Thermodynamics of event horizons in (2+1)-dimensional gravity

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Although gravity in $2+1$ dimensions is very different in nature from gravity in $3+1$ dimensions, it is shown that the laws of thermodynamics for event horizons can be manifested also for (2+1)-dimensional gravity. The validity of the classical laws of horizon mechanics is verified in general and exemplified for the (2+1)-dimensional analogues of Reissner-Nordström and Schwarzschild–de Sitter spacetimes. We find that the entropy is given by $\frac{1}{4}L$, where L is the length of the horizon. A consequence of having consistent thermodynamics is that the second law fixes the sign of Newton's constant to be positive.

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I. INTRODUCTION

The theory of general relativity in $2+1$ dimensions retains the same formal structure as the (3+1)-dimensional theory. The metric g_{ab}, connection Γ^c_{ab}, and Riemann tensor R_{abcd} are defined in the same way, and the Einstein equation still holds. However, this similarity is misleading; the nature of the theory in $2+1$ dimensions is quite different [1].

In $2+1$ dimensions, the number of independent components of R_{abcd} is six, the same as that of the Ricci tensor R_{ab}. The Weyl tensor C_{abcd} vanishes identically. Consequently, the Riemann tensor can be expressed in terms of the stress tensor and the metric alone. Therefore, in contrast to gravity in any higher dimension, the stress tensor has only a local effect on the curvature, i.e., the curvature at a point is nonzero if and only if the stress tensor is nonzero. It is this which changes dramatically the nature of the theory.

Because curvature requires a source, one does not have a solution of the $2+1$ vacuum Einstein equations which would be the analogue of a $3+1$ black hole. In fact, in $2+1$ dimensions the "Schwarzschild" and "Kerr" solutions [2] give rise only to a globally nontrivial geometry; locally, except at the source spacetime is flat. Furthermore, the "Newtonian" limit of this theory breaks down [1,2] and does not fix the value or the sign of its analogue of Newton's constant.

In $3+1$ dimensions there is a well-known connection between the classical laws for event-horizon mechanics and the laws of thermodynamics [3-5] which has led to some far-reaching suggestions on the nature of the theory of quantum gravity [6-8]. Therefore, if this connection holds also for $2+1$ dimensions, then thermodynamics may be of significance to the quantization of $2+1$ gravity. The purpose of this work is to show that one can still manifest the thermodynamical laws for event horizons with all of its ramifications (entropy, Hawking radiation, etc.), even though the nature of gravity in $2+1$ dimensions is very different.

Pure gravity is flat in $2+1$ dimensions; however, by coupling gravity to some extended source, one finds classical solutions with event horizons [11–14]. We examine the general validity of the classical laws of horizon mechanics. Following Bekenstein [3,4], we will then regard these to be the laws of thermodynamics. As a consequence of this procedure, we can conclude that, if (2+1)-dimensional gravity is to have consistent laws of horizon thermodynamics, then the sign of Newton's constant must be fixed as positive. As examples, we study the thermodynamics of the (2+1)-dimensional analogues of the Reissner-Nordström and Schwarzschild–de Sitter spacetimes. If gravity is coupled to other fields [9,10], similar behavior is expected.

The plan of the paper is as follows. In Sec. II, we describe some classical solutions of the Einstein equation in $2+1$ dimensions, which possess event horizons. In Sec. III, the first law of event-horizon mechanics is constructed and the entropy and temperature are identified. Finally, in Sec. IV we verify the general validity of the second law and discuss its relation to the sign of Newton's constant. We shall adopt units such that $\hbar=k_B=c=1$, and use a metric with a signature $(-,+,+)$.

II. EVENT HORIZONS IN 2+1 DIMENSIONS

In this section we shall discuss some examples of exact solutions of Einstein's equation in $2+1$ dimensions coupled to an extended source. Provided that Newton's constant is positive, the resulting spacetimes possess an outer event horizon similar to a cosmological horizon.

