Four-dimensional quantum gravity in the conformal sector

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We study the trace-anomaly-induced dynamics of the conformal factor of four-dimensional (4D) quantum gravity. The resulting effective scalar theory is ultraviolet renormalizable, and possesses a nontrivial, infrared stable fixed point. The exact anomalous scaling dimension of the conformal factor at the critical point is derived. We argue that this theory describes 4D gravity at large distances and provides a framework for a dynamical solution of the cosmological-constant problem.

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I. INTRODUCTION

In this paper we propose and study the theory of the conformal sector of four-dimensional (4D) quantum gravity. The conformal sector is described by the conformal factor $e^{2\sigma(x)}$ in spacetime metrics of the form

$$g_{ab}(x)=e^{2\sigma(x)}\bar{g}_{ab}(x) , \tag{1.1}$$

where $\bar{g}_{ab}(x)$ is a fixed fiducial metric.

The basic motivation for this study is the cosmological-constant problem [1]. The cosmological "constant," if not identically zero, is the smallest fundamental mass scale in nature, at least some 120 orders of magnitude smaller than its "natural" scale in Planck units. In contrast with other fine-tuning problems, an adjustment of its value once is not sufficient to explain its smallest at all epochs in the evolution of the Universe. For instance, gauge theories of the fundamental interactions will generate *effective* nonzero (and large) vacuum energies through spontaneous symmetry breaking, even if no such term is present in the underlying Lagrangian. Thus any attempt at an explanation cannot rely on an exact symmetry of the fundamental quantum theory of gravitation (such as supersymmetry), if that symmetry is ultimately broken in the low-energy effective theory. Instead, the fact that the effective cosmological "constant" is dynamically dependent on the vacuum state of all quantum fields in nature implies that not the physics of the Planck scale, but the low-energy or *infrared* dynamics of gravity is essential to a resolution of the problem.

Support for this point of view comes from earlier studies of quantum fields in de Sitter spacetime [2–4]. This classical spacetime is the natural candidate for the ground state of Einstein gravity with a positive cosmological term because of its maximal symmetry. Yet the existence of exponentially diverging geodesic world lines and a cosmological event horizon lead to rather peculiar long-distance kinematics for massless field theories in de Sitter space. One indication of the infrared problems encountered is that the graviton propagator grows without bound at large distances [3,4]. Moreover, the spin-0 or conformal part of the propagator provides the dominant contribution [4]. These infrared divergences indicate that classical de Sitter spacetime is *not* the ground state of the quantum theory with a cosmological constant and suggest that the infrared dynamics of the conformal factor should be treated exactly. However, in the pure Einstein theory, the conformal part of the metric is constrained and cannot propagate [5]; i.e., it has no dynamics.

The situation in two dimensions is similar. The Einstein-Hilbert action is proportional to the Euler number, a total derivative. Hence the classical limit of 2D gravity has no dynamics either. As soon as we consider quantized matter fields, the situation changes dramatically and discontinuously [6]. The quantum trace anomaly of the matter fields generates an effective nonlocal action for 2D gravity, which in the covariant, conformal coordinatization of (1.1) becomes a local kinetic term for σ. The Liouville field must now be treated as a quantum field in its own right, and its dynamics may possibly hold the key to a more complete understanding of string theory.

Encouraged by this progress in 2D gravity, we carry out the same program in four dimensions and study the dynamical theory of the conformal factor induced by the quantum trace anomaly. Of course, in two dimensions this analysis is exact, there being at most a finite number of modes remaining in the fiducial metric $\bar{g}_{ab}$ related to the global topology. In four dimensions the freezing of $\bar{g}_{ab}$ represents a severe truncation of the full theory. However, our aim is the infrared dynamics of gravity, and this we expect to be controlled by the conformal sector. Moreover, this truncated theory is a *bona fide*, nontrivial field theory in its own right and, as such, already much richer than the "minisuperspace" truncation to a finite number of degrees of freedom [7]. There is no reason, in principle, why the effects of spin-2 graviton modes cannot be considered afterward in a more complete theory.

The starting point for our analysis is the conformal anomaly of matter fields in a fixed spacetime metric. From the general form of this trace anomaly in four dimensions, we deduce the dependence of the effective action on the conformal part of the metric. Although this action is nonlocal in terms of the full metric, remarkably, it becomes local in conformal coordinates [Eq. (1.1)], in analogy with the 2D case. In four dimensions the action contains terms up to four derivatives in σ. Nevertheless, the negative-norm state which appears in the four-derivative theory should not cause a violation of unitarity, if it is eliminated by the constraints in the same way as the conformal factor in the Einstein theory [5,8]. The Euclidean form of the induced action is bounded from below, unlike the 2D case, where matter fields contribute negatively.

The resulting theory of the σ field has nontrivial infrared dynamics. We first study this dynamics when the fiducial metric $\bar{g}_{ab}$ is taken to be conformally flat. In this case there is an exact global symmetry of the effective σ theory, which is a remnant of the diffeomorphism invariance of the full theory, and is generated by the rigid conformal transformations of SO(4,2). Hence general covariance of the full theory guarantees the absence of quantum anomalies in these symmetries. This implies that the β functions of all couplings must vanish, and we are led to study the critical scaling behavior of the σ theory. We should emphasize that in two dimensions, also, the requirement of vanishing β functions and resulting anomalous scaling behavior follows from the global conformal symmetry group SO(2,2), independently of the infinite-dimensional Virasoro algebra. Moreover, again, as in two dimensions, the global conformal symmetry determines uniquely the form of the action, which is the same as that obtained from the trace anomaly and guarantees its renormalizability. Our principle result is the existence of a nontrivial, infrared stable critical point, which we calculate to all orders of perturbation theory. In particular, we find the anomalous scaling dimension of the conformal field e^{σ} in one-loop perturbation theory and show that the one-loop result is exact, just as in two dimensions [9,10].

The Ricci scalar R also acquires an anomalous scaling dimension under rigid dilations. This means that the expectation value of R may be used as an order parameter for the spontaneous breaking of global scale invariance. The cosmological-constant problem may be formulated in a succinct and well-defined way in terms of this order parameter. It reduces to the question of whether or not in the ground state of the effective σ theory the exact conformal symmetry is spontaneously broken. We propose a physically plausible mechanism by which this symmetry, apparently spontaneously broken at the classical level, may be restored by quantum effects and the *effective* cosmological constant thereby forced to vanish. This is analogous to the restoration of global symmetries in two dimensions due to infrared divergences in the massless propagators of would-be Goldstone bosons [11]. A preliminary examination of the behavior of the correlation function of Ricci scalars at different points suggests that the effective cosmological "constant" decays as a power of the geodesic distance at scales larger than the horizon.

The outline of the paper is as follows. In the next section we discuss the general form of the effective theory of the σ field, as deduced from the trace anomaly in four dimensions, and discuss its conformal symmetries. In Sec. III we calculate the anomalous scaling dimension of the conformal factor. We further analyze the consequences of scale invariance and find a nontrivial relationship between coupling constants at the infrared stable fixed point. The details of the diagrammatic analysis necessary for this section are contained in the Appendix. In Sec. IV we apply the theory to de Sitter spacetime and describe the mechanism by which quantum fluctuations may restore the conformal symmetry. We conclude with a discussion of the possible broader validity of our results in the complete theory and suggest some directions for future research.

