Quantum mechanics of a solid-state bar gravitational antenna

Leonid P. Grishchuk
Joint Institute for Laboratory Astrophysics, University of Colorado, Boulder, Colorado 80309-0440
and Sternberg Astronomical Institute, Moscow University, 119899 Moscow, V-234, U.S.S.R.
(Received 23 September 1991)

A quantum-mechanical treatment of a bar gravitational antenna is presented. The theory takes into account the crystalline structure of the bar and collective behavior of its mass elements. The low-frequency and high-frequency (Debye) modes of oscillations are considered. It is shown that the quantum-mechanial derivation of the absorption cross section of the gravitational antenna agrees totally with the classical result. The recent claims of a significant (six orders of magnitude) quantum-mechanical enhancement of the cross section are shown to be incorrect.

PACS number(s): 04.80.+z, 04.30.+x, 63.20.−e

I. INTRODUCTION

The use of solid-state bars is a current technique aimed at detecting gravitational waves produced by violent astrophysical events. The first bar detectors were built and operated by Weber in 1965 and since then their performance has been gradually improving. Currently, the best sensitivity of bar detectors is close to $h \approx 3\times 10^{-19}$ [1] which is approximately a factor of 2 or 3 better than the sensitivity of existing prototypes of laser-interferometer gravitational detectors.

A solid bar represents a mass-quadrupole mechanical oscillator. The fundamental mode of a bar, usually at the frequency of the order of $\omega_1/2\pi \approx 10^3$ Hz, is assumed to be nearly in resonance with characteristic frequencies of the incoming bursts of gravitational radiation. For a bar with mass M and length L, the absorption cross section $\sigma(\omega)$ integrated over the antenna's bandwidth and divided by ω_1 is of the order of

$$\sigma_1 \sim \frac{G}{c^3} ML^2\omega_1 \tag{1}$$

(see, for example, Refs. [2–4] and Sec. II below). The numerical value of σ_1 determines the excitation energy of the antenna acquired under the action of a burst of gravitational radiation with duration Δt and characteristic frequency $\omega \approx 2\pi/\Delta t$; eventually, it determines the detector's sensitivity.

Expression (1) has been recently questioned by Weber [5] who claims that the quantum-mechanical treatment of the antenna provides a considerable enhancement factor increasing with the number of atoms in the antenna. Weber discusses the bar antenna as a mechanical system consisting of N planar slabs each having one-atom thickness and mass m. The total mass of the bar is $M = mN$. Each pair of slabs located symmetrically with respect to the center of the bar form an oscillator of their own with the same fundamental frequency ω_1, so that the bar can be viewed as an ensemble of almost identical oscillators. When the bar oscillates at the fundamental frequency, each pair of slabs also oscillates with the same frequency. The absorption cross section of each individual isolated pair of slabs would be approximately equal to $Gc^{-3}mL^2\omega_1$. However, referring to quantum-mechanical considerations, Weber says that all such oscillators absorb the incoming gravitational radiation coherently, so that the quantum-mechanical amplitude of the transition from the ground state to the first excited state of the antenna is proportional to N and, hence, the probability of transition, the absorbed energy, and the cross section of the entire system are proportional to N^2. As a result, he arrives at the expression for σ_1 which is N times larger than (1). Numerically, his value of σ_1, for presently operating antennas, is about six orders of magnitude larger than the conventional expression (1) (and his own estimate [4] published earlier).

A similar question of a possible large quantum-mechanical enhancement of σ_1 over its classical value has been raised by Preparata [6]. In his first paper, Preparata disagrees with a particular mechanism suggested by Weber. He says that the validity of the classical and quantum-mechanical derivation of expression (1) is based on the assumption of the direct interaction of a gravitational wave with the fundamental vibrational mode of the antenna. Instead, in the second paper, he endeavors to show that there exists a different interaction mechanism of a gravitational burst with the elements of the antenna which increases the cross section (1) by many orders of magnitude. In his view, this mechanism is associated with the very high-frequency collective oscillations of an ensemble of atoms of the antenna. A typical frequency of such oscillations is of the order of the Debye frequency $\omega_D/2\pi \approx 10^{13}$ Hz, the largest acoustic frequency available in solids. In other words, instead of the lowest acoustic frequency in the antenna, which is Weber's primary concern, Preparata emphasizes on the highest frequencies which are about ten orders of magnitude larger than ω_1 and characteristic frequencies of the incoming gravitational waves. As a result, Preparata arrives at a new form of the cross section which is different from Weber's, but is, again, six orders of magnitude larger numerically than expression (1). In this way, Preparata tries to explain the correlations between the neutrino and gravity-wave detectors claimed to be recorded [7] during the supernova 1987A explosion. He concludes by saying that "Weber's intuition that the coherent behavior of the atoms in the bar greatly enhances their gravitational interaction is seen to be confirmed, though through a different logic."

If the statements of Weber and Preparata were correct they would drastically change the existing attitude toward past experiments with solid-state bar gravitational detectors and the prospects for their use in future experiments. Unfortunately, as we will show below, these statements are not correct. We will present a detailed quantum-mechanical theory of a model of the bar antennas which takes into account their crystalline structure and all acoustic frequencies available. We will explicitly indicate the assertions in the Weber and Preparata papers which are, we believe, in error. Our conclusion is that the quantum-mechanical properties of bar antennas do not enhance their absorption cross section (1) regardless of whether the low-frequency or high-frequency oscillations of their elements are considered.

