# Strong $CP$ violation and the neutron electric dipole moment reexamined

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The relation between the $CP$-violating operator $\theta F\tilde{F}$ and its effective version $L_{CP}^{\text{eff}}$ is reconsidered on the basis of the anomalous Ward-Takahashi (WT) identity. The consistency of the previous phenomenological calculations of the neutron electric dipole moment (NEDM) with the WT identity is critically examined. We demonstrate a consistent evaluation of the $O(N_c^0)$ contribution to the NEDM and the result is compared with the leading term in the chiral expansion which is $O(N_c^{-1})$.

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## I. INTRODUCTION

The absence of the $\theta F\tilde{F}$ term is one of the longstanding problems in QCD, which has stimulated the search for a rationale such as the Peccei-Quinn symmetry [1] and its variations (see the review, Ref. [2]). Experimentally, the most stringent bound on $\theta$ comes from the measurement of the neutron electric dipole moment [3] (NEDM), which should vanish if there is no $CP$ violation. (For a recent review of the general electric dipole moments, see Ref. [4].) On the theoretical side, the precise calculation of the NEDM is rather difficult since it requires the nonperturbative evaluation of the nucleon matrix element of $F\tilde{F}$. Instead of taking such a matrix element, it has been proposed to replace $F\tilde{F}$ by an effective $CP$-violating operator $L_{CP}^{\text{eff}}$ written only by the quark fields [5], which allows us to apply the current algebra techniques and the effective Lagrangians of QCD at low energies. So far, a number of papers have been published based on this idea. (See the reviews in Refs. [2,6].) One of the problems of these approaches is that it is not clear whether the results and the approximations adopted are consistent with the anomalous Ward-Takahashi (WT) identity. Actually, some of the previous calculations break the constraint from the WT identity as we will show.

The purpose of this paper is to clarify the above issue in view of a general principle and also to point out the existence of a consistent tree-level contribution to the NEDM [7]. First, on the basis of the WT identity, we generalize and develop the argument by Aoki, Gocksch, Manohar, and Sharpe [8], where the danger in the naive use of $L_{CP}^{\text{eff}}$ instead of $F\tilde{F}$ is pointed out. The analysis is useful to clarify problematic points in previous calculations of the NEDM. Then, in Sec. III, we reanalyze the NEDM based on an effective Lagrangian that is consistent with the WT identity. A contribution, which is leading in the large-$N_c$ expansion (but not leading in the chiral perturbation), is estimated and compared to the term in the opposite limit (i.e., the term which is leading near the chiral limit but subleading in large $N_c$). Section IV is devoted to the summary and concluding remarks.

## II. ANOMALOUS WT IDENTITY

We first assume the following anomalous WT identity for QCD without a $\theta$ term:

$$\left\langle \left[\partial^\mu J_\mu^5(x) - 2m\bar{\psi}i\gamma_5\psi(x) - N_f\frac{g^2}{16\pi^2}F_a^{\mu\nu}\tilde{F}^a_{\mu\nu}(x)\right]\mathcal{O}\right\rangle + \langle \delta_x \mathcal{O}\rangle = 0 , \quad (2.1)$$

where $J_\mu^5$ is the (flavor-singlet) axial-vector current, $m$ is the quark mass [9], $g$ is the QCD coupling constant, $N_f$ is the number of flavors, $\tilde{F}_{\mu\nu} = \frac{1}{2}\epsilon_{\mu\nu\alpha\beta}F^{\alpha\beta}$ with $\epsilon_{0123} = 1$, and $\delta_x$ is the (infinitesimal) variation under the local chiral rotation at site $x$, $\mathcal{O}$ is an arbitrary operator, and $\langle\cdot\rangle$ denotes the average over all the configuration of quarks and gluons [10]. If one recalls the derivation of the axial anomaly [11], it is easy to see that the above WT identity indeed implies the following two independent equations when $\mathcal{O}$ contains the quark operators:

$$\langle [\partial^\mu J_\mu^5(x) - 2m\bar{\psi}i\gamma_5\psi(x)]\mathcal{O}\rangle_{\text{dis}} = N_f\frac{g^2}{16\pi^2}\langle F_a^{\mu\nu}\tilde{F}^a_{\mu\nu}(x)\mathcal{O}\rangle , \quad (2.2)$$

$$\langle [\partial^\mu J_\mu^5(x) - 2m\bar{\psi}i\gamma_5\psi(x)]\mathcal{O}\rangle_{\text{con}} = -\langle \delta_x\mathcal{O}\rangle , \quad (2.3)$$

where we have decomposed Eq. (2.1) into the disconnected and connected pieces following Aoki, Gocksch, Manohar, and Sharpe [8]; $\langle A\mathcal{O}\rangle_{\text{dis}}$ denotes the insertion

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of $A$ into the quark loop, and $\langle A\mathcal{O}\rangle_{\rm con}$ denotes the insertion of $A$ into the operator $\mathcal{O}$. The well-known triangle diagram, which produces the axial anomaly, is a former example. In other words, Eq. (2.2) means that if we connect the fermion and antifermion operators in $A$ to create the fermion loop, the sum of $\partial^\mu J^5_\mu$ and $2m\bar{\psi}i\gamma_5\psi$ produces the anomaly at the lowest order ($g^2$), and the radiative corrections to these diagrams are always canceled between $\partial^\mu J^5_\mu$ and $2m\bar{\psi}i\gamma_5\psi$ (Adler-Bardeen theorem [12]). Equation (2.3) says that the sum of the insertions of $\partial^\mu J^5_\mu$ and $2m\bar{\psi}i\gamma_5\psi$ into the fermion line starting from or ending at the operator $\mathcal{O}$ is always equal to the local chiral variation of the operator $\mathcal{O}$.

Now we consider the space-time integration of the anomalous WT identity Eqs. (2.2–2.3). Since there are no massless particles that couple to the axial-vector current $J^5_\mu$ as long as $m\neq 0$, we can safely use integration by parts:

$$\int d^4x\,\langle\partial^\mu J^5_\mu(x)\mathcal{O}\rangle_{\rm dis}=\int d^4x\,\langle\partial^\mu J^5_\mu(x)\mathcal{O}\rangle_{\rm con}=0\ . \qquad (2.4)$$

Thus we obtain

$$\int d^4x\; 2m\langle\bar{\psi}i\gamma_5\psi(x)\mathcal{O}\rangle_{\rm dis}=-2N_f\langle Q\mathcal{O}\rangle\ , \qquad (2.5)$$

$$\int d^4x\; 2m\langle\bar{\psi}i\gamma_5\psi(x)\mathcal{O}\rangle_{\rm con}=\int d^4x\,\langle\delta_x\mathcal{O}\rangle\ , \qquad (2.6)$$

where the topological charge $Q$ is defined as

$$Q=\frac{g^2}{32\pi^2}\int d^4x\, F^{\mu\nu}_a\tilde{F}^{\,a}_{\mu\nu}(x)\equiv\int d^4x\, q(x)\ . \qquad (2.7)$$

Equation (2.5) implies that the *disconnected* insertion of $\int d^4x\; m\bar{\psi}i\gamma_5\psi$ is identical to the insertion of $-N_fQ$.