II. EFFECTIVE THEORY OF THE CONFORMAL FACTOR

In order to determine the Lagrangian which describes the dynamics of the conformal factor in four dimensions, let us first review the situation in two dimensions. In that case the general form of the trace anomaly of the energy-momentum tensor for matter in a background gravitational field is[1]

$$\begin{aligned} T_a^{a\,(\text{matter})} &= \frac{c_m}{24\pi} R \\ &= \frac{c_m}{24\pi} e^{-2\sigma}(\bar{R} - 2\bar{\Box}\sigma), \quad d=2\ , \end{aligned} \tag{2.1}$$

in the decomposition (1.1). The coefficient $c_m = N_s + N_f$ for N_s scalar and N_f (Dirac) fermion fields. Considering next the quantization of gravity, we may regard (1.1) as a gauge condition with $\bar{g}_{ab}$ fixed. Then the anomaly coefficient receives a contribution of -26 from the corresponding reparametrization ghosts and $+1$ from the σ field itself. Thus we should replace c_m by

$$c = N_s + N_f - 25\ , \tag{2.2}$$

The trace anomaly may be derived from a nonlocal effective action, which becomes, local in the conformal coordinates (1.1) [6],

$$\begin{aligned} T_a^a &= \frac{1}{\sqrt{-g}}\frac{\delta}{\delta\sigma(x)} S_{\text{anom}}\ , \\ S_{\text{anom}} &= -\frac{c}{96\pi}\int d^2x\sqrt{-g}\int d^2x'\sqrt{-g}' \\ &\qquad \times R(x)\Box^{-1}(x,x')R(x') \\ &\to \frac{c}{24\pi}\int d^2x\sqrt{-\bar{g}}\,(-\sigma\bar{\Box}\sigma + \bar{R}\sigma)\ , \end{aligned} \tag{2.3}$$

where a σ-independent term has been dropped. To this induced action we should add the classical action for 2D gravity:

$$\begin{aligned} S_{\text{cl}} &= \int d^2x\sqrt{-g}\,(\gamma R - 2\lambda) \\ &= 4\pi\gamma\chi - 2\lambda\int d^2x\sqrt{-\bar{g}}\,e^{2\sigma}\ , \end{aligned} \tag{2.4}$$

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[1] We use the Lorentzian metric and curvature conventions of Ref. [12] throughout the paper.

where χ is the Euler number. Note that the Einstein term alone describes no dynamics, since the scalar curvature is a total derivative. Hence the anomaly-induced dynamics of the effective 2D gravity theory is very different from the classical theory, even in the infrared, where one's usual prejudice is that quantum gravity or trace anomaly effects are "higher order" and irrelevant.

Let us now consider this same line of reasoning for $d=4$. In four dimensions the general form of the trace anomaly of the energy-momentum tensor is a linear combination of the four terms R^2, $R_{ab}R^{ab}$, $R_{abcd}R^{abcd}$, and $\Box R$ [13]. Each of these corresponds to local counterterms in the dimensionally regulated effective action. Actually, since $\Box R$ is a total divergence in any number of dimensions, adding it to the effective action gives no contribution to the trace of the energy-momentum tensor. Hence there can be at most three independent local counterterms possible in dimensional regularization near $d=4$, and the trace anomaly for matter may be written in the form [14]

$$T_a^{a(\mathrm{matter})}=b\left(F+\tfrac{2}{3}\Box R\right)+b'G+b''\Box R,\quad d=4\ , \tag{2.5}$$

where

$$F=R_{abcd}R^{abcd}-2R_{ab}R^{ab}+\tfrac{1}{3}R^2 \tag{2.6}$$

is the square of the Weyl tensor in four dimensions and

$$G=R_{abcd}R^{abcd}-4R_{ab}R^{ab}+R^2 \tag{2.7}$$

is the Gauss-Bonnet integrand. In four (but only four) dimensions, G is a total divergence, and its integral is a topological invariant. The coefficients b and b' have been computed for scalar, Dirac fermion, and vector fields at one-loop order, with the results

$$\begin{aligned} b&=\frac{1}{120(4\pi)^2}(N_s+6N_f+12N_v)\ ,\\ b'&=-\frac{1}{360(4\pi)^2}(N_s+11N_f+62N_v). \end{aligned} \tag{2.8}$$

If the matter-field Lagrangian is invariant under local conformal (Weyl) trasnformations classically, then the coefficient b'' vanishes at one-loop order [14]. However, in principle, it corresponds to an arbitrary parameter, since $\Box R$ is the variation of a local action:

$$\sqrt{-g}\,\Box R=-\frac{1}{6}g_{ab}\frac{\delta}{\delta g_{ab}}\int d^4x\sqrt{-g}R^2\ , \tag{2.9}$$

which receives divergent contributions in general. Thus we allow for $b''\neq 0$. In contrast with this term, the first two terms on the right side of (2.5) do not arise from local effective actions in four dimensions. However, in the conformal parametrization of the metric [Eq. (1.1)], the anomaly may be derived from a local effective action, just as in the two-dimensional case [15]. In fact, the quantity $\sqrt{-g}F$ becomes independent of σ (actually vanishing if $\bar{g}_{ab}$ is conformally flat), and the combination $\sqrt{-g}(G-\frac{2}{3}\Box R)$ is particularly simple, being only linear in σ in analogy with Eq. (2.1):

$$\begin{aligned} e^{4\sigma}\left(G-\tfrac{2}{3}\Box R\right)=\;&4\bar{\Box}^2\sigma+8\bar{R}^{ab}\bar{\nabla}_a\bar{\nabla}_b\sigma-\tfrac{8}{3}\bar{R}\bar{\Box}\sigma\\ &+\tfrac{4}{3}(\bar{\nabla}^a\bar{R})(\bar{\nabla}_a\sigma)+\bar{G}-\tfrac{2}{3}\bar{\Box}\bar{R}\ . \end{aligned} \tag{2.10}$$

Here $\bar{\nabla}$ is the covariant derivative operator with respect to the fiducial metric $\bar{g}_{ab}$, $\bar{\Box}=\bar{\nabla}_a\bar{\nabla}^a$, and all indices are raised and lowered by this fiducial metric tensor. Thus, if we rewrite the trace anomaly (2.5) in the form

$$\begin{aligned} T_a^{a(\mathrm{matter})}&=bF+b'\left(G-\tfrac{2}{3}\Box R\right)+\left[b''+\tfrac{2}{3}(b+b')\right]\Box R\\ &=\frac{1}{\sqrt{-g}}\frac{\delta}{\delta\sigma(x)}S_{\mathrm{anom}}[\sigma]\ , \end{aligned} \tag{2.11}$$

the second term in (2.11) gives a contribution to the effective action which is quadratic in σ, and the full anomaly induced action reads

$$\begin{aligned} S_{\mathrm{anom}}[\sigma]=\;&b\int d^4x\sqrt{-\bar{g}}\,\bar{F}\sigma+b'\int d^4x\sqrt{-\bar{g}}\,\{\sigma[2\bar{\Box}^2+4\bar{R}^{ab}\bar{\nabla}_a\bar{\nabla}_b-\tfrac{4}{3}\bar{R}\bar{\Box}+\tfrac{2}{3}(\bar{\nabla}^a\bar{R})\bar{\nabla}_a]\sigma+(\bar{G}-\tfrac{2}{3}\bar{\Box}\bar{R})\sigma\}\\ &-\tfrac{1}{12}\left[b''+\tfrac{2}{3}(b+b')\right]\int d^4x\sqrt{-\bar{g}}\,[\bar{R}-6\bar{\Box}\sigma-6(\bar{\nabla}_a\sigma)(\bar{\nabla}^a\sigma)]^2\ , \end{aligned} \tag{2.12}$$

where we have used the expression

$$R=e^{-2\sigma}[\bar{R}-6\bar{\Box}\sigma-6(\bar{\nabla}_a\sigma)(\bar{\nabla}^a\sigma)]\ , \tag{2.13}$$

for the Ricci scalar in conformal coordinates. We have omitted a σ-independent term in S_{anom} proportional to F. This local action in the conformal parametrization of the metric corresponds to the fully covariant nonlocal action:

$$\begin{aligned} S_{\mathrm{anom}}=\;&-\frac{1}{4b'}\int d^4x\sqrt{-g}\int d^4x'\sqrt{-g'}\left[bF+b'\left(G-\tfrac{2}{3}\Box R\right)\right]_x\\ &\times\left[2\Box^2+4R^{ab}\nabla_a\nabla_b-\tfrac{4}{3}R\Box+\tfrac{2}{3}(\nabla^aR)\nabla_a\right]^{-1}_{xx'}\left[bF+b'\left(G-\tfrac{2}{3}\Box R\right)\right]_{x'}\\ &-\tfrac{1}{12}\left[b''+\tfrac{2}{3}(b+b')\right]\int d^4x\sqrt{-g}R^2\ . \end{aligned} \tag{2.14}$$