The paper is organized as follows. In Sec. II we present an elementary classical theory of the interaction of a gravitational wave with mechanical oscillators of different frequencies. We derive the cross section (1) and discuss the changes which arise when the proper frequency of the oscillator is much larger than the frequency of the incoming burst. In Sec. III we consider a one-dimensional model of a crystal: a chain of masses coupled by the internal springlike restoring forces. This model is quite adequate to describe a realistic antenna which, essentially, behaves like a one-dimensional system under the influence of a plane gravitational wave with a wavelength much larger than the size of the antenna. We derive the proper modes, equations of motion, and the Hamiltonian of the system. Section IV contains the quantum-mechanical analysis of the system. We calculate the transition probabilities and the absorbed energy in the antenna and demonstrate the agreement between the classical and quantum approaches. Section V is devoted to the discussion of the results.

II. AN ELEMENTARY CLASSICAL THEORY OF A QUADRUPOLE GRAVITATIONAL ANTENNA

The simplest model of a gravitational antenna is a mechanical oscillator consisting of two masses, each of mass m, connected by a spring. Before arrival of the wave, the distance between the masses is L. The action of the wave produces a "tidal" driving force which displaces a given mass from its initial equilibrium position by the amount $\xi(t)$. The main equation governing the interaction of the oscillator with the wave has the form

$$\ddot{\xi}+\frac{2}{\tau}\dot{\xi}+\omega_l^2\xi=g_l\cos(\omega t+\varphi)\ , \tag{2}$$

where ω is the frequency of the wave, g_l is the amplitude of the exerted acceleration, ω_l is the proper frequency of the oscillator, and τ is its damping time related to the "quality factor" Q by $\tau=Q/\omega_l$.

In order to make contact with the rest of the paper we consider oscillators with various frequencies ordered by an integer number l. We assume that ω may be close to ω_1, but $\omega \ll \omega_l$ for large l. The coefficient g_l is arbitrary for the time being; it will be shown to be proportional to l^{-2} for the lth mode of oscillation in a free extended bar. We begin our analysis with a monochromatic gravitational wave acting on the oscillator. However, we will also give estimates for a short burst of waves by considering the action of the driving force in a finite interval of time $\Delta t=t_f-t_i\approx 2\pi/\omega$.

The general solution to Eq. (2) has the form

$$\xi=e^{-t/\tau}\left(C_1\cos\sqrt{\omega_l^2-\tau^{-2}}\,t+C_2\sin\sqrt{\omega_l^2-\tau^{-2}}\,t\right)-\frac{g_l}{R}\left[(\omega^2-\omega_l^2)\cos(\omega t+\varphi)-\frac{2\omega}{\tau}\sin(\omega t+\varphi)\right]\ , \tag{3}$$

where

$$R\equiv(\omega^2-\omega_l^2)^2+4\omega^2\tau^{-2}$$

and C_1, C_2 are arbitrary constants determined by the initial conditions. The part of solution (3) with constants C_1 and C_2 is the general solution to the homogeneous form of Eq. (2) (with no right-hand side). If initially, at $t=0$, the oscillator was at rest, i.e., $\xi(0)=\dot{\xi}(0)=0$, then

$$C_1=\frac{g_l}{R}\left[(\omega^2-\omega_l^2)\cos\varphi-\frac{2\omega}{\tau}\sin\varphi\right]\ ,$$

$$C_2=\frac{-g_l}{R}\frac{1}{\sqrt{\omega_l^2-\tau^{-2}}}\left[\frac{1}{\tau}(\omega^2+\omega_l^2)\cos\varphi+\omega(\omega^2-\omega_l^2+2\tau^{-2})\sin\varphi\right]\ . \tag{4}$$

The energy of the oscillator is zero initially but by the time t, it will be equal to

$$E(t)=2\tfrac{1}{2}m(\dot{\xi}^2+\omega_l^2\xi^2)\ , \tag{5}$$

where the factor 2 reflects the presence of two masses. To find $E(t)$ one should use solution (3).

Normally, the damping time τ is much larger than the period $T_l=2\pi/\omega_l, \omega_l\tau\gg 1$, and by the time $t\gg\tau$, the part of solution (3) with constants C_1, C_2 decreases significantly and can be neglected, so that only the oscillations at frequency ω persist. The solution to Eq. (2) can also be presented in terms of a Green's function

$$\xi=\mathrm{Re}\left[-\frac{i}{\omega}\int_{-\infty}^{t}e^{i\omega_l(t-t')}e^{-(t-t')/\tau}F(t')dt'\right]\ ,$$

where $F(t)$ stands for the right-hand side of Eq. (2). This is the established regime of the forced oscillations. In this case, the $E(t)$, averaged over a period of oscillations $T=2\pi/\omega$, takes the value

$$\bar{E}=\frac{1}{2}mg_l^2\frac{\omega^2+\omega_l^2}{R}\ . \tag{6}$$

The energy E_d dissipated by the damping force $F_d=(2m/\tau)\dot{\xi}$ during a period T has the value

$$E_d=2\int F_d d\xi=\frac{4m}{\tau}\int_0^T\dot{\xi}^2dt=T\frac{2m}{\tau}\frac{g_l^2}{R}\omega^2 . \quad (7)$$

In the established regime of forced oscillations, the rate of dissipation of energy, dE_d/dt, should be equal to the rate at which energy is absorbed from the gravitational-wave flux I:

$$\frac{dE_d}{dt}=\sigma(\omega)I ,$$

where $\sigma(\omega)$ is the absorption cross section at frequency ω; $\sigma(\omega)$ has the dimensionality of cm^2. For a gravitational wave with the amplitude h and frequency ω, the coefficient $g_l=a_l h\omega^2 L$, where a_l is a constant depending on l, and the energy flux $I=h^2\omega^2c^3/32\pi G$. Thus, one can obtain for $\sigma(\omega)$:

$$\sigma(\omega)=64\pi\frac{G}{c^3}a_l^2\frac{mL^2}{\tau}\frac{\omega^4}{(\omega^2-\omega_l^2)^2+4\omega^2\tau^{-2}} . \quad (8)$$

For $\omega_l\tau\gg 1$, the maximum value of $\sigma(\omega)$ is achieved at $\omega_l\approx\omega$ (the case of resonance). However, this maximum value drops already by a factor of 2 for slightly different frequencies, such that $\delta\omega=\omega-\omega_l\approx\pm 1/\tau$ [8].

