Remarks on the weak states of neutrinos

C. Giunti
Department of Physics and Astronomy, The Johns Hopkins University, Baltimore, Maryland 21218 and Istituto di Fisica Nucleare, I-10125 Torino, Italy

C. W. Kim
Department of Physics and Astronomy, The Johns Hopkins University, Baltimore, Maryland 21218 and Center for Theoretical Physics, Seoul National University, Seoul 151, Korea

U. W. Lee
Department of Physics and Astronomy, The Johns Hopkins University, Baltimore, Maryland 21218
(Received 12 August 1991)

It is shown that the weak states which are used in the standard treatment of neutrino oscillations do not, in general, describe correctly the neutrinos produced and detected in weak-interaction processes. It is also shown that it is impossible to construct a Fock space of weak states. However, neutrino oscillations can be described by defining appropriate "weak-process states," which are superpositions of mass eigenstates weighted by their transition amplitudes in the process under consideration. In the extreme relativistic limit, the weak-process states reduce to the usual weak states. Some numerical examples are given to illustrate the magnitude of nonrelativistic corrections to the standard results.

PACS number(s): 14.60.Gh, 12.15.Ff

I. INTRODUCTION

As in the case of the quark sector, if neutrinos are massive, the mass matrix in the weak basis of the neutrino fields can be nondiagonal. The mass matrix can be diagonalized by defining a mass basis for the neutrino fields. The left-handed neutrino fields in the weak basis $\nu_{\alpha L}(x)$ and the left-handed neutrino fields in the mass basis $\nu_{aL}(x)$ are related by a unitary transformation

$$\nu_{\alpha L}(x)=\sum_a \mathcal{U}_{\alpha a}\nu_{aL}(x) \tag{1}$$

(throughout this paper the greek indices α,β refer to the weak basis, whereas the latin indices a,b refer to the mass basis). In the usual treatment of neutrino oscillations [1], the flavor neutrinos are described by the "weak states" $|\nu_\alpha\rangle$ given by

$$|\nu_\alpha\rangle=\sum_a \mathcal{U}^*_{\alpha a}|\nu_a\rangle \ , \tag{2}$$

where $|\nu_a\rangle$ are mass eigenstates. In Eq. (2), it is not obvious which values must be assigned to the momentum and energy of the mass eigenstates. In the literature, the most popular choice has been to assume that the mass eigenstates have the same momentum $\mathbf{p}$ but different energies E_a, given by $E_a=\sqrt{|\mathbf{p}|^2+m_a^2}$, where m_a are the mass eigenvalues. On the other hand, as noted in Ref. [2], from energy-momentum conservation in the process in which the neutrinos are produced, the mass eigenstates must have different momenta $\mathbf{p}_a$ as well as different energies E_a (with $E_a^2=|\mathbf{p}_a|^2+m_a^2$). For extremely relativistic neutrinos this problem is irrelevant because both approaches lead to the standard oscillation probability. In this paper we will show how to calculate the neutrino oscillation probability that is valid for nonrelativistic as well as relativistic neutrinos. The nonrelativistic corrections to the standard oscillation probability depend on the processes in which the neutrinos are produced and detected through the corresponding transition amplitudes of the mass-eigenstate neutrinos. Hence, in order to calculate the transition amplitudes, we will assume energy-momentum conservation and the propagating mass eigenstates will have different momenta as well as different energies.

In this paper we will emphasize that the weak states $|\nu_\alpha\rangle$, which are used in the standard treatment of neutrino oscillations, do not, in general, describe correctly the neutrinos produced and detected in weak-interaction processes. In order to elucidate this point, let us consider, for example, the weak charged-current process $\nu+X_i\to X_f+e^-$ in which a neutrino is detected through the production of an electron. If the neutrino were correctly described by the weak state $|\nu_e\rangle$, as given by Eq. (2), the transition amplitudes

$$\begin{aligned}\langle e^-|\bar{e}\gamma^\rho(1+\gamma_5)\nu_e|\nu_\alpha\rangle h_\rho(X_i,X_f)\\ =\sum_a \mathcal{U}_{ea}\mathcal{U}^*_{\alpha a}\langle e^-|\bar{e}\gamma^\rho(1+\gamma_5)\nu_a|\nu_a\rangle h_\rho^{(a)}(X_i,X_f)\end{aligned} \tag{3}$$

should vanish for $\alpha\neq e$ [$h_\rho^{(a)}(X_i,X_f)$ are the matrix elements of the X part of the process and the superscript (a) indicates that they depend on the mass eigenvalue m_a because of energy-momentum conservation]. Instead Eq. (3) is, in general, not proportional to $\delta_{e\alpha}$. In fact, even though the mixing matrix $\mathcal{U}$ is unitary, i.e., $\sum_a \mathcal{U}_{ea}\mathcal{U}^*_{\alpha a}=\delta_{e\alpha}$, the factors $\langle e^-|\bar{e}\gamma^\rho(1+\gamma_5)\nu_a|\nu_a\rangle h_\rho^{(a)}(X_iX_f)$, which depend on the index a through the different masses m_a, spoil the diagonality in the flavor indices (e and α) of Eq. (3). In the limit in which the differences between the mass eigenvalues are negligible compared with the neutrino momentum (in particular for extremely relativistic neutrinos), the factors $\langle e^- | \bar{e}\gamma^\rho(1+\gamma_5)\nu_a | \nu_a \rangle h_\rho^{(a)}(X_i, X_f)$ can be taken out of the sum in Eq. (3) and the transition amplitudes vanish for $\alpha \neq e$. Only in this case, the weak states $|\nu_\alpha\rangle$ can be used to describe approximately the neutrinos detected in the weak-interaction process under consideration.

