Schwinger terms in charge-density commutators

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(Received 3 December 1991)

The necessity for q-number Schwinger terms in the equal-time commutators between flavor charge densities for chiral fermions in $3+1$ dimensions is shown. The charge densities here are not coupled to any gauge field. The number of quark species (colors) gives the central charge of an infinite-dimensional Lie algebra. The result is obtained by considering the analogue of the double spectral function in current-current correlation functions.

PACS number(s): 11.40.Dw, 11.30.Rd

An abiding mystery in the standard model concerns electroweak symmetry breaking, whose origins probably lie beyond the dynamics of elementary scalar fields [1–3]. Any such new dynamics that will do the job is described by fields with stronger interactions than those mediated by the gauge fields of the standard model. In this respect, the situation is akin to that surrounding the nature of strong interactions in the 1960s. A fruitful technique in common use to investigate the strong interactions involved the algebra of the electroweak currents [4,5]. In this procedure, it is unnecessary to consider the currents to be the actual sources of electroweak gauge bosons, since these interactions are relatively weaker than the dynamics being studied at the scales under consideration. The resultant plethora of sum rules in many ways helped to define eventually quantum chromodynamics (QCD).

There has been a revival in the use of similar sum rules to study electroweak symmetry breaking [6]. How can electroweak current algebra be a useful guide in our present dilemma? In answering this question, one must first elucidate the structure of the algebra. The major difference between a current algebra and its charge counterpart is the presence of Schwinger terms in the local algebra, whose nature is dependent on the dynamics being studied through the sum rules. We shall show in this paper that in addition to the Schwinger terms examined earlier in the 1960s, there must also be similar q-number terms present in charge-density commutators, whose properties are controlled by the dynamics of the fermion fields which make up the currents in question.

Let $J^{\mu a}(\mathbf{x},t)$ be a local current, with Lorentz index μ, and flavor index a. The current algebra we shall be discussing here takes the form

$$[J^{0a}(\mathbf{x},x^0),J^{b\mu}(\mathbf{y},y^0)]|_{x^0\to y^0} = f^{ab}{}_c J^{c\mu}(\mathbf{x},x^0)\delta^3(\mathbf{x}-\mathbf{y})+S^{ab\mu i}\partial_i\delta^3(\mathbf{x}-\mathbf{y}) \ . \quad (1)$$

Here, the quantity $f^{ab}{}_c$ is the structure constant of the algebra in question. The quantity $S^{ab\mu i}$, i denoting spatial components, will be referred to as the Schwinger term in what follows. Its presence is not supposed to alter the algebraic structure of the charges obtained by spatially integrating Eq. (1).

The necessity for such terms was first noted by Schwinger [7], who considered the case of $\mu=i$ in Eq. (1). The left-hand side survives in the vacuum expectation value (VEV), since it is given by the integral over the spectral function defined by the currents. Since this integral cannot be zero on grounds of positivity of the spectrum, the existence of the Schwinger term on the right is thereby established.

Actually, there is a direct check when the currents consist of fermion bilinears. Since the VEV is evaluated at equal times, the Schwinger term is given by the absorptive part of the vacuum-polarization diagram. Perturbatively, this diagram is divergent, and is given by an integral over the positive-energy states of the fermion-antifermion pair. The presence of other states in the spectrum can only add to this result.

Several remarks can be made about this exercise. Firstly, the VEV's considered are sensitive to c-number Schwinger terms only. Secondly, there is no similar necessity for c-number Schwinger terms in the time-time component in Eq. (1). Whether there are q-number terms remains an open question, since these will only appear in matrix elements of Eq. (1) between excited states, or in higher-point current amplitudes, which can mimic the presence of such excited states.

Notice that for the charge and charge-density commutator not to be disturbed, $S^{0,i}$ in Eq. (1) must be automatically divergenceless, and given by the curl of another vector. When the currents are made up of Dirac fermion bilinears, it is therefore safe to ignore such terms in the time-time component, since they will not affect the amplitudes of interest. On the other hand, electroweak interactions involve chiral fermions, and in principle these kind of terms can appear in physical amplitudes.

We show in this paper that in fact such terms are q numbers, and must be present in any charge-density algebra involving chiral fermion bilinears [8]. Furthermore, unlike in the case of the c-number Schwinger terms, they are generally not canceled by seagull terms in applications to low-energy processes, and so control both high- and low-energy behavior. They are explicitly dependent on the number of quark species of the underlying theory, and can distinguish between $(V-A)$ currents and $(V+A)$-type currents. This extended algebra could therefore constrain the underlying dynamics in a finer way than the older algebra ever could.