Let us consider in more detail the case of resonance: $\omega=\omega_l$, $l=1$, $a_l^2=1/16$. In the established regime of forced oscillations at frequency $\omega_1=\omega$, the energy of the oscillator $E(t)$ is a constant $E=\frac{1}{4}mg_1^2\tau^2$. The rate of dissipation of energy is $dE_d/dt=\frac{1}{2}m\tau g_1^2$, and the absorption cross section at resonance $\sigma(\omega_1)$ is equal to

$$\sigma(\omega_1)=\pi\frac{G}{c^3}mL^2\omega_1^2\tau . \quad (9)$$

For a burst of radiation with a sufficiently broad bandwidth, $\Delta\omega\approx\omega$, the flux of radiation changes little in the vicinity of the resonance peak, that is for frequencies $\omega=\omega_1+\delta\omega$, $\delta\omega=\pm 1/\tau$. One can integrate $\sigma(\omega)$ over the resonance bandwidth [or simply multiply expression (9) by $2\delta\omega$] and then divide the result by ω_1, in order to find the cross section appropriate for bursts of radiation. In this way one obtains, for an oscillator with total mass $M=2m$, the quoted standard expression:

$$\sigma_1=\frac{1}{\omega_1}\sigma(\omega_1)\frac{2}{\tau}=2\pi\frac{G}{c^3}ML^2\omega_1 . \quad (10)$$

Similar to $\sigma(\omega_1)$, the quantity σ_1 has the dimensionality of a surface area but, obviously, $\sigma(\omega_1)$ and σ_1 have a different meaning.

By order of magnitude, formula (10) can be derived directly from solution (3) assuming that the force is applied during a finite interval of time $\Delta t\approx 2\pi/\omega$, and the frequency is of order ω_1 but not necessarily equal to ω_1. In this case one cannot neglect the homogeneous part of solution (3) (with constants C_1, C_2) since all the terms give comparable contributions to ξ and $\dot{\xi}$. One can estimate the energy accumulated in the oscillator by time t_f:

$$\Delta E_1\sim mh^2L^2\omega^2 . \quad (11)$$

This energy is absorbed from a burst carrying

$$U\approx I\Delta t\sim\frac{c^3}{G}h^2\omega$$

ergs per cm^2. From the relationship $\Delta E_1=\sigma_1 U$ one derives the estimate for σ_1 which agrees with expression (10).

Now we will turn to oscillators with proper frequencies $\omega_l\gg\omega$. The characteristics of the gravitational burst are supposed to be the same as above. Our goal is to show that energy ΔE_l stored in an oscillator with proper frequency $\omega_l\gg\omega$ is much smaller than ΔE_1 [expression (11)], i.e., energy stored in the oscillator with $\omega_1\approx\omega$. To do this, one should consider (3) neglecting ω in comparison with ω_l. It can easily be seen that

$$\Delta E_l\sim a_l^2mh^2L^2\omega^2\left[\frac{\omega}{\omega_l}\right]^2 . \quad (12)$$

[It is worth noting that although the kinetic energy associated with forced oscillations is an extra factor $(\omega/\omega_l)^2$ smaller than (12), the potential energy is of the order of (12). The homogeneous terms of solution (3) give a comparable or a smaller contribution to (12).] Thus, expression (12) contains a small factor $(\omega/\omega_l)^2$ in comparison with (11). In addition, as was mentioned above and will be shown in full detail below, for the lth mode of oscillations of an extended body, the factor a_l^2 is l^4 times smaller than a_1^2, which reduces (12) further.

The estimate (12) is based on an analysis of a periodic force acting for a finite interval of time $\Delta t=t_f-t_i$. As can be expected, this leads to a significant rôle for the boundary conditions—the ways in which the force is turned on and turned off. The estimate (12) is rather the maximum estimate for ΔE_l since it includes a case of a sudden turn on of the force at $t=t_i=0$, encoded in an arbitrary phase φ. Or, equivalently, it includes an assumption of a sudden release of the oscillator, previously held at rest, at $t=0$ in the field of a continuous periodic wave. In practice, for a burst with the time scale much longer than the period of proper oscillations, one can expect the opposite—a smooth turning on and turning off of the interaction. This is the case of the adiabatic interaction which is known to cause a much smaller excitation of the oscillator.

We will illustrate the results for the adiabatic case by assuming that the right-hand side of Eq. (2) vanishes at the ends of the interaction interval, that is, at initial $t_i=0$ and at finite t_f, and is zero outside the interaction interval. This leads to the conditions

$$\cos\varphi=0, \quad \sin\omega t_f=0 .$$

For simplicity, we take $\tau\to\infty$ and reduce the exact solution (3) to the form

$$\xi=\pm\frac{g_l}{\omega^2-\omega_l^2}\left[\sin\omega t-\frac{\omega}{\omega_l}\sin\omega_l t\right] .$$

Calculation of the energy of the oscillator gives

$$E(t_f)=2ma_l^2h^2L^2\omega^6\frac{1}{(\omega^2-\omega_l^2)^2}(1\mp\cos\omega_l t_f) . \quad (13)$$

By comparing (13) and (12) one can see that, at most,

$$E_l(t_f)\sim\Delta E_l\left[\frac{\omega}{\omega_l}\right]^2,$$

i.e., $E_l(t_f)$ is a factor $(\omega/\omega_l)^2$ smaller than in the previous case [expression (12)].