More generally, in Sec. II it will be shown that it is impossible to define appropriate creation and destruction operators of weak states and hence a Fock space of weak states do not exist in a rigorous sense. However, in some special case (in particular for extremely relativistic neutrinos) an *approximate Fock space of weak states* can be constructed.

The observation that a Fock space of weak states does not exist does not cause any difficulty in calculating usual cross sections and decay rates of processes in which neutrinos participate as external particles. The reason is that these neutrinos must be described by the mass eigenstates $|\nu_a\rangle$ and each mass eigenstate corresponds to a separate, incoherent process, as discussed in detail in Ref. [3]. In Sec. III, it will be shown that neutrino oscillations can also be described by using only the mass eigenstates, without any reference to weak states. It will also be shown that, in order to establish an analogy with the usual treatment of neutrino oscillations, it is possible to *define* appropriate "weak-process" states that can be used instead of the usual weak states in order to obtain the correct oscillation probability. Since the concept of flavor neutrinos has a physical meaning only in connection with their weak interactions, in order to properly define the weak-process states, one must consider specific physical processes in which neutrinos are produced or detected. Namely, the weak-process states are given by superpositions of mass eigenstates weighted by their transition amplitudes. It will be shown that in the extreme relativistic limit the weak-process states reduce to the weak states given in Eq. (2). Thus, this treatment differs from the standard one in that the nonrelativistic corrections to the weak-interaction processes in which neutrinos are produced and detected are properly taken into account whereas the standard treatment is valid strictly for the extremely relativistic case. Since the present direct experimental upper limit of the neutrino masses are quite poor ($m_1 \lesssim 10$ eV, $m_2 \lesssim 250$ keV, and $m_3 \lesssim 35$ MeV), it is possible that nonrelativistic corrections may not be negligible in some experiments that search neutrino oscillations.

II. WEAK STATES

In a free-field theory only the neutrino fields in the mass basis have a physical meaning. The concept of flavor neutrinos (i.e., electron, muon, and tau neutrinos) arises when weak interactions of the neutrino fields are introduced. So far, it appears that the existence of the weak states $|\nu_\alpha\rangle$ has been assumed without investigating the possibility of quantizing the neutrino fields in the weak basis by building a Fock space of weak states. The reason for this assumption is the fact that the canonical anticommutation relations of the fields are preserved by the unitary transformation that relates the weak and the mass bases of the neutrino fields. In fact, the canonical anticommutation relations of the fields depend only on the kinetic part of the Lagrangian (through the definition of the conjugate momentum), which is invariant under any unitary transformation. However, the physical content of the theory is established by the full Lagrangian, which, due to the presence of the mass term, is not invariant under unitary transformations of the fields. In the canonical formulation of quantum field theory, one must solve the field equations and implement the canonical anticommutation relations of the fields by defining appropriate creation and destruction operators of one-particle states. In the case of mixed neutrinos, the field equations can be solved in the mass basis (in which they are independent) and the neutrino fields in the mass basis can be quantized by building a Fock space of mass eigenstates $|\nu_a\rangle$. It will be shown below that, contrary to the general belief, in the weak basis it is impossible to define operators that obey the canonical anticommutation relations and thus can be interpreted as creation and annihilation operators of weak states. Hence, a Fock space of weak states does not exist.

We consider the case of Majorana neutrinos, which is predicted by a large class of gauge theories. The free-field equations for the two-component fields $\Phi_a(x)$ in the mass basis are[1]

$$(i\partial_0 - i\boldsymbol{\sigma}\cdot\nabla)\Phi_a(x) + m_a i\sigma^2\Phi_a^*(x) = 0 \ . \tag{4}$$

The plane-wave solutions of the field equations are [5]

$$\Phi_a(x) = \int \frac{d\mathbf{p}}{\sqrt{(2\pi)^2}} \sum_{h=\pm1} \left[\left[\frac{E_a - hP}{2E_a} \right]^{1/2} a_a(\mathbf{p},h)\omega(\mathbf{p},h) e^{-iE_a t + i\mathbf{p}\cdot\mathbf{x}} - h \left[\frac{E_a + hP}{2E_a} \right]^{1/2} a_a^\dagger(\mathbf{p},h)\omega(\mathbf{p},-h) e^{iE_a t - i\mathbf{p}\cdot\mathbf{x}} \right] , \tag{5}$$

where $P \equiv |\mathbf{p}|$, $E_a \equiv \sqrt{P^2 + m_a^2}$, $h = \pm 1$ denotes the helicity and $\omega(\mathbf{p},h)$ are two-component orthonormal helicity eigenstate spinors [in the present case, it is convenient to quantize the neutrino fields as helicity eigenstates because the spinors $\omega(\mathbf{p},h)$ do not depend on the neutrino mass]. The operators $a_a(\mathbf{p},h)$ and $a_a^\dagger(\mathbf{p},h)$ obey the canonical anticommutation relations and can be interpreted as destruction and creation operators of one-particle states with definite helicity in the mass basis $|\nu_a(\mathbf{p},h)\rangle$ (mass eigenstates).