The simplest solution of the Einstein equation

$$G_{ab}=R_{ab}-\tfrac{1}{2}g_{ab}R=8\pi G T_{ab}-\Lambda g_{ab} \tag{2.1}$$

with a horizon is obtained by setting $T_{ab}=0$ and $\Lambda>0$. The solution, known as the de Sitter spacetime, is given in static coordinates as

$$ds^2=-(1-\Lambda r^2)dt^2+(1-\Lambda r^2)^{-1}dr^2+r^2d\theta^2 \,. \tag{2.2}$$

The singularity of the metric at $r^2=1/\Lambda$ corresponds to the cosmological event horizon.

When a massive point source is added, we obtain the (2+1)-dimensional analogue of the Schwarzschild–de

Sitter spacetime which reads [11]

$$ds^2 = -\left[1-\frac{\Lambda}{\alpha^2}r^2\right]dt^2+\alpha^{-2}\left[1-\frac{\Lambda}{\alpha^2}r^2\right]^{-1}dr^2+r^2d\theta^2 \; , \tag{2.3}$$

where $\alpha=1-4Gm$ and m is the mass of the source. Unlike the solution in $3+1$ dimensions, there is no black-hole horizon. The effect of the mass is to shift the horizon's location to $r^2=\alpha^2/\Lambda$. It can be shown [11] that the mass changes the global structure of this spacetime from a sphere to a sphere minus a wedge with the edges identified.

Normalizing the Killing vector $\xi=\partial/\partial t$ to have unit length at $r=0$, the surface gravity κ, defined by

$$\xi^a{}_{;b}\xi^b=\kappa\xi^a \; , \tag{2.4}$$

takes on the horizon the value

$$\kappa_{\rm SD}=\sqrt{\Lambda} \; . \tag{2.5}$$

Now, consider the coupling of Einstein's equation to a charged massive pointlike source. Then, in addition to (2.1), we have the Maxwell equations

$$F^{ab}{}_{;b}=2\pi j^a \; , \tag{2.6}$$

$$F_{[ab;c]}=0 \; . \tag{2.7}$$

The stress tensor is the sum of the stress tensor of the field F_{ab}

$$T^a_{Fb}=(1/2\pi)(F^a{}_cF_b{}^c-\tfrac{1}{4}g^a{}_bF^2) \tag{2.8}$$

and the stress tensor of the matter.

The solution of Eq. (2.1), (2.6), and (2.7), with $\Lambda=0$, is given by [12–14]

$$ds^2=-\left[1-\frac{\beta}{\alpha^2}\ln\frac{r}{r_0}\right]dt^2 +\alpha^{-2}\left[1-\frac{\beta}{\alpha^2}\ln\frac{r}{r_0}\right]^{-1}dr^2+r^2d\theta^2 \; , \tag{2.9}$$

where $\beta=4Ge^2$, e is the charge of the source and r_0 is some constant. This spacetime is the $(2+1)$-dimensional analogue of the Reissner-Nordström solution.

The metric (2.9) possesses a regular singularity at $r_H=r_0e^{\alpha^2/\beta}$, which (provided that $G>0$) corresponds to an outer event horizon. A second singularity exists at the location of the source ($r=0$). Since the curvature scalar, $R^{ab}R_{ab}=\beta^2/r^4$, diverges at this point, $r=0$ is a real singularity. However, it is the infinite electric field generated by the charged point source which is responsible for this singularity. The singularity can be avoided by replacing the point charge by a smooth charge distribution.

The full structure of this spacetime is obtained by an extension to Kruskal coordinates. In these coordinates the metric reads [13,14]

$$ds^2=-4\alpha^2e^{2\alpha^2/\beta}e^{-\Sigma(r)}dU\,dV+r^2d\theta^2 \; . \tag{2.10}$$

The function $\Sigma(r)$ is regular everywhere except at the source. The coordinates U and V are related to the Schwarzschild coordinates r,t through the relations

$$UV=[(\beta/\alpha^2)\ln(r/r_0)-1]e^{\Sigma(r)} \; , \tag{2.11}$$

$$V/U=-\exp[(\beta/\alpha r_0)e^{-\alpha^2/\beta}t] \; . \tag{2.12}$$

It is interesting to note that, if Newton's constant is negative, the outer horizon becomes an inner horizon and the spacetime (obtained also by rotating the U-V axis by 90°) becomes very similar to that of the (extended) Schwarzschild spacetime, i.e., to that of a black hole. However, as we shall see, this spoils the validity of the second law of thermodynamics.