The nonlocal term in (2.14) is the four-dimensional analogue of the nonlocal form of the 2D Polyakov action (2.3). As in two dimensions, we add the anomaly-induced action (2.12) to the classical Einstein-Hilbert action:

$$S_{\rm cl}=\frac{1}{2\kappa}\int d^4x\sqrt{-g}\,(R-2\Lambda)$$
$$=\frac{1}{2\kappa}\int d^4x\sqrt{-\bar g}\,e^{2\sigma}[\bar R-6\bar\Box\sigma-6(\bar\nabla_a\sigma)(\bar\nabla^a\sigma)]-\frac{\Lambda}{\kappa}\int d^4x\sqrt{-\bar g}\,e^{4\sigma}\ , \quad (2.15)$$

where $\kappa=8\pi G$.

In the case that the fiducial metric is conformally flat, i.e., $\bar g_{ab}=e^{2\bar\sigma}\eta_{ab}$, the effective action for σ may be obtained by a translation $\sigma\to\sigma+\bar\sigma$ of the simple flat-space action

$$S_{\rm eff}=S_{\rm anom}+S_{\rm cl}$$
$$=\int d^4x\left[2b'(\Box\sigma)^2-[3b''+2(b+b')][\Box\sigma+(\partial_a\sigma)^2]^2+\frac{3}{\kappa}e^{2\sigma}(\partial_a\sigma)^2-\frac{\Lambda}{\kappa}e^{4\sigma}\right]\ , \quad (2.16)$$

up to a surface term. When $\bar g_{ab}$ is conformally flat, the effective action (2.16) may be derived by a completely different method, based on global conformal symmetry, which we now explain.

Conformal flatness implies that the fiducial metric has a set of exactly $d+1$ special conformal Killing vectors, $\xi_a^{(i)}$, satisfying

$$(L\xi^{(i)})_{\rm ab}\equiv\bar\nabla_a\xi_b^{(i)}+\bar\nabla_b\xi_a^{(i)}-\frac{2}{d}\bar g_{ab}\bar\nabla^c\xi_c^{(i)}$$
$$=0,\quad i=1,\dots,d+1\ . \quad (2.17)$$

On the other hand, the decomposition, (1.1) is certainly invariant under the relabeling (Weyl) transformation

$$\bar g_{ab}(x)\to\bar g_{ab}(x)e^{2\omega(x)}\ ,$$
$$\sigma(x)\to\sigma(x)-\omega(x)\ , \quad (2.18)$$

for an arbitrary $\omega(x)$. Lastly, any theory of gravity must be invariant under coordinate transformations. The infinitesimal form of a combined coordinate plus relabeling transformation is

$$\delta\bar g_{ab}=\bar\nabla_a\xi_b+\bar\nabla_b\xi_a+2\omega\bar g_{ab}\ ,$$
$$\delta\sigma=\xi^a\bar\nabla_a\sigma-\omega\ .$$

Hence, by choosing

$$\xi_a=\xi_a^{(i)}\ ,$$
$$\omega=-\frac{1}{d}\bar\nabla^c\xi_c^{(i)}\ , \quad (2.19)$$

we find that

$$\delta^{(i)}\bar g_{ab}=(L\xi^{(i)})_{ab}=0\ ,$$
$$\delta^{(i)}\sigma=\xi^{(i)a}\bar\nabla_a\sigma+\frac{1}{d}\bar\nabla^a\xi_a^{(i)} \quad (2.20)$$

must be a symmetry transformation of the σ Lagrangian which leaves the fiducial metric unchanged. In flat space the special conformal Killing vectors are

$$\xi^{(b)a}=2x^ax^b-\eta^{ab}x^cx_c,\quad i=b=1,\dots,d\ ,$$
$$\xi^{i(a)}=x^a,\quad i=d+1\ . \quad (2.21)$$

The last of these plays a special role since it generates dilations, $x^a\to e^sx^a$. It serves to define the scaling dimension Δ_ϕ of the general field ϕ by

$$\delta_s\phi\equiv(x\cdot\partial+\Delta_\phi)\phi\ . \quad (2.22)$$

This definition implies that e^σ has classical scaling dimension equal to unity (as does $\partial\sigma$) under the global dilation $\xi^{(d+1)}$. The σ field itself does not have a well-defined scaling dimension, since it transforms inhomogeneously under δ_s:

$$\delta_s\sigma=x\cdot\partial\sigma+1\ . \quad (2.23)$$

Since the global conformal symmetry for σ is a remnant of the coordinate invariance of the full theory with total metric g_{ab}, the effective action of the conformal factor must respect this symmetry. Invariance under dilations along (2.23) determine the 4D action (2.16) up to four derivatives, leaving only the freedom of introducing two different coefficients for the fourth-order terms $\Box\sigma(\partial\sigma)^2$ and $(\partial\sigma)^4$. Their ratio is fixed to be 2 to 1 from the invariance under the remaining global conformal transformations (2.21).

It is remarkable that global conformal invariance determines uniquely the form of the effective action for σ. Normally, we would expect that this classical scale invariance will be violated in the quantum theory. This is precisely what the trace anomaly of matter fields in a fixed gravitational background geometry [Eq. (2.1)] reflects. Now that we are considering the quantum theory of the geometry itself, and the global conformal transformations are simply particular coordinate transformations, an anomaly in this symmetry would imply a breakdown of general covariance at the quantum level. The invariance of the σ theory under the rigid scale transformations generated by $\xi^{(5)}$ implies that the total trace anomaly of the conformal factor plus matter plus ghosts must vanish. In particular, this implies that the β functions of rigid dilations must vanish identically, corresponding to a critical point of the effective action.

This conclusion will seem less startling if reconsider the 2D case in the covariant conformal parametrization. In the language of string theory, if the target-space dimension N_s+N_f in Eq. (2.2) is different from 26, then the Liouville mode does not decouple and must be quantized. The fact that all two-dimensional metrices are conformally flat guarantees the existence of residual symmetries [Eq. (2.20)]. The conformal group in two dimensions is infinite dimensional and gives rise to the Virasoro algebra. However, the *global* scale invariance generated by the finite-dimensional subgroup SO(2,2) or SO(3,1)$\simeq$SL(2,C) is sufficient to enforce the vanishing of the total trace anomaly.

