

No Starobinsky inflation from self-consistent semiclassical gravity

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The same theory of semiclassical gravity that predicts Starobinsky inflation (de Sitter-like solutions driven only by higher-order curvature terms) also predicts flat space to be unstable to small perturbations. When semiclassical gravity is modified in a way suggested by and consistent with the perturbative nature of its derivation, flat space is predicted to be stable, in accord with observation, but Starobinsky inflation is no longer a solution. The modified semiclassical theory, constrained to only solutions perturbatively expandable in \hbar , has the same dynamical degrees of freedom as the classical gravitational field, despite the presence of fourth-order derivatives in the field equations. There are no de Sitter or de Sitter-like self-consistent solutions except in the presence of a cosmological constant, so inflation generated purely by curvature is not predicted. Furthermore, linearized gravitational perturbations in a de Sitter background (with a cosmological constant) show no signs of instability from quantum effects.

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I. INTRODUCTION

Quantum corrections to general relativity are expected to be important at early times in the evolution of the Universe. Semiclassical approximations to these corrections can be calculated by techniques developed over the last two decades. The form of these semiclassical corrections is now well known for many spacetimes [1], e.g., for conformally flat spacetimes in four dimensions, for which the corrections to Einstein's equation are proportional to curvature-squared terms [given below by Eqs. (2)–(6)].

One of the most interesting predictions of semiclassical gravity is Starobinsky inflation [2], a class of de Sitter and de Sitter-like solutions to the semiclassical field equations. In Starobinsky inflation higher-order curvature terms can remain nearly constant, thus mimicking the effects of a nearly constant, nonzero scalar field. This might allow an inflationary epoch without requiring additional matter fields.

Unfortunately, the same semiclassical theory that predicts Starobinsky inflation suffers from severe problems. Perhaps the worst of these problems is the instability of flat space. The same higher-order curvature terms that would drive Starobinsky inflation also predict that flat space is unstable in a number of ways, including the production of Planck-energy γ rays, Planck-scale tidal forces, tachyonic propagation of gravitational particles, and violation of the positive-energy theorem [3,4].

It was shown in earlier work [5] that it is possible, and indeed desirable, to modify semiclassical gravity in a way suggested by and consistent with the perturbative nature of its derivation. The effective action and field equations of semiclassical gravity are perturbative expansions (formally, asymptotic expansions) in powers of \hbar , truncated at first order in \hbar . All behavior higher order and nonperturbative in \hbar has already been lost in the process of deriving the (approximate) effective action and field equations.

Self-consistency then requires that only the solutions that are also asymptotic expansions in powers of \hbar , truncated to first order, will be approximations to solutions of the full, nonperturbative effective action. Solutions not in this form are likely to be unphysical and should be excluded. A simple model, presented below, will demonstrate that retaining nonperturbative solutions to a perturbatively derived action results in false predictions. The nonperturbatively expandable solutions are spurious artifacts arising from the higher derivatives appearing in the perturbative correction, and will be referred to as pseudosolutions. For convenience, perturbatively expandable solutions will sometimes be referred to as physical, since only they correspond to predictions of the self-consistent semiclassical theory. For semiclassical gravity, it has been shown that the physical solutions show no signs of any instability of flat space (to first order in \hbar) [5].

In this work, the predictions of the constrained semiclassical theory (i.e., the theory constrained to include only physical solutions) are applied to homogeneous, isotropic solutions in the presence of a cosmological constant, and to small perturbations on those spacetimes. We find de Sitter solutions only when the cosmological constant is nonzero and positive. All other de Sitter-like solutions that might lead to inflation are found to be spurious, and therefore not physical predictions of the semiclassical theory. This, unfortunately, rules out Starobinsky inflation as a physical solution.