Finally, normalizing the Killing vector to have unit length at $r=r_0$, the surface gravity and electric potential $\Phi=a_c\xi^c$ are given on the horizon by

$$\Phi_{\rm RN}=(1-4Gm)/4Ge \; , \tag{2.13}$$

$$\kappa_{\rm RN}=\frac{1}{2r_0}\frac{\beta}{\alpha}\exp\left[-\frac{\alpha^2}{\beta}\right] . \tag{2.14}$$

III. THERMAL RADIATION AND THE FIRST LAW

The presence of an event horizon implies that our $(2+1)$-dimensional spacetime should have a temperature and particles should be created with a thermal distribution. In order to calculate the details of this process, and, in particular, the temperature of the radiation, one could use the standard method of Bogolubov transformations or the path integral method [15,16]. However, the simplest way to derive the temperature is to examine the analyticity of the Green's function $G(x',x)$ with respect to the Schwarzschild time coordinate in the complex plane [17].

The Green's function $G(x',x)$ is analytic with respect to the Kruskal coordinates U and V. However, an observer limited to the static region $r<r_H$ should be described by the Schwarzschild coordinates r and t. By Eq. (2.12), the Schwarzschild imaginary time coordinate is a multivalued function of U and V with a period $P=2r_0(\alpha/\beta)\exp(\alpha^2/\beta)$. Therefore, $G(x',x)$ is also periodic, in these coordinates, with the same period. This implies that, with respect to an observer who sits at $r=r_0$ of the Reissner-Nordström spacetime, the Green's function describes a thermalized state whose temperature T is given by

$$T_{\rm RN}=\frac{1}{2\pi P}=\frac{1}{4\pi r_0}\frac{\beta}{\alpha}\exp\left[-\frac{\alpha^2}{\beta}\right]=\frac{\kappa_{\rm RN}}{2\pi} \; . \tag{3.1}$$

By the same procedure, the temperature of the Schwarzschild–de Sitter space time, with respect to an observer at $r=0$ is

$$T_{\rm SD}=\sqrt{\Lambda}/2\pi=\kappa_{\rm SD}/2\pi \; . \tag{3.2}$$

Therefore, the factor of proportionality between the temperature and the surface gravity (which is constant over the horizon) is given by $1/2\pi$.

In order to identify the entropy related to this temperature, we shall derive the first law of horizon mechanics [18] and subsequently identify it as the first law of thermodynamics. We shall consider a spacetime with an outer horizon, since, provided that $G>0$, black-hole horizons do not exist. Also, as will be argued later, $G>0$ is required for consistent thermodynamics. For simplicity, we assume that there is no angular momentum.

Let S be a two-surface tangent to the timelike Killing vector ξ and bounded by the horizon. Then by the identity

$$\xi^{a;b}{}_{;b}=-R^{a}{}_{b}\xi^{b} , \tag{3.3}$$

we get

$$\frac{1}{8\pi}\oint_H \xi^{a;b}dS_{ab}=-\int_S T^a{}_b\xi^b d\Sigma_a-\frac{1}{16\pi G}\int_S R\xi^a d\Sigma_a . \tag{3.4}$$

The integral on H is taken over the intersection of the two-surface S and the horizon. The stress tensor is given by $T^{ab}=T^{ab}_M+T^{ab}_F$, where T^{ab}_M is the matter part and T^{ab}_F is given by (2.8). Substituting the electromagnetic stress tensor and using the constancy of the surface gravity and the electric potential on the horizon we get

$$-\frac{\kappa}{8\pi}L-\Phi^H Q_H=\int_S(T^a_{Mb}\xi^b-j^a A_c\xi^c)d\Sigma_a+\frac{1}{16\pi G}\int_S(R+2GF^2)\xi^a d\Sigma_a . \tag{3.5}$$