[1]In the chiral representation of the γ matrices, the two-component fields $\Phi_a(x)$ are related to the four-component fields $\nu_{aL}(x)$ by $\nu_{aL}^T(x) = (\Phi_a^T(x), 0)$. For details, see Ref. [4].

From Eqs. (1) and (5), the two-component neutrino fields in the weak basis $\Phi_\alpha(x)$ can be expanded as

$$\Phi_\alpha(x)=\int \frac{d\mathbf{p}}{\sqrt{(2\pi)^3}} \sum_{h=\pm1} [A_\alpha(\mathbf{p},h,t)\omega(\mathbf{p},h)e^{i\mathbf{p}\cdot\mathbf{x}}-hB_\alpha^\dagger(\mathbf{p},h,t)\omega(\mathbf{p},-h)e^{-i\mathbf{p}\cdot\mathbf{x}}] , \tag{6}$$

where

$$A_\alpha(\mathbf{p},h,t)=\sum_a \mathcal{U}_{\alpha a}\left[\frac{E_a-hP}{2E_a}\right]^{1/2} a_a(\mathbf{p},h)e^{-iE_a t} ,$$
$$B_\alpha(\mathbf{p},h,t)=\sum_a \mathcal{U}^*_{\alpha a}\left[\frac{E_a+hP}{2E_a}\right]^{1/2} a_a(\mathbf{p},h)e^{-iE_a t} . \tag{7}$$

The crucial observation here is that, in general, the operators $A_\alpha(\mathbf{p},h,t)$ and $B_\alpha(\mathbf{p},h,t)$ do not obey the canonical anticommutation relations. For example, at equal time, one has

$$\{A_\alpha(\mathbf{p},h,t), A_\beta^\dagger(\mathbf{p}',h',t)\} =\delta(\mathbf{p}-\mathbf{p}')\delta_{hh'}\sum_a \mathcal{U}_{\alpha a}\mathcal{U}^*_{\beta a}\frac{E_a-hP}{2E_a} \tag{8}$$

and

$$\{A_\alpha(\mathbf{p},h,t), B_\beta^\dagger(\mathbf{p}',h',t)\} =\delta(\mathbf{p}-\mathbf{p}')\delta_{hh'}\sum_a \mathcal{U}_{\alpha a}\mathcal{U}_{\beta a}\frac{m_a}{2E_a} . \tag{9}$$

In general, these anticommutation relations are nondiagonal in the flavor indices. In fact, even though $\mathcal{U}$ is unitary, i.e., $\sum_a \mathcal{U}_{\alpha a}\mathcal{U}^*_{\beta a}=\delta_{\alpha\beta}$, the terms $(E_a-hP)/(2E_a)$ spoil the diagonality of the anticommutation relation given in Eq. (8). Similarly, the anticommutation relation given in Eq. (9) is not diagonal because, in general, $\sum_a \mathcal{U}_{\alpha a}\mathcal{U}_{\beta a}\neq\delta_{\alpha\beta}$ and because of the presence of the term $m_a/(2E_a)$. Therefore, one cannot construct a Fock space of weak states. This proof is general and also applies to Dirac neutrinos, because a Dirac neutrino field is made of two Majorana neutrino fields with degenerate mass and opposite *CP* phases. It can also be easily extended to any set of mixed particles, either fermions or bosons.

Even though one cannot construct an exact Fock space of weak states, an *approximate Fock space of weak states* can be defined in the following two cases in which the anticommutation relations given in Eqs. (8) and (9) become diagonal in the flavor indices.

(1) In the *extremely relativistic* limit (i.e., for $P\gg\max\{m_a\}$). Since, in this case, $(E_a-hP)/(2E_a)\to(1-h)/2$ and $m_a/(2E_a)\to0$, the anticommutation relation given in Eq. (8) becomes proportional to $\delta_{\alpha\beta}$ and the one given in Eq. (9) vanishes. In the extremely relativistic limit the operators $A_\alpha(\mathbf{p},+1,t)$ and $B_\alpha(\mathbf{p},-1,t)$ are suppressed and the surviving operators can be written as

$$A_\alpha(\mathbf{p},-1,t)\simeq e^{-iPt}a_\alpha(\mathbf{p},-1) ,$$
$$a_\alpha(\mathbf{p},-1)=\sum_a \mathcal{U}_{\alpha a}a_a(\mathbf{p},-1) ,$$
$$B_\alpha(\mathbf{p},+1,t)\simeq e^{-iPt}a_\alpha(\mathbf{p},+1) ,$$
$$a_\alpha(\mathbf{p},+1)=\sum_a \mathcal{U}^*_{\alpha a}a_a(\mathbf{p},+1) . \tag{10}$$

The operators $a_\alpha(\mathbf{p},-1)$ and $a_\alpha(\mathbf{p},+1)$ can be interpreted as destruction operators of an approximate Fock space of weak states with negative and positive helicity, respectively.