If the theory described by Einstein gravity plus higher-order curvature terms is not considered as a semiclassical theory, but as a fundamental theory (i.e., the higher-order curvature terms are taken to be purely classical, not arising from quantum corrections), then the theory cannot legitimately be modified in this way. By definition, all solutions to the field equations would be physical, including Starobinsky inflation. By the same token, however, flat space would then suffer from the same instabilities described above, clearly conflicting with our everyday experience [6].

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The structure of this paper is as follows. First is a short derivation of the semiclassical field equations for conformally flat spacetimes. Following is an example of Starobinsky inflation, and a summary of the effects, when all solutions to the semiclassical field equations are retained, on the stability of flat space, and a discussion of the flaws of that analysis. The next section is a brief, pedagogical presentation on higher-derivative expansions, including a simple model to demonstrate the basic concepts. A method for finding only the physical solutions is shown. Applications to semiclassical gravity are emphasized. Finally, the physical semiclassical homogeneous, isotropic solutions with a cosmological constant and no matter are derived. It is found that the semiclassical corrections to the classical solutions are small, and remain so for all times. In short, an exponential increase in the scale factor requires a cosmological constant (or matter), and Starobinsky inflation is not a physical solution.

II. HISTORICAL SEMICLASSICAL GRAVITY

We use the conventions $c=1$, $\eta_{\mu\nu}=(-+++)$, $R_{\mu\alpha\nu}^\lambda=\partial_\alpha\Gamma_{\mu\nu}^\lambda+\dots$, and $R_{\mu\nu}^\lambda=R_{\nu\mu}^\lambda$. The semiclassical field equations of general relativity (including a cosmological constant) take the form

$$R_{\mu\nu}-\frac{1}{2}Rg_{\mu\nu}+\Lambda g_{\mu\nu}=\kappa\langle T_{\mu\nu}\rangle, \quad (1)$$

where $\langle T_{\mu\nu}\rangle=O(\hbar)$ is the expectation value or transition amplitude of the matter stress-energy tensor. For convenience, we consider only massless, conformally coupled fields (of arbitrary spin). We may reasonably restrict the form of $\langle T_{\mu\nu}\rangle$ to obey Wald's physical axioms [7]: (1) covariant conservation; (2) causality; (3) standard results for "off-diagonal" matrix elements; (4) standard results in Minkowski space. Wald showed that any $\langle T_{\mu\nu}\rangle$ that obeys the first three axioms is unique up to the addition of a local, conserved tensor. Furthermore, any local, conserved tensor can reasonably be considered part of the geometrical dynamics and so be written on the left-hand side of the field equations. We shall do so, rewriting (1) as

$$R_{\mu\nu}-\frac{1}{2}Rg_{\mu\nu}+\Lambda g_{\mu\nu}+\Omega_{\mu\nu}=\kappa\langle T_{\mu\nu}\rangle, \quad (2)$$

where $\Omega_{\mu\nu}$ is conserved and purely local; i.e., it is constructed purely from the metric, the curvature, and (a finite number of) its covariant derivatives.

Only terms in $\Omega_{\mu\nu}$ that are first order in \hbar will be considered, consistent with the semiclassical approximation. Any term with a constant coefficient proportional to \hbar must have dimensions of (length)⁻⁴, since the only length scale is the Planck length l_{Pl} and $\hbar=l_{\text{Pl}}^2$ in units where $G=1$. This restricts the form of $\Omega_{\mu\nu}$ for conformally flat spacetimes in four dimensions, such that there are exactly three possible contributing terms [8]:

$$\begin{aligned} {}^{(1)}H_{\mu\nu} &= \frac{1}{\sqrt{g}} \frac{\delta}{\delta g^{\mu\nu}} \int d^4x \sqrt{g} R^2 \\ &= \frac{1}{2}R^2 g_{\mu\nu} - 2RR_{\mu\nu} - 2\Box Rg_{\mu\nu} + 2\nabla_\mu\nabla_\nu R, \end{aligned} \quad (3)$$

