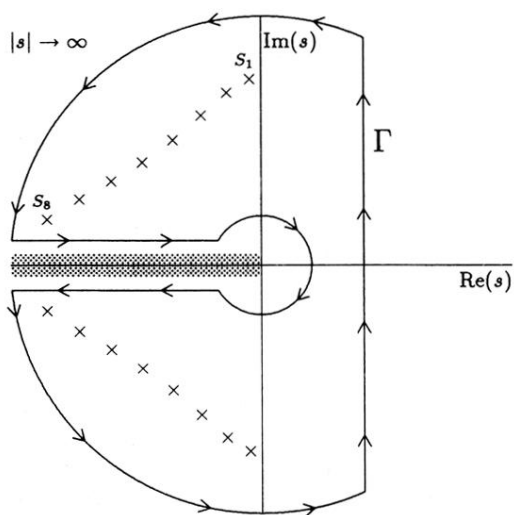


FIG. 1. Path of integration Γ in the complex plane for the inverse Laplace transformation. S_1, \dots, S_8 are the first eight poles of the Green's function. The shaded region indicates a branch cut of the Green's function along the negative real axis including $\{0\}$.