Since the q-number Schwinger terms contain ϵ symbols, they show up only in the parity-odd parts of products of four or more currents. We shall examine the relevant commutators through the double spectral function for the four-point correlation amplitude, defined by having all internal fermions in the box diagrams on the mass shell. In essence, we shall compute various vacuum expectation values of charge-density operators at equal times, and then assemble these pieces directly to form multiple commutators for the algebra. This procedure is to be contrasted with what happened in the original Schwinger term, for which only the single spectral function need be considered [7].

We begin with the vacuum expectation value of the product of four left-handed charges. For massless quarks, the left-handed $(V-A)$ currents can be written in terms of the left-handed two components of the fermions. $L^a=(V^a-A^a)/2=:\psi^\dagger T^a\psi(x):$, where ψ is the two-component left-handed fermion field, $\sigma^\mu_+\partial_\mu\psi=0$, $(\partial_t\mp\boldsymbol{\sigma}\cdot\nabla)\equiv\sigma^\mu_\pm\partial_\mu$. Here, $\{T^a\}$, $a=1,\ldots,N^2$, is a basis for $N\times N$ anti-Hermitian matrices, normalized to $\mathrm{tr}(T^aT^b)=-\delta^{ab}/2$. They represent the generators of the $U_L(N)$ left-handed flavor symmetries. We encounter two kinds of terms in evaluating the amplitude by Wick contractions:

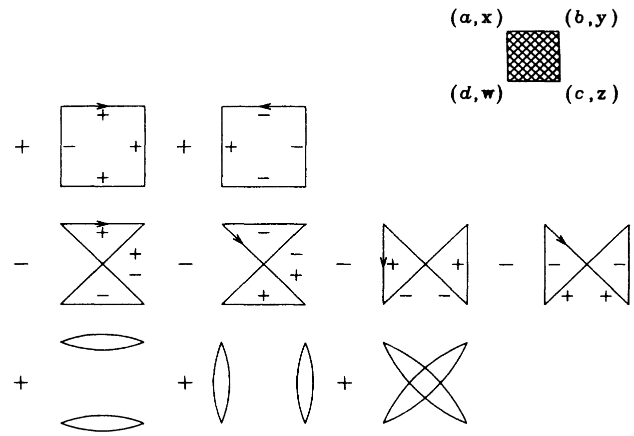

FIG. 1. Contractions for the product of four charge densities.

$$\langle 0|\psi^\dagger(y)\otimes\psi(x)|0\rangle=S_-(x-y)\ ,$$
$$\langle 0|\psi(x)\otimes\psi^\dagger(y)|0\rangle=S_+(x-y)\ , \tag{2}$$

where

$$S_\pm(x)=\pm\frac{i}{(2\pi)^2}\sigma^\mu_-\partial_\mu[(t\pm i\epsilon)^2-\mathbf{x}^2]^{-1}\ . \tag{3}$$

Let $T^{abc\cdots}\equiv\mathrm{tr}(T^aT^bT^c\cdots)$ and

$$A[x(\pm_1)y(\pm_2)z(\pm_3)\cdots]=\mathrm{tr}[S_{\pm_1}(x-y)S_{\pm_2}(y-z)S_{\pm_3}\cdots]\ .$$

There are nine different contractions contributing to $\langle 0|L^a(x)L^b(y)L^c(z)L^d(w)|0\rangle$ (cf. Fig. 1):

$$\begin{aligned}\langle 0|L^a(x)L^b(y)L^c(z)L^d(w)|0\rangle\\ =T^{abcd}A[x(+)y(+)z(+)w(-)]+T^{adcb}A[x(+)w(-)z(-)y(-)]-T^{acdb}A[x(+)z(+)w(-)y(-)]\\ -T^{abdc}A[x(+)y(+)w(-)z(-)]-T^{adbc}A[x(+)w(-)y(+)z(-)]-T^{acbd}A[x(+)z(-)y(+)w(-)]\\ +T^{ab}T^{cd}A[x(+)y(-)]A[z(+)w(-)]+T^{ac}T^{bd}A[x(+)z(-)]A[y(+)w(-)]\\ +T^{ad}T^{bc}A[x(+)w(-)]A[z(+)z(-)]\ .\end{aligned} \tag{4}$$