$$\begin{aligned} {}^{(2)}H_{\mu\nu} &= \frac{1}{\sqrt{g}} \frac{\delta}{\delta g^{\mu\nu}} \int d^4x \sqrt{g} R^{\alpha\beta} R_{\alpha\beta} \\ &= \frac{1}{2}R^{\alpha\beta} R_{\alpha\beta} g_{\mu\nu} - \Box R_{\mu\nu} - \frac{1}{2}\Box Rg_{\mu\nu} \\ &\quad + \nabla_\mu\nabla_\nu R - R^{\alpha\beta} R_{\alpha\mu\beta\nu}, \end{aligned} \quad (4)$$

$${}^{(3)}H_{\mu\nu} = -\frac{1}{12}R^2 g_{\mu\nu} + R^{\alpha\beta} R_{\alpha\mu\beta\nu}. \quad (5)$$

The first two expressions are conserved automatically from their variational definition. The last expression is conserved, but not as a result of a variational derivation, nor as the limit of a conserved quantity in nonconformally flat spacetimes [9]. Also it is second order in derivatives of the metric, whereas the first two are fourth order. Nevertheless, it is allowed by Wald's axioms, and, in general, contributes to the conformal anomaly. The most general expression for $\Omega_{\mu\nu}$ under these conditions can be written

$$\Omega_{\mu\nu} = \alpha {}^{(1)}H_{\mu\nu} + \beta {}^{(2)}H_{\mu\nu} + \gamma {}^{(3)}H_{\mu\nu} + O(\hbar^2). \quad (6)$$

Values of α , β , and γ are predicted by specific matter couplings and regularization schemes, but we will treat them as free parameters. They are all proportional to \hbar .

We are particularly interested in self-consistent solutions of isotropic, homogeneous spacetimes. In these spacetimes, we can find a state for which there are initially no particles, and which has no particle creation or nonlocal vacuum polarization. For this state,

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} + \Lambda g_{\mu\nu} + \Omega_{\mu\nu} = 0 + O(\hbar^2). \quad (7)$$

Because of the higher-order derivatives in ${}^{(1)}H_{\mu\nu}$ and ${}^{(2)}H_{\mu\nu}$, and because ${}^{(3)}H_{\mu\nu}$ is quadratic in second-order derivatives, there appear to be more solutions to (7) than to the classical Einstein equation. Some of these extra solutions lead to de Sitter-like behavior, even in the absence of Λ . This makes them interesting as possible candidates in inflationary scenarios and is known as Starobinsky inflation [2].

Examples of Starobinsky inflation are straightforward to find. We look for isotropic, homogeneous spacetimes in the coordinate system given by

$$ds^2 = -dt^2 + a^2(t) \left[\frac{1}{1-kr^2} dr^2 + r^2 d\Omega_2^2 \right], \quad (8)$$

where $d\Omega_2$ is the line element of a two-sphere. The semiclassical Einstein equations (7) give

$$\frac{\dot{a}^2}{a^2} + \frac{k}{a^2} - \frac{\Lambda}{3} - 6(\alpha+\beta) \left[2\frac{\ddot{a}\dot{a}}{a^2} - \frac{\ddot{a}^2}{a^2} + 2\frac{\ddot{a}\dot{a}^2}{a^3} - 3\frac{\dot{a}^4}{a^4} - 2\frac{k\dot{a}^2}{a^4} + \frac{k^2}{a^4} \right] + \gamma \left[\frac{\dot{a}^4}{a^4} + 2\frac{k\dot{a}^2}{a^4} + \frac{k^2}{a^4} \right] = O(\hbar^2). \quad (9)$$

This is a fourth-order system and has more solutions than Einstein's equation. One solution, for $k=1$, $\Lambda=0$, and $\gamma < 0$, is

$$a = \sqrt{|\gamma|} \cosh \left[\frac{t}{\sqrt{|\gamma|}} \right]. \quad (10)$$

This is exact de Sitter space, with an effective cosmological constant of $3/|\gamma|$ and constant scalar curvature $R = 12/|\gamma|$. Solutions for which inflation halts after a finite number of e -foldings can also be found [2].