(2) For *almost degenerate mass eigenvalues* (i.e., if the differences between the mass eigenvalues are much smaller than the momentum P), with a *real mixing matrix* $\mathcal{U}=\mathcal{U}^*$. The reality of the mixing matrix is necessary in order to have $\sum_a \mathcal{U}_{\alpha a}\mathcal{U}_{\beta a}=\delta_{\alpha\beta}$. Since $E_a\simeq E$, independent from the index a, one has

$$A_\alpha(\mathbf{p},h,t)\simeq\left[\frac{E-hP}{2E}\right]^{1/2} e^{-iEt}a_\alpha(\mathbf{p},h) ,$$
$$B_\alpha(\mathbf{p},h,t)\simeq\left[\frac{E+hP}{2E}\right]^{1/2} e^{-iEt}a_\alpha(\mathbf{p},h) \tag{11}$$

with $a_\alpha(\mathbf{p},h)=\sum_a \mathcal{U}_{\alpha a}a_a(\mathbf{p},h)$. The operators $a_\alpha(\mathbf{p},h)$ can be interpreted as destruction operators of an approximate Fock space of weak states with definite helicity. Note that in this case the Majorana character of the creation and destruction operators is preserved in the weak basis.

The anticommutation relations given in Eqs. (8) and (9) become diagonal in the flavor indices also in the extremely nonrelativistic limit (i.e., for $P\ll\min\{m_a\}$), with a real mixing matrix. However, in this case it is impossible to define time-independent creation and destruction operators of weak states because of the different phase factors $e^{-im_a t}$ that cannot be factorized out of the sum over the mass eigenstates in the operators given in Eq. (7).

The anticommutation relation given in Eq. (9) is characteristic of Majorana neutrinos: in the Dirac case there is only one nontrivial anticommutation relation, analogous to that given in Eq. (8). Therefore, for Dirac neutrinos an approximate Fock space of weak states is well defined in the extremely relativistic limit and for almost degenerate mass eigenvalues.

In order to understand the physical meaning of the reality condition for the mixing matrix $\mathcal{U}$ in the case of Majorana neutrinos, let us consider, for example, the lepton-number- (L-)violating process $X_i\to X_f+e^-+\nu$ in which a neutrino is created together with an electron. If the neutrino were correctly described by the weak state $|\nu_e\rangle$, given by Eq. (2), the transition amplitudes

$$\langle e^-,\nu_\alpha|\bar{e}\gamma^\rho(1+\gamma_5)\nu_e|0\rangle h_\rho(X_i,X_f)=\sum_a \mathcal{U}_{\alpha a}\mathcal{U}_{ea}\langle e^-,\nu_a|\bar{e}\gamma^\rho(1+\gamma_5)\nu_a|0\rangle h_\rho^{(a)}(X_i,X_f) \tag{12}$$

should vanish for $\alpha \neq e$. Instead, if the mixing matrix $\mathcal{U}$ is not real, Eq. (12) is not proportional to $\delta_{\alpha e}$, even if the mass eigenvalues are almost degenerate. This means that the weak states given in Eq. (2) do not describe the Majorana neutrinos produced in L-violating processes. One must instead use the "antineutrino" definition of weak states[2]:

$$|\nu_{\bar{\alpha}}\rangle = \sum_a \mathcal{U}_{\alpha a} |\nu_a\rangle \ . \tag{13}$$

This is a simple example in which the weak states appropriate for the description of the neutrinos produced in a specific process differ from the usual weak states $|\nu_\alpha\rangle$ given in Eq. (2). The states $|\nu_{\bar{\alpha}}\rangle$ can be used for the study of L-violating neutrino oscillations. Because of a helicity mismatch analogous to the one that occurs in the double-β decay, in the extremely relativistic limit the L-violating oscillation amplitudes are suppressed by m_a/E_a. By using the states $|\nu_{\bar{\alpha}}\rangle$, one obtains the well-known [6] L-violating oscillation probabilities

$$\begin{aligned} P_{\bar{\alpha}\to\beta}(t) &\sim \left| \sum_a \frac{m_a}{E_a} \langle \nu_\beta(t)|\nu_a(t)\rangle \langle \nu_a(t)|\nu_a(0)\rangle \langle \nu_a(0)|\nu_{\bar{\alpha}}(0)\rangle \right|^2 \\ &\sim \left| \sum_a \frac{m_a}{E_a} \mathcal{U}_{\beta a} \mathcal{U}_{\alpha a} e^{-iE_a t} \right|^2 \end{aligned} \tag{14}$$

in which the CP-violating phases of the mixing matrix $\mathcal{U}$ are observable.