We need the parity-odd part of this quantity. Three out of the nine contractions, i.e., the last three terms, can be factorized into contributions of two-point functions which cannot contain the ϵ symbol. Since we are interested in the equal-time commutators, we set the "propagators" $S_\pm(x)$ to be also at equal time, and proceed to extract the parity-odd parts from the remaining terms. The three-dimensional Fourier transform of $S_\pm(\mathbf{x})$ has a very simple form $\tilde{S}_\pm(\mathbf{p})=(1+\boldsymbol{\sigma}\cdot\hat{\mathbf{p}})/2$ and can be regarded as projection operators to the $\pm$ energy states. In this space, we can classify the terms by their degree of divergence under momentum integration. As an example, consider the term where $S_\pm$ is strung together in the order $(++--)$. Write the three-momentum flow from the first vertex to the second as $\mathbf{p}_{12}$ and $\tilde{S}_\pm(\mathbf{p}_2)\equiv\tilde{S}_{12\pm}$, and so on. Using the identity $\tilde{S}_\pm\tilde{S}_\mp=0$,

$$\mathrm{tr}(\tilde{S}_{12+}\tilde{S}_{23+}\tilde{S}_{34-}\tilde{S}_{41-})=\mathrm{tr}[\tilde{S}_{12+}\tilde{S}_{23+}(\tilde{S}_{34-}-\tilde{S}_{23-})(\tilde{S}_{41-}-\tilde{S}_{12-})]\ .$$

The first two factors contribute a degree of divergence of 0 while that for the last two factors has been reduced to -1 each. The total degree of divergence under momentum integration then is $3-2=1$. The same result is obtained for $(+++-)$, $(+---)$. For $(+-+-)$,

$$\mathrm{tr}(\tilde{S}_{12+}\tilde{S}_{23-}\tilde{S}_{34+}\tilde{S}_{41-})=\mathrm{tr}[\tilde{S}_{12+}(\tilde{S}_{23-}\tilde{S}_{12-})(\tilde{S}_{34+}-\tilde{S}_{23+})(\tilde{S}_{41-}-\tilde{S}_{12-})]$$

so the degree of divergence here is $3-3=0$. Now, integrals with linear degrees of divergence depend on momentum shifts, while those with zero degrees do not. To be explicit, we do the integration $\int d^3l$ in a solid ball with a fixed origin and the radius taken to be infinity. $\mathbf{p}_{12}=\mathbf{l}+\mathbf{k}_{12}$, $\mathbf{k}_{23}-\mathbf{k}_{12}=\mathbf{k}_2$, etc. The parity-odd (PO) part of the integral is given by

$$\mathrm{PO}\frac{1}{(2\pi)^3}\int d^3l\,\mathrm{tr}(\tilde{S}_{12\pm_1}\tilde{S}_{23\pm_2}\tilde{S}_{34\pm_3}\tilde{S}_{41\pm_4})=(\pm_1)(\pm_2)(\pm_3)(\pm_4)\frac{i}{48\pi^2}\times[\pm_1(\mathbf{k}_{23}\times\mathbf{k}_{34})\cdot\mathbf{k}_{41}\pm_2(\mathbf{k}_{34}\times\mathbf{k}_{41})\cdot\mathbf{k}_{12}\pm_3(\mathbf{k}_{41}\times\mathbf{k}_{12})\cdot\mathbf{k}_{23}\pm_4(\mathbf{k}_{12}\times\mathbf{k}_{23})\cdot\mathbf{k}_{34}]\ . \tag{5}$$

It is now a trivial matter to check that the orderings with linear divergences are not invariant with respect to shifts of the momenta, $\mathbf{k}_{i,i+1}\to\mathbf{k}_{i,i+1}+\mathbf{d}$, while the orderings, such as $(+-+-)$, are invariant under momentum shifts, consistent with their zero divergence degrees. We will adopt a minimal regularization procedure with requisite momentum shifts so that the four noninvariant integrals vanish. The remaining two types of integrals are unambiguous, and in momentum space become

$$\frac{i}{48\pi^2}[T^{adbc}(\mathbf{k}_1\times\mathbf{k}_4)\cdot\mathbf{k}_2+T^{acbd}(\mathbf{k}_1\times\mathbf{k}_3)\cdot\mathbf{k}_2]\ . \tag{6}$$

So in coordinate space, we finally have

$$\mathrm{PO}\langle 0|L^a(\mathbf{x})L^b(\mathbf{y})L^c(\mathbf{z})L^d(\mathbf{w})|0\rangle=\frac{1}{48\pi^2}(T^{acbd}-T^{adbc})[\nabla\delta^3(\mathbf{x}-\mathbf{w})\times\nabla\delta^3(\mathbf{y}-\mathbf{w})]\cdot\nabla\delta^3(\mathbf{z}-\mathbf{w})\ . \tag{7}$$

It is obvious from the calculation that if the fermions are repeated m times, the results will be multiplied by m.