We can see from (10) that for smaller γ (i.e., smaller \hbar , in regimes where quantum effects should be less important), the rate of expansion is larger. In fact, when the field equations (9) are evaluated at this solution, the semiclassical terms are of the same order as the classical terms. Strictly speaking this means that the semiclassical approximation has broken down and that solutions such as (10) should not be taken too seriously. Historically it was hoped, despite this admitted breakdown, that these solutions might still be qualitatively indicative of the behavior of quantum gravity. This approach can be made more rigorous by considering models with large numbers of conformally coupled matter fields, making the semiclassical breakdown less extreme.

Nevertheless, there is nothing in that approach which addresses the issue of the semiclassical field equations being higher-order equations than the classical field equations. Furthermore, on dimensional grounds, each term of increasing order in \hbar may generically contain even higher-order time derivatives, since $\hbar = l_{\text{Pl}}^2$ in units where $G=1$. If all solutions to these higher and higher order differential equations are considered, the dimension of the solution space appears to grow at every higher order of the expansion in \hbar , qualitatively changing the nature of the theory according to order.

The semiclassical theory described by *all* the solutions to (7) has other problems as well. Perhaps its worst feature is that it predicts that flat space is plagued by unavoidable instabilities, independent of the specific values of α , β , and γ [3]. This theory cannot describe the spacetime in which we live, since small perturbations to nearly flat space do remain small. It appears necessary to modify the semiclassical theory to avoid these unphysical predictions.

As demonstrated in a previous paper, however, there is an important reason why the theory described by all solutions to (7) should fail. For an action or field equations derived as a perturbative expansion in powers of \hbar , all nonperturbative (not Taylor expandable in \hbar as $\hbar \rightarrow 0$) behavior of the theory has already been discarded by the perturbative approximation of the action. Nonperturbative solutions to the perturbative field equations are not expected to be related to nonperturbative solutions to the nonperturbative field equations. In general, they are not. These pseudosolutions must be discarded for the sake of self-consistency. Only solutions that are also perturbative expansions in powers of \hbar can be expected to approximate the full theory.

Once the spurious pseudosolutions are excised, the size of the solution space remains constant, order by order. Furthermore, the pseudosolutions are often associated

with predictions of unnatural behavior, such as kinetic energy unbounded from below. This behavior itself is not reason enough to justify excising them as solutions, but it can drastically change the quality of the solutions of the theory. One example of this behavior is the loss of stability of flat space. Flat space is known to be stable in the classical theory of general relativity, as demonstrated in the positive-energy theorem [10], but not in the case of unconstrained semiclassical gravity [3].

III. HIGHER-DERIVATIVE EXPANSIONS

The formalism of asymptotic expansions is mathematically well established [11]. We will present a brief overview of some of the more important concepts.

The "is of order" symbol, " $=O(\cdot)$," is defined as follows: if $\phi_\epsilon < A\psi_\epsilon$ for some constant A , independent of ϵ , for all ϵ in a given region, then we say $\phi_\epsilon = O(\psi_\epsilon)$. Technically the " $=$ " is only part of the whole symbol and does not represent a true equality, but in practice this is not usually a problem, and some typographic abuse is allowable. In this paper, the region of interest will always be the neighborhood around $\epsilon=0$.

An N th-order asymptotic power series is an asymptotic expansion of the form

$$f(\epsilon) = a_0 + \epsilon a_1 + \epsilon^2 a_2 + \cdots + \epsilon^N a_N + O(\epsilon^{N+1}), \quad (11)$$

which, technically, is an abuse of the notation, and should be written

$$f(\epsilon) - a_0 - \epsilon a_1 - \epsilon^2 a_2 - \cdots - \epsilon^N a_N = O(\epsilon^{N+1}). \quad (12)$$

In general f and the a_n are functions of more variables than just ϵ .