One can use the above relations to evaluate the effect of small $\theta$ on the physical quantities:

$$\langle\mathcal{O}_\theta\rangle=\langle\mathcal{O}\exp(-i\theta Q)\rangle\simeq\langle\mathcal{O}\rangle-i\theta\langle Q\mathcal{O}\rangle=\langle\mathcal{O}\rangle+i\theta\frac{m}{N_F}\int d^4{}_x\langle\bar{\psi}i\gamma_5\psi(x)\mathcal{O}\rangle_{\rm dis}$$

$$\simeq\left\langle\mathcal{O}\exp\left[i\theta\frac{m}{N_f}\int\bar{\psi}i\gamma_5\psi\right]\right\rangle_{\rm dis}\simeq\left\langle\mathcal{O}\exp\left[im\int\bar{\psi}(e^{i\theta\gamma_5/N_f}-1)\psi\right]\right\rangle_{\rm dis}\ . \qquad (2.8)$$

Equation (2.8) represents the precise meaning of the usual statement that the effect of $-i\theta Q$ can be replaced by the complex fermion mass term. (Note that one can prove this equivalence even if $\theta$ is not infinitesimal.) However one should remark that the *connected* insertion of $\int d^4x\; 2m\bar{\psi}i\gamma_5\psi(x)$ has nothing to do with the $\theta$ term as Eq. (2.6) tells us; this produces only the chiral variation of the external fermions in the operator $\mathcal{O}$. This chiral rotation cannot produce any physical effect as the equivalence theorem [13] tells us (we show it explicitly in the last part of this section), which means that *the connected insertions are essentially zero for the physical process* once the on-shell conditions are imposed on the external lines. Note that our argument here using the anomalous WT identity is applicable only to the first-order effect of $\theta$. The advantage to assuming small $\theta$ (which is actually true in the real world) is that one can treat the $\theta$ term by the first-order perturbation theory and not have to worry about the change of the hadron propagators in the $\theta$ vacuum. For general $\theta$, one has to calculate the two-point functions, e.g., the $\theta$ dependence of the nucleon propagator to extract the neutron electric dipole moments [8].

It will be useful here for the phenomenological evaluation of the NEDM to comment on the possible source of the violation of the above constraint. If the *connected* insertion of $\int d^4x\; 2m\bar{\psi}i\gamma_5\psi(x)$ is considered and the finite results are obtained for the NEDM, one can imagine several possible origins of such a fake result. (i) On the lattice, the violation of the WT identity by the finite lattice spacing may cause the fake result [14,8]. (ii) If the chiral-noninvariant intermediate states were used to calculate the NEDM, it would break the equality (2.6), and a nonzero result would be obtained. In fact, there always exist contributions from the other intermediate states which ensure the chiral invariance and tend to cancel the fake contribution. (Note that the identity operator $1=\sum_n|n\rangle\langle n|$ is chiral invariant.) (iii) If the correct one-shell condition for the nucleon is not imposed (which often occurs when one uses the extended models of the nucleon that do not have an exact covariance), a fake contribution remains since the right-hand side (RHS) of Eq. (2.6) does not vanish.

Here let us briefly see what happens when one takes the chiral limit ($m\rightarrow 0$). We are not allowed to take the limit separately in Eqs. (2.5) and (2.6) since each matrix element can have a massless pole, while it does not appear in the sum [15]

$$\int d^4x\; 2m\langle\bar{\psi}i\gamma_5\psi(x)\mathcal{O}\rangle=-2N_f\langle Q\mathcal{O}\rangle+\int d^4x\,\langle\delta_x\mathcal{O}\rangle\ . \qquad (2.9)$$

By taking the chiral limit in this formula, one gets

$$2N_f\langle Q\mathcal{O}\rangle=\int d^4x\,\langle\delta_x\mathcal{O}\rangle\ , \qquad (2.10)$$

which is a well-known relation telling us that the effect of the $\theta$ term is equivalent to the chiral rotation with the angle $-\theta/2N_f$ in the chiral limit. However, it does not imply that there is no effect of $\theta$ in the matrix elements. In fact, off-shell matrix elements may be nonzero for the chiral noninvariant operators, e.g.,

$$\langle \bar{\psi} i\gamma_5 \psi(y)\rangle_\theta \propto \int d^4x \langle \delta_x(\bar{\psi} i\gamma_5\psi(y))\rangle$$
$$= -2\langle \bar{\psi}\psi(y)\rangle \neq 0 \ . \quad (2.11)$$

Of course, as we mentioned and will show soon, the *on-shell* quantities for $\int d^4x\, \delta_x \mathcal{O}$ are always zero, which implies that one cannot observe the physical effect of $\theta$ in the chiral limit. This gives a constraint on the structures of the low-energy effective theory of QCD; the terms induced by the $U_A(1)$ anomaly of QCD should satisfy, e.g., (2.10), which is violated in some previous calculations of the NEDM (e.g., Ref. [16]). We will return to this point in Sec. III.

Now we give a brief proof of the vanishing of $\delta\mathcal{O}$ for on-shell physical quantities as we promised [17]. [We have defined $\delta = \int d^4x\, \delta_x$ as the U(1) global chiral rotation.] Let us consider the following general operator:

$$\mathcal{O} = \prod_{i=1}^{l} B_i \prod_{j=1}^{m} F_j \prod_{k=1}^{n} \bar{F}_k \ , \quad (2.12)$$

where $B_i$ is a bosonic composite operator made of quarks, which produces a physical meson with mass $M_i$, and $F_j$ ($\bar{F}_j$) is a fermionic composite operator made of quarks, which produces a physical fermion (antifermion) with mass $m_j$ [18]. For example, $B_i = \bar{\psi}\gamma_5\psi$ or $\bar{\psi}\psi$, $F_j = \psi\psi\psi$ and $\bar{F}_j = \bar{\psi}\bar{\psi}\bar{\psi}$ [19].

The on-shell amplitude $S(\mathcal{O})$ is proportional to the pole residue $T(\mathcal{O})$, which is obtained from $\langle \mathcal{O}\rangle$ by cutting the external legs and multiplying the external wave functions with proper on-shell conditions according to the reduction formula.

