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General class of inhomogeneous perfect-fluid solutions

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We present a general class of solutions to Einstein's field equations with two spacelike commuting Killing vectors by assuming the separation of variables of the metric components. The solutions can be interpreted as inhomogeneous cosmological models. We show that the singularity structure of the solutions varies depending on the different particular choices of the parameters and metric functions. There exist solutions with a universal big-bang singularity, solutions with timelike singularities in the Weyl tensor only, solutions with singularities in both the Ricci and the Weyl tensors, and also singularity-free solutions. We prove that the singularity-free solutions have a well-defined cylindrical symmetry and that they are generalizations of other singularity-free solutions obtained recently.

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I. INTRODUCTION

The study of the Universe as a whole has been an outstanding scientific subject since the appearance of general relativity. Until very recently, the exact solutions used for that study have been the so-called spatially homogeneous models (Bianchi or Friedmann-Robertson-Walker), which admit at least a three-parameter group of isometrics. However, in order to study inhomogeneous epochs of the Universe, which apparently are necessary for the formation of large-scale structures, it is necessary to use *exact* inhomogeneous solutions to Einstein's field equations. Of course, the general inhomogeneous metric does not have any symmetry at all, but the complexity of Einstein's equations is so high that some simplifications must be assumed. The simplest inhomogeneous models are those with two spacelike commuting Killing vectors, known as orthogonally transitive G_2 cosmologies [1–3]. Very few solutions of this type for a perfect-fluid energy-momentum tensor are known up to now. The first class of solutions was given by Wainwright and Goode [4], and new metrics were later found in [5] and [6] and recently in [7].

Most of these metrics present a universal spacelike big-bang singularity in the finite past. It was thought, therefore, that this would be the usual singularity in general models. However, the solution presented by one of us in [6] had no curvature singularity at all and a well-defined cylindrical symmetry. In fact, it has been shown in [8] that this solution is singularity-free in the sense that all causal curves are complete. Further investigation in this sense was thus needed. In this paper, we present a very general class of inhomogeneous perfect-fluid metrics and study the character of the possible singularities they can have. The result we obtain is rather surprising because the solutions can have *all* types of singularities, and the big-bang type is just one among them. Furthermore, we find a very large class of singularity-free and cylindrically symmetric solutions, which generalizes the metric of [6]. The properties of this singularity-free family are very similar to that shown in [8] for the particular metric found in Ref. [6].

We start in Sec. II with the basic equations and formulas of the line element and we prove that the metrics are generated by the solutions to a system of coupled first-order ordinary differential equations. The general solution of this system has not been found, but many interesting particular cases can be completely integrated providing solutions with all types of behaviors. The found explicit solutions are presented in Sec. III. Among these solutions there appear the above-mentioned singularity-free family of solutions, and we describe its properties in detail in Sec. IV. In Sec. V we prove that this family of solutions is the only one without singularities and, by means of a qualitative analysis of the differential equations, we give the general behavior of the most general metric. Finally, we devote Sec. VI to perform a brief discussion of the results.

II. THE LINE ELEMENT AND BASIC EQUATIONS

We want to study orthogonally transitive diagonal G_2 cosmologies [1]. From a geometrical point of view, this means that the spacetime admits two commuting spacelike Killing vector fields, both of which are hypersurface orthogonal, and that the Einstein field equations are satisfied for an energy-momentum tensor of a perfect fluid. Under these assumptions, it can be shown [2] that the line element takes the so-called generalized Einstein-Rosen form [3]

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$$ds^2 = -F_0 dt^2 + F_1 dx^2 + F_2(F_3 dy^2 + F_3^{-1} dz^2) , \tag{1}$$

where the positive functions $F_\alpha(\alpha,\beta,\ldots=0,1,2,3)$ depend on the coordinates t and x, the Killing vectors are $\partial/\partial y$ and $\partial/\partial z$ and the unit velocity vector of the fluid is given by

$$\mathbf{u} = -F_0^{1/2} dt . \tag{2}$$

At this stage, the coordinates $\{t,x,y,z\}$ are generic and nothing can be said about the range of values they take. As we shall presently see, there are cases in which (1) represents cylindrically symmetric spacetimes and cases in which the coordinates can be thought as Cartesian-like, among others.

In this paper, we only consider metrics of type (1) such that the functions F_α are separable, that is, $F_\alpha = T_\alpha(t) X_\alpha(x)$. When in addition to this it is assumed that a $p=\gamma\rho$ $(0<\gamma<1)$ equation of state holds, it can be shown [1] that there exist two different classes of solutions. In the first class, F_3 does not depend on x (so that the three-slices orthogonal to the fluid flow are conformally flat) and the explicit general solution has been given by Wainwright and Goode [4]. On the other hand, it can be proven that the second class is characterized by the fact that all the functions $T_\alpha(t)$ are powers of a single function $T(t)$. This is the class we shall study here, despite the fact that we shall not make any *a priori* assumption on the form of the equation of state.

Hence, we restrict the line element (1) to the form

$$ds^2 = T^{2m}F^2(-dt^2 + H^2 dx^2) + TG(T^n P dy^2 + T^{-n} P^{-1} dz^2) , \tag{3}$$

$$\mathbf{u} = -T^m F\, dt , \tag{4}$$

where T is a function of only t, F,G,P, and H are functions of only x, and m and n are constants. Here, we have used the freedom in choosing the coordinate t to set equal powers of the function T in the first two terms of (3). The corresponding freedom in the coordinate x could also be used to set $H(x)=1$, but we shall not do so now because the function $H(x)$ will enable us to integrate the field equations in terms of elementary functions in some occasions. Particular cases of the metric (3) have been studied earlier [9], and some explicit solutions have been given in [5–7].