Asymptotic power series may be added, subtracted, and multiplied freely. Division is only permitted when the a_0 of the denominator is nonzero. In algebraic language, the system is a commutative ring with zero divisors (expressions with $a_0=0$), where the role of the zero element is played by $O(\epsilon^{N+1})$ [12]. Note that this implies that if $f(x) + \epsilon g(x) = 0 + O(\epsilon^2)$, and f and g are both zeroth order in ϵ , then both f and g must vanish independently. Note also that the vanishing of the product of two terms does not guarantee that either must vanish [e.g., $\epsilon \times \epsilon = 0 + O(\epsilon^2)$]. In semiclassical gravity, the role of ϵ is played by \hbar .

It is because of restrictions such as these that a fourth- or higher-order field equation may only have a two-parameter family of perturbative solutions. This occurs when not all solutions to the perturbative field equation are themselves perturbatively expandable (expressible in an asymptotic series in the expansion parameter). Jaén, Llosa, and Molina [12] have shown that, to any order, the same amount of initial data suffices for all solutions analytic in the perturbative expansion parameter, for any system of the form

$$L = \frac{1}{2} \sum_{\alpha=1}^N m_\alpha \dot{q}_\alpha^2 + \sum_{l=0}^n \epsilon^l V_l \left[q, \frac{dq}{dt}, \dots, \frac{d^l q}{dt^l} \right] + O(\epsilon^{n+1}), \quad (13)$$

where ε is the perturbative expansion parameter, m_α is the mass of particles $\alpha=1, \dots, N$, and the matrices $\partial^2 V_1 / \partial q_\alpha^{(l)} \partial q_\beta^{(l)}$ are regular. Their proof demonstrates that all but N of the functions that determine the canonical momenta in terms of the generalized velocities cannot be inverted within the formalism of perturbative expansions. This noninvertibility corresponds to the presence of constraints, which are then shown to be second class (in Dirac's terminology). The constrained system has the same number of degrees of freedom, N , for any order of expansion n . This result can be generalized to more complicated systems, such as minisuperspace cosmological models of gravity and linearized gravity. The proof (to first order) for linearized gravity in flat space has been done [5], and the proof for homogeneous isotropic solutions to semiclassical gravity is described below. For the case of semiclassical gravity, $n=2$, $\varepsilon^2 = \hbar$, $\Omega_{\mu\nu}$ in the field equations is generated by V_2 , and $V_1=0$ (but only roughly speaking, since there are also first-order constraints, e.g., from the lapse and shift).

A sketch of the proof of Jaén, Llosa, and Molina runs as follows. The Euler equation derived by varying the Lagrangian of (13) takes the form

$$\sum_{\beta=1}^N q_\beta^{(2n)} \varepsilon^n \frac{\partial^2 V_n}{\partial q_\beta^{(n)} \partial q_\alpha^{(n)}} + f_\alpha(q, \dots, q^{(2n-1)}; \varepsilon) = O(\varepsilon^{n+1}). \quad (14)$$

Since we may not divide by ε , the Hessian matrix $\varepsilon^n \partial^2 V_1 / \partial q_\alpha^{(l)} \partial q_\beta^{(k)}$ cannot be inverted within the perturbative formalism. The noninvertibility of the Hessian matrix signifies a primary constraint, given by the inner product of the null vectors of the Hessian with (14). In this case, any vector proportional to ε is a null vector, so the primary constraints are

$$\varepsilon f_\alpha(q, \dots, q^{(2n-1)}; \varepsilon) = O(\varepsilon^{n+1}) \quad (15)$$

and there are no others (from the invertibility of $\partial^2 V_1 / \partial q_\alpha^{(l)} \partial q_\beta^{(l)}$). Secondary constraints are given by time differentiating (15). There are $2n-2$ sets of N constraints in all, and these may be rearranged to take the form

$$q_\alpha^{(r)} - m_\alpha^{-1} \sum_{s=0}^n \varepsilon^s B_{\alpha,r,s}(q, \dot{q}) = O(\varepsilon^{n+1}), \quad r=2, \dots, 2n-1. \quad (16)$$