To complete our construction of the effective theory of the conformal factor in four dimensions, we should discuss the covariant measure to be used in the path integral and the ghost contribution(s) to the σ action. This follows along the same lines as the 2D case [16]. The conformal parametrization (1.1) amounts to a choice of coordinates on the space of metrics. The covariant meaure on the function space of metrics is defined by means of the DeWitt supermetric on this space. The change of variables from g_{ab} to σ and $\bar{g}_{ab}$ results in a Jacobian factor in the measure:

$$J = \det{}'^{1/2}(L^\dagger L) , \qquad (2.24)$$

where $L^\dagger$ is the Hermitian adjoint of L as defined in (2.17). Explicitly,

$$(L^\dagger L)_a^b = -2(\delta_a^b \Box + \tfrac{1}{2}\nabla_a \nabla^b + R_a^b) . \qquad (2.25)$$

The prime in (2.24) indicates that the zero modes of L must be excluded from J and treated separately. The inner product on the full space of metrics induces a Weyl-invariant inner product on the cotangent space of conformal deformations [8]:

$$\langle \delta\sigma, \delta\sigma \rangle_W = \int d^4x \sqrt{-\bar{g}}\, e^{4\sigma} (\delta\sigma)^2 . \qquad (2.26)$$

After dividing out by the infinite volume of the diffeomorphism group, we are left with the measure $J[\mathcal{D}\delta\sigma]_W$, where

$$\int [\mathcal{D}\delta\sigma]_W \exp\left[-\frac{i}{2} \langle \delta\sigma, \delta\sigma \rangle_W \right] = 1 \qquad (2.27)$$

is the σ-dependent normalization condition.

It is more convenient to work with a σ-independent, translationally invariant measure $[\mathcal{D}\delta\sigma]_s$ defined with respect to the usual scalar inner product:

$$\langle \delta\sigma, \delta\sigma \rangle_s = \int d^4x \sqrt{-\bar{g}}\, (\delta\sigma)^2 . \qquad (2.28)$$

Changing from the Weyl-invariant measure to the translationally invariant one results in another Jacobian J', which can be written as the exponential of a renormalizable effective action. In fact, it should be of the same form as the effective action S_{anom} found above, by reasoning exactly parallel to the two-dimensional case [10,16]. Likewise, the anomalous scaling of the ghost determinant (2.24) may be computed, as well as the quantum contribution of the fourth-order action of the σ field itself. The inclusion of these effects leaves us finally with an action of the same form as (2.12) with altered numerical coefficients, a translationally invariant measure for the effective σ theory, and the remaining ghost determinant (2.24). The detailed calculation of the ghost and σ contributions will be presented in a separate paper [17].

The above construction of the effective theory for σ in four dimensions has led us inevitably to a fourth-order action. At linearized order such an action always gives rise to two excitations, one of which has negative metric. It is the role of the ghost determinant (2.24), which is the covariant equivalent of the reparametrization constraints, to enforce unitarity at the quantum level. Recall that the σ field has a negative-metric kinetic term already at the level of the classical Einstein action (2.15). In two dimensions as well, the induced Polyakov action contains two derivatives more than the classical action, and σ has negative metric when c_m exceeds 26. Even in that case we know that the theory is unitary [18]. In both two and four dimensions, the higher-derivative terms in the σ action emerge from the nonlocal terms induced by the trace anomaly (with calculable coefficients). This should be contrasted with local fourth-order gravity actions which contain a massive spin-2 ghost, associated with the local F term in the action with arbitrary coefficient. Of course, the four-dimensional case is quite different in containing spin-2 excitations which couple to the σ field. For this reason it may not be meaningful to require unitarity of the σ theory alone. Rather, it should probably be viewed as an effective theory from which only limited information can be extracted.

III. ANOMALOUS SCALING AND THE INFRARED FIXED POINT

In this section we study the consequences of the quantization of the conformal factor with the four-dimensional action (2.16). The trace of the energy-momentum tensor of this effective scalar theory must vanish, because of the coordinate invariance of the full theory, as discussed previously. This implies that the β functions of all couplings must vanish, but does not preclude the possibility of nontrivial anomalous scaling dimensions. In order to allow for this possibility, let us assume that the field $\phi = e^\sigma$, which has classical weight equal to unity, acquires a scaling dimension $\Delta_\phi \equiv \alpha$ under the transformation (2.22). Then the transformation of the σ field is modified to

$$\delta_s \sigma = x \cdot \partial\sigma + \alpha , \qquad (3.1)$$

while the rescaled field $\hat{\sigma} \equiv \sigma/\alpha$ transforms exactly as in (2.22). In order to determine the anomalous scaling dimension α, we study the counterterm structure of the rescaled action:

$$\begin{aligned} S = &-\frac{Q^2}{(4\pi)^2} \int d^4x (\Box\sigma)^2 \\ &- \zeta \int d^4x [2\alpha(\partial_a\sigma)^2 \Box\sigma + \alpha^2 (\partial_a\sigma)^4] \\ &+ \gamma \int d^4x\, e^{2\alpha\sigma} (\partial_a\sigma)^2 - \frac{\lambda}{\alpha^2} \int d^4x\, e^{4\alpha\sigma} , \end{aligned} \qquad (3.2)$$

where we have introduced the notations

$$\begin{aligned} &\frac{Q^2}{(4\pi)^2} = 2b + 3b'' , \\ &\zeta = 2b + 2b' + 3b'' , \\ &\gamma = \frac{3}{\kappa} , \\ &\lambda = \frac{\Lambda}{\kappa} , \end{aligned} \qquad (3.3)$$

replaced S by $\alpha^2 S$, and dropped the caret on $\hat{\sigma}$. Simple power-counting arguments, combined with the existence of global symmetry [Eq. (2.20)], show that the fourth-order Lagrangian (3.2) is renormalizable in four dimensions in σ perturbation theory. The Feynman rules for the quartic propagator and vertices derived from this Lagrangian are given in the Appendix, as well as the detailed results of all the relevant graphs which we discuss in this section.

Let us first consider the effect of the higher-derivative cubic and quartic interaction terms parametrized by the coupling constant ζ, which does not receive any renormalizations from the lower-dimension operators in (3.2). Perturbatively in ζ, there are two divergent diagrams at order ζ^2, illustrated in Fig. 1. Because of the propagator involving p^4, these diagrams contain only logarithmic divergences. The first of these renormalizes the three-point function in the ζ interaction, while the second renormalizes the four-point function. Because of conformal symmetry [Eq. (2.20)], the two are not independent. Using dimensional regularization, we easily evaluate the one-loop ζ counterterm

$$\zeta_b = \mu^{4-d}\left[\zeta_r + \frac{80\pi^2}{4-d}\,\frac{\alpha^2}{Q^4}\zeta_r^2\right] , \tag{3.4}$$

where ζ_b and ζ_r are the bare and renormalized couplings, respectively. From this we find the one-loop β function for the coupling ζ:

$$\beta_\zeta = \mu\frac{d}{d\mu}\zeta_r = \frac{80\pi^2\alpha^2}{Q^4}\zeta_r^2 \geq 0 . \tag{3.5}$$

Hence $\zeta_r = 0$ is an infrared stable perturbative fixed point (as in ordinary $\lambda\phi^4$ theory), if $\zeta \geq 0$, the same sign corresponding to boundedness of the Euclidean action. Since we are interested in this effective theory at its IR fixed points, we shall set $\zeta_r = 0$ henceforth.

At this fixed point we see from (3.3) that

$$Q^2 = -32\pi^2 b', \quad \zeta = 0 . \tag{3.6}$$

Recall that b' was the coefficient of the Gauss-Bonnet term in the trace anomaly (2.5), which is negative definite for all known matter fields [Eqs. (2.8)]. This corresponds again to a bounded Euclidean action, in contrast with the two-dimensional case [Eqs. (2.2) and (2.3)]. This quantity has been proposed as the four-dimensional analogue of the two-dimensional central charge c for which the Zamolodchikov theorem applies [19]. It is worth remarking also that the requirement of a IR fixed point at $\zeta = 0$ fixes the indeterminanteness of the b' term, which is usually present in purely local higher-derivative theories of gravity.