Now we will prove that $S(\delta\mathcal{O})=0$ holds provided that the external momenta satisfy the *same* on-shell conditions as those of $S(\mathcal{O})$. From Eq. (2.12),

$$\delta\mathcal{O} = \sum_i \delta B_i \mathcal{O}_{B_i} + \sum_j \delta F_j \mathcal{O}_{F_j} + \sum_k \delta \bar{F}_k \mathcal{O}_{\bar{F}_k} \ , \quad (2.13)$$

where $\mathcal{O}_A$ is defined as $\mathcal{O} = A\mathcal{O}_A$ with $A = B_i, F_j$, or $\bar{F}_k$. The $U_A(1)$ chiral rotation among the fields $B_i$, $F_j$, and $\bar{F}_k$ can be written using the $c$-number matrices $C$, $D$, and $D'$: $\delta B_i = \sum_{j\neq i} C_{ij} B_j$, $\delta F_i = \gamma_5 D_{ii} F_i + \sum_{j\neq i} D_{ij} F_j$, $\delta\bar{F}_i = D'_{ii}\bar{F}_i\gamma_5 + \sum_{j\neq i} D'_{ij}\bar{F}_j$. The nonzero off-diagonal matrix elements of $C$, $D$, and $D'$ mean the presence of the chiral multiplet. The explicit construction of these operators for the nucleon can be seen in Ref. [20].

To prove $S(\mathcal{O})=0$, it is enough to prove $S(\delta A\mathcal{O}_A)=0$ for an arbitrary $A = B_i$, $F_j$, or $\bar{F}_k$. If the chiral symmetry is not broken, $S(\delta A \mathcal{O}_A)$ vanishes trivially since $\langle \mathcal{O}(A)\rangle = \langle \mathcal{O}(A+\delta A)\rangle$. If the chiral symmetry is (explicitly or spontaneously) broken, the masses of the particles in the chiral multiplet become different, i.e., $M_i \neq M_j$ and $m_i \neq m_j$ for $i\neq j$ [21]. On the other hand, the infinitesimal chiral rotation turns the original operator into the one creating a particle with different mass or into the same particle with extra $\gamma_5$. Therefore it is easy to see that the on-shell condition for the original particle kills the matrix element $S(\delta\mathcal{O})$, e.g.,

$$S(\delta B_i \mathcal{O}_{B_i}) \sim \sum_{i\neq j} C_{ij} \frac{p_i^2 - M_i^2}{p_i^2 - M_j^2} T(B_j \mathcal{O}_{B_i}) = 0 \ , \quad (2.14)$$

for the boson operators. (The same thing holds for fermions if one notes $\{\gamma_5, \gamma_\mu\}=0$.)

Thus, $S(\delta A\mathcal{O}_A)=0$ for all $A$ irrespective of the realization of the chiral symmetry, which completes the proof. As we mentioned before, the assumption of small $\theta$ makes it possible to use the on-shell conditions at $\theta=0$ in the proof.

## III. NEUTRON ELECTRIC DIPOLE MOMENT

The measurement of the neutron electric dipole moment (NEDM) gives the most stringent bound on the $\theta$ parameter at present. To extract a bound on $\theta$ from experiment, we need to calculate the following matrix element in a reliable way:

$$i\int d^4x \langle n(p')|T[J_\mu(0) L_{CP}(x)]|n(p)\rangle$$
$$= -D_n \bar{u}(p')\sigma_{\mu\nu}\gamma_5 k^\nu u(p) \ , \quad (3.1)$$

where $L_{CP}(x) = -\theta q(x)$ with $q(x)$ being the topological charge density. (We have already used the first-order perturbation for $\theta$.) Various authors have calculated the LHS of Eq. (3.1) using the low-energy effective theories of QCD (see the review, Ref. [2]) or the numerical simulation on the lattice [14]. As discussed in Sec. II, however, one should keep in mind that only the disconnected part has to be taken into account in the calculation when one uses the effective operator $L_{CP}^{\text{eff}}$ in place of $-\theta q$.

In the following, we give a direct estimate of the LHS of Eq. (3.1), i.e., the matrix element of $-\theta Q$ using an effective meson-baryon Lagrangian. We will first calculate $D_n$ in the two-flavor case to show the essential points. Then we examine the three-flavor case and show that there exists a tree-level $O(N_c^0)$ contribution to the NEDM which has not been addressed so far. Essential constraints from the WT identity discussed in Sec. II are (i) the vanishing of the matrix element in the chiral limit and/or in the absence of the axial anomaly [22] and (ii) the independence of the result under the change of variables.

The meson part of the low-energy effective action inspired by the large-$N_c$ QCD reads [23]

$$\mathcal{L}_M(U,\theta) = \frac{f_\pi^2}{4}\text{Tr}(\partial^\mu U \partial_\mu U^\dagger) + \frac{f_\pi^2}{4}\text{Tr}(MU + M^\dagger U^\dagger)$$
$$+ \frac{i}{2} q(x)\text{Tr}(\ln U - \ln U^\dagger)$$
$$+ \frac{1}{a f_\pi^2} q^2(x) - \theta q(x) \ , \quad (3.2)$$

where $f_\pi = 93$ MeV, $M$ is the explicit chiral-symmetry breaking [$M = \text{diag}(m_\pi^2, m_\pi^2)$ for flavor SU(2) and $M = \text{diag}(m_\pi^2, m_\pi^2, 2m_K^2 - m_\pi^2)$ for SU(3)], $a$ is a constant that controls the strength of anomaly, and $U$ is a chiral field defined by

$$U = \exp(i\phi^a \lambda^a / f_\pi) \ , \quad (3.3)$$

with $\lambda^a$ ($a = 0 \sim N_f$) being the flavor matrices normalized as $\text{Tr}(\lambda^a\lambda^b) = 2\delta_{ab}$. $\mathcal{L}_M$ has $U_L(N_f)\otimes U_R(N_f)$ invariance except for the explicit mass terms in the large-$N_c$ limit.

The breaking of the $U_A(1)$ symmetry due to the anomaly, which occurs in the next leading order of the large-$N_c$ expansion, is summarized in the third term of Eq. (3.2) representing the mixing of $F\tilde{F}$ with $\phi^0$. In the $N_f=2$ case ($\phi^{1,2,3}=\pi^{1,2,3}$, $\phi^0=\eta$), the equation of motion at the tree level [24] yields the partially conserved U(1) current ($\mathrm{PCU_1C}$) relation [25]

$$q(x)=af_\pi\eta(x)\ , \tag{3.4}$$

and the parameter $a$ is written as

$$a=\tfrac{1}{2}(m_\eta^2-m_\pi^2)\ , \tag{3.5}$$

which is $O(1/N_c)$ and vanishes in the absence of anomaly.