We are now ready to see what this result has to say about charge-density commutators. Consider the two quantities

$$\langle 0|\{L^c(\mathbf{z}),\{L^b(\mathbf{y}),[L^a(\mathbf{x}),L^d(\mathbf{w})]\}\}|0\rangle\ ,$$

$$\langle 0|[L^c(\mathbf{z}),[L^b(\mathbf{y}),[L^a(\mathbf{x}),L^d(\mathbf{w})]]]|0\rangle\ .$$

From Eq. (7),

$$\langle 0|\{L^c(\mathbf{z}),\{L^b(\mathbf{y}),[L^a(\mathbf{x}),L^d(\mathbf{w})]\}\}|0\rangle=-\frac{m}{48\pi^2}f^{bce}d^{ead}\nabla\delta^3(\mathbf{x}-\mathbf{w})\cdot[\nabla\delta^3(\mathbf{y}-\mathbf{w})\times\nabla\delta^3(\mathbf{z}-\mathbf{w})]\ . \tag{8}$$

The flavor indices (a,d) appear *symmetrically*. Naive algebras can only give results antisymmetric in (a,d). So, there must be a new term in $[L^a(\mathbf{x}),L^d(\mathbf{w})]$ which is symmetric in (a,d). We can take this new term to be of the form

$$(1/48\pi^2)d^{ade}[\nabla\times\mathbf{b}^e(\mathbf{x})]\cdot\nabla\delta^3(\mathbf{x}-\mathbf{w})\ ,$$

with $\mathbf{b}^e$ having then to satisfy

$$\langle 0|\{L^c(\mathbf{z}),\{L^b(\mathbf{y}),\nabla\times\mathbf{b}^e(\mathbf{w})\}\}|0\rangle=-mf^{bce}[\nabla\delta^3(\mathbf{y}-\mathbf{w})\times\nabla\delta^3(\mathbf{z}-\mathbf{w})]\ . \tag{9}$$

Our main conclusion therefore is that the naive commutator must be extended to [9]

$$[L^a(\mathbf{x}),L^b(\mathbf{y})]=f^{abc}L^c(\mathbf{x})\delta^3(\mathbf{x}-\mathbf{y})+\frac{1}{48\pi^2}d^{abc}[\nabla\times\mathbf{b}^c(\mathbf{x})]\cdot\nabla\delta^3(\mathbf{x}-\mathbf{y})\ . \tag{10}$$

The precise form of the operators $\mathbf{b}^a$ cannot be determined without further dynamical input, for much the same reason that the usual spectral function constraints cannot completely determine the form of the ordinary Schwinger terms. However, its commutation relations with the charge density can be surmised.

For this purpose, consider next the triple commutator. Using Eq. (7) once again, we find

$$\langle 0|[L^c(\mathbf{z}),[L^b(\mathbf{y}),[L^a(\mathbf{x}),L^d(\mathbf{w})]]]|0\rangle=\frac{m}{48\pi^2}f^{bce}d^{ead}\nabla\delta^3(\mathbf{x}-\mathbf{w})\cdot[\nabla\delta^3(\mathbf{y}-\mathbf{w})\times\nabla\delta^3(\mathbf{z}-\mathbf{w})]\ . \tag{11}$$

This verifies the result for the triple commutator found recently from an effective anomaly functional [10]. For the right-handed currents, the right-hand side of Eqs. (8), (11) will have sign flip.

The constraint expressed in Eq. (11) cannot yet by itself completely determine the Lie algebra of the charge densities. However, minimally, taking into account Eq. (10), we may posit the following infinite-dimensional Lie algebra which is consistent with Eqs. (8) and (11):

$$[\rho^a(\mathbf{x}),\rho^b(\mathbf{y})]=f^{abc}\rho^c(\mathbf{x})\delta^3(\mathbf{x}-\mathbf{y})+\frac{1}{48\pi^2}d^{abc}[\nabla\times\mathbf{b}^c(\mathbf{x})]\cdot\nabla\delta^3(\mathbf{x}-\mathbf{y})\ ,$$