The calculation of the Einstein tensor for the metric (3) in the orthonormal tetrad

$$\theta^0 = T^m F\, dt,\ \theta^1 = T^m FH\, dx ,$$
$$\theta^2 = T^{(1+n)/2}\sqrt{GP}\, dy,\ \theta^3 = T^{(1-n)/2}\sqrt{G/P}\, dz , \tag{5}$$

yields for the nonvanishing components

$$G_{00} = T^{-2m}F^{-2}\left[\frac{1}{H^2}\left[\frac{F'G'}{FG} - \frac{G''}{G} + \frac{1}{4}\frac{G'^2}{G^2} + \frac{G'H'}{GH} - \frac{1}{4}\frac{P'^2}{P^2}\right] + \frac{4m-n^2+1}{4}\frac{\dot{T}^2}{T^2}\right] , \tag{6}$$

$$G_{01} = T^{-2m}F^{-2}H^{-1}\frac{\dot{T}}{T}\left[\frac{F'}{F} + \frac{2m-1}{2}\frac{G'}{G} - \frac{n}{2}\frac{P'}{P}\right] , \tag{7}$$

$$G_{11} = T^{-2m}F^{-2}\left[\frac{1}{H^2}\left[\frac{F'G'}{FG} + \frac{1}{4}\frac{G'^2}{G^2} - \frac{1}{4}\frac{P'^2}{P^2}\right] - \frac{\ddot{T}}{T} + \frac{4m-n^2+1}{4}\frac{\dot{T}^2}{T^2}\right] , \tag{8}$$

$$G_{22} = T^{-2m}F^{-2}\left[\frac{1}{H^2}\left[\frac{F''}{F} - \frac{F'^2}{F^2} - \frac{F'H'}{FH} + \frac{1}{2}\frac{G''}{G} - \frac{1}{4}\frac{G'^2}{G^2} - \frac{1}{2}\frac{G'H'}{GH} - \frac{1}{2}\frac{G'P'}{GP} + \frac{1}{2}\frac{P'H'}{PH} - \frac{1}{2}\frac{P''}{P} + \frac{3}{4}\frac{P'^2}{P^2}\right] - \frac{2m-n+1}{2}\frac{\ddot{T}}{T} + \frac{4m-n^2+1}{4}\frac{\dot{T}^2}{T^2}\right] , \tag{9}$$

$$G_{33} = G_{22} + T^{-2m}F^{-2}\left[\frac{1}{H^2}\left[\frac{G'P'}{GP} - \frac{P'H'}{PH} + \frac{P''}{P} - \frac{P'^2}{P^2}\right] - n\frac{\ddot{T}}{T}\right] , \tag{10}$$

where dots and primes stand for derivatives with respect to t and x, respectively. We seek solutions of the Einstein field equations for an energy-momentum tensor of a perfect fluid:

$$T_{\alpha\beta} = (\rho+p)u_\alpha u_\beta + p g_{\alpha\beta} \tag{11}$$

with u_α given in (4), and where ρ is the energy density, p the pressure, and $g_{\alpha\beta}$ the metric tensor. From (11) and (4) and due to Einstein's equations it follows that G_{01} vanishes. Then, expression (7) implies ($\dot{T}\neq 0$)

$$F^2 = G^{1-2m}P^n , \tag{12}$$

so that there only remain two independent functions of x. In a similar way, the field equations together with (11) and (4) provide $G_{22}=G_{33}$, so that formula (10) above yields

$$\frac{1}{H^2}\left[\frac{G'P'}{GP} - \frac{P'H'}{PH} + \frac{P''}{P} - \frac{P'^2}{P^2}\right] = n\frac{\ddot{T}}{T} . \tag{13}$$

The left-hand side of this equation is a function of x, whereas the right-hand side is a function of t. Therefore, both of them must be equal to a separation constant. Thus, we have

$$\frac{\ddot{T}}{T} = \epsilon a^2,\quad \epsilon = 0, \pm 1 \tag{14}$$

or, equivalently,

$$T(t)=\begin{cases} A\cosh(at)+B\sinh(at) & \text{if } \epsilon=1\ , \\ At+B & \text{if } \epsilon=0\ , \\ A\cos(at)+B\sin(at) & \text{if } \epsilon=-1\ , \end{cases} \tag{15}$$

where A and B denote arbitrary constants of integration. In addition to this, from (13) we must have

$$\frac{P''}{P}-\frac{P'^2}{P^2}+\frac{G'P'}{GP}-\frac{P'H'}{PH}=\epsilon n a^2 H^2\ . \tag{16}$$

Analogously, G_{22} has to be equal to G_{11}. Then, from (8) and (9) and making use of (12), (14), and (16) we obtain

$$(1-m)\left[\frac{G''}{G}-\frac{G'H'}{GH}\right]+\frac{4m-3}{2}\frac{G'^2}{G^2}-n\frac{G'P'}{GP}+\frac{1}{2}\frac{P'^2}{P^2}=\frac{2m-1-n^2}{2}\epsilon a^2H^2\ . \tag{17}$$

The last information we can extract from Einstein's equations is the expression of the density and pressure, which can be obtained from (11), (4), (14), (6), and (8) and read

$$\chi\rho=T^{-2m}F^{-2}\left[\frac{1}{H^2}\left[\frac{F'G'}{FG}-\frac{G''}{G}+\frac{1}{4}\frac{G'^2}{G^2}+\frac{G'H'}{GH}-\frac{1}{4}\frac{P'^2}{P^2}\right]+\frac{4m-n^2+1}{4}\frac{\dot{T}^2}{T^2}\right]\ , \tag{18}$$

$$\chi p=T^{-2m}F^{-2}\left[\frac{1}{H^2}\left[\frac{F'G'}{FG}+\frac{1}{4}\frac{G'^2}{G^2}-\frac{1}{4}\frac{P'^2}{P^2}\right]-\epsilon a^2+\frac{4m-n^2+1}{4}\frac{\dot{T}^2}{T^2}\right]\ , \tag{19}$$

where χ is the gravitational constant.

We see that, in order to get a solution of the field equations, we only need to solve equations (16) and (17). Given a pair of functions G and P solutions of these equations, we can get F via expression (12). This, together with expressions (15) for T, allows us to obtain the full metric and the density and pressure given in (18) and (19). Thus, equations (16) and (17) constitute the fundamental system of equations of the problem under consideration.

As we can see directly from (17), a very special case arises when $m=1$, because the second derivative of G does not appear in the equations. However, as is easily checked, the general solution in this particular case provides metrics with a stiff fluid equation of state $p=\rho$ (apart from very singular metrics with negative pressure). But the stiff fluid solutions have been previously obtained, and it is very well known that they can be generated starting from the vacuum general solution by means of a very simple transformation (see Refs. [3] and [10]).