The remaining theory is a Liouville-like theory, which is actually easy to study in perturbation theory because of the quartic propagator. In fact, it is superrenormalizable and contains only a finite set of divergent diagrams. Let us denote by V_γ the number of vertices of the general graph with the exponential interactions generated by the Einstein term and by V_λ the vertices generated by the cosmological term. Then elementary power counting shows that the degree of divergence of the graph is given by

$$\begin{aligned} D &= 4L - 4I + 2V_\gamma , \\ &= 4 - 2V_\gamma - 4V_\lambda , \end{aligned} \tag{3.7}$$

since the number of loops L is related to the number of internal propagators I by the usual topological relation

$$L = I - V_\gamma - V_\lambda + 1 . \tag{3.8}$$

Thus it is clear that there are only three classes of divergent diagrams:

$$\begin{aligned} &\text{(i)} \quad V_\lambda = 0, \quad V_\gamma = 1, \quad D = 2 , \\ &\text{(ii)} \quad V_\lambda = 0, \quad V_\gamma = 2 , \quad D = 0 , \\ &\text{(iii)} \quad V_\lambda = 1, \quad V_\gamma = 0, \quad D = 0 . \end{aligned} \tag{3.9}$$

The graphs of type (i) are the rosette graphs illustrated in Fig. 2, with $L = I$, obtained by expanding the exponential interaction in the Einstein term. The divergences of this set of graphs are removed by one counterterm determined at the one-loop level ($L = I = 1$). Analyzing the one-loop graphs in detail (Fig. 3), we observe that there are two types of contributions. Figure 3(a), where the derivatives act on the external legs, denotes the contribution to the renormalization of the γ vertex, which is only logarithmically divergent. In Fig. 3(b) the derivatives act on the internal lines and give rise to a quadratically divergent $e^{2\alpha\sigma}$ term, not present in the original Lagrangian. This dangerous term will be cancelled by graphs of type (ii), as we shall see in a moment.

The logarithmic divergence of the graph represented in Fig. 3(a) gives rise to the following β function for the γ coupling:

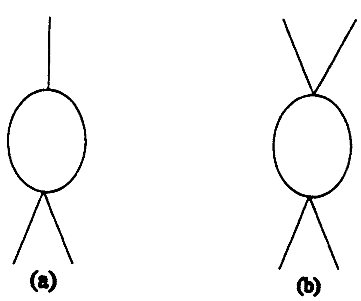

FIG. 1. Two one-loop divergent diagrams at order ζ^2 contributing to the renormalization of the cubic and quartic vertices, respectively, of the ζ interaction in (3.2).

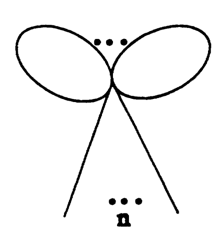

FIG. 2. Generic rosette graph with any number of loops and any number of external legs emerging from a single exponential vertex $V_\gamma = 1$.

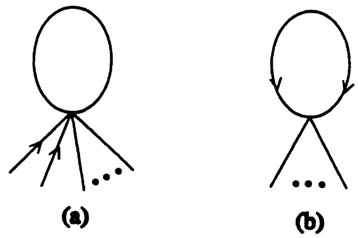

FIG. 3. One-loop diagrams of type (i) considered in detail. The arrows on two of the legs of the γ vertex denote derivatives operating on the corresponding σ fields, according to the Feynman rules of the Appendix.

$$\beta_\gamma=\mu\frac{d}{d\mu}\gamma_R=\left[2-2\alpha+\frac{2\alpha^2}{Q^2}\right]\gamma_R\ . \tag{3.10}$$

The first two terms correspond to the "classical" scaling dimension of the γ coupling, for $\alpha\neq1$, while the α^2 term arises from the one-loop counterterm. The vanishing of this β function for $\gamma_R\neq0$ yields a quadratic relation for the anomalous scaling of e^σ:

$$1-\alpha+\frac{\alpha^2}{Q^2}=0\ , \tag{3.11}$$

with the solutions

$$\alpha_\pm=\frac{1\pm(1-4/Q^2)^{1/2}}{2/Q^2}\ . \tag{3.12}$$

This result is completely analogous to the anomalous scaling obtained for the Liouville mode in two dimensions, except that in that case it is the β function of λ which determines the scaling relation [9,10]. The classical scaling $\alpha=1$ is obtained from the negative branch of the square root, α_-, in the limit $Q^2\to\infty$:

$$\alpha_-\to1+\frac{1}{Q^2},\quad Q^2\to\infty\ . \tag{3.13}$$

Unlike two dimensions, this limit is obtained by sending the number of matter fields to $+\infty$, rather than $-\infty$. The value $Q^2=Q^2_{\rm cr}=4$ corresponds to $c=1$ in two dimensions, where the theory could exhibit a phase transition or qualitatively new phenomena. However, it seems from (2.8) and (3.6) that the physically relevant case in four dimensions is always $Q^2>4$, corresponding to $c<1$ in two dimensions.

We have checked the one-loop result for anomalous scaling [Eq. (3.12)] by a method independent of the Feynman graph analysis and any use of conformal symmetry. Treating the lower-derivative terms in the Lagrangian (3.2) as *arbitrary* functions of the σ field, the background field calculation for the vanishing of the β function of the analogue of the "tachyon" field T of string theory gives

$$\left[2-\frac{\partial}{\partial\sigma}+\frac{1}{2Q^2}\frac{\partial^2}{\partial\sigma^2}\right]T(\sigma)=0\ , \tag{3.14}$$

which is satisfied by $T=e^{2\alpha\sigma}$ with α given by (3.12) above. Actually, this one-loop result is exact, as we argued above from the highly convergent Feynman diagrams with a quartic propagator.

The generic diagrams of type (ii), which contribute new primitive divergences, are illustrated in Fig. 4 and are logarithmically divergent. The one-loop graph [Fig. 5(a)] gives a sum of two terms:

$$\frac{8\pi^2\gamma^2}{Q^4}\frac{\mu^{4-d}}{4-d}e^{4\alpha\sigma}-\frac{16\pi^2\gamma^2}{Q^4}\frac{\mu^{4-d}}{4-d}e^{2\alpha\sigma}\ . \tag{3.15}$$

The first term is an additive renormalization of the λ coupling. The second part of (3.15), corresponding to a new term not present in the original Lagrangian (3.2), is precisely that needed to cancel a similar contribution from the graph of Fig. 3(b), which was naively quadratically divergent. That these terms with different power-counting divergences could end up canceling is a consequence of the fact that dimensional regularization near $d=4$ is sensitive to quadratic divergences only through the additional power of the mass parameter γ multiplying the pole. The cancellation is not an artifact of the regularization scheme, since it is essential to use a regulator that respects coordinate invariance, such as dimensional regularization. The remaining higher-loop graphs of type (ii) in Fig. 4 contribute only to the renormalization of λ. In fact, an analysis of the divergence of these graphs shows that the only new primitive divergences arise at the two- and three-loop level, illustrated in Figs. 5(b) and 5(c) and cataloged in the Appendix.