In this example the force goes to zero at the ends of the interaction interval but the first time derivative of the force has a finite jump at $t=t_i$ and $t=t_f$. If the first derivative also went to zero at t_i and t_f, the energy of excitation would be smaller than the above formula for $E_l(t_f)$ by the extra factor $(\omega/\omega_l)^2$. In general, the higher the degree of adiabaticity, the smaller the energy of excitation. The cases considered above will be shown to be in full agreement with the quantum-mechanical treatment.

III. EXTENDED ONE-DIMENSIONAL ANTENNA

A real antenna is a solid bar with a complicated crystalline structure. We will approximate this structure by a chain of masses connected by a springlike restoring force. For our purposes it is sufficient to consider a one-dimensional system. This model is often used in the theory of solids (see, for example, Ref. [9]). In the context of gravity-wave research it was recently considered by Thorne [10].

The chain consists of $N+1$ particles and N springs between them, where N is an odd number. The mass of each particle is m and the distance between the neighbors is $a=L/N$. The total mass of the system is $M=m(N+1)$ and the total length is L. The equilibrium position of the nth particle is at $x_n=n(a/2)$, where n is an odd number running from $n=-N$ to $n=N$. The center of the entire system is at $x=0$. The displacement of nth particle from its equilibrium position is denoted by $\xi_n(t)$.

If a is of order of the size of a cell in the crystalline lattice ($a\sim10^{-8}$ cm), then N is a huge number ($N\sim10^{10}$) for any macroscopic antenna ($L\sim10^2$ cm). One can think of particles in this model as plane slabs in a bar antenna.

We assume that the restoring force acting on the nth particle is produced by its neighbors only, that is by the particles $n+2$ and $n-2$, and is proportional to $\xi_{n+2}-\xi_n$ and $\xi_{n-2}-\xi_n$. Therefore, the equation of motion of the nth particle, in the absence of gravitational waves and internal damping, is

$$\ddot{\xi}_n+\omega_D^2(2\xi_n-\xi_{n+2}-\xi_{n-2})=0\ . \tag{14}$$

The general solution to this equation is

$$\xi_n(t)=e^{-i\omega t}[\alpha e^{ikn(a/2)}+\beta e^{-ikn(a/2)}]+e^{i\omega t}[\alpha^* e^{-ikn(a/2)}+\beta^* e^{ikn(a/2)}]\ , \tag{15}$$

where ω and k are related by the equality

$$\omega^2=2\omega_D^2(1-\cos ka) \tag{16}$$

and α, β are arbitrary complex constants. Solution (15) and (16) give the displacements of a set of discrete particles ordered by the number n. However, one can easily go over to the limit of a continuous elastic body $\xi(t,x)$ by considering $x_n=n(a/2)$ as a continuous variable x.

Now we need to impose some boundary conditions at the ends of the chain. Usually, one deals with an elastic bar antenna having zero strain at the ends of the bar, that is $\partial\xi(x)/\partial x=0$ at $x=\pm L/2$. We will impose similar boundary conditions in the case of the chain by requiring $\partial\xi_n/\partial n=0$ at $n=\pm N$. Since chains are not precisely continuous bodies and their descriptions coincide only in the limit of large N, our procedure will require an adjustment of the solution, though negligibly small for $N\gg1$, which we will perform later on. The boundary conditions select a discrete set of frequencies allowed in the chain and reduce the number of arbitrary constants. The conditions we have adopted, as in the case of a continuous body, reduce the general form of solution (15) and (16) to

$$\xi_n(t)=\sum_{l=0}^{N}\xi_{n,l},\quad \xi_{n,l}=A_l\cos(\omega_l t+\varphi_l)\mathrm{cs}\frac{l\pi n}{2N}\ , \tag{17}$$

$$\omega_l^2=2\omega_D^2\left[1-\cos\frac{l\pi}{N}\right],\quad l=0,1,2,\ldots,N\ , \tag{18}$$

where $\mathrm{cs}(l\pi n/2N)$ stands for the function equal to $\cos l\pi n/2N$ if l is an even number, and equal to $\sin l\pi n/2N$, if l is an odd number; A_l, φ_l are arbitrary real constants. Thus, every particle can oscillate at N different nonzero frequencies (18).

The $N+1$ oscillating particles in the chain form a collection of coupled oscillators. For ease of calculation and for subsequent quantization one needs to introduce a set of normal (orthogonal and normalized) modes and, in this way, reduce the dynamical system to a set of decoupled oscillators. The functions $\mathrm{cs}(l\pi n/2N)$ are not quite orthogonal for the reasons mentioned above. This leads to the necessity of a slight modification of solution (17) and (18), negligibly small for $N\gg1$. Instead of the functions $\mathrm{cs}(l\pi n/2N)$ one should use the functions $\mathrm{cs}[l\pi n/2(N+1)]$, i.e., N is replaced by $N+1$. One can check that the following conditions of orthogonality are satisfied:

$$\sum_{n=-N}^{N}\cos\frac{l\pi n}{2(N+1)}\cos\frac{l'\pi n}{2(N+1)}=\frac{1}{2}(N+1)\delta_{ll'}\ ,$$

$$\sum_{n=-N}^{N}\cos\frac{l\pi n}{2(N+1)}\sin\frac{l'\pi n}{2(N+1)}=0\ ,$$

$$\sum_{n=-N}^{N}\sin\frac{l\pi n}{2(N+1)}\sin\frac{l'\pi n}{2(N+1)}=\frac{1}{2}(N+1)\delta_{ll'}\ .$$