Therefore, we assume $m\neq 1$ from now on. Then, by defining

$$\alpha\equiv\frac{P'}{P},\quad \beta\equiv\frac{G'}{G} \tag{20}$$

we rewrite the fundamental system (16) and (17) as a system of two coupled nonlinear first-order differential equations, which read

$$\alpha'+\alpha\beta-\alpha\frac{H'}{H}=\epsilon n a^2H^2\ , \tag{21}$$

$$\beta'+\frac{2m-1}{2(1-m)}\beta^2-\frac{n}{1-m}\beta\alpha+\frac{1}{2(1-m)}\alpha^2-\beta\frac{H'}{H}=\frac{2m-1-n^2}{2(1-m)}\epsilon a^2H^2\ , \tag{22}$$

where H is, in principle, a disposable function. By using expression (12) for F and Eq. (22), and introducing the new functions (20) in (18) and (19), the density and pressure become now

$$\chi\rho=T^{-2m}F^{-2}\left[\frac{m+1}{m-1}\mu+\frac{4m+1-n^2}{4}\left[\frac{\dot{T}^2}{T^2}-\epsilon a^2\right]\right]\ , \tag{23}$$

$$\chi p=T^{-2m}F^{-2}\left[\mu+\frac{4m+1-n^2}{4}\left[\frac{\dot{T}^2}{T^2}-\epsilon a^2\right]\right]\ , \tag{24}$$

where we have put

$$\mu\equiv\frac{1}{H^2}\left[\frac{n}{2}\alpha\beta-\frac{1}{4}\alpha^2+\frac{3-4m}{4}\beta^2+\frac{4m-3-n^2}{4}\epsilon a^2H^2\right]\ . \tag{25}$$

From these expressions it follows immediately that the fluid will satisfy a barotropic equation of state $p=p(\rho)$ (apart from the special stiff fluid case) if and only if either

$$4m+1=n^2,\quad m\geq -\tfrac{1}{4}\ , \tag{26}$$

or

$$T(t)=e^{at},\quad \epsilon=1\ . \tag{27}$$

In that case, the equation of state is necessarily linear, that is

$$p=\gamma\rho\ , \tag{28}$$

where the constant γ is given by

$$\gamma=\frac{m-1}{m+1}\ . \tag{29}$$

In general, a much simpler expression for the density and pressure can be given by noting that

$$\frac{\mu'}{\mu}=\frac{1}{m-1}[(2m-1)\beta-n\alpha]\ ,$$

or, equivalently, taking into account (12) and (20),

$$\frac{\mu'}{\mu}=\frac{2}{1-m}\frac{F'}{F}\Longrightarrow\mu=cF^{2/(1-m)}\ , \tag{30}$$

where c is a constant. Therefore, we can also write

$$\chi\rho = T^{-2m}F^{-2}\left[\frac{m+1}{m-1}cF^{2/(1-m)} + \frac{4m+1-n^2}{4}\left[\frac{\dot{T}^2}{T^2}-\epsilon a^2\right]\right], \quad (31)$$

$$\chi p = T^{-2m}F^{-2}\left[cF^{2/(1-m)} + \frac{4m+1-n^2}{4}\left[\frac{\dot{T}^2}{T^2}-\epsilon a^2\right]\right]. \quad (32)$$

In fact, the last two formulas could have also been obtained directly from the conservation equation $\nabla_\alpha T^{\alpha\beta}=0$ and the fact that the metric is given in separated form. In the particular case in which the relations (26) or (27) are satisfied, that is to say, when the equation of state (28) is verified, expressions (31) and (32) reduce to

$$\chi p = cT^{-2m}F^{2m/(1-m)} = cT^{-2(1+\gamma)/(1-\gamma)}F^{-(1+\gamma)/\gamma}, \quad (33)$$

with γ given in (29). The previous expression for p (and ρ) is very important with regard to the existence of *physical* singularities, because these singularities will appear depending only on the from of the functions T and F and the value of the parameter m. We shall make a detailed study of these matters in Sec. V.

To end this section, we shall now give the expressions for the kinematical quantities and the Weyl tensor. With regard to the kinematical properties of the velocity vector (4) in the metric (3), a straightforward calculation leads to the following expression for the expansion, acceleration, and shear of the fluid

$$\theta = (m+1)\frac{1}{T^m F}\frac{\dot{T}}{T}, \quad (34)$$

$$a_1 = \frac{1}{T^m FH}\frac{F'}{F}, \quad (35)$$

$$\sigma_{11} = \frac{2m-1}{3}\frac{1}{T^m F}\frac{\dot{T}}{T}, \quad (36)$$

$$\sigma_{22} = \frac{1+3n-2m}{6}\frac{1}{T^m F}\frac{\dot{T}}{T}, \quad (37)$$

$$\sigma_{33} = \frac{1-3n-2m}{6}\frac{1}{T^m F}\frac{\dot{T}}{T}, \quad (38)$$

where all other components as well as the rotation tensor vanish, and all the components have been computed in the orthonormal tetrad (5). As for the nonvanishing components of the Weyl tensor, we perform the computation in the null tetrad,

$$\mathbf{k} = \frac{1}{\sqrt{2}}(\theta^0-\theta^1), \quad \mathbf{l} = \frac{1}{\sqrt{2}}(\theta^0+\theta^1), \quad \mathbf{m} = \frac{1}{\sqrt{2}}(\theta^2+i\theta^3), \quad (39)$$

and we get (using standard notation, see for example Ref. [11])

$$\Psi_0+\Psi_4 = T^{-2m}F^{-2}\left[\frac{1}{H^2}\left[\alpha\frac{F'}{F}-\frac{1}{2}\alpha\beta+\frac{1}{2}\alpha\frac{H'}{H}-\frac{1}{2}\alpha'\right] + mn\frac{\dot{T}^2}{T^2}-\frac{n}{2}\epsilon a^2\right], \quad (40)$$

$$\Psi_0-\Psi_4 = T^{-2m}F^{-2}H^{-1}\frac{\dot{T}}{T}\left[\frac{2m-1}{2}\alpha+n\frac{F'}{F}-\frac{n}{2}\beta\right], \quad (41)$$

$$\Psi_2 = \frac{1}{6}T^{-2m}F^{-2}\left[\frac{1}{H^2}\left[\frac{F''}{F}-\frac{F'^2}{F^2}-\frac{F'H'}{FH}-\frac{1}{2}\beta' + \frac{1}{2}\beta\frac{H'}{H}-\frac{1}{2}\alpha^2\right] + \frac{2m+n^2-1}{2}\frac{\dot{T}^2}{T^2}+\frac{1-2m}{2}\epsilon a^2\right]. \quad (42)$$

It is clear that these expressions could be further simplified by using (12) and the main equations (21) and (22) so that there would not appear any derivative of the functions α and β. We shall not do that here for the sake of brevity (see Sec. V). Moreover, from (40)–(42) it follows that the metric will be, in general, of Petrov type I, at least in generic points. Very special cases could arise in which the Weyl tensor is of Petrov types D or 0, but these cases are those of very well-known metrics with higher symmetry. However, we shall prove that in some relevant cases the solutions contain regions where the Petrov type specializes to the D type, and this will be of some importance for the avoidance of singularities.