Finally, we have the graphs of type (iii), whose primitive divergences is obtained from the one-loop diagram illustrated in Fig. 6. This gives an additional contribution to the λ renormalization. Collecting all terms from graphs of types (ii) and (iii) that make such contributions yields the following β function for the λ coupling:

$$\beta_\lambda=\mu\frac{d}{d\mu}\lambda_R=\left[4-4\alpha+\frac{8\alpha^2}{Q^2}\right]\lambda_R -\frac{8\pi^2\alpha^2}{Q^4}\gamma_R^2\left[1+\frac{4\alpha^2}{Q^2}+\frac{6\alpha^4}{Q^4}\right]\ . \tag{3.16}$$

As for β_γ, the first two terms in β_λ represent the classical scaling dimension of this coupling, while the remaining terms are the loop corrections of Figs. 6 and 5, respec-

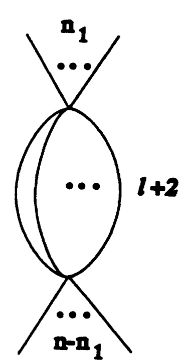

FIG. 4. Generic multiloop graphs of type (ii), with two γ vertices and $l+1$ internal loops. The integers n_1 and $n-n_1$ denote the possible number of external legs emerging from each vertex to be summed over.

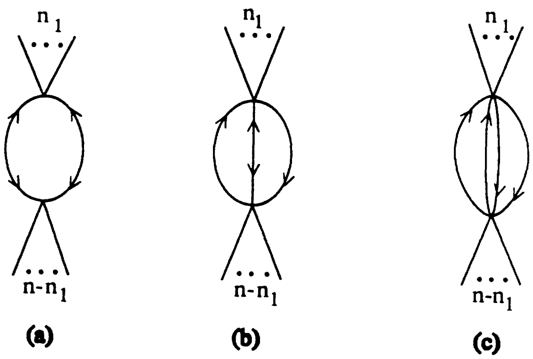

FIG. 5. One-, two-, and three-loop graphs of type (ii), respectively, with $V_\gamma=2$. These graphs are responsible for the only primitive divergences in graphs of this type. The arrows denote derivatives on the external lines as before.

tively. Since α has already been determined by (3.12), setting $\beta_\lambda=0$ gives a nontrivial relation for the cosmological constant in Planck units:

$$\frac{\lambda}{\gamma^2}-9\kappa\Lambda=\frac{2\pi^2}{Q^2}\left[1+\frac{4\alpha^2}{Q^2}+\frac{6\alpha^4}{Q^4}\right], \tag{3.17}$$

which is a function of the only parameter remaining, viz., Q^2.

There is an equivalent way of deriving the scaling relations in four dimensions, which is closer in spirit to the methods of conformal field theory in two dimensions, once ζ has been set to zero. This is based on the observation that the operator $e^{p\sigma}$, which has conformal weight p classically, acquires a quantum contribution from the graph of Fig. 6:

$$\Delta(e^{p\sigma})=p-\frac{p^2}{2Q^2}, \tag{3.18}$$

which may be computed also by its operator-product expansion with the energy-momentum tensor derived from the quartic Lagrangian (2.12):

$$\begin{aligned}T^{ab}=\frac{Q^2}{(4\pi)^2}\{&-4\Box\sigma\partial^a\partial^b\sigma-\tfrac{2}{3}\partial^a\partial^b(\partial\sigma)^2\\&+2\Box(\partial^a\sigma\partial^b\sigma)-\tfrac{2}{3}\partial^a\partial^b\Box\sigma\\&+\eta^{ab}[-\tfrac{1}{3}\Box(\partial\sigma)^2+(\Box\sigma)^2+\tfrac{2}{3}\Box^2\sigma]\}\end{aligned}, \tag{3.19}$$

in the flat-space limit. In Wick contracting the σ fields, the propagator

$$G_{\text{flat}}(x,x')=-\frac{1}{2Q^2}\ln[\mu^2(x-x')^2] \tag{3.20}$$

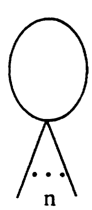

FIG. 6. One-loop diagram of type (iii), which contributes the only primitive divergence of the λ renormalization.

of the free quartic action should be used. The anomalous scaling relation (3.12) may be obtained by requiring that the operator $\sqrt{-g}R=-6e^{2\alpha\sigma}[(\partial\sigma)^2+\Box\sigma]$ have conformal weight equal to 4. Indeed, using Eq. (3.18) and the fact that $\partial\sigma$ has only a classical weight equal to 1, one finds

$$\Delta(\sqrt{-g}R)=2\alpha-\frac{2\alpha^2}{Q^2}+2=4. \tag{3.21}$$

This is the necessary condition for the Einstein term to be invariant under conformal diffeomorphism symmetry [Eq. (2.20)]. Once this condition has been imposed, note that one can no longer insist that the cosmological term $e^{4\alpha\sigma}$ have the same conformal weight:

$$\Delta(e^{4\alpha\sigma})=4\alpha-\frac{8\alpha^2}{Q^2}=4(2-\alpha)\neq 4. \tag{3.22}$$

Instead, as Eq. (3.16) shows, there is a nontrivial mixing between the λ and γ couplings, so that invariance can be enforced only by the relation (3.17).

Since renormalizability and anomalous scaling are properties of the short-distance behavior of the Lagrangian, we expect that the results of this section are valid in a general background, although they have been derived with Feynman rules in flat space. Actually, flat space is not a solution of the classical equations of motion following from the Lagrangian (3.2). Hence, in our flat-space calculations, we implicitly assumed the presence of a linear term in σ to cancel tadpole contributions at both the classical and quantum levels. The one calculation we have performed which does not make this assumption is based on the background field method and leads to Eq. (3.14), which is fully consistent with the flat-space result.

IV. de SITTER SPACE AND THE COSMOLOGICAL CONSTANT

The formal development of the previous sections shows that the effective theory of the conformal factor in four-dimensional gravity has some very remarkable properties. It is renormalizable and has a nontrivial, infrared stable fixed point and anomalous scaling dimensions. Once the fiducial metric $\bar{g}_{ab}$ has been frozen, the analogies with two-dimensional gravity are manifest. However, in four dimensions, this truncation of the full theory is serious. Under what circumstances can we expect the σ theory to give a reliable approximation to nature?

Certainly, a necessary condition for neglecting the transverse graviton modes of the metric is that we are discussing phenomena at energies far below the Planck scale: We can apply the σ theory only to the infrared aspects of metric fluctuations. However, quantum fluctuations of the metric seem to be negligible at all accessible energy scales below the Planck scale. At such scales would expect the classical Einstein theory to provide an excellent phenomenological description.

The only situation where this might not be the case is when the classical background spacetime described by $\bar{g}_{ab}$ has associated infrared divergences. Such is known to be the case in de Sitter spacetime, which is of particular interest both because it is the maximally symmetric spacetime which solves Einstein's equations with a positive cosmological term and because of its role in inflationary cosmological models. Massive matter is redshifted away exponentially rapidly in a background with a cosmological term, becoming essentially noninteracting, cold, and decoupled in a few expansion times. Hence de Sitter space is approached rapidly in the classical picture, and treating the remaining massless fields as conformally invariant would seem to be valid. Then the conditions needed for the applicability of the effective theory of the conformal factor have been satisfied. Let us attempt to apply the σ theory to this case.

The first observation we can make is that if the expectation value of the Ricci scalar is different from zero, then the global conformal symmetry discussed in the previous sections must be spontaneously broken. This will be clear from the expression for the conformal weight for the operator R, using (3.18):

$$\Delta(R)=-2\alpha-\frac{2\alpha^2}{Q^2}+2=4(1-\alpha)\ . \tag{4.1}$$

In the semiclassical limit, when the anomaly-induced fluctuations are suppressed by $1/Q^2$ for large Q^2, $\alpha\to 1$, and the weight of R is zero; i.e., it transforms like a scalar under global conformal transformations. If α differs from unity, then this is no longer the case. This should not be viewed with alarm if we recall that in two dimensions physical matter-field operators become "dressed" by their interactions with the Liouville mode and acquire modified scaling dimensions [9]. In four dimensions the Ricci scalar acquires an anomalous scaling from the self-interactions of the conformal part of gravity itself. This implies that $\langle R\rangle$ becomes an order parameter for the spontaneous breaking of global conformal symmetry, in sharp contrast with the classical situation in which $\langle R\rangle$ can take on any value consistent with the symmetry. Therefore, the cosmological-"constant" problem reduces to the question of whether this symmetry remains spontaneously broken or is restored in the quantum theory.