After this modification, we present the solution to Eq. (14) in the final form:

$$\xi_n(t)=q_0+\sqrt{2}\sum_{l=1}^{N}q_l(t)\mathrm{cs}\frac{l\pi n}{2(N+1)}\ , \tag{19}$$

$$\omega_l^2=2\omega_D^2\left[1-\cos\frac{l\pi}{N+1}\right]. \tag{20}$$

If one multiplies Eq. (14) by $\mathrm{cs}[l\pi n/2(N+1)]$ and takes the sum over all n from $-N$ to N, one derives the expected equation of free motion for every l mode:

$$\ddot{q}_l + \omega_l^2 q_l = 0 \ . \tag{21}$$

The proper frequencies ω_l satisfy the continuum-model acoustic relationship

$$\omega_l \approx \omega_D \frac{l\pi}{N+1}$$

for $l \ll N$, and approach the maximum value $2\omega_D$ for $l = N$.

The kinetic, T, and potential, W, energies of the chain are equal, by definition, to

$$T = \frac{1}{2} m \sum_{n=-N}^{N} \dot{\xi}^2, \quad W = \frac{1}{2} m \omega_D^2 \sum_{n=-N}^{N-2} (\xi_{n+2} - \xi_n)^2 \ .$$

By substituting (19) into the above definitions, one can derive

$$T = \frac{1}{2} M \sum_{l=1}^{N} \dot{q}_l^2, \quad W = \frac{1}{2} M \sum_{l=1}^{N} \omega_l^2 q_l^2 \ .$$

Thus, the Hamiltonian of the system is given by

$$H = \frac{1}{2M} \sum_{l=1}^{N} p_l^2 + \frac{M}{2} \sum_{l=1}^{N} \omega_l^2 q_l^2 \ , \tag{22}$$

where $p_l = M\dot{q}_l$.

Now we will derive the interaction Hamiltonian H_{int} of the chain which is acted upon by a gravitational wave. The force acting on the nth particle is given by

$$F_n = -\tfrac{1}{2} m h \omega^2 x_n f(t) \ , \tag{23}$$

where $f(t) = \cos(\omega t + \varphi)$ for a monochromatic wave with frequency ω. The H_{int} includes the interaction energy of all the particles in the chain with the wave:

$$H_{\text{int}} = \frac{1}{2} m h \omega^2 f(t) \sum_{n=-N}^{N} x_n \xi_n(t) \ . \tag{24}$$

As one can expect from symmetry considerations, all the even-l terms in (24), containing the factors $\cos[l\pi n/2(N+1)]$, cancel out. The final expression for H_{int} reduces to

$$H_{\text{int}} = \frac{1}{2\sqrt{2}} m h \omega^2 \frac{L}{N} f(t) \times \sum_{l=1,3,\ldots}^{N} (-1)^{(l-1)/2} q_l(t) ctg^2 \frac{l\pi}{2(N+1)} \ . \tag{25}$$

The reasonably accurate approximate form of (25) is

$$H_{\text{int}} \approx FNf(t) \sum_{l=1,3,\ldots}^{N} (-1)^{(l-1)/2} \frac{1}{l^2} q_l(t) \ , \tag{26}$$

where

$$F \equiv \frac{\sqrt{2}}{\pi^2} m h \omega^2 L \ .$$

In a similar way one can derive the equation of motion, which takes into account the driving force produced by the gravitational wave, for every l mode. If one uses (23) on the right-hand side of Eq. (14), multiplies the equation by $\text{cs}[l\pi n/2(N+1)]$, and sums over all n, one sees again that only the odd modes interact with the wave. The equation of motion acquires the form

$$\ddot{q}_l + \omega_l^2 q_l \approx -\sqrt{2} h \omega^2 L f(t) (-1)^{(l-1)/2} \frac{1}{l^2} \ . \tag{27}$$

Thus, the proper high-frequency modes of oscillations in the chain interact very little with the wave as the right-hand side of Eq. (27) is proportional to l^{-2} (compare with the discussion in Sec. II) [11,12].

IV. QUANTUM THEORY OF THE EXTENDED ANTENNA

We can develop now the quantum treatment of the chain of masses. The Hamiltonian (22) describes a set of N decoupled oscillators, each having mass M. The pair q_l, p_l represent the generalized coordinate and momentum for every l. In the quantum treatment, q_l and p_l are operators satisfying the commutation relation $[\hat{q}_l, \hat{p}_{l'}] = i\hbar \delta_{ll'}$. The position operator $\hat{\xi}_n(t)$ for every nth particle can be written as

$$\hat{\xi}_n(t) = \sqrt{2} \sum_{l=1}^{N} \hat{q}_l(t) \text{cs} \frac{l\pi n}{2(N+1)}$$

(we have omitted the zero-frequency term). In the usual manner, one can introduce the creation and annihilation operators b_l^+, b_l according to the rule

$$q_l = \sqrt{\hbar/2\omega_l M}\,(b_l + b_l^\dagger), \quad p_l = -i\sqrt{\hbar M \omega_l/2}\,(b_l - b_l^\dagger) \ .$$

The operators b_l, $b_l^\dagger$ satisfy the commutation relation $[b_l, b_{l'}^\dagger] = \delta_{ll'}$.