In the next section, we try to solve the fundamental system of equations (21) and (22) in order to get explicit solutions. Although the general solution has not been obtained in terms of simple functions, some special cases, each of them with different properties, are presented in closed form.

III. EXPLICIT SOLUTIONS

Our task now is trying to solve Eqs. (21) and (22) for α and β (or P and G), bearing in mind that H is a choosable function of x. We shall devote attention only to those families of solutions which provide nonpreviously known metrics. In this sense, we shall not consider the cases yielding stiff fluid solutions (for example, when $\mu=0$) which can be completely integrated.

Hitherto, we have been able to produce four different classes of solutions, all of them for particular values or special relations of the parameters m, n, and ϵ. We present them separately in what follows and omit the details of the calculations.

Case 1. The simplest family of solutions is characterized by

$$\epsilon=0 \Longrightarrow T(t)=At+B. \quad (43)$$

In this case, Eqs. (20)–(22) can be integrated completely if $n^2+3-4m \geq 0$, and the general solution, after rescaling of spurious constants, is given by

$$P^{k+l}=C^{(3-4m)/(1-2m)}+N, \quad G=P^k C^{2(1-m)/(2m-1)}, \quad H=j\frac{C'}{P^l}, \tag{44}$$

where N and j are arbitrary constants, $C(x)$ is an arbitrary function, and

$$k=\frac{n+(n^2+3-4m)^{1/2}}{4m-3}, \quad k+l=-\frac{(n^2+3-4m)^{1/2}}{1-m}.$$

The pressure and energy density read as

$$\chi p=(At+B)^{-2m}C^{2(1-m)}P^{(k-l)(1-m)}\left[\frac{(3-4m)(1-m)^2}{(2m-1)^2k^2}\frac{N}{C^2P^{k-l}}+\frac{4m+1-n^2}{4}\frac{A^2}{(At+B)^2}\right], \tag{45}$$

$$\chi \rho=(At+B)^{-2m}C^{2(1-m)}P^{(k-l)(1-m)}\left[\frac{(3-4m)(m^2-1)}{(2m-1)^2k^2}\frac{N}{C^2P^{k-l}}+\frac{4m+1-n^2}{4}\frac{A^2}{(At+B)^2}\right]. \tag{46}$$

The limit case $A=0$ provides static solutions with $p=\gamma\rho$, which can be identified as the cylindrically symmetric static solutions given first by Bronnikov [12] and later by Kramer [13]. In the general case, the solutions have ρ and p positive everywhere for adequate choices of the constants and they present singularities at $t=-B/A$ (big bang) and also where $P=0$ or ∞ depending on the parameters. However, this last singularity does not exist if $k=l \Longleftrightarrow 2m=1+n$. In particular, for example, the $p=\frac{1}{3}\rho$ solution with $n=3$, $m=2$, $N=-1$, and $C(x)=(1+x^2)^{3/5}$ has been recently obtained by Davidson [7].

The above general solution holds for general values of m and n except for the exceptional cases $m=\frac{3}{4}$, $m=\frac{1}{2}$, and $4m=n^2+3$ $(k+l=0)$. We now give the solutions for these exceptional cases:

$$m=\tfrac{3}{4}:\ P^{4n}=N+ax, \quad G=P^{1/2n}e^{-ax}, \quad H=e^{-ax}P^{1/2n-4n},$$

$$m=\tfrac{1}{2}:\ P^{k+l}=C(x)+N, \quad G=\frac{P^k}{C(x)}, \quad H=j\frac{C'(x)}{C(x)P^l}, \quad k=-n-(n^2+1)^{1/2}, \quad k+l=-2(n^2+1)^{1/2},$$

$$4m=n^2+3:\ P=\exp(ax^{2n^2/(1+n^2)}), \quad G=P^{1/n}(ax)^{(1-n^2)/(1+n^2)}, \quad H=P^{1/n}.$$

The expressions of ρ and p for these special solutions can be easily obtained from (23)–(25).

Case 2. For $\epsilon\neq 0$ there are several integrable cases. The first of them is given by

$$\epsilon=1, \quad n^2=\frac{(3m-2)^2}{2m-1}, \quad m>\tfrac{1}{2}, \tag{47}$$

which can be fully solved in terms of hypergeometric equations. There is, however, a particular solution involving elementary functions only. This solution is

$$H=1, \quad P=\exp\left[\frac{m-1}{2\sqrt{2m-1}}ax\right]f^{\sqrt{2m-1}},$$

$$G=\exp\left[-\frac{m-1}{2(2m-1)}ax\right]f, \tag{48}$$

$$f(x)=A_1\exp\left[\frac{5m-3}{2(2m-1)}ax\right]-A_2\exp\left[-\frac{5m-3}{2(2m-1)}ax\right],$$

where A_1 and A_2 are arbitrary constants. The explicit expressions for the density and pressure are easily obtained from (23)–(24) with (47) by noting that now

$$\mu=a^2A_2f^{-1}\frac{(m-1)^2(5m-3)}{2(2m-1)^2}\exp\left[-\frac{5m-3}{2(2m-1)}ax\right].$$

Among this family of solutions, the case $A=B$ ($T=Ae^{at}$) provides, as always, a $p=\gamma\rho$ equation of state with γ given in (29). In addition, the special solution with $m^2-10m+5=0 \Longrightarrow m=5+2\sqrt{5}$ has the equation of state

$$p=\gamma\rho, \quad \gamma=\frac{1+\sqrt{5}}{4}.$$

In general, these solutions have singularities at $x=0$ if the density and pressure are chosen positive everywhere. These specific properties will be studied in Sec. V.

Case 3. The third case is characterized by the relations

$$\epsilon=\pm 1, \quad n^2=\frac{1}{5-4m}, \quad m<\tfrac{5}{4}, \tag{49}$$

where now ϵ can be positive or negative. A particular solution for Eqs. (20)–(22) is now

$$H=1, \quad P=f^{\sqrt{5-4m}}, \quad G=ff'^{\,2(1-m)/(2m-1)},$$

$$\frac{f''}{f}=\frac{2m-1}{5-4m}\epsilon a^2, \tag{50}$$

unless for the special value $m=\frac{1}{2}$ $(n=1/\sqrt{3})$. The above differential equation for $f(x)$ can be straightforwardly integrated in terms of hyperbolic or trigonometric functions depending on whether $[(2m-1)/(5-4m)]\epsilon$ is positive or negative, respectively. The expressions for ρ and p are now given by (23), (24) with (49), and

$$\mu=a^4\left[\frac{f}{f'}\right]^2\frac{(3-4m)(1-m)^2}{(5-4m)^2} .$$

As always, there are two cases with a $p=\gamma\rho$ equation of state: when $A=B$ and for the special value $m=(1+\sqrt{2})/2$, which produces a value $\gamma=(4\sqrt{2}-5)/7$.