To address this question, let us consider the σ theory at the $\zeta=0$ fixed point, in a maximally symmetric space. The effective Lagrangian becomes

$$L_{\text{eff}}=-\frac{Q^2}{(4\pi)^2}\left[\sigma\bar{\Box}\left[\bar{\Box}-\frac{\bar{R}}{6}\right]\sigma+\frac{\bar{R}^2}{12}\sigma\right]+\gamma e^{2\sigma}\left[(\bar{\nabla}\sigma)^2+\frac{\bar{R}}{6}\right]-\lambda e^{4\sigma}\ . \tag{4.2}$$

At the fixed point, the theory behaves like a "free" theory with an anomalous dimension. This can be seen by shifting σ by a constant σ_0, under which the partition function scales according to

$$Z(\gamma)=Z(\gamma e^{2\alpha\sigma_0})e^{-Q^2\chi\sigma_0}\ , \tag{4.3}$$

where χ is the Euler number ($\chi=2$ for the sphere) and we have used the translational invariance of the measure. The partition function depends only on γ, because of the relation (3.16). From (4.3) we see that even this dependence is trivial at the fixed point, and we can scale it away by taking an appropriate limit ($\sigma_0\to-\infty$). This is completely analogous to string theory, restricted to the world-sheet topology of the sphere. Hence, for calculating correlation functions, we may replace the effective Lagrangian (4.2) by the free quartic one, with $\gamma=\lambda=0$.

The propagator corresponding to the quartic term $\bar{\Box}(\bar{\Box}-\bar{R}/6)$ is dominated by the $\bar{\Box}^2$ term at short distances and by the $(-\bar{R}/6)\bar{\Box}$ term at large distances. However, in de Sitter space [4],

$$\bar{\Box}^{-1}_{xx'}=-\frac{1}{8\pi^2}\left[\frac{2}{s^2(x,x')}-H^2\ln\left[\frac{H^2s^2(x,x')}{4}\right]\right]\ ,\qquad \bar{R}=12H^2\ , \tag{4.4}$$

where $s(x,x')$ is the geodesic distance between the two points. Hence the quartic propagator has the *same* logarithmic behavior in both limits:

$$\langle\sigma(x)\sigma(x')\rangle=G_\sigma(x,x')\to-\frac{1}{2Q^2}\begin{cases}\ln\mu^2s^2 & \text{as } s\to 0\ ,\\ \ln H^2s^2, & \text{as } s\to\infty\ ,\end{cases} \tag{4.5}$$

with the ultraviolet scale μ replaced by the inferred horizon scale H. This means that the fixed-point calculations we performed in the preceding section, using the short-distance behavior of the theory in dimensional regularization or the operator-product expansion, in flat space, can be carried out in the infrared with the same results, in de Sitter space. This is not surprising, as it is known from critical phenomena that there is a close interplay between ultraviolet and infrared behavior in systems with conformal symmetry. Note that unlike a purely local R^2 action whose effects are unimportant in the infrared, the effects of the σ propagator arising from the anomaly-induced effective action (4.2) grow logarithmically at large distances and dominate the physics.

Consider now the correlation function of Ricci scalars $\langle R(x)R(x')\rangle$ at two different points. Using the expression (2.13) with σ replaced by $\alpha\sigma$ and the free action which describes the theory at its critical point, we find

$$\langle R(x)R(x')\rangle\to\bar{R}^2\langle e^{-2\alpha\sigma(x)}e^{-2\alpha\sigma(x')}\rangle\to\bar{R}^2e^{4\alpha^2G_\sigma(x,x')}=\bar{R}^2\left[\frac{Hs(x,x')}{2}\right]^{-4\alpha^2/Q^2}\ ,\qquad |s(x,x')|\to\infty\ , \tag{4.6}$$

where only the dominant infrared behavior has been retained.

The result (4.6) states that the *effective* cosmological "constant" goes to zero with a definite power-law behavior for large distances. In other words, there is screening in the infrared of the effective value of vacuum energy at larger and larger scales. The value of the power is *universal,* depending only on Q^2, which counts the effective number of massless degrees of freedom. In particular, it depends neither on the classical value of the background curvature $\bar{R}$ nor on the Planck scale. This is essential for a scale-invariant, phenomenologically acceptable solution of the cosmological-constant problem.

We may compare this case to that of spontaneous breaking of a continuous symmetry in 1+1 dimensions [11]. Consider a complex scalar field with a tree-level potential-energy function of $|\phi|^2$ corresponding to symmetry breaking. When the field is quantized, the corresponding massless Goldstone boson has a propagator which grows logarithmically at large distances. This infrared divergence implies instability of the spontaneously broken vacuum to quantum fluctuations. Because of this instability of the ordered state, the system becomes disordered, the U(1) symmetry is restored at the quantum level, and the Mermin-Wagner-Coleman theorem is satisfied. Locally, there are regions of broken symmetry in which the classical description remains valid. However, as we consider regions of larger and larger size, the local classical vacuum executes a random walk in the phase angle and averages to zero. The quantitative description of this phenomenon is given by the power-law falloff of the correlation function $\langle \phi(x)\phi^\dagger(x')\rangle$. Introducing the nonlinear polar field decomposition $\phi=\rho\exp(i\theta)$ and neglecting the fluctuations of the massive ρ field in the infrared, one finds that the angular field θ may be treated as a free field with the propagator

$$\langle \theta(x)\theta(x')\rangle = -\frac{1}{4\pi\rho^2}\ln[\mu^2(x-x')^2] , \tag{4.7}$$

so that the model is now equivalent to the x-y model in two dimensions. In this infrared scaling limit, the correlation function for ϕ has a power-law behavior analogous to (4.6):

$$\begin{aligned}\langle \phi(x)\phi^\dagger(x')\rangle &\to \rho^2\langle e^{i\theta(x)}e^{(-i\theta(x'))}\rangle \\ &\to \rho^2\exp\langle \theta(x)\theta(x')\rangle \\ &= \rho^2|\mu(x-x')|^{-1/2\pi\rho^2} , \\ &\qquad |x-x'|\to\infty . \end{aligned}\tag{4.8}$$

In this case the power law is nonuniversal, as it depends on the classical value of the background field ρ.

One might ask if there is an analogue of the massless Goldstone pole of this two-dimensional model in the four-dimensional σ theory. The answer is yes, but its role is quite subtle. In the case of conformally flat fiducial metrics $\bar{g}_{ab}$, there are conserved Noether currents corresponding to global scale invariance:

$$J^{a(i)}=\xi_b^{(i)}T^{ab}, \quad i=1,\ldots,d+1 . \tag{4.9}$$

The covariant conservation of $J^{a(i)}$ follows from the covariant conservation and tracelessness of T^{ab} with the use of (2.17). The Ward identities corresponding to these conserved currents may be derived by the usual methods. In fact, using the transformation (2.20) modified by the anomalous scaling (3.12), we find that the self-energy function $\Gamma^{(2)}$ satisfies

$$\int d^4x\sqrt{-\bar{g}}\,\bar{\nabla}^a\xi_a^{(i)}(x)\Gamma^{(2)}(x,x')=0 , \tag{4.10}$$

at the extremum of the effective action, $\Gamma^{(1)}=0$. It follows that the self-energy function of σ has a zero when evaluated at the conformal Killing field $\bar{\nabla}^a\xi_a^{(i)}(x)$. Equivalently, since

$$\left[\bar{\Box}+\frac{\bar{R}}{3}\right]\bar{\nabla}^a\xi_a^{(i)}(x)=0 , \tag{4.11}$$

in maximally symmetric backgrounds, $\Gamma^{(2)}$ contains at least one factor of $\bar{\Box}+\bar{R}/3$. The corresponding "Goldstone" pole describes a tachyonic mode in de Sitter space, with an infrared behavior even more divergent than the logarithmic growth of (4.5) [4].