The interaction Hamiltonian (26) can be presented in the form

$$H_{\text{int}} = \sum_l \hat{H}_l, \qquad \hat{H}_l \approx FNf(t)(-1)^{(l-1)/2} \frac{1}{l^2} \sqrt{\hbar/2\omega_l M}\,(b_l + b_l^\dagger) \ . \tag{28}$$

We assume that the system is initially in its ground state, i.e., the state vector of the system is a product of the ground-state vectors of all the oscillators. According to first-order perturbation theory, under the action of the perturbation (28), each of the oscillators can only go from its ground state to the first excited state with the energy $\hbar\omega_l$. The transition amplitude is given by the formula

$$C_l(t) \approx -\frac{i}{\hbar} FN (-1)^{(l-1)/2} \frac{1}{l^2} \sqrt{\hbar/2\omega_l M} \times \int_0^t f(t') e^{i\omega_l t'} dt' \ . \tag{29}$$

Note that the $N+1$ particles participating in the process (for every l) give a large coherent factor N in the numerator of this expression. However, for the very same reason, the coherent behavior, the spread of the ground-state wave function $q_l \sim (\hbar/2\omega_l M)^{1/2}$ depends on the total mass of the chain M, not the mass m of an individual particle (slab). This gives a large factor $N^{1/2}$ in the denominator of expression (29).

The probability of transition is defined by

$$|C_l|^2 \approx \frac{1}{2\hbar} f^2 N \frac{1}{m\omega_l} \frac{1}{l^4} |d(t)|^2 , \quad (30)$$

where

$$|d(t)|^2 = \left| \int_0^t f(t') e^{i\omega_l t'} dt' \right|^2 . \quad (31)$$

For a particular form of $f(t)$, namely $f(t) = \cos(\omega t + \varphi)$, the calculation of $|d(t)|^2$ gives

$$|d(t)|^2 = \frac{\sin^2[(\omega_l - \omega)/2]t}{(\omega_l - \omega)^2} + \frac{\sin^2[(\omega_l + \omega)/2]t}{(\omega_l + \omega)^2} + 2\frac{\sin[(\omega_l - \omega)/2]t \sin[(\omega_l + \omega)/2]t}{\omega_l^2 - \omega^2} \cos(\omega + 2\varphi)t . \quad (32)$$

We will begin the analysis of the transition probabilities from the lowest frequency mode $l = 1$. For a monochromatic gravitational wave one can have the case of resonance $\omega \approx \omega_1$. According to (32), the resonance term in $|d|^2$ is

$$|d|^2 \approx \frac{1}{4} \frac{\sin^2[(\omega_l - \omega)/2]t}{[(\omega_l - \omega)/2]^2} . \quad (33)$$

In practice, the unlimited growth of $|d(t)|^2$ with t is restricted by the damping time τ. In a standard way (see, for example, Ref. [13]), one can introduce the transition probability per unit time

$$\frac{\overline{W}}{\tau} = \frac{1}{\tau} \int_0^\infty |C_1|^2 \frac{1}{\tau} e^{-t/\tau} dt . \quad (34)$$

By using (30) and (33) one can calculate (34) and obtain

$$\frac{\overline{W}}{\tau} = \frac{1}{\hbar} F^2 N \frac{1}{m\omega_1} \tau . \quad (35)$$

The rate of absorption of energy is $dE/dt = \hbar\omega_1 \overline{W}/\tau = \sigma I$ where σ is the cross section on resonance and I is the flux of the incoming radiation. By inserting here all the necessary definitions one can derive an expression for $\sigma(\omega_1)$ that is in full agreement with the classical estimate (9). Similar to the classical case, one can also find the cross section σ_1 appropriate for bursts. This quantity does, again, fully correspond to (10).

Of course, one can derive an estimate for σ_1 directly, by considering the interaction Hamiltonian H_1 (28) during a finite interval of time $\Delta t \approx 2\pi/\omega$. In this case, the square of the matrix element $|d|^2$ is of the order of ω_1^{-2}, as follows from (32). The absorbed energy ΔE_1 is $\Delta E_1 = |C_1|^2 \hbar\omega_1$. By combining these formulas one returns to the same estimate (10) for σ_1.

Thus, we see that regardless of whether the antenna is treated as an elementary oscillator with two masses of order of M each, or as a chain of coupled mass m, the classical and quantum-mechanical analyses give the same results.

We should also notice that for the real experimental situation the number of gravitons absorbed by the bar's fundamental mode is much larger than one. Strictly speaking, first-order perturbation theory is not sufficient and one should take into account the higher-order corrections. However, as is well known, this more rigorous treatment results simply in the observation that the actual quantum state of the excited antenna is a coherent state $|\alpha\rangle$ with the complex amplitude

$$\alpha = -\frac{i}{\omega} \int_{-\infty}^t e^{i\omega_l(t-t')} e^{-(t-t')/\tau} F(t') dt' .$$

(Compare with the Green's function form of the classical solution discussed in Sec. II. That form of the solution is the classical version of the quantum coherent-state formula presented here.) Obviously, the conclusions regarding the absorption cross section remain valid.

Now we turn to the high-frequency modes $\omega_l \gg \omega$, where ω_l is of order of ω_D. There are many such modes, and their number is of the same order of magnitude as the total number of the allowed modes, i.e., N. And they all participate in the absorption of radiation. However, we will show that the energy absorbed by all these modes is much smaller than ΔE_1.