Before considering the baryon sector, we will examine an implication of the anomalous WT identity in this effective action. By changing the variables as $\eta\to\eta-f_\pi\theta/2$, we can eliminate $-\theta q(x)$ in (3.2) and get

$$\begin{aligned}\mathcal{L}_M(Ue^{-i\theta\lambda^0/2},0)=&\frac{f_\pi^2}{4}\mathrm{Tr}(\partial^\mu U\partial_\mu U^\dagger)\\&+\frac{f_\pi^2}{4}\mathrm{Tr}(MUe^{-i\theta\lambda^0/2}+M^\dagger e^{i\theta\lambda^0/2}U^\dagger)\\&+\frac{i}{2}q(x)\mathrm{Tr}(\ln U-\ln U^\dagger)\\&+\frac{1}{af_\pi^2}q^2(x)\ ,\end{aligned} \tag{3.6}$$

where $\theta$ appears only in the explicit symmetry-breaking term corresponding to the operator $m\bar{\psi}e^{-i\theta\gamma_5/2}\psi$ in the quark level. Both effective actions $\mathcal{L}_M(Ue^{-i\theta\lambda^0/2},0)$ and $\mathcal{L}_M(U,\theta)$ should give the same result since the charge of variables does not affect the physical quantities [13]. Therefore we can conclude immediately that the *CP* violation in the physical quantities should vanish in the absence of anomaly ($a=0$) and/or in the chiral limit ($m_\pi^2=0$).

Let us demonstrate this explicitly in the case of the $\eta\to2\pi$ decay. From (3.3) and (3.4), the matrix element for this process is given by the zero-momentum insertion of $-\theta q(x)=-\theta f_\pi a\eta(x)$,

$$-i\theta f_\pi a\int d^4x\langle\eta|\eta(x)|\pi^+\pi^-\rangle\ , \tag{3.7}$$

at the tree level, while from (3.6), we obtain

$$i\theta\frac{f_\pi}{2}m_\pi^2\int d^4x\left[\langle\eta|\eta(x)|\pi^+\pi^-\rangle-\frac{1}{f_\pi^2}\langle\eta|\eta\pi^+\pi^-(x)|\pi^+\pi^-\rangle\right]. \tag{3.8}$$

Since, at the tree level,

$$\int d^4x\langle\eta|\eta|\pi^+\pi^-\rangle=\frac{m_\pi^2}{f_\pi^2m_\eta^2}$$

and

$$\int d^4x\langle\eta|\eta\pi^+\pi^-|\pi^+\pi^-\rangle=1\ ,$$

both (3.7) and (3.8) give the same result

$$i\theta\frac{a}{f_\pi}\frac{m_\pi^2}{m_\eta^2}\ , \tag{3.9}$$

which has the correct property mentioned above. If one takes the approximation $m_\pi/m_\eta\ll1$, which may be regarded as the large-anomaly limit ($a\to\infty$) or the chiral perturbation for $m_\pi$ (but not for $m_\eta$), (3.9) reduces to $i\theta m_\pi^2/2f_\pi$ being consistent with the result of Ref. [26]. Therefore, the formula (3.9) has the correct behavior both for $a\to0$ and $a\to\infty$.

If there exists a small isospin-breaking term,

$$\frac{f_\pi^2}{4}\delta m^2\mathrm{Tr}[\lambda^3(U+U^\dagger)]\ ,$$

it causes the mixing between $\eta$ and the neutral pion ($\pi^0$). Then the matrix element is modified as

$$-i\theta f_\pi a\int d^4x(\cos\phi\langle\eta|\eta|\pi^+\pi^-\rangle-\sin\phi\langle\eta|\pi^0|\pi^+\pi^-\rangle)\ , \tag{3.10}$$

where

$$\int d^4x\langle\eta|\eta|\pi^+\pi^-\rangle=\frac{(m_\eta^2-m_\pi^2)2\sin^2\phi+m_\pi^2(1+2\cos^2\phi)+\delta m^2\sin2\phi}{3f_\pi^2m_\eta^2}$$

and

$$\int d^4x\langle\eta|\pi^0|\pi^+\pi^-\rangle=\frac{(m_\eta^2-m_\pi^2)\sin2\phi-m_\pi^2\sin2\phi+\delta m^2\cos2\phi}{3f_\pi^2m_{\pi^0}^2}\ .$$

Here $m_\pi$ is the charged-pion mass and the mixing angle $\phi$ satisfies $\tan2\phi=\delta m^2/a$ with $m_\eta^2=m_\pi^2+a+\sqrt{a^2+(\delta m^2)^2}$ and $m_{\pi^0}^2=m_\pi^2+a-\sqrt{a^2+(\delta m^2)^2}$. [Therefore the quantity $a$ is expressed in terms of the meson masses: $a=(m_\eta^2+m_{\pi^0}^2-2m_\pi^2)/2$.] It is also easy to check that both (3.2) and (3.6) give the same result.

The baryon sector in $N_f=2$ is written in terms of the baryon field transforming as a fundamental representation of SU(2):

$$\begin{aligned}\mathcal{L}_B(B,U)=&\bar{B}(i\gamma\cdot\partial-\alpha U_5^\dagger)B-\gamma\bar{B}MB\\&+\bar{B}\mu\sigma_{\mu\nu}U_5^\dagger B(\partial^\nu A^\mu)\ .\end{aligned} \tag{3.11}$$

Here the first term is manifestly chiral invariant since $U_5=\exp(i\gamma_5\phi^a\lambda^a/f_\pi)$ with $\lambda^a=(1,\boldsymbol{\lambda})$; the second term breaks U(2) in the same way with the meson sector, and the last term denotes the phenomenological (anomalous) magnetic moment of the baryon with $\mu$ being a $2\times2$ matrix and $A_\mu$ being the photon field. A few comments are in order. First, we have constructed this effective action to make $\mathcal{L}_B$ $U_A(1)$ invariant. [See $U_5^\dagger$ in the last term. It can be $U_5^\dagger=\exp(-i\gamma_5\phi^0\lambda^0/f_\pi)$, but the difference does not matter in the tree-level calculation of the NEDM.] This is because the electromagnetism does not break $U_A(1)$ although it breaks SU(2). [In principle, one can add terms that break $U_A(1)$ but vanish in the chiral limit. Here we have neglected these terms to make the argument simple. For the realistic $SU_f(3)$ case, see the later discussion.] Second, we did not introduce the usual vector coupling $A_\mu[\bar{B}\gamma_\mu(\lambda_3+1)B]/2$ since it does not contribute to the NEDM at the tree level because of its vector nature.