III. WEAK-PROCESS STATES

In the previous section, we have demonstrated that the standard "weak states" are, in general, ill defined. In spite of this problem, it will be shown in this section that it is possible to describe neutrino oscillations by defining appropriate "weak-process states" which depend on the processes by which neutrinos are produced and detected.

As a simple example, we consider the neutrino oscillation process

$$\begin{aligned} &\pi^+ \to \mu^+ + \nu \\ &\qquad\searrow \\ &\qquad \nu + \mathcal{N}_i(A, Z-1) \to \mathcal{N}_f(A,Z) + e^- \end{aligned} \tag{15}$$

in which a neutrino is produced by pion decay, propagates between the source and the detector and is detected by nuclear capture. Since one observes the initial and final particles but not the intermediate neutrino, the oscillation probability is given by[3]

$$P_{\mu\to e}(\mathbf{x},t) \sim \left| \sum_a \langle e^-, \mathcal{N}_f | \mathcal{L}_W(0) | \mathcal{N}_i, \nu_a \rangle e^{i\mathbf{p}_a\cdot\mathbf{x} - iE_a t} \langle \nu_a, \mu^+ | \mathcal{L}_W(0) | \pi^+ \rangle \right|^2 , \tag{16}$$

where $\mathcal{L}_W(0)$ is the weak-interaction Lagrangian evaluated at the origin, E_a and $\mathbf{p}_a$ are the energies and momenta of the mass eigenstates and $\mathbf{x}$ is the direction of propagation of the intermediate neutrino (for simplicity, the details on the energies, momenta and spins of the particles involved have been omitted).

Even though the oscillation probability given in Eq. (16) contains all the information necessary for the calculation of the neutrino oscillation process given in Eq. (15), in order to establish an analogy with the standard treatment of neutrino oscillations, we can *define* the following "weak-process states" corresponding, respectively, to the production and detection processes:

$$\begin{aligned} |\nu_\mu\rangle_{\rm WP} &\sim \sum_a |\nu_a\rangle \langle \nu_a, \mu^+ | \mathcal{L}_W(0) | \pi^+ \rangle \ , \\ |\nu_e\rangle_{\rm WP} &\sim \sum_a |\nu_a\rangle \langle \nu_a, \mathcal{N}_i | \mathcal{L}_W(0) | \mathcal{N}_f, e^- \rangle \ . \end{aligned} \tag{17}$$

It is obvious that the definition of the weak-process states depend on the specific process under consideration. By using these weak-process states, the oscillation probability given in Eq. (16) can be written as

$$P_{\mu\to e}(\mathbf{x},t) \sim \left| \sum_a {}_{\rm WP}\langle \nu_e|\nu_a\rangle e^{i\mathbf{p}_a\cdot\mathbf{x} - iE_a t} \langle \nu_a|\nu_\mu\rangle_{\rm WP} \right|^2 \tag{18}$$

which is analogous to the standard oscillation probability, with the usual weak states replaced by the corresponding weak-process states. In the following, we will show how the weak-process states defined in Eq. (17) are

[2]For Dirac antineutrinos $|\bar{\nu}_\alpha\rangle = \sum_a \mathcal{U}_{\alpha a} |\bar{\nu}_a\rangle$.

[3]A more rigorous treatment, with localized wave packets in the framework of quantum field theory, will be discussed elsewhere [7].

different from the standard weak states given in Eq. (2) (in the extremely relativistic limit they become equivalent and one recovers the standard formulation of neutrino oscillations).

Let us consider the production process $\pi^+ \to \mu^+ + \nu$. In the rest frame of the pion, because of angular momentum conservation, the muon and the neutrino must be emitted with the same helicity. In the limit of zero neutrino mass they both have negative helicity, but, for a finite neutrino mass, the decay into positive helicities is also allowed. The transition amplitudes $\langle \nu_a(\mathbf{p}_a,h),\mu^+(-\mathbf{p}_a,h)|\mathcal{L}_W(0)|\pi^+\rangle \sim \mathcal{U}^*_{\mu a}\mathcal{P}_a(h)$, for the two helicity cases $h=\pm 1$, can be calculated by using a definite representation of the γ matrices. The result is

$$\begin{aligned}\mathcal{P}_a(h) &= \frac{G_F \cos\vartheta_C}{\sqrt{2}} f_\pi \overline{v_\mu}(-\mathbf{p}_a,h)[m_\mu(1+\gamma_5)-m_a(1-\gamma_5)]u_\nu(\mathbf{p}_a,h) \\ &= -G_F\cos\vartheta_C f_\pi\left(m_\mu\sqrt{m_\pi^2-m_\mu^2-m_a^2-h\lambda_a^{1/2}}+m_a\sqrt{m_\pi^2-m_\mu^2-m_a^2+h\lambda_a^{1/2}}\right) , \qquad (19)\end{aligned}$$