The proof then shows that the $2n-2$ constraints are second class, leaving only one pair of canonical variables per particle freely specifiable as initial conditions. The analogous theorem for the slightly more general system where $m_\alpha = m_\alpha(q)$ can be proven in an identical matter, whenever $m_\alpha \neq 0$. For the semiclassical gravitational case (9), the constraints will take the form

$$\begin{aligned} \ddot{a} &= \ddot{a}_{cl}(a, \dot{a}) + \hbar \ddot{a}_1(a, \dot{a}), \\ \ddot{a} &= \ddot{a}_{cl}(a, \dot{a}) + \hbar \ddot{a}_1(a, \dot{a}). \end{aligned} \quad (17)$$

Since proof of Jaén, Llosa, and Molina is somewhat technical, an easily followed example which demonstrates its usefulness is in order. The model is a nonlocal har-

monic oscillator (for a fuller treatment, including quantization, see Simon [13]). This model simply displays the appearance of higher derivatives in a perturbative expansion, and it has the important advantage of being exactly soluble. The model's equation of motion is

$$\ddot{x}(t) = -\omega_0^2 \int_0^\infty ds e^{-s\frac{1}{2}} [x(t+\varepsilon s) + x(t-\varepsilon s)], \quad (18)$$

where $\varepsilon\omega_0 < 1$. This is a harmonic oscillator with a nonlocal potential, in the sense that the force is linear in displacement, but it depends not only on the position of the spring at a specific instant, but also on the position in the past and future (with heavier weighting of times near the present). In the limit $\varepsilon \rightarrow 0$, we regain the simple harmonic-oscillator equation $\ddot{x} = -\omega_0^2 x$.

The two-parameter family of exact solutions is given by

$$x = A \cos(\omega t + \phi), \quad (19)$$

where A and ϕ depend on the initial conditions and

$$\begin{aligned} \omega^2 &= \omega_0^2 \left(\frac{1}{2} + \frac{1}{2} \sqrt{1 + 4\varepsilon^2 \omega_0^2} \right)^{-1} \\ &= \omega_0^2 (1 - \varepsilon^2 \omega_0^2 + 2\varepsilon^4 \omega_0^4 + \dots) \end{aligned} \quad (20)$$

is the new effective frequency due to nonlocal effects.

One may also solve the system perturbatively and compare the result with the exact solution. Since both the equation of motion and the general solution are perturbatively expandable in ε , there should be no obstacles. The equation of motion becomes

$$\ddot{x} = -\omega_0^2 (x + \varepsilon^2 \ddot{x} + \varepsilon^4 x^{(4)} + \varepsilon^6 x^{(6)} + \dots). \quad (21)$$

There appears to be an arbitrarily high number of degrees of freedom due to the infinite sum of higher derivatives. In fact, we know that the exact solution has only two arbitrary parameters, so all other degrees of freedom not corresponding to A and ϕ must be excluded implicitly by demanding that the sum converge. If we truncate at any finite order, though, we lose the implicit constraints, and we must then explicitly exclude nonperturbative solutions. Truncating (21) at ε^0 or ε^2 and solving gives no trouble because the equation of motion remains second order and gives the correct answers

$$\begin{aligned} \varepsilon^0: \quad x &= A \cos(\omega_0 t + \phi), \\ \varepsilon^2: \quad x &= A \cos(\omega_2 t + \phi) \end{aligned} \quad (22)$$

to the appropriate order in ε , where

$$\omega_2^2 = \omega_0^2 (1 - \varepsilon^2 \omega_0^2 + \dots) = \omega^2 + O(\varepsilon^4)$$

is an easily calculable function of ε and ω_0 . Truncating (21) at higher orders, however, gives an extra pseudosolutions that are not perturbatively expandable in ε ,