The exceptional solution for $m=\frac{1}{2}$ can be also integrated similarly. The solution reads

$$H=1,\quad P=(A_1x+A_2)^{\sqrt{3}},\quad G=(A_1x+A_2)e^{(\epsilon a^2/6A_1^2)(A_1x+A_2)^2},\quad F=(A_1x+A_2)^{1/2}, \tag{51}$$

with A_1, A_2 constants. The expressions for the density and pressure are then trivially obtained from (31)–(32) where $c=a^4/36A_1^2$.

Case 4. The last and perhaps most interesting case is defined by the relation

$$2m=1+n . \tag{52}$$

The general solution for Eqs. (20)–(22) can be found and is given by $(n\neq\frac{1}{2})$

$$H=M\frac{C'}{P},\quad G=PC^{(1-n)/n},\quad P^2=\epsilon n^2a^2M^2C^2+NC^{(2n-1)/n}-K\geq 0 , \tag{53}$$

where $C(x)$ is an arbitrary function and M, N, and K are constants. The calculation of the density and pressure yields

$$\chi p=T^{-(1+n)}C^{1-n}\left[\frac{(n-1)^2(2n-1)}{4n^2M^2}\frac{K}{C^2}+\frac{(3-n)(n+1)}{4}\left[\frac{\dot{T}^2}{T^2}-\epsilon a^2\right]\right], \tag{54}$$

$$\chi\rho=T^{-(1+n)}C^{1-n}\left[\frac{(n-1)(2n-1)(n+3)}{4n^2M^2}\frac{K}{C^2}+\frac{(3-n)(n+1)}{4}\left[\frac{\dot{T}^2}{T^2}-\epsilon a^2\right]\right] . \tag{55}$$

The important thing about this family is that the metric has a well-defined cylindrical symmetry and that it contains a very large subfamily of *singularity-free* solutions. Because of its importance, we devote the next section to the study of this subfamily.

IV. A FAMILY OF SINGULARITY-FREE SOLUTIONS

Let us consider the subfamily of the solution (53) defined by

$$\epsilon=1,\quad n\geq 3,\quad T(t)=\cosh(at),\quad C(x)=\cosh(nax) . \tag{56}$$

This class of solutions has very interesting properties, as we shall see immediately. By relabeling the coordinates appropriately, the line element can be written in the form

$$\begin{aligned} ds^2=&\cosh^{1+n}(at)\cosh^{n-1}(nar)\left[-dt^2+\frac{\sinh^2(nar)}{P^2}dr^2\right]\\ &+\cosh^{1+n}(at)\frac{P^2}{n^2a^2L^2\cosh^{(n-1)/n}(nar)}d\phi^2+\frac{\cosh^{1-n}(at)}{\cosh^{(n-1)/n}(nar)}dz^2 , \end{aligned} \tag{57}$$

where

$$L\equiv K-\frac{K-1}{2n},\quad P^2=\cosh^2(nar)+(K-1)\cosh^{(2n-1)/n}(nar)-K , \tag{58}$$

and the range of coordinates is taken to be

$$-\infty<t,\quad z<\infty,\quad 0\leq r<\infty,\quad 0\leq\phi\leq 2\pi$$

such that ϕ and $\phi+2\pi$ are identified. With the above choice of constants the metric (57) has a regular axis of symmetry $r=0$ at all times and the so-called elementary flatness [11] on the vicinity of the axis is satisfied. The expressions (54) and (55) for the pressure and energy density become now

$$\chi p=\frac{a^2}{4}\cosh^{-(1+n)}(at)\cosh^{1-n}(nar)\left[\frac{(n-1)^2(2n-1)K}{\cosh^2(nar)}+\frac{(n+1)(n-3)}{\cosh^2(at)}\right], \tag{59}$$

$$\chi\rho=\frac{a^2}{4}\cosh^{-(1+n)}(at)\cosh^{1-n}(nar)\left[\frac{(n-1)(2n-1)(n+3)K}{\cosh^2(nar)}+\frac{(n+1)(n-3)}{\cosh^2(at)}\right] . \tag{60}$$

From these equations we see that both density and pressure are positive and such that $p<\rho$ if $K>0$. Furthermore, ρ and p are *regular* over the whole spacetime, so that there is no physical singularity in the solution. Let us note that the maximum value of the density (and pressure), which occurs when $t=r=0$, is represented by the constants K and a, and then we can choose this maximum as large as we like. It is also obvious from (59) and (60) that ρ and p approach zero when either $t\to\pm\infty$ or $r\to\infty$.

From (34) it follows that the expansion is

$$\theta=a\frac{n+3}{2}\cosh^{-(n+1)/2}(at)\cosh^{(1-n)/2}(nar)\sinh(at) , \tag{61}$$

from where we learn that there is a contracting phase for $t<0$ and an expanding phase for $t>0$. Moreover, θ is regular for all possible values of r and t, and it goes to zero when $r,|t|\to\infty$. The spacelike surface $t=0$ is a maximal hypersurface in the sense that the expansion vanishes there. In a similar way, from (35)–(38) it is easily seen that all the kinematical quantities are well behaved everywhere. In fact, solution (57) has no curvature singularity at all, as can be directly checked by computing the Weyl tensor given in (40)–(42). The Petrov type of the Weyl tensor is I in generic points, but it specializes to type D at the axis of symmetry $r=0$.

The particular case $n=3$ of the metric (57) has a realistic equation of state for radiation-dominated matter:

$$p=\tfrac{1}{3}\rho . \tag{62}$$

The special subcase $n=3$, $K=1$ is the solution found previously by one of us [6], which has been shown to be singularity-free in the sense that all causal curves are complete [8]. A reasoning similar to that of Ref. [8] can be used to prove that the general solution (57) is singularity-free too, so that all causal curves can be extended to arbitrary values of the affine parameter.

In the next section we shall prove that this family of solutions is unique in the sense that any other solution contained in (3) has singularities.