Let us evaluate the matrix element (3.1) in two different ways that will clarify how the general discussion in Sec. II is realized and also what is wrong in some of the previous calculations (in particular Ref. [16]). Here we use (3.2) as an effective action for mesons. If we expand $\mathcal{L}_B$ up to the first order of the meson fields, we get

$$\mathcal{L}_B^{\rm I}=\bar{B}(i\gamma\cdot\partial-m_N)B+\bar{B}\mu\sigma_{\mu\nu}B(\partial^\nu A^\mu)+g_s\bar{B}i\gamma_5\lambda^a B\phi^a -\bar{B}\mu\sigma_{\mu\nu}i\gamma_5\lambda^a B(\phi^a/f_\pi)(\partial^\nu A^\mu)\ , \qquad (3.12)$$

where $g_s=\alpha/f_\pi$ and $m_N=\alpha+\gamma m_\pi^2$ are the pion-nucleon coupling constants and the physical nucleon mass, respectively. Another form of $\mathcal{L}_B$ is obtained by defining the chiral-invariant nucleon $N=\sqrt{U_5^\dagger}B$:

$$\mathcal{L}_B^{\rm II}=\bar{N}(i\gamma\cdot\partial-m_N)N+\bar{N}(\sqrt{U_5^\dagger}\,i\gamma_\mu\partial^\mu\sqrt{U_5})N -g_w\bar{N}\lambda^a i\gamma_5 N\phi^a+\bar{N}\mu\sigma_{\mu\nu}N(\partial^\nu A^\mu)\ , \qquad (3.13)$$

where $g_w=\gamma m_\pi^2/f_\pi$. One should note that this change of variables does not alter the final result [13]. If one uses (3.12) (case I), we have a pseudoscalar (PS) nucleon-meson coupling and an additional contact photon-meson-baryon coupling. On the other hand, in (3.13) (case II), we have a PS-coupling from the explicit symmetry breaking and a pseudovector (PV) coupling from the invariant term.

Our $CP$-violating operator $L_{CP}=-\theta q$ is written by the gluon field carrying zero momentum. Therefore the PV coupling does not contribute at the tree level and only two diagrams shown in Fig. 1 are relevant in case II. The explicit evaluation of them gives

$$D_n=\theta(af_\pi)\frac{1}{m_\eta^2}\frac{1}{m_N}g_w\mu_n\ . \qquad (3.14)$$

The meaning of each term in $D_n$ is self-evident. $af_\pi$ denotes the coupling strength of $q(x)$ to the physical $\eta$ meson. Rewriting it by the physical quantities, we obtain

$$D_n=\theta\frac{m_\eta^2-m_\pi^2}{2m_N^2}\frac{m_\pi^2}{m_\eta^2}\tilde{\gamma}\mu_n\ , \qquad (3.15)$$

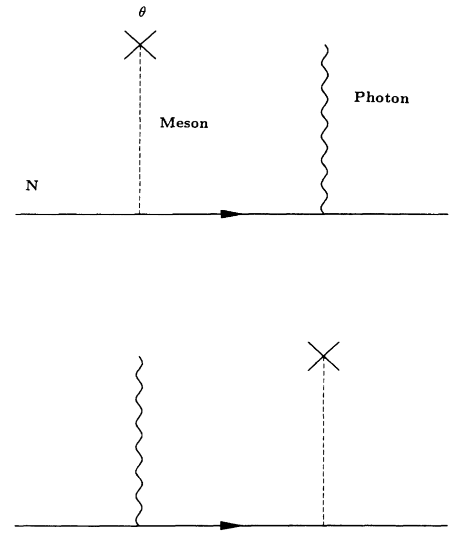


FIG. 1. The tree-level contributions to the NEDM in the effective Lagrangian in case II. There is no contact photon-meson-nucleon coupling in this case.

where we have defined a dimensionless number $\tilde{\gamma}=\gamma m_N$ characterizing the strength of the explicit symmetry breaking. Equation (3.15) shows that $D_n$ vanishes in the absence of anomaly ($m_\eta\to m_\pi$) and/or in the chiral limit ($m_\pi\to0$), which is consistent with our constraint. The former aspect ensures that we correctly took into account only the disconnected piece, and the latter is related to the fact $\langle\delta\mathcal{O}\rangle|_{\rm on\ shell}=0$. One should note that if one adopts the first order of the chiral perturbation for $m_\pi$ or takes $a\to\infty$, $a$ disappears from the formula: $D_n\to\theta(m_\pi^2/2m_N^2)\tilde{\gamma}\mu_n$.

Let us now examine the large-$N_c$ behavior of our $D_n$ in (3.15). $m_N$ and $\mu_n$ are known to be $O(N_c)$ quantities as the quark model tells us [27]. $\gamma$ is also $O(N_c)$, since it is proportional to the $\pi$-$N$ $\sigma$ term. Therefore, together with the fact that $m_\eta^2-m_\pi^2=O(N_c^{-1})$, we conclude that our $D_n$ is an $O(N_c^0)$ quantity. (Here one should not confuse the artificial limit $a\to0$ with the large-$N_c$ limit.)

If we perform the same calculation using $\mathcal{L}_B^{\rm I}$ we have three diagrams at the tree level (Fig. 2). The sum of the first two gives

$$D_n^{12}=-\theta(af_\pi)\frac{1}{m_\eta^2}\frac{1}{m_N}g_s\mu_n=-\theta a\frac{1}{m_\eta^2}\frac{\alpha}{f_\pi m_N}\mu_n\ , \qquad (3.16)$$

which does not vanish in the chiral limit. In other words, *if one were not to make $\mathcal{L}_B$ $U_A(1)$ invariant, a wrong result is achieved.* The remedy comes from the third term related to the $U_A(1)$-invariant magnetic moment which gives

$$D_n^3=\theta a\frac{1}{m_\eta^2}\mu_n\ . \qquad (3.17)$$

This term cancels a part of $D_n^{12}$ and recovers the desired

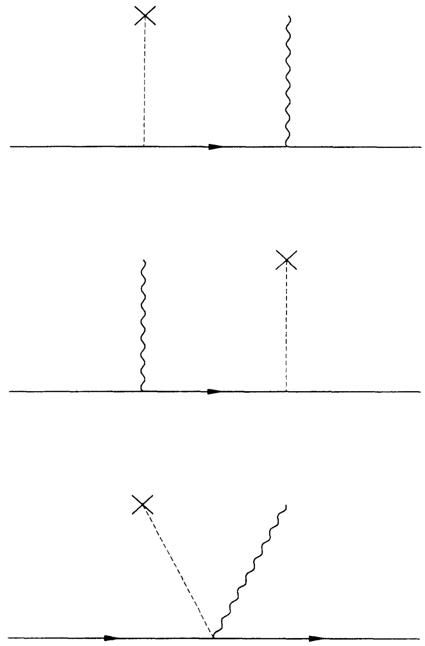

FIG. 2. The tree-level contributions to the NEDM in the effective Lagrangian in case I. There is a contact photon-meson-nucleon coupling from the $U_A(1)$-invariant magnetic moment of the neutron.

result, Eq. (3.15).

Since it now becomes clear how to perform a consistent evaluation of the NEDM by the above exercise, let us give a diagnosis of the previous calculations. They are classified in two categories: (i) the direct evaluation of $-\theta Q$ using the effective Lagrangians with an explicit gluon field [28,29] or with the instanton-induced vertex [30], and (ii) calculations using $L_{CP}^{\text{eff}} \sim \theta m \bar{\psi} i\gamma_5 \psi$ in place of $-\theta Q$ [5,16,31–35].