The length of the horizon is denoted by L, Q_H is the total charge confined within the horizon, and j^a is the electric current. From (3.5) we can obtain a differential form of the energy conservation law (i.e., the first law of event-horizon mechanics). The variation of the right-hand side with respect to the metric vanishes and gives rise only to surface terms, which cancel with the variation of the terms on the left-hand side with respect to $\delta\kappa$ and $\delta\Phi^H$. Therefore, only the variations with respect to the matter and electromagnetic field degrees of freedom are left. The final result for the first law is

$$-\int_S \delta T^a_{Mb}\xi^b d\Sigma_a=\frac{\kappa}{8\pi}\delta L+\Phi^H\delta Q_H+\int_S \xi^c A_c(\delta j^a)d\Sigma_a+\int_S j^{[c}\xi^{a]}(\delta A_c)d\Sigma_a . \tag{3.6}$$

The term on the left-hand side is the variation in the matter energy-momentum tensor within the horizon. The two first terms on the right-hand side are to be identified as the $TdS+\phi dQ$ terms of the first law. The third term represents the change in the self-interaction of the charge density, while the last term will vanish for a static source. The minus sign occurring on the left-hand side of (3.6) (see also, Ref. [16]) does not mean a change in the first law since $\int\delta T^a_{Mb}\xi^b d\Sigma_a$ corresponds to a variation in the mass (i.e., the mass energy associated with T^{ab}_M) within the region S. Dynamically, the first law states that a decrease of the mass in S will cause an increase of the horizon's length. An observer who sits in region S and drops some of the mass beyond the horizon is an example of such a process.

We have verified that $\kappa/2\pi$ plays the role of temperature, therefore, we suggest that in $2+1$ dimensions it is $\frac{1}{4}L$ which plays the role of entropy. Following Gibbons and Hawking [16] we interpret the entropy $\frac{1}{4}L$ as the information hidden beyond the outer horizon to an observer who sits in the region S.

Let us apply the general results for the spacetimes discussed in Sec. II. For the $(2+1)$-dimensional Reissner-Nordström spacetime, which is generated by a charged pointlike source, the two last terms on the right-hand side of (3.6) vanish. Therefore, the first law takes the form

$$-\delta m=T\delta S+\Phi\delta e , \tag{3.7}$$

where the temperature is given by (3.1) and the entropy by

$$S_{\rm RN}=\frac{\pi}{2}r_0\exp\left[\frac{\alpha^2}{\beta}\right]=\frac{\pi}{2}r_0\exp\left[\frac{(1-4Gm)^2}{4Ge^2}\right] . \tag{3.8}$$

As a check, we calculate Φ and T directly from Eq. (3.8). Inverting Eq. (3.8), we get a formula for the mass in terms of the entropy and the charge

$$m(S,e)=\frac{1}{4G}-\frac{1}{4G}\left[4Ge^2\ln\frac{2S}{\pi r_0}\right]^{1/2} . \tag{3.9}$$

Then, the temperature and the electric potential are given by

$$T=-\frac{\partial m}{\partial S}=\frac{1}{4\pi r_0}\frac{\beta}{\alpha}e^{-\alpha^2/\beta}=\frac{1}{2\pi}\kappa_{\rm RN} , \tag{3.10}$$

$$\Phi=-\frac{\partial m}{\partial e}=\frac{1-4Gm}{4Ge}=\Phi_{\rm RN} \tag{3.11}$$

in agreement with our former results.

For the Schwarzschild–de Sitter spacetime, the first law reads

$$-\delta m=T\delta S . \tag{3.12}$$

The temperature is given by (3.2) and the entropy by [20]

$$S_{\rm SD}=(\pi/2)(1-4Gm)/\sqrt{\Lambda} . \tag{3.13}$$

Finally, one can verify that the entropy (3.8) or (3.13) cannot decrease by a classical process that changes the total mass or charge in the region $r<r_H$. For example, suppose a particle of charge e_0 and mass m_0 is dropped beyond the horizon of the Reissner-Nordström spacetime. If this process increases the charge e of the source, it seems, by Eq. (3.8), that the entropy will go down. However, not all values of e_0 and m_0 are allowed. The radial "effective potential" equation for the particle is

$$\frac{1}{\alpha^2}\left[\frac{dr}{d\tau}\right]^2=\left[m_0+e_0\frac{e}{\alpha}\ln\frac{r}{r_0}\right]^2-\left[1-\frac{\beta}{\alpha^2}\ln\frac{r}{r_0}\right]\left[1+\frac{L^2}{r^2}\right] , \tag{3.14}$$

where τ is the proper time and L is the angular momentum. A particle will cross the horizon provided that $dr/d\tau\geq 0$ at the horizon. Therefore, since $\delta e=-e_0$ and $\delta m=-m_0$,

$$-\delta m\geq\delta e(e/\alpha)\ln(r_H/r)=\Phi\delta e \tag{3.15}$$

must be satisfied, which leads, by (3.7), to $\delta S\geq 0$. This is an example for the validity of the second law. The general validity of the second law will be studied in the following section.