V. BEHAVIOR OF THE SOLUTIONS AND THEIR SINGULARITIES

The families of solutions found in the previous two sections are representative of all the possible metrics. We have seen that the singularity structure of the solutions is very rich, in the sense that there are solutions with big-bang singularity only, solutions with big-bang and timelike singularities, solutions with timelike singularities only, and also singularity-free solutions. We shall now give an exhaustive study of all these different possibilities for the general line element (3). Although the most general solution cannot be found in closed form, the analysis of the behavior of the metric and the physical quantities can be performed by using the theory of ordinary differential equations applied to the fundamental system (20)–(22). This method has been used by other authors lately (see [1] and [14] and references therein). Our aim is to find the possible singularity-free solutions and to identify them among the whole class of solutions. It will turn out that the singularity-free solutions are given by case 4 above and that they constitute, so to speak, the separation in the space of metrics between two families with very different properties from the singularity point of view.

Of course, we restrict our study to physically realistic solutions in the sense that the energy conditions be satisfied. Therefore, we shall consider only the cases with both density and pressure non-negative everywhere. This condition will permit us to dismiss a great lot of solutions by simple inspection of the physical quantities.

To begin with, we see that, as remarked in Sec. II, the possible physical singularities (that is, singularities in ρ or p) can only come from the function $T(t)$ or from the function $F(x)$ due to the fact that the metric is given in separated form. With regard to the function $T(t)$, we can see from expressions (31) and (32) that, in general, the solutions will contain big-bang (and/or big-crunch) spacelike universal singularities at $T(t)=0$ or ∞. There are, however, four possibilities without this type of singularity. To see it, let us note that by means of a shift and a rescaling in the coordinate t we can always set $T(t)=\cos(at)$ if $\epsilon=-1$; $T(t)=t$ if $\epsilon=0$; and $T(t)=\cosh(at)$, e^{at}, $\sinh(at)$ if $\epsilon=1$ for $A^2>B^2$, $A^2=B^2$, and $A^2<B^2$, respectively. From expressions (31) and (32) and assuming $c\neq0$ (if $c=0$ then $p=\rho$), we identify the above-mentioned four cases as

$$T(t)=t,\quad \sinh(at),\quad \epsilon=0,1,\quad m=0,\quad n^2=1 , \tag{63}$$

$$T(t)=e^{at},\quad \epsilon=1,\quad m=0 , \tag{64}$$

$$T(t)=\cos(at),\quad \epsilon=-1,\quad m\le0,\quad 4m+1-n^2=0 , \tag{65}$$

$$T(t)=\cosh(at),\quad \epsilon=1,\quad m\ge0 . \tag{66}$$

The cases (63) and (64) have $\chi\rho=-\chi p=-c$, and then we do not consider them here. On the other hand, case (65) produces

$$\chi p=cF^{2m/(1-m)}\cos^{2|m|}(at),\quad p=\frac{m-1}{m+1}\rho$$

where also $0\ge m=(n^2-1)/4\ge-\frac{1}{4}$. Therefore, in this case the pressure and the density cannot be both positive at the same time and the energy conditions are not satisfied.

Thus, it only remains case (66) which is a little bit more complex than the other cases. From now on, we assume that relations (66) hold. Hence, the singular behavior of the solutions can only come from the function $F(x)$ [or equivalently $\mu(x)$] and the value of the parameters m and n. In order to proceed, it is convenient to fix the variable x by putting

$$H=\frac{1}{a}\ , \tag{67}$$

and to define new dependent variables as

$$u=\frac{1}{2\sigma}(\alpha-n\beta),\quad v=\beta \tag{68}$$

where

$$\sigma=\begin{cases}\frac{1}{2}\sqrt{n^2+3-4m} & \text{if } n^2+3-4m>0\ ,\\ 1 \ \text{if } n^2+3-4m=0\ , & \\ \frac{1}{2}\sqrt{4m-n^2-3} & \text{if } n^2+3-4m<0\ .\end{cases} \tag{69}$$

With these definitions the density (23) and pressure (24) become

$$\chi\rho=a^2\cosh^{-2m}(at)F^{-2}\left[\frac{m+1}{m-1}\sigma^2\hat{\mu}+\frac{n^2-1-4m}{4\cosh^2(at)}\right]\ , \tag{70}$$

$$\chi p=a^2\cosh^{-2m}(at)F^{-2}\left[\sigma^2\hat{\mu}+\frac{n^2-1-4m}{4\cosh^2(at)}\right]\ , \tag{71}$$

where

$$\hat{\mu}\equiv\frac{\mu}{a^2\sigma^2}=\begin{cases}v^2-u^2-1 & \text{if } n^2+3-4m>0\ ,\\ -u^2 & \text{if } n^2+3-4m=0\ ,\\ 1-v^2-u^2 & \text{if } n^2+3-4m<0\ .\end{cases} \tag{72}$$

From (71) and (72) it follows that the pressure is negative when $n^2+3-4m=0$. Analogously, when $n^2+3-4m<0$ we see from (71) and (72) that the second term inside the parentheses is always negative and reaches the value $-(1+\sigma^2)$ at $t=0$. But the first term inside the parentheses is obviously less than or equal to σ^2. Therefore, it follows that in this case there will always be regions (for some values of t) with negative pressures. Then, a necessary condition to keep $p>0$ is

$$n^2+3-4m>0 \tag{73}$$

which we shall assume from now on. Consequently, the first formulas in (69) and (72) are assumed to hold in the rest of this section. Finally, a new inspection of (70) and (71) in the case (73) tells us that the pressure and energy density are both non-negative everywhere if either

$$\sigma\geq 1,\quad m>1,\quad \hat{\mu}\equiv v^2-u^2-1\geq 0\ , \tag{74}$$

or

$$0<\sigma<1,\quad m>1,\quad \hat{\mu}\geq\max\left\{\frac{1-\sigma^2}{\sigma^2},\frac{m-1}{m+1}\frac{1-\sigma^2}{\sigma^2}\right\}>0\ . \tag{75}$$

The first two relations in each of these cases are conditions on the parameters m, n, whereas the third inequality in (74) and (75) is a condition on the functions u and v. Hence, we need to show that there exist *solutions* to the fundamental system of equations which verify those inequalities.

Thus, we cannot proceed any further without studying the system of differential equations, which taking into account the first equality in (69) and the definitions (68) writes now

$$u'=\frac{n\sigma}{m-1}(v^2-u^2-1)-uv\ , \tag{76}$$

$$v'=\frac{2m-1-n^2}{2(m-1)}(v^2-u^2-1)-u^2\ . \tag{77}$$

This system of equations (SOE's) has a series of properties which will allow us to continue our study. Next, we formulate and comment on these properties separately so that our analysis goes on with clear and simple steps.