In the former approach, there arises no contradiction to the general principle since only the disconnected part is treated from the beginning. In Ref. [28], the photoproduction of $\eta'$ off the nucleon is adopted to estimate $D_n \sim \langle n | J_\mu F\tilde{F} | n \rangle$. In Ref. [29], the pion-loop contribution, which is a dominant term in the chiral limit because of the factor $\ln(m_N^2/m_\pi^2)$, is adopted following Ref. [32]. In Ref. [30], the instanton-induced 't Hooft–type vertex [36] is used to calculate the NEDM in the $N_f=2$ case.

In approach (ii), one has to make sure that the direct insertion defined in Sec. II is properly eliminated. For example, direct insertion of $L_{CP}^{\text{eff}}$ to the quark field gives a finite result on the lattice [14], which is an artifact due to the violation of the WT identity by the finite lattice spacing [8].

In the bag model, the nonzero result of the NEDM is obtained by the insertion of $L_{CP}^{\text{eff}}$ into the *valence* quarks in the lowest order of the bagged perturbation theory [5,31]. If one assumes that the valence quark in the bag model is the same with the current quark in the QCD Lagrangian, the contribution to the NEDM in the above calculation should vanish. Hence, the claim of the nonzero result should be fake either due to the use of the chiral-noninvariant intermediate states or the inappropriate on-shell condition for the nucleon. If the valence quark is a much more complicated object than the current quark, there is no systematic way to single out only the disconnected pieces to calculate the matrix element of $L_{CP}^{\text{eff}}$. Therefore, the result of the bag model is not reliable also in this case because of the uncontrollable contamination of the connected piece.

In Ref. [16] it is claimed that a rather large $|D_n|$ is obtained at the tree level. However, the result has serious drawbacks. First, the contamination from direct insertion is unavoidable since the matrix element $\langle 0 | \bar{q} i\gamma_5 q | \eta' \rangle$, which has both a connected and a disconnected piece, is evaluated (note that $\eta'$ has both quark and gluon contents due to the axial anomaly). Second, the $U_A(1)$-noninvariant magnetic moment of the neutron is introduced, which gives an erroneous result being nonvanishing in the absence of anomaly as we already remarked in Eq. (3.16). Once one introduces the $U_A(1)$-invariant magnetic moment in this approach and uses the $\eta'-N$ coupling from the quark model, the NEDM vanishes due to the contact photon-$\eta'$-nucleon coupling [37].

In Ref. [32], current algebra and chiral perturbation theory are used to evaluate the *CP*-violating pion-nucleon coupling $\bar{g}_{\pi N}$ induced by $L_{CP}^{\text{eff}}$, which gives rise to a NEDM from the one-pion loop. At first look, the result does not seem to vanish when $m_{\eta'} \to m_\pi$. However, if one treats the *CP*-violating pion-nucleon coupling $\bar{g}_{\pi N}$ more carefully, one finds that the factor $a/(1+a\,\mathrm{Tr}M^{-1})$ should be multiplied when $a$ is finite. The factor reduces to 1 in the limit $m_\pi/m_{\eta'} \ll 1$ but vanishes in the limit $a \to 0$. (See the derivation of $D_n$ in Ref. [29].) Therefore, keeping in mind this correction (numerically, however, this is a minor correction), the result is acceptable. One remaining question, although it has nothing to do with the consistency of the WT identity, is whether this leading term near the chiral limit is really a main term in the real world where $m_\pi = 140$ MeV. This point was properly addressed by Cea and Nardulli [16] and we will come to this point later. $\bar{g}_{\pi N}$ is also calculated by using the chiral quark model [33], the chiral bag model [34,31], and the Skyrme model [35] which are essentially the variants of Ref. [32].

It is instructive to consider more about the relation between the approaches (i) and (ii) within our effective Lagrangian. In approach (i) we have to use the total action

$$\mathcal{L}_M(U,\theta)+\mathcal{L}_B(B,U) , \tag{3.18}$$

while in approach (ii) there are several choices for $\mathcal{L}_B$ with the mesonic part given by (3.6). The most naive choice is

$$\mathcal{L}_M(Ue^{-i\theta\lambda^0/2},0)+\mathcal{L}_B(B,Ue^{-i\theta\lambda^0/2}) , \tag{3.19}$$

where

$$\mathcal{L}_B(B,Ue^{-i\theta\lambda^0/2})=\bar{B}(i\gamma\cdot\partial-\alpha U_5^\dagger e^{i\theta\gamma_5\lambda^0/2})B-\gamma\bar{B}MB + \bar{B}\mu\sigma_{\mu\nu}U_5^\dagger e^{i\theta\gamma_5\lambda^0/2}B(\partial^\nu A^\mu) . \tag{3.20}$$

At the quark level, this choice is rather strange since the quarks in the baryon are unchanged while the quarks in the meson are transformed under the $U_A(1)$ rotation. Furthermore, the CP-violating effect appears in several places: the complex baryon mass, the CP-violating baryon-meson interaction, the explicit electric dipole moment of the baryon, and so on. Each term does not have a definite physical meaning; only the sum of all effects has physical relevance. A more reasonable and simpler choice is

$$\begin{aligned}\mathcal{L}_B(e^{-i\theta\gamma_5\lambda^0/4}B, Ue^{-i\theta\lambda^0/2}) &= \bar{B}(i\gamma\cdot\partial-\alpha U_5^\dagger)B-\gamma\bar{B}Me^{-i\theta\gamma_5\lambda^0/2}B \\ &\quad +\bar{B}\mu\sigma_{\mu\nu}U_5^\dagger B(\partial^\nu A^\mu)\,, \end{aligned} \tag{3.21}$$

where the CP-violating term appears only in the explicit symmetry-breaking term. Another choice adopted in Ref. [29] is a little complicated:

$$\mathcal{L}_M(Ue^{-ib\theta\lambda^0},\theta)+\mathcal{L}_B(e^{ic\theta\gamma_5\lambda^0/4}B, Ue^{-ib\theta\lambda^0})\,, \tag{3.22}$$

where $b=a/m_\eta^2=(m_\eta^2-m_\pi^2)/(2m_\eta^2)$ and $c=-2b\alpha/m_N$. This is chosen so that the $\theta\eta^0$ term is absent in $\mathcal{L}_M$, and the baryon mass is $\gamma_5$ free. (Here we consider the $N_f=2$ case for simplicity, although $N_f=3$ in Refs. [29,32].) In this choice of variables, the CP-violating coupling $\bar{B}\lambda^a B\pi^a$ appears from the original CP-preserving interaction $\bar{B}\lambda^a B\pi^a\eta$ by the replacement $\eta\to\eta-(2b+c)\theta f_\pi/2$, and this factor becomes $\gamma m_\pi^2 f_\pi/(2m_N)$ $(b=\frac{1}{2})$ in the large-$a$ limit. The CP-violating interaction in this limit was used to calculate the NEDM [29,32].