$$\begin{aligned} \varepsilon^4: \quad x &= A \cos(\omega_4 t + \phi) + B \cos(\gamma t + \psi), \quad \gamma \sim \frac{1}{\varepsilon} \frac{1}{\varepsilon \omega_0}, \\ \varepsilon^6: \quad x &= A \cos(\omega_6 t + \phi) + B_+ \cos(\gamma_+ t + \psi_+) \\ &\quad + B_- \cos(\gamma_- t + \psi_-), \quad \gamma_\pm \sim \frac{1}{\varepsilon} \frac{1}{\sqrt{\pm i \varepsilon \omega_0}}, \end{aligned} \quad (23)$$

and so on, where ω_{2n} is a (calculable) function of ε and ω_0

in each case and $\omega_{2n}^2 = \omega^2 + O(\varepsilon^{2n+2})$, where ω is defined in (20). The extra pseudosolutions found here, being nonanalytic in ε , are quite similar to the Starobinsky inflationary solution found already in (10), which is nonanalytic in \hbar .

Thus, the simple model is an explicit example of how abandoning the perturbative formalism for the solution simply gives the wrong answer. Retaining the perturbative formalism (that is, excluding, by the appropriate constraints, all nonperturbative results) gives the correct answer, to any order. We see that when the order of derivatives grows with the order of expansion, it alerts us that the higher derivatives do not represent dynamical degrees of freedom but are artifacts of the expansion. Keeping only perturbative solutions is the only self-consistent path available [14].

Solving for all exact solutions of the truncated expansion and then discarding all solutions not perturbatively expandable, while a valid procedure, is computationally wasteful and may not always be possible. A more feasible prescription is to solve the equations of motion while remaining, at every step, strictly within the perturbative formalism. This guarantees that only perturbatively expandable solutions are found. For systems which are derived from a Lagrangian in the form of (13), it is guaranteed that all such solutions can be found.

As an example, we solve the model oscillator system introduced above, truncated to powers of ε^4 , remaining at all times strictly within the perturbative formalism. The equation of motion is

$$\ddot{x} + \omega_0^2 x + \varepsilon^2 \omega_0^2 \ddot{x} + \varepsilon^4 \omega_0^2 x^{(4)} = O(\varepsilon^6). \quad (24)$$

Dividing by ε^4 is forbidden if the equation is to remain a perturbative expansion to $O(\varepsilon^4)$. Instead we multiply by ε^4 ,

$$\varepsilon^4 \ddot{x} + \varepsilon^4 \omega_0^2 x = O(\varepsilon^6), \quad (25)$$

take two time derivatives,

$$\varepsilon^4 x^{(4)} + \varepsilon^4 \omega_0^2 \ddot{x} = O(\varepsilon^6), \quad (26)$$

and substitute back into (24) to get

$$\ddot{x}(1 + \varepsilon^2 \omega_0^2 - \varepsilon^4 \omega_0^4) + \omega_0^2 x = O(\varepsilon^6). \quad (27)$$

We are still forbidden to divide by any expression containing ε , but we may multiply by the reciprocal if it exists. Since

$$(1 + \varepsilon^2 \omega_0^2 - \varepsilon^4 \omega_0^4)(1 - \varepsilon^2 \omega_0^2 + 2\varepsilon^4 \omega_0^4) = 1 + O(\varepsilon^6), \quad (28)$$

the final form of the equation of motion is

$$\ddot{x} + \omega_0^2(1 - \varepsilon_2 \omega_0^2 + 2\varepsilon^4 \omega_0^4)x = O(\varepsilon^6). \quad (29)$$

Compare this with (20) to see that this gives the correct answer to the full equation of motion (to order ε^4), and compare with the first line of (23) to see that this also agrees with the method of first solving for all solutions and afterwards excising all nonperturbative pseudosolutions. From the latter comparison we see that all perturbative solutions were found [this was guaranteed by there being a Lagrangian in the form of (13)].