Property 1. The system (76) and (77) has the following symmetries. First, if $\{u(x),v(x)\}$ is a solution of the SOE's for the values (m,n) of the parameters, then $\{-u(x),v(x)\}$ is a solution for the values $(m,-n)$. Actually, this is a simple consequence of the fact that the change $n\to -n$ provides the same metric with the coordinates y,z interchanged. From a practical point of view, this symmetry restricts n to non-negative values: $n\geq 0$.

Second, if $\{u(x),v(x)\}$ is a solution of the SOE's, then $\{-u(-x),-v(-x)\}$ is also a solution for the same values of the parameters. This property assures us that we can construct the solutions in the whole $\{u-v\}$ plane if we know them, say, in the half-plane $v>0$.

Property 2. It is immediate from (76) and (77) that $\hat{\mu}=v^2-u^2-1=0$ is a particular solution of the system for every possible value of the parameters. This is a crucial point, because this particular solution divides all the solutions of the SOE's into two classes depending on whether $\hat{\mu}$ is positive or negative. But we know from (74) and (75) that $\hat{\mu}\geq 0$ is a necessary condition to obtain positive density and pressure. In the $\{u\text{-}v\}$ plane, $\hat{\mu}=0$ is a hyperbola whose asymptotes are $v=\pm u$. Because of the property 1 second symmetry, the region we are interested in is that "above" one of the branches of this hyperbola, because it is enough to consider $v\geq 1$. We restrict our study to this region, defined by

$$\hat{\mu}=v^2-u^2-1\geq 0,\quad v\geq 1\ . \tag{78}$$

Let us remark that the limit solutions $\hat{\mu}=0$ have a stiff fluid equation of state $p=\rho$.

Property 3. The singular points [15] of the SOE's are $u=0$, $v=\pm 1$, and two others located in the unphysical region $\hat{\mu}<0$. In region (78), the only singular point is $u=0$, $v=1$. This is a very important property, because the solutions of the SOE's can cross each other only at singular points [15]. Consequently, the solutions of the SOE's in region (78) are either fully contained in $\hat{\mu}>0$ or they reach the singular point $u=0$, $v=1$ if they treat to escape from region (78).

Property 4. In the range of values of m,n that we are considering, we have

$$s^2\equiv\frac{n^2+1-2m}{2(m-1)}>0,\quad q^2\equiv\frac{n\sigma}{m-1}>0\ , \tag{79}$$

so that from Eq. (77) it follows that

$$v' = -s^2\hat{\mu} - u^2 \leq 0 \; . \tag{80}$$

Therefore, v' is negative in all the region $\hat{\mu}>0$, and it vanishes only at the singular point $u=0$, $v=1$. In other words, all the solutions of the SOE's within region (78) reach the singular point and are such that the function v is always decreasing.

This behavior proves that all solutions satisfying $0<\sigma<1$ will present negative pressures for some values of x, for (75) does not hold in a neighborhood of the singular point where $\hat{\mu}=0$. As a consequence, hereafter we limit our study to case (74).

Property 5. By means of a very simple reasoning, it is easily shown that the independent variable x is bounded below for the solutions of the SOE's we are interested in. In addition, these solutions go to the singular point when $x\to+\infty$. Then, we can always redefine x so that its range of values is $0\leq x\leq+\infty$.

In order to show the last properties of the SOE's it is convenient to choose, as dependent variables,

$$\hat{\mu}=v^2-u^2-1, \quad \psi=\frac{u}{v} \Longleftrightarrow u=v\psi, \quad v=\left[\frac{1+\hat{\mu}}{1-\psi^2}\right]^{1/2} . \tag{81}$$

This change of variables is obviously well defined in the physical region (78). Let us also note that the new variable ψ is such that $|\psi|<1$ in that region. With these new variables the SOE's take the form

$$\hat{\mu}' = -2v(q^2\psi+s^2)\hat{\mu} \; , \tag{82}$$

$$\psi' = \frac{1}{v}\{\hat{\mu}[(s^2-1)\psi+q^2]-\psi\} \; . \tag{83}$$

Property 6. Starting from the singular point and going along the solution curves of the SOE's in inverse sense (from greater to lower values of x), we see that the curves try to approach $\psi=1$ initially, but they never reach these points. Instead, they turn back and end up coming towards $\psi=-1$, where they terminate. This behavior is represented in Fig. 1, where we have plotted three different families of solutions of the SOE's by solving the equations numerically.

In order to know the behavior of the metrics under consideration, we learn from this property that the possible singularities may appear *only* at the extreme points of the solutions, that is, $x=0$ and $x\to+\infty$. At any other point the functions $\hat{\mu}$ and ψ (or u,v) are regular, and therefore, so are the density, pressure and Weyl scalars.

With regard to $x\to+\infty(\hat{\mu}=0, v=1, \psi=0)$, from expressions (70) and (71), and taking into account (74) and (30), it is obvious that the density and pressure are well behaved at this point. Furthermore, from equations (34)–(38) it follows that all the kinematical quantities are also regular at this point. It remains to see what is the behavior of the Weyl tensor. By using the definitions (81) and (68), we can rewrite relations (40)–(42) as

$$\Psi_0+\Psi_4=\frac{a^2}{2}\cosh^{-2m}(at)F^{-2}\{[n(n^2+3-4m)\psi^2+2\sigma(2n^2-2m+1)\psi+n(n^2+1-2m)]v^2+2n[m\tanh^2(at)-1]\} \; , \tag{84}$$

$$\Psi_0-\Psi_4=\frac{a^2}{2}\cosh^{-2m}(at)\tanh(at)F^{-2}[2\sigma(n^2-1+2m)\psi+n(n^2-1)]v \; , \tag{85}$$

$$\begin{aligned}\Psi_2=-\frac{a^2}{12(m-1)}\cosh^{-2m}(at)F^{-2}\{&[(2m-1)(n^2+3-4m)\psi^2+\sigma n(m-1)\psi+n^2(m-2)+m(2m-1)]v^2\\ &-(m-1)(n^2-1+2m)\tanh^2(at)+n^2-2m+1\} \; .\end{aligned} \tag{86}$$

It is therefore a matter of checking to see that all these Weyl scalars vanish when $x\to\infty$. Thus, we have proven that the metrics are regular at this particular point.

The behavior at the other point of conflict $x=0$ is given by the following property.