We come to the point of evaluating a contribution to the NEDM which survives even in the large-$N_c$ limit or equivalently in the tree level of hadron fields in the realistic $N_f=3$ case. As we have already demonstrated in the $N_f=2$ case, the $U_A(1)$-invariant magnetic moment is an essential ingredient. Equation (3.2) is our meson Lagrangian with

$$q(x)=(\tfrac{3}{2})^{1/2}af_\pi\eta^0,\quad a=\tfrac{1}{3}(m_{\eta'}^2+m_\eta^2-2m_K^2)\,,$$

and (3.23)

$$M=\mathrm{diag}(m_\pi^2,m_\pi^2,2m_K^2-m_\pi^2)\,,$$

where we have neglected the small isospin breaking. We introduce the baryon field by $B=\frac{1}{2}(1+\gamma_5)B_R+\frac{1}{2}(1-\gamma_5)B_L$, where $B_L$ and $B_R$ transform as $(3,3^*)$ and $(3^*,3)$ representation of the $U_L(3)\otimes U_R(3)$ flavor, respectively. Then $\mathcal{L}_B$ is written as

$$\begin{aligned}\mathcal{L}_B &= \mathrm{Tr}(\bar{B}i\gamma\cdot\partial B)-\alpha\,\mathrm{Tr}(\bar{B}_L UB_R U+\mathrm{H.c.})+[\delta\,\mathrm{Tr}(\bar{B}_L UB_R M)+\delta'\mathrm{Tr}(\bar{B}_L UB_R UMU)+\mathrm{H.c.}] \\ &\quad +[\gamma\,\mathrm{Tr}(\bar{B}_L MB_R U)+\gamma'\mathrm{Tr}(\bar{B}_L UMUB_R U)+\mathrm{H.c.}]+\epsilon\,\mathrm{Tr}(MU+MU^\dagger)\mathrm{Tr}(\bar{B}_L UB_R U+\bar{B}_R U^\dagger B_L U^\dagger) \\ &\quad +\epsilon'\mathrm{Tr}(MU-MU^\dagger)\mathrm{Tr}(\bar{B}_L UB_R U-\bar{B}_R U^\dagger B_L U^\dagger)+(\text{magnetic moment})\,. \end{aligned} \tag{3.24}$$

The $\delta$, $\gamma$, $\epsilon$, $\delta'$, $\gamma'$, and $\epsilon'$ terms denote the U(3) breaking. The last term denotes the magnetic moment of the octet baryons, which contains $U_A(1)$ invariant parts and also terms of $O(M)$. Note that our $\alpha$, $\delta$, and $\gamma$ correspond to $f_\pi/\sqrt{2}$ times $\alpha$, $\delta$, and $\gamma$ in Ref. [29]. A more general form of the $U_L(3)\otimes U_R(3)$-invariant part of the effective Lagrangian has $D$ and $F$ terms for the PV coupling [38]. In our case, however, such details are irrelevant since PV couplings do not contribute. On the other hand, in the approaches calculating the one-loop diagram, PV couplings are important to get the observed $g_{\pi N}$ at the tree level, i.e., the Goldberger-Treiman relation $g_{\pi N}=(F+D)m_N/f_\pi$.

To evaluate the NEDM, it is convenient to define the chiral-invariant nucleon field by

$$B_L=\sqrt{U}N_L\sqrt{U}\quad\text{and}\quad B_R=\sqrt{U^\dagger}N_R\sqrt{U^\dagger}\,. \tag{3.25}$$

Using this, expanding $U$ up to the first order of the meson field, and retaining only the part relevant for the neutron $(n)$, we get

$$\begin{aligned}\mathcal{L}_B &\to \bar{n}(i\gamma\cdot\partial-m_N)n+(\text{PV coupling}) \\ &\quad -(\eta' g_{\eta'}+\eta g_\eta)\bar{n}i\gamma_5 n+\mu_n\bar{n}\sigma_{\mu\nu}n(\partial^\nu A^\mu) \\ &\quad +[O(M)\text{ magnetic moment}]\,, \end{aligned} \tag{3.26}$$

with

$$\begin{aligned} g_{\eta'} &= \frac{1}{f_\pi}[\sqrt{2}\Delta(2m_K^2-m_\pi^2)\cos(\phi+\phi_0) \\ &\quad +\Gamma m_\pi^2\sin(\phi+\phi_0)]\,, \\ g_\eta &= \frac{1}{f_\pi}[-\sqrt{2}\Delta(2m_K^2-m_\pi^2)\sin(\phi+\phi_0) \\ &\quad +\Gamma m_\pi^2\cos(\phi+\phi_0)]\,, \end{aligned} \tag{3.27}$$

where $\Delta=\delta-\delta'-2\epsilon'$ and $\Gamma=\gamma-\gamma'-4\epsilon'$ and

$$\begin{aligned} m_N &= \alpha-2\epsilon(2m_K^2+m_\pi^2) \\ &\quad -(\delta+\delta')(2m_K^2-m_\pi^2)-(\gamma+\gamma')m_\pi^2\,, \end{aligned}$$

$\mu_n$ is the neutron magnetic moment, and we have used a relation between the flavor eigenstates ($\eta^0$ and $\eta^8$) and the mass eigenstates ($\eta$ and $\eta'$):

$$\begin{aligned} \eta^0 &= \eta'\cos\phi-\eta\sin\phi\,, \\ \eta^8 &= \eta\cos\phi+\eta'\sin\phi\,, \end{aligned} \tag{3.28}$$

with the empirical mixing angle [39] $\phi=-11^\circ\sim-20^\circ$ and $\phi_0=54.73^\circ$ $(\tan\phi_0=\sqrt{2})$. Owing to the fact that we introduce the magnetic moment as an $U_A(1)$-invariant form, the contact photon-$\eta^0$-neutron coupling does not arise in the choice of the variables here (as in case II in our previous example).