IV. THE SECOND LAW AND THE SIGN OF NEWTON'S CONSTANT

We wish to demonstrate the second law of event-horizon mechanics, i.e., that the length of the horizon never decreases. Consider a general spacetime with a horizon defined as follows [16]: Let λ stand for an observer's world line which has an infinite length in the future direction and does not run into a singularity. Then, the event horizon is defined as $\dot{I}^-(\lambda)$ (I^- is the chronological past), the boundary of the past of the timelike curve λ. This observer-dependent definition is suitable for the spacetimes discussed previously. For example, we can regard the world line of an observer who sits in the Reissner-Nordström spacetime at a constant distance from the source. We shall assume that the horizon is predictable; that is, the portion of the event horizon $\dot{I}^-(\lambda)\cap J^+(\mathcal{S})$ is contained in the future Cauchy development, $D^+(\mathcal{S})$, of the two-surface $\mathcal{S}$.

The horizon is generated by null geodesics. Therefore, to demonstrate the second law, it will suffice to show that the generators of a predictable event horizon cannot be converging. To this end, let us consider a smooth one-parameter subfamily $\gamma_s(\tau)$ of null geodesics in a congruence, and the vector field k^a of tangents. The physical class of deviations spans, in general, an $(n-2)$-dimensional space denoted by $\hat{V}_p$. The projection of a tensor into this space will be denoted as "hatted." In particular, the metric g_{ab} gives rise to the hatted metric $\hat{h}_{ab}$.

The expansion of null geodesics is described by Raychaudhuri's equation [19]. Since, in 2+1 dimensions the subspace of deviations $\hat{V}_p$ is one dimensional, the shear and twist vanish [21] and Raychaudhuri's equation is given simply by

$$k^c\nabla_a\theta=d\theta/d\tau=-\theta^2-R_{ab}k^ak^b\ , \tag{4.1}$$

where $\theta=\hat{h}^{ab}\nabla_{\hat{b}}k_a$ is the expansion. By the Einstein equation, the right-hand side of Eq. (4.1) is negative provided that

$$R_{ab}k^ak^b=8\pi G T_{ab}k^ak^b\geq 0\ , \tag{4.2}$$

A sufficient condition for this inequality to hold is that the stress tensor satisfies the weak or strong energy condition (i.e., for any timelike vector v^a, $T_{ab}v^av^b\geq 0$ or $T_{ab}v^av^b\geq T^a_av^bv_b$, respectively). If the stress tensor T_{ab} is diagonalizable, with eigenvalues μ and p_i $(i=1,2)$, standing for the energy density and the principle pressures, respectively, the strong energy condition is satisfied if $\sum_i p_i\geq 0$ and $\mu+p_i\geq 0$ $(i=1,2)$.

The validity of Raychaudhuri's equation for null geodesics is of great importance. Assuming the inequality (4.2) holds, one can show that the null geodesics, which generate a predictable event horizon, must have $\theta\geq 0$. An immediate consequence of the positivity of the expansion is the following mechanical law for event horizons, which will be regarded as the second law of thermodynamics: The length of any connected one surface in a predictable event horizon cannot decrease with time.

Notice that, since this result depends on the validity of the inequality (4.2), the second law would not necessarily hold if Newton's constant would have been taken as negative. In fact, for the Reissner-Nordström and Schwarzschild–de Sitter spacetimes discussed above, one can show that, if G is negative, the second law fails. Therefore, even though in 2+1 dimensions Newton's constant cannot be determined from a Newtonian limit, if the theory under question is to satisfy the second law of thermodynamics, the sign of Newton's constant must be fixed to be positive.

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