Property 7. If $s^2>q^2$ then $\hat{\mu}\to+\infty$ when $\psi\to-1$, whereas if $s^2<q^2$ then $\hat{\mu}\to0$ when $\psi\to-1$. The limiting case $s^2=q^2$ corresponds to $2m=1+n$ and $\hat{\mu}$ goes to a positive finite value when $\psi\to-1$. This last case is that of the singularity-free solutions of Sec. IV. All three possibilities are given in Fig. 1, where we have plotted some appropriate representative integral curves of the SOE's for each case.

In the first possibility, when $s^2>q^2$ or equivalently $2m>1+n$, it is enough to take a look at expressions (70), (71), and (84)–(86) to realize that these metrics have singularities in the Ricci and Weyl tensors at $x=0$. However, in the second possibility $s^2<q^2$ (or $2m<1+n$) from (70) and (71) it follows that the density and pressure vanish at $x=0$. In addition, it is straightforward to show that if $s^2<q^2$, then

$$\hat{\mu}\sim(1+\psi)^{q^2-s^2}\sim v^{-2(q^2-s^2)} \; , \tag{87}$$

when $\psi\to-1$. Unfortunately, by combining (87) with (30) and (86) we see that, for example,

$$\Psi_2 \sim v^2 F^{-2} \propto v^2 \hat{\mu}^{m-1} \sim v^{2-2(m-1)(q^2-s^2)} \tag{88}$$

when $x=0$. But for these values of the parameters we have

$$0<(m-1)(q^2-s^2)=n\sigma+m-\tfrac{1}{2}(n^2+1)<1 , \tag{89}$$

so that when $x=0$, $\Psi_2 \to +\infty$. We conclude, therefore, that these metrics have a Weyl curvature singularity at $x=0$ but no physical (or Ricci) singularity.

Summarizing, we have proven that the only singularity-free solutions are those presented previously in Sec. IV. We have also shown that, so to speak, these singularity-free solutions form the boundary in the space of metrics between the solutions with timelike Weyl singularities only, and those with both timelike Weyl and Ricci singularities. The possible meaning of this result is, as yet, not clear to us.

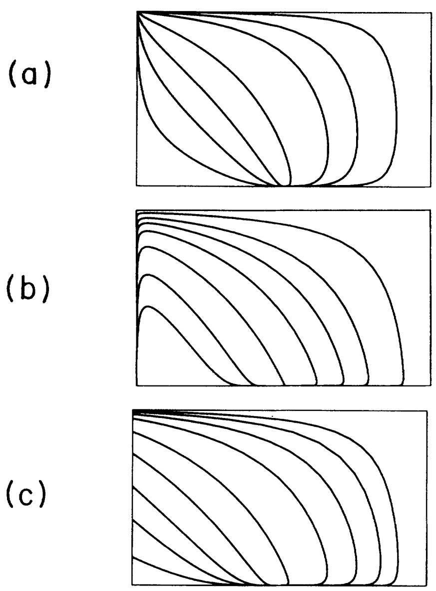

FIG. 1. Three families of integral curves of the SOE's, plotted in the $\{\psi - \mu/(\mu+1)\}$ plane. The right and left vertical lines correspond to $\psi=1$ and $\psi=-1$, respectively, whereas the top and bottom horizontal lines correspond to $\mu=+\infty$ and $\mu=0$, respectively. The shown curves start at the point $(\psi=0,\mu=0)$ in all cases. (a) These are some solutions for the values of the parameters $m=6$ and $n=5$ $(s^2>q^2)$. As we can see, all these curves end at the top left corner, that is, at $(\psi=-1,\ \mu=+\infty)$, where the metrics have Weyl and Ricci singularities. (b) Integral curves of the SOE's for the values $m=\frac{21}{16}$ and $n=\frac{5}{2}(s^2<q^2)$. Now the curves terminate at the bottom left corner $(\psi=-1,\mu=0)$, where the Ricci tensor is regular but the Weyl tensor has a singularity. (c) Finally, these are solutions of the SOE's for the values $m=2$ and $n=3(s^2=q^2)$. This family ends at any point of the left vertical line, that is to say, at $\psi=-1$ with μ finite and positive. The metrics in this case are singularity-free and correspond to the family presented in Sec. IV. Figures (a), (b), and (c) collect the three possible different types of behavior of the solutions of the SOE's and therefore of the related metrics as well.

VI. DISCUSSION

The main result of this paper is that the very simple family of inhomogeneous perfect-fluid solutions to Einstein's equations given by (3) displays an unusual richness with respect to the singularities that the curvature can have. This fact poses some questions about the current views on the subject. It is very well known that the isotropic and homogeneous Friedmann-Robertson-Walker models must have a universal big-bang singularity if the energy conditions are satisfied. In the same way, homogeneous Bianchi models have to have singularities if reasonable energy conditions hold [16,17]. Very few inhomogeneous models are available up to now, but the general singularity theorems [17,18] seemed to imply that all of them should have singularities as well. We have seen that for the very simple line element (3), many solutions do have singularities. However, some of them cannot be interpreted as big-bang singularities, due to their timelike character. Even more importantly there are singularity-free solutions satisfying the stronger energy and causality conditions (see the discussion of [8]). This fact proves that these two conditions, usually thought of as crucial in the singularity theorems, are not determinant by themselves for the appearance of singularities. It is interesting to remark that the singularity-free family of solutions of Sec. IV have a cylindrical symmetry, so that this type of symmetry could have some relevance for the avoidance of singularities. In any case, it is obvious that completely general inhomogeneous models will or will not be singular, but the singularities they may have are of a very different kind, and in many cases they will not be of big-bang type. There were indications that this should be the case (see, for example, [19]), but the lack of inhomogeneous solutions had not allowed any explicit checking. The study we have made opens new questions too. The most important is how many reasonable singularity-free solutions are there. From the discussion of [8] it follows that the lack of causally trapped sets [17] is enough for the avoidance of singularities, and then the question is whether or not these types of set are essential in real situations. In any case, we believe that further analysis of general inhomogeneous explicit solutions is needed to get a clear view on these matters.

Another important result is that the mere qualitative analysis of the differential equations can lead to very strong conclusions. For example, we have proven in Sec. V that the singular points of the SOE's are particular solutions of the field equations which govern the asymptotic behavior of the general solutions. These ideas, already contained in [1], can be of great help in the study of general inhomogeneous spacetimes. Finally, we want to stress that the singularity-free solutions of Sec. IV are the separation between families with very different properties. Thus, the study of the space of metrics could also be useful in order to know the singularities of inhomogeneous models.

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