where G_F is the Fermi constant, ϑ_C is the Cabibbo angle, f_π is the pion decay constant, and $\lambda_a \equiv \lambda(m_\pi^2, m_\mu^2, m_a^2)$ is the usual kinematical function $[\lambda(x,y,z)\equiv x^2+y^2+z^2-2xy-2yz-2zx]$. For each value of h, we can define a weak-process state $|\nu_\mu(h)\rangle_{\rm WP}$ given by

$$|\nu_\mu(h)\rangle_{\rm WP} = \sum_a \mathcal{U}^*_{\mu a}\widetilde{\mathcal{P}}_a(h)|\nu_a(\mathbf{p}_a,h)\rangle \qquad (20)$$

with $\widetilde{\mathcal{P}}_a(h) = \mathcal{P}_a(h)/\mathcal{P}_0(-1)$, where $\mathcal{P}_0(-1) = -\sqrt{2}G_F\cos\vartheta_C f_\pi m_\mu \sqrt{m_\pi^2 - m_\mu^2}$ is the production amplitude in the massless limit.

The weak-process states $|\nu_\mu(h)\rangle_{\rm WP}$ are quite different from the usual muon-neutrino weak states $|\nu_\mu(h)\rangle$ with definite helicity, given by Eq. (2), for, in Eq. (20), the production amplitudes $\mathcal{U}^*_{\mu a}\widetilde{\mathcal{P}}_a(h)$ of the mass eigenstates $|\nu_a(\mathbf{p}_a,h)\rangle$ are not simply given by the elements of the mixing matrix of the fields $\mathcal{U}$ alone, but depend on the specific process under consideration. The weak-process states have not been normalized to one particle in order to take into account the variation of the neutrino flux as a function of the mass eigenvalues; however, they have been normalized in such a way that the negative-helicity weak-process state $|\nu_\mu(-1)\rangle_{\rm WP}$ becomes a usual muon-neutrino state in the extremely relativistic limit. In fact, if a mass eigenstate is extremely relativistic, one has $\mathcal{P}_a(+1)\to 0$ and $\mathcal{P}_a(-1)\to\mathcal{P}_0(-1)$, independent of the index a, thus $\widetilde{\mathcal{P}}_a(+1)\to 0$ and $\widetilde{\mathcal{P}}_a(-1)\to 1$. Therefore, if all the mass eigenstates are extremely relativistic, $|\nu_\mu(+1)\rangle_{\rm WP}$ is suppressed and $|\nu_\mu(-1)\rangle_{\rm WP}$ becomes a usual muon-neutrino weak state with negative helicity $|\nu_\mu(-1)\rangle$, as given by Eq. (2).

The weak-process states $|\nu_\mu(h)\rangle_{\rm WP}$ have neither a definite momentum nor a definite energy: the energies and momenta of the mass eigenstates $|\nu_a(\mathbf{p}_a,h)\rangle$ are both determined by energy-momentum conservation in the production process [in the pion rest frame, $E_a=(m_\pi^2-m_\mu^2+m_a^2)/2m_\pi$]. Although the production amplitudes $\mathcal{P}_a(h)$ have been calculated in the pion rest frame, they are Lorentz-invariant quantities and the weight factors of the superposition of mass eigenstates in the weak-process states $|\nu_\mu(h)\rangle_{\rm WP}$ do not depend on the frame of the observer; hence the weak-process states $|\nu_\mu(h)\rangle_{\rm WP}$ describe correctly the neutrinos propagating between the two interaction processes given in Eq. (15).

In Fig. 1, for an illustrative purpose, we have plotted $\widetilde{\mathcal{P}}_a(h)$ for $h=\pm 1$ as functions of the mass eigenvalue m_a. It can be seen that $\widetilde{\mathcal{P}}_a(+1)$ is non-negligible for $m_a \gtrsim 1$ MeV and $\widetilde{\mathcal{P}}_a(-1)$ deviates from unity for $m_a \gtrsim 10$ MeV. For the maximum value $m_a = m_\pi - m_\mu$, the mass eigenstate ν_a is emitted with zero momentum and the production amplitudes for the two helicity states have the same value.

To be more specific, let us consider the following two numerical examples. (1) We take ν_3 to be the as-yet unconfirmed but still persistent 17-keV neutrino. In this case we have $\widetilde{\mathcal{P}}_3(-1)\cong 1$ and $\widetilde{\mathcal{P}}_3(+1)=4\times 10^{-4}$ for the negative- and positive-helicity states, respectively, as can be seen in Fig. 1. That is, the standard definition of the weak eigenstates contains inherent errors of up to 0.04%. (2) On the other hand, if one takes m_3 to be, say, 10 MeV which is not yet ruled out by experiment, one has $\widetilde{\mathcal{P}}_3(-1)=0.998$ and $\mathcal{P}_3(+1)=0.224$. In this case, the standard treatment, i.e., the use of Eq. (12), can lead to a result which is wrong by as much as 22%.