$$[\rho^a(\mathbf{x}),\mathbf{b}^b(\mathbf{y})]=f^{abc}\mathbf{b}^c(\mathbf{x})\delta^3(\mathbf{x}-\mathbf{y})+k\delta^{ab}\nabla\delta^3(\mathbf{x}-\mathbf{y})\ , \tag{12}$$

$$[b^{ia}(\mathbf{x}),b^{jb}(\mathbf{y})]=0,\quad [\rho^a(\mathbf{x}),k]=0,\quad [\mathbf{b}^a(\mathbf{x}),k]=0\ ,$$

where ρ^a is the chiral charge density $L^a(R^a)$ [11]. By construction, the current algebras defined by fermions of distinct chiralities are distinct from and commute with each other. Substituting this algebra into Eq. (11), we find the value of the central charge $k=+m$ $(-m)$ for left- (right-)handed currents.

It should be clear from the calculations leading up to Eqs. (8), (11) that these results are valid even in the absence of gauge fields. That means that $\mathbf{b}^a$ should be expressible entirely in terms of fermion fields. Indeed, for free fields, we may take it to be of the form $\mathbf{b}^a=\Lambda^{-2}{:}\bar{\psi}\sigma T^a\psi{:}$, where Λ is a momentum cutoff. It is important that this cutoff is not set to infinity prematurely, since that would have masked the presence of the extension term. Indeed, Eqs. (8) and (11) imply that the central charge k is equal to the dimensionless quantity Σ^2/Λ^2, where Σ^2 is the coefficient of the ordinary Schwinger term in the commutator between the charge and current densities. This coefficient is divergent for free quarks. As a result, the limits on Σ and Λ cannot be taken independently for finite nonvanishing central charges. Generally, $\mathbf{b}^a$ is sensitive to the interactions among the fermions, so a study of the sum rules of current correlation functions controlled asymptotically by matrix elements of this quantity can be used to probe what is actually going on at the deeper level.

In summary, we have established that the naive charge-density algebra is inadequate in the studies of Green's functions of currents, especially for four or more of them, and additional terms are necessary. The result follows from explicit forms for the quantity Eq. (8). We have also verified in the quark model a formula for the triple commutator which had been found indirectly in a previous work [10]. In general, in dealing with (time-ordered as well as ordinary) products of more than four currents by Wick's theorem, terms involving $\langle 0|L^a(\mathbf{x})L^b(\mathbf{y})L^c(\mathbf{z})L^d(\mathbf{w})|0\rangle$ will inevitably appear in the contractions. In such cases, commutators of the form displayed in Eq. (12) become significant, and can be used to constrain the underlying dynamics.

This work was supported in part by DOE under Contract Nos. DE-AC02-76ER13065 and DE-AS05-80ER10713.

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[9] These commutators should not be confused with those considered by L. D. Faddeev [Phys. Lett. **145B**, 81 (1984)]. We are looking at the algebra generated by the quark flavor currents directly, without prejudice as to whether they are actually coupled to any gauge fields. Faddeev is concerned with the equal-time commutator of Gauss law constraints in a quantized chiral gauge theory, and points to a difficulty in the quantization of the gauge fields if there is a Schwinger term in these commutators. One may attempt to verify these relations perturbatively [S. Jo, Nucl. Phys. **B259**, 616 (1985)], using the Bjorken-Johnson-Low (BJL) limit [J. Bjorken, Phys. Rev. **148**, 1467 (1966); K. Johnson and F. E. Low, Prog. Theor. Phys. Suppl. **37-38**, 74 (1966)]. In this way, one can constrain the commutators of the fermion charge densities. The Schwinger terms in $3+1$ dimensions obtained in this way disappear in the limit of vanishing gauge fields, as does the vacuum value of the commutator in Eq. (12). But what concerns us here are functions involving the product of four fermion currents. One might be tempted to use Eq. (2.18) in Jo (see above) to study the triple commutator, but that would give zero, and one would have missed obtaining Eq. (11) entirely. The point is that although it is justifiable in the context of verifying Faddeev's result to take one BJL limit, because the other two vertices are linked to gauge fields, it is not clear in practice in what order the three BJL limits should be taken in order to use this procedure to find the triple commutator.
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[11] These relations look somewhat like those between the Kac-Moody algebras and fermions in $1+1$ dimensions. These algebras have proved to be remarkably useful in two-dimensional conformal field theories and string theories. See *Kac-Moody and Virasoro Algebras: a Reprint Volume for Physicists,* edited by P. Goddard and D. Olive (World Scientific, Singapore, 1988). They only have c-number Schwinger terms, however, although the central charges do behave similarly.