In the case of $\omega_l \gg \omega$, the numerical value of $d(t)$ depends significantly on the behavior of $f(t)$ at the ends of the integration interval. If $f(t)$ does not vanish at $t = t_i$ and $t = t_f$, then $|d|^2$ can have the maximum value of order of ω_l^{-2}. However, if $f(t)$ does vanish at the ends of the interval, the integral for $d(t)$ can be evaluated by parts and reduced to

$$d(t) \equiv \int_0^t f(t') e^{i\omega_l t'} dt' = -\frac{1}{i\omega_l} \int_0^t e^{i\omega_l t'} \frac{d}{dt'} f(t') dt' .$$

In this case a typical numerical value of $d(t)$ is of the order of $\omega\omega_l^{-2}$ and $|d|^2 \sim \omega^2 \omega_l^{-4}$. This corresponds to the adiabatic change of the interaction Hamiltonian (see, for example, Ref. [14]). One can illustrate this by considering $f(t) = \cos(\omega t + \varphi)$. If $\cos\varphi = 0$ and $\sin\omega t_f = 0$, the calculation of (32) gives

$$|d(t_f)|^2 = \frac{8\omega^2}{(\omega_l^2 - \omega^2)^2} (1 \pm \cos\omega_l t_f) ,$$

that is $|d|^2 \sim \omega^2 \omega_l^{-4}$. Essentially, this analysis repeats the classical considerations (see Sec. II).

We take ω_l^{-2} as the maximum estimate for $|d|^2$ and use it in the calculation of $\Delta E_l = |C_l|^2 \hbar\omega_l$. One can easily obtain

$$\Delta E_l \sim h^2 M L^2 \omega^2 \frac{1}{l^4} \left[\frac{\omega}{\omega_l} \right]^2$$

which means that $\Delta E_l \sim \Delta E_1 (\omega/\omega_l)^2 (1/l^4)$. In order to find an estimate for the energy ΔE stored in all modes with frequencies of the order of the Debye frequency ω_D, one should take ΔE_l for $\omega_l \approx \omega_D$ and multiply it by N, which is an approximate number of all such modes. The result is disappointingly small:

$$\Delta E = \Delta E_D N \sim \Delta E_1 \frac{1}{N^3} \left(\frac{\omega}{\omega_D} \right)^2 .$$

Thus, the Debye modes can by no means change the amount of energy absorbed by the antenna from a gravitational burst with characteristic frequency ω.

V. DISCUSSION

We have presented classical and quantum-mechanical treatments of a crystalline antenna interacting with gravitational waves. We did not find any reasons for quantum-mechanical enhancement of the antenna's cross section. In view of this result it is instructive to compare our analysis with explicit and implicit statements of papers [5] and [6].

Let us start with papers of Weber [5]. He considers the case of $l=1$, $\omega_1 \approx \omega$. His interaction Hamiltonian is essentially the same as our expression (26) for $l=1$. In our notations, we can write

$$H_1 \sim mh\omega_1^2 LNq_1 .$$

This gives the probability amplitude

$$C_1 \sim \frac{1}{\hbar} mh\omega_1 LNq_1 ,$$

where the factor N is a manifestation of "summation of the probability amplitude over all possible absorber mass elements [5]." However, at the next step, Weber makes a mistake. He uses $q_1 \sim (\hbar/2\omega_1 m)^{1/2}$ instead of a correct value $q_1 \sim (\hbar/2\omega_1 M)^{1/2}$. This gives him an extra factor N, $N \sim 10^{10}$, in the transition probability $|C_1|^2$, in the absorbed energy $\Delta E_1 = |C_1|^2 \hbar\omega_1$ and, finally, in the cross section σ_1. (Weber speaks also of "large damping corrections" which effectively reduce his enhancement factor from 10^{10} to 10^6. A critical discussion of these damping corrections is given in Ref. [10].)

In order to see better (in addition to a direct derivation performed in Sec. IV) why the normalization coefficient should be equal to $(\hbar/2\omega_1 M)^{1/2}$ and not to $(\hbar/2\omega_1 m)^{1/2}$ one can make the following simple calculation. We accept the picture of N particles (slabs) oscillating together at the fundamental frequency ω_1 of the bar. Classically, the total energy of the bar is

$$E \approx m \sum_n \dot{\xi}_n^2 \approx mN\dot{q}_1^2 \approx mN\omega_1^2 q_1^2 .$$

Quantum mechanically, the mean energy of the system is $\langle E \rangle = \bar{n}\hbar\omega_1$ where $\bar{n}$ is the occupation number. From equality of the energies it follows that the classical $q_1^2 \approx \bar{n}\hbar\omega_1/mN\omega_1^2$. On the other hand, the operator $\hat{q}_1$ has the form $\hat{q}_1 = K(b+b^\dagger)$ where K is a constant which we want to determine. The mean number of the operator $\hat{q}_1^2$ is $\langle \hat{q}_1^2 \rangle \approx K^2 \bar{n}$. This quantity should be equal to the classical q_1^2. From their comparison one finds $K \approx (\hbar/mN\omega_1)^{1/2}$, i.e., $\hat{q}_1 \sim (\hbar/\omega_1 M)^{1/2}(b+b^\dagger)$.