Let us now consider the detection process; for the pur-

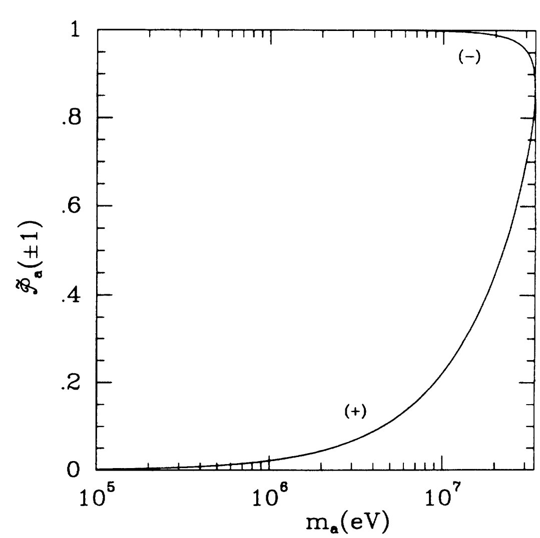

FIG. 1. Plot of the production amplitudes $\widetilde{\mathcal{P}}_a(h)$ for $h=\pm 1$ as functions of the mass eigenvalue m_a. The symbols (−) and (+) correspond to negative and positive helicities, respectively.

pose of calculating the detection weak-process states, we consider a simple nuclear neutrino capture $\nu_a(p_a)+\mathcal{N}_i(A,Z-1;p_i;0^+)\to\mathcal{N}_f(A,Z;p_f;0^+)+e^-(p_e)$ in which the nuclear transition is between states with spin-parity 0^+. The initial nucleus $\mathcal{N}_i$ is assumed to be at rest. Neglecting the recoil of the nucleus, the hadronic matrix element can simply be expressed in terms of a vector form factor F_V as [8]

$$\langle \mathcal{N}_f(A,Z;p_f;0^+)|J_\rho(0)|\mathcal{N}_i(A,Z-1;p_i;0^+)\rangle \simeq MF_V g_{\rho 0} \tag{21}$$

where $M\simeq M_i\simeq M_f$ is the nuclear mass. The final electron can be emitted in any direction. However, from energy-momentum conservation, for each direction, the absolute value $P_e\equiv|\mathbf{p}_e|$ of the momentum of the final electron can be determined from the absolute value $P_a\equiv|\mathbf{p}_a|$ of the momentum of the incoming neutrino. The helicity of the final electron is not fixed and will be denoted by h_e. The transition amplitudes $\langle e^-(p_e,h_e),\mathcal{N}_f(A,Z;p_f;0^+)|\mathcal{L}_W(0)|\mathcal{N}_i(A,Z-1;p_i;0^+),\nu_a(p_a,h)\rangle\sim\mathcal{U}_{ea}\mathcal{D}_a(h,\theta,h_e)$, which depend on the helicities $h=\pm1$ and $h_e=\pm1$ of the incoming neutrino and the outgoing electron, respectively, and on the angle θ between $\mathbf{p}_a$ and $\mathbf{p}_e$, can be calculated by using a definite representation of the γ matrices. The result is

$$\begin{aligned}\mathcal{D}_a(h,\theta,h_e)&=\frac{G_F\cos\vartheta_C}{\sqrt{2}}\overline{u_e}(p_e,h_e)\gamma^\rho(1+\gamma_5)u_\nu(p_a,h)\langle\mathcal{N}_f(A,Z;p_f;0^+)|J_\rho(0)|\mathcal{N}_i(A,Z-1;p_i;0^+)\rangle\\&=\frac{G_F\cos\vartheta_C MF_V}{2\sqrt{2}}\sqrt{(E_a-hP_a)(E_e-h_eP_e)}\left[(1+h_eh)\cos\left[\frac{\theta}{2}\right]+(h_e-h)\sin\left[\frac{\theta}{2}\right]\right].\end{aligned} \tag{22}$$

For each value of h, θ, and h_e, we can define a weak-process state $|\nu_e(h,\theta,h_e)\rangle_{\mathrm{WP}}$ given by

$$|\nu_e(h,\theta,h_e)\rangle_{\mathrm{WP}}=\sum_a\mathcal{U}_{ea}^*\widetilde{\mathcal{D}}_a(h,\theta,h_e)|\nu_a(\mathbf{p}_a,h)\rangle \tag{23}$$

with $\widetilde{\mathcal{D}}_a(h,\theta,h_e)=\mathcal{D}_a(h,\theta,h_e)/\mathcal{D}_0(-1,\theta,h_e)$, where $\mathcal{D}_0(-1,\theta,h_e)$ is the detection amplitude in the massless limit.