Here several comments are in order. (i) The term proportional to $\epsilon$ in (3.24) is irrelevant in our calculation since it does not produce Yukawa coupling. (ii) The $U_A(1)$ breaking due to anomaly is solely governed by the $F\tilde{F}$-$\eta^0$ mixing in $\mathcal{L}_M$, since we have used the effective action motivated by the large-$N_c$ QCD. Therefore, the PS coupling of $\eta^0$ with neutron in Eq. (3.26) takes the value in the large-$N_c$ limit. (iii) As already mentioned in the $N_f=2$ case, one has in general magnetic moments of baryons which break $U_A(1)$ invariance by the quark masses, such as $\mathrm{Tr}(\bar{B}\sigma_{\mu\nu}MB)\partial^\mu A^\nu$. One can determine some of such terms by the observed magnetic moments of the octet baryons. The detailed examination of this effect will be given in a separate publication [40]. (iv) To determine the parameters in Eq. (3.24), one needs information of not only the mass splitting and magnetic moments of octet baryons but also the $\pi$-$N$ and $K$-$N$ scattering length and other observables. Recently Pich and de Rafael wrote down most general terms that can contribute to the NEDM in the tree level [7], which is an extension of our previous paper in Ref. [7]. The six parameters $(c_0,c_1,c_2,b_0,b_1,b_2)$ in their paper correspond to a combination of the six parameters $(\delta,\gamma,\epsilon,\delta',\gamma',\epsilon')$ in our Eq. (3.24). The parameters $\delta_i$ in their paper correspond to the chiral-invariant and -noninvariant magnetic-moment terms discussed in (iii) above.

If one uses PCU$_1$C, the problem is now reduced to evaluating the matrix element

$$-i\theta(af_\pi)\int d^4x\,\langle n(p')|T[\eta'(x)\cos\phi-\eta(x)\sin\phi]J_\mu(0)|n(p)\rangle\ , \tag{3.29}$$

which is given by diagrams similar to those in Fig. 1. To get a rough estimate of the NEDM, let us take only the chiral-invariant magnetic moment. Then, after a straightforward calculation, one gets

$$D_n=\theta\frac{m_{\eta'}^2+m_\eta^2-2m_K^2}{3}\left[\frac{3}{2}\right]^{1/2}\left[\frac{f_\pi}{m_N}\right]\times\left[\frac{g_{\eta'}}{m_{\eta'}^2}\cos\phi-\frac{g_\eta}{m_\eta^2}\sin\phi\right]\mu_n\ , \tag{3.30}$$

which is proportional to $a$ and vanishes in the chiral limit since $g_{\eta'(\eta)}$ is proportional to the current masses. Note that the values of $g_{\eta'(\eta)}$ will be effectively modified if one takes into account the magnetic moment with explicit $U_A(1)$ breaking due to $M$. Note also that our $D_n$ is an $O(N_c^0)$ contribution, while $D_n$ due to the pion loop is $O(1/N_c)$. The latter can easily be seen if one remembers that the pion-nucleon PS coupling $g_{\pi N}$ is $O(N_c^{3/2})$ and the parity-violating pion-nucleon coupling $\bar{g}_{\pi N}$ is $O(N_c^{-3/2})$ due to the multiplicative factor $a/(1+a\,\mathrm{Tr}M^{-1})$ which we mentioned before.

If we set $\delta'=\gamma'=\epsilon'=0$ and determine $(\delta,\gamma,\epsilon)$ by the baryon mass splitting as was done in Ref. [29], we get an estimate for the tree-level contribution:

$$D_n=-7.2\times10^{-16}\ \theta e\ \mathrm{cm}\quad\text{for }\phi=-11^\circ$$
$$=-3.9\times10^{-16}\ \theta e\ \mathrm{cm}\quad\text{for }\phi=-20^\circ\ , \tag{3.31}$$

which is different in sign [41] and comparable in magnitude with the one-loop result ($+3.6\times10^{16}\theta\ e$ cm) in Ref. [29]. This result suggests that one cannot neglect either of the following two contributions: one is a leading term in the large-$N_c$ limit having a structure $N_c^0M$ like our $D_n$ and another is the leading term in the chiral expansion having the structure $N_c^{-1}M\ln M$. However, a complete calculation taking into account all the parameters in the tree level will be necessary to draw a definite conclusion [42].

Here we should mention briefly a similar calculation for the $CP$-violating condensation parameter or the topological susceptibility $K$ which is defined by

$$\theta K=\langle -q(x)\rangle_\theta\simeq\theta\int d^4y\,\langle Tq(x)q(y)\rangle\ . \tag{3.32}$$

If the disconnected part is properly treated in the calculation, $K$ has to be proportional to both $a$ and the current quark mass. The evaluation using the current algebra and the chiral perturbation, however, gives only the latter dependence, as can be seen in Ref. [26]. Although $K$ is an off-shell quantity, (3.2) and (3.6) give the same result, since $q$ is a purely gluonic operator being chiral invariant:

$$K=\frac{f_\pi^2}{2}a\left[1-2a\left[\frac{\cos^2\phi}{m_\eta^2}+\frac{\sin^2\phi}{m_{\pi^0}^2}\right]\right], \tag{3.33}$$

in $N_f=2$. Here we took into account the isospin violation. In the large-$a$ or small current quark mass limit, (3.33) reduces to the result of Ref. [26]:

$$K=\frac{f_\pi^2}{4}m_\pi^2\ . \tag{3.34}$$

As can be seen in the above examples, once one adopts the *lowest* order of the chiral perturbation, one always misses the explicit factor $a$. The reason is easy to understand. Schematically every tree amplitude is proportional to the combination

$$N_f\frac{a}{m_S^2}\ , \tag{3.35}$$

in the flavor-symmetric case with $m_S$ being the mass of the flavor-singlet pseudoscalar meson. However, Eq. (3.35) reduces to unity in the chiral limit.

## IV. CONCLUSION

In this paper we have extensively examined the consistency of the calculation for the low-energy matrix element of $\theta F\tilde{F}$.

The anomalous WT identity tells us that the insertion of $\theta F\tilde{F}$ in the hadronic process is related to the *disconnected* insertion of $L_{CP}^{\mathrm{eff}}$, while the connected insertion of $L_{CP}^{\mathrm{eff}}$ is merely the change of variables $\delta\mathcal{O}$, which vanishes for the on-shell amplitude.

This simple observation gives a constraint on the use of the effective theory to calculate the NEDM and also makes it possible to diagnose the previous calculations. In fact, one cannot construct effective theories without $U_A(1)$ invariance in the chiral limit. This also implies all the *CP*-violating matrix elements have to be proportional to the current quark masses and the strength of the anomaly. We pointed out, in Sec. III, the problematic points of the previous calculations of the NEDM. Then we gave a new calculation of the NEDM that is consistent with the general constraints above and does not vanish in the large-$N_c$ limit. The explicit inclusion of the $U_A(1)$-invariant magnetic moment of the neutron is the essential ingredient of the calculation.

Finally we should mention that it is desirable to calculate the NEDM from first principle. One possibility is a lattice QCD calculation, although there are several problems to be overcome, as the recent analysis shows [8]. Another interesting problem related to the operator $F\tilde{F}$ is the spin content of the nucleon [43], where the nucleon matrix element of $F\tilde{F}$ plays a crucial role [44]. Extensive studies using the effective Lagrangians and the lattice QCD consistent with the WT identity are also called for on this problem.

ACKNOWLEDGMENTS

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