This pole exists even in Einstein theory, localized about the de Sitter background. However, in that case we know that it does not propagate because of the constraints: The equation of motion tells us that $R=4\Lambda$ and the scalar modes are frozen. In the covariant path integral, this comes about by means of the nontrivial Jacobian (2.24) discussed in Sec. II. When the effective action induced by the trace anomaly is included, the scalar curvature is no longer constrained to be a constant by the equations of motion, and propagating modes survive. However, it seems from (4.5) that the tachyonic divergence does not survive, but a subleading logarithmic divergence is physically relevant. This is precisely the behavior found previously for the physical transverse gravitons of de Sitter space.

To summarize, we believe that the effective theory of the conformal factor presented here provides a useful framework for studying the infrared behavior of gravity in four dimensions and addressing the cosmological-constant problem. The anomalous scaling of the conformal factor may be the key to understanding why $\langle R\rangle=0$ in the observed Universe. However, many unanswered questions remain.

First, we relied heavily on conformal flatness of the metric and resulting conformal symmetry [Eq. (2.20)]. Because the analysis of Sec. III was purely local, it is not clear that this is a necessary restriction, but neither is it immediately obvious how to derive the results in a more general context. What is lacking in the approach presented here is a controlled approximation scheme in which the conformal symmetry is *not* exact and the fixed point is approached in a well-defined manner. This is essential for demonstrating that the scaling behavior found in this paper survives in a more complete theory. It is important for the application to the cosmological constant as well, since with the present methods we cannot analyze the transition from the regime in which the classical Einstein equations are valid to that where the conformal fluctuations predominate, and asymptotic power-law behavior [Eq. (4.6)] applies. If the theoretical framework proposed in this paper withstands a more thorough analysis, then it should have dramatic effects on the dynamics of the Universe in its early inflationary epoch(s) and consequences for observational cosmology, large-scale structure, and the missing-matter puzzles.

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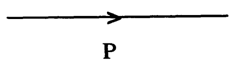
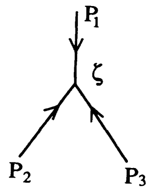
APPENDIX

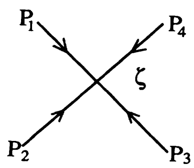
Here we give the Feynman rules of the Lagrangian (3.2), as well as the divergent contributions of all relevant graphs used in Sec. III. We use dimensional regularization in $d=4-2\varepsilon$ dimensions of the Euclidean continuation of the action: $S_E=-S$.



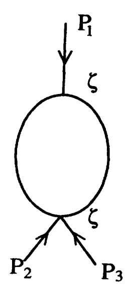
$$=\frac{1}{2\left[\dfrac{Q^2}{16\pi^2}P^4-\gamma P^2+8\lambda\right]}\ , \tag{A1}$$



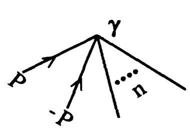
$$=-4\alpha\zeta[P_1^2(P_2\cdot P_3)+P_2^2(P_1\cdot P_3)+P_3^2(P_1\cdot P_2)]\ , \tag{A2}$$



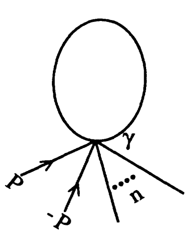
$$=-8\alpha^2\zeta[(P_1\cdot P_2)(P_3\cdot P_4)+(P_1\cdot P_3)(P_2\cdot P_4)+(P_1\cdot P_4)(P_2\cdot P_3)]\ , \tag{A3}$$



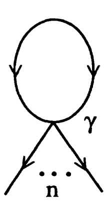
$$+\text{symmetrizations}=160\pi^2\frac{\alpha^2\zeta^2}{Q^4}\frac{1}{\varepsilon}[P_1^2(P_2\cdot P_3)+P_2^2(P_1\cdot P_3)+P_3^2(P_1\cdot P_2)]\ , \tag{A4}$$



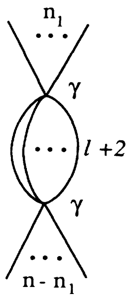
$$=2\gamma(2\alpha)^nP^2\ , \tag{A5}$$



$$=\frac{2\alpha^2\gamma}{Q^2}\frac{1}{\varepsilon}\mu^{2\varepsilon}(2\alpha)^nP^2\ , \tag{A6}$$



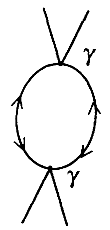
$$=8\pi\frac{\gamma}{Q^4}\frac{1}{\varepsilon}\mu^{2\varepsilon}(2\alpha)^n, \tag{A7}$$



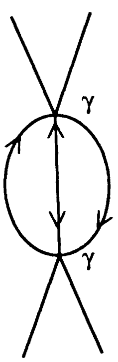
$$=\frac{1}{2}\gamma^2(2\alpha)^{n+2l}\frac{A_l}{(l!)^2}\left[\delta_{l,0}\sum_{n_1=1}^{n-1}+(1-\delta_{l,0})\sum_{n_1=0}^{n}\right]\begin{bmatrix} n \\ n_1 \end{bmatrix}$$
$$=\frac{1}{2}\gamma^2(4\alpha)^n(2\alpha)^{2l}\frac{A_l}{(l!)^2}-\gamma^2(2\alpha)^n A_0\delta_{l,0} \ . \tag{A8}$$

Equation (A8) follows from the observation that in the one-loop case ($l=0$) the number of external lines n_1 (or $n-n_1$) cannot vanish, since in that case there is no vertex. A_l counts for the result of the internal loop integration, together with the combinatoric factor associated to the $l+2$ internal lines.

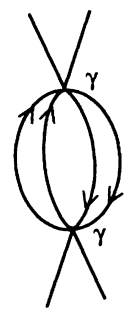
Since the γ-vertex contain only two derivatives and the σ propagator (A1) is quartic, is simple power argument shows that no new divergences arise beyond three loops. The primitive divergences of the remaining graphs are listed below:



$$A_0=\frac{8\pi^2}{Q^4}\frac{1}{\varepsilon}\mu^{2\varepsilon} \ , \tag{A9}$$

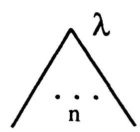


$$A_1=\frac{4\pi^2}{Q^6}\frac{1}{\varepsilon}\mu^{4\varepsilon}, \tag{A10}$$

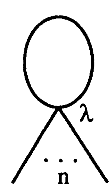


$$A_2=\frac{4\pi^2}{Q^8}\frac{1}{\varepsilon}\mu^{6\varepsilon} \ . \tag{A11}$$

The arrows in the above graphs (A9)–(A11) denote the action of the derivatives from the γ vertex. A comparison of Eqs. (A7)–(A9) shows that the terms multiplying $(2\alpha)^n$, which corresponds to a new Lagrangian term $e^{2\alpha\sigma}$, are canceled.



$$=\frac{\lambda}{\alpha^2}(4\alpha)^n \ , \tag{A12}$$



$$=-\frac{4\lambda}{Q^2}(4\alpha)^n\frac{1}{\varepsilon}\mu^{2\varepsilon} \ . \tag{A13}$$

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