In the standard treatment, it is straightforward to derive the standard oscillation probability for extremely relativistic neutrinos by using the weak states defined in Eq. (2). In our treatment, which *is valid for nonrelativistic as well as relativistic neutrinos,* the neutrino oscillation probability must be calculated by taking into account the corrections due to the production and detection processes (strictly speaking, a process-independent oscillation probability does not exist, in general). From Eqs. (18), (20), and (23), the oscillation probability for the process given in Eq. (15) is given by

$$\begin{aligned}P_{\mu\to e}(\mathbf{x},t)&\sim\sum_{h,h_e=\pm1}\int d\theta\left|\sum_a {}_{\mathrm{WP}}\langle\nu_e(h,\theta,h_e)|\nu_a(p_a,h)\rangle e^{i\mathbf{p}_a\cdot\mathbf{x}-iE_at}\langle\nu_a(p_a,h)|\nu_\mu(h)\rangle_{\mathrm{WP}}\right|^2\\&\sim\sum_{h,h_e=\pm1}\int d\theta\left|\sum_a\widetilde{\mathcal{D}}_a(h,\theta,h_e)\widetilde{\mathcal{P}}_a(h)\mathcal{U}_{ea}\mathcal{U}_{\mu a}^*e^{i\mathbf{p}_a\cdot\mathbf{x}-iE_at}\right|^2.\end{aligned} \tag{24}$$

This oscillation probability is a generalization of the standard expression, which is valid only for extremely relativistic neutrinos, the difference being the presence of the terms $\widetilde{\mathcal{D}}_a(h,\theta,h_e)\widetilde{\mathcal{P}}_a(h)$ that take into account the nonrelativistic corrections due to the production and detection processes.

To demonstrate potential significance of the nonrelativistic corrections in Eq. (18), let us consider the electron neutrinos with the energy of about 1 MeV. For definiteness, we assume $m_3=100$ keV. If one ignores details on dynamics and kinematics of the production and detection processes, one roughly expects

$$\begin{aligned}&\widetilde{\mathcal{D}}_3(-1)\sim\widetilde{\mathcal{P}}_3(-1)\sim1-\epsilon,\\&\widetilde{\mathcal{D}}_3(+1)\sim\widetilde{\mathcal{P}}_3(+1)\sim\epsilon'\end{aligned} \tag{25}$$

with $\epsilon\sim(m_3/E)^2\sim0.01$ and $\epsilon'\sim(m_3/E)\sim0.1$. Equation (25) indicates that the use of the standard formula in this example can lead to errors of up to several percent in the oscillation probability. The exact values, of course, depend on specific production and detection mechanisms. At present, the nonrelativistic corrections we have discussed so far are negligibly small and are of academic interest. However, when oscillations are observed in the future and their precision analysis becomes necessary, the corrections discussed in this paper must be taken into account for nonrelativistic neutrinos.

IV. CONCLUSIONS

We have shown that the weak states which are used in the standard treatment of neutrino oscillations do not, in general, describe correctly the neutrinos produced and detected in weak-interaction processes. We have also shown that it is impossible to construct a Fock space of weak states. However, neutrinos oscillations can be described by defining appropriate "weak-process states" in which the superposition of mass eigenstates is weighted by their transition amplitudes in the process under consideration. In the extreme relativistic limit, the weak-process states reduce to the usual weak states.

The present discussion can be applied to any set of mixed particles with a significant mass difference. In the K^0-$\overline{K^0}$ case, the relative mass difference $|M_{K_L}-M_{K_S}|/|M_{K_L}+M_{K_S}|\sim10^{-15}$ is too small to produce any measurable effect. In supersymmetric models in which superparticles are mixed, there can be large effects.

ACKNOWLEDGMENTS

The authors would like to thank G. Feldman for his interest in this work and for many helpful discussions. This work was supported in part by the Fondazione Angelo Della Riccia (Italy) and the Korea Science and Engineering Foundation. One of the authors (C.W.K.) wishes to thank members of the Center for Theoretical Physics, Seoul National University for their warm hospitality.

[1] S. M. Bilenky and B. Pontecorvo, Phys. Rep. **41**, 225 (1978); S. M. Bilenky and S. T. Petcov, Rev. Mod. Phys. **59**, 671 (1987); P. H. Frampton and P. Vogel, Phys. Rep. **82**, 339 (1982); F. Boehm and P. Vogel, *Physics of Massive Neutrinos* (Cambridge University Press, Cambridge, England, 1987); B. Kayser, *The Physics of Massive Neutrinos,* Lecture Notes in Physics Vol. 25 (World Scientific, Singapore, 1989); R. N. Mohapatra and P. B. Pal, *Massive Neutrinos in Physics and Astrophysics* (World Scientific, Singapore, 1991).

[2] R. G. Winter, Lett. Nuovo Cimento **30**, 101 (1981).

[3] R. E. Shrock, Phys. Lett. **96B**, 159 (1980); Phys. Rev. D **24**, 1232 (1981); **24**, 1275 (1981).

[4] J. Schechter and J. W. F. Valle, Phys. Rev. D **22**, 2227 (1980).

[5] K. M. Case, Phys. Rev. **107**, 307 (1957).

[6] J. Schechter and J. W. F. Valle, Phys. Rev. D **23**, 1666 (1981).

[7] C. Giunti, C. W. Kim, and U. W. Lee (in preparation).

[8] C. W. Kim and H. Primakoff, in *Mesons in Nuclei,* edited by M. Rho and D. H. Wilkinson (North-Holland, Amsterdam, 1979), p. 68.