Now we will compare our results with those of Preparata [6]. He considers the opposite case of very large l and ω_l of the order of ω_D. In terms of our notations, he calculates $|C_l|^2$. However, in doing this he totally neglects the factor l^{-4} representing the boundary conditions, and, like Weber, uses $q_l \sim (\hbar/2\omega_l m)^{1/2}$ instead of $q_l \sim (\hbar/2\omega_l M)^{1/2}$. On the other hand, when calculating ΔE, the energy absorbed and stored in the Debye oscillations, he does not multiply ΔE_D by the number of modes N. This procedure, through the definition $\Delta E = \sigma_D U$, would lead to the cross section $\sigma_D \approx (G/c^3) ML^2 \omega_1 N(\omega/\omega_D)^2$ showing (if calculated correctly) how much energy is stored in the Debye modes. This cross section would be irrelevant for operating antennas since their instrumentation monitors the fundamental-mode displacements or strains. However, Preparata multiples σ_D by the factor $\Delta\omega/\omega$, $\Delta\omega/\omega \sim 10^{-3}$, which he associates with the bandwidth of the antenna. He claims that the resulting expression

$$\sigma_{\rm RB} \approx \frac{G}{c^3} ML^2 \omega_1 N \left(\frac{\Delta\omega}{\omega} \right) \left(\frac{\omega}{\omega_D} \right)^2$$

gives a cross section applicable for calculating the energy stored in the fundamental mode of the resonating bar. It is worth emphasizing that Preparata does not propose any mechanism at all for transferring energy from the Debye modes to the fundamental mode but simply asserts that a portion $\sim \Delta\omega/\omega$ of the energy gets transferred. This expression, even without the factor l^{-4}, is still 10^{13} times smaller than σ_1. However, Preparata substitutes in this expression $N \approx 3 \times 10^{29}$, instead of $N \approx 10^{10}$ which makes the $\sigma_{\rm RB}$ six orders of magnitude larger than σ_1. This last calculation is equivalent to the assumption that in a three-dimensional lattice, with the total volume $\sim L^3$, all the Debye modes available are equally sensitive to the impinging gravitational wave. This assumption, which ignores all the selection rules, can hardly be true. It is obvious, for example, that the modes corresponding to the oscillations along the propagation direction of the wave do not interact with the wave and do not contribute to the cross section. Thus, we see the origin of Preparata's claim of the quantum-mechanical enhancement of the cross section σ_1 but we cannot accept his logic and disagree with his calculations.

In addition, it is necessary to make a comment with regard to the experimental problem arising in connection with the Weber and Preparata statements. If the absorption cross section of a bar antenna were as large as Weber and Preparata suggest, it would likely be applicable to the calibrating signals applied to the antenna. However, all known laboratory measurements of $\sigma(\omega_1)$ are in agreement with the old value of $\sigma(\omega_1)$, not the new one. Any attempts to reconcile the laboratory measurements with the proposed enhanced cross section seem to lead to inconsistencies [10].

ACKNOWLEDGMENTS

I am grateful to Kip S. Thorne for valuable discussions and suggestions, and to the Joint Institute for Laboratory Astrophysics of the National Institute of Standards and Technology and University of Colorado for financial support.

[1] W. Hamilton, invited talk, in *Sixth Marcel Grossmann Meeting on General Relativity,* Proceedings, Kyoto, Japan, 1991, edited by T. Nakamura (World Scientific, Singapore, 1992).

[2] C. W. Misner, K. S. Thorne, and J. A. Wheeler, *Gravitation* (Freeman, San Francisco, 1973).

[3] R. Ruffini and J. A Wheeler, in *Relativistic Cosmology and Space Platforms,* Proceedings of the Conference on Space Physics (ESRO, Paris, France, 1971).

[4] J. Weber, Phys. Rev. **117**, 306 (1960).

[5] J. Weber, Found. Phys. **14**, 1185 (1984); in *Gravitational Radiation and Relativity,* Proceedings of the Sir Arthur Eddington Centenary Symposium, Nagpur, India, 1984, edited by J. Weber and T. M. Karade (World Scientific, Singapore, 1986), Vol. 3; in *Recent Advances in General Relativity,* edited by A. Janis and J. Porter (Birkhauser, Boston, 1991).

[6] G. Preparata, Nuovo Cimento **101**, 625 (1988); Mod. Phys. Lett. A **5**, 1 (1990).

[7] M. Aglietta *et al.*, Nuovo Cimento C **12**, 75 (1989).

[8] By the way, this consideration does not imply that a mechanical oscillator is necessarily a narrow-band detector sensitive to a narrow interval of frequencies only. The sensitivity of a detector is determined by the signal-to-noise ratio. Off the resonance peak, the response of the oscillator is much smaller but if the noise is weak the signal can still be detected.

[9] Ch. Kittel, *Introduction to Solid State Physics* (Wiley, New York, 1966); H. Poulet and J. P. Mathieu, *Vibration Spectra and Symmetry of Crystals* (Gordon and Breach, New York, 1976); B. D. Bartolo, *Optical Interactions in Solids* (Wiley, New York, 1968).

[10] K. S. Thorne, in *Recent Advances in General Relativity [5].*

[11] It is worth mentioning that the factor l^{-2} in the equation of motion (27) and in the interaction Hamiltonian (26) is associated with the boundary conditions of "free ends" adopted in our calculations. In the theory of solids, the periodic boundary conditions are more often used. They would lead to the factor l^{-1}, instead of l^{-2}, in (27) and (26) and to subsequent modification of the cross section. For situations where the volume effects are more important than the effects on boundaries, the periodic boundary conditions are more appropriate. See, e.g., [12], where the periodic boundary conditions have been used in order to calculate the cross section of a crystal interacting with the very-high-frequency part of relic gravitational waves, $\nu \approx 10^{10}-10^{11}$ Hz.

[12] L. P. Grishchuk and A. S. Il'chenko, in *Abstracts of Sixth Soviet Gravitational Conference* (in Russian) (Moscow State Pedagogical Institute, Moscow, 1984), p. 303; L. P. Grishchuk, Usp. Fiz. Nauk **156**, 297 (1988) [Sov. Phys. Usp. **31**, 940 (1988)].

[13] R. H. Dicke and J. P. Wittke, *Introduction to Quantum Mechanics* (Addison, Reading, MA, 1960).

[14] A. S. Davydov, *Quantum Mechanics* (translated from the Russian) (NEO, Ann Arbor, MI, 1966).