In a similar way, it is straightforward to analyze the case of linearized semiclassical gravity near flat space. This was done explicitly in a previous paper [5]. Since semiclassical gravity is a perturbative approximation to the full effective theory of quantum gravity, there is no reason to believe that nonperturbative solutions to the semiclassical field equations are any more valid than the nonperturbative pseudosolutions found in the above example. For self-consistency, only perturbatively expandable solutions should be considered physical. It was found that, to first order in \hbar , the physical solutions of linearized semiclassical gravity near flat space are identical to the solutions of linearized gravity near flat space (i.e., the semiclassical corrections to the classical solutions all vanish on the flat background). Since we know classical gravity is stable near flat space (from the positive-energy theorem [10]), then semiclassical gravity is also stable, to first order in \hbar . Its stability is not proven to all orders, but neither are there any indications to the contrary. If the pseudosolutions were also to be interpreted as physical, then the many indications of its stability [3] would all be valid, resulting in a theory that cannot describe the physics of our Universe.

So, for two reasons, self-consistency and experiment, we should consider perturbative semiclassical theory as the ‘‘correct’’ semiclassical gravity, or at least as a potentially correct theory. Semiclassical gravity that does not exclude pseudosolutions cannot be considered even a potentially correct theory. Next follow some of the cosmological consequences.

IV. SEMICLASSICAL GRAVITY AND DE SITTER SPACE

First we review the classical de Sitter solution. For a metric given by (8), the classical field equation is

$$0 = \frac{\dot{a}^2}{a^2} + \frac{k}{a^2} - \frac{\Lambda}{3}, \quad (30)$$

and for $k = 1$, Λ must be positive. The most general spatially closed solution is

$$a = \left[\frac{3}{\Lambda} \right]^{1/2} \cos \left[\left[\frac{\Lambda}{3} \right]^{1/2} (t - t_0) \right]. \quad (31)$$

Matterless classical general relativity cannot undergo exponential expansion without a cosmological constant.

When semiclassical corrections are added, we might expect the behavior of solutions to not alter drastically, assuming we examine only physical solutions to the semiclassical field equations. This turns out to be the case (it may happen that for some solutions the semiclassical approximation breaks down [15]). We begin by solving (9) for all solutions that are perturbatively expandable in \hbar . The method used is essentially the same as used in the higher-derivative harmonic-oscillator system above. The first step is to multiply (9) by \hbar (recall that α , β , and γ are all proportional to \hbar):

$$\hbar \left[\frac{\dot{a}^2}{a^2} + \frac{k}{a^2} - \frac{\Lambda}{3} \right] = O(\hbar^2). \quad (32)$$

We also need the first and second time derivatives of this equation:

$$\hbar \left[2 \frac{\ddot{a}\dot{a}}{a^2} - 2 \frac{\dot{a}^3}{a^3} - 2 \frac{k\dot{a}}{a^3} \right] = O(\hbar^2), \quad (33)$$

$$\hbar \left[2 \frac{\ddot{a}\dot{a}}{a^2} + 2 \frac{\ddot{a}^2}{a^2} - 10 \frac{\ddot{a}\dot{a}^2}{a^3} + 6 \frac{\dot{a}^4}{a^4} - 2 \frac{k\ddot{a}}{a^3} + 6 \frac{k\dot{a}^2}{a^4} \right] = O(\hbar^2). \quad (34)$$

Using (32)–(34) to simplify (9) gives

$$\frac{\dot{a}^2}{a^2} + \frac{k}{a^2} - \frac{\Lambda}{3} - 6(\alpha + \beta)(0) + \gamma \left[\frac{\Lambda^2}{9} \right] = O(\hbar^2) \quad (35)$$

or

$$\frac{\dot{a}^2}{a^2} + \frac{k}{a^2} - \frac{\tilde{\Lambda}}{3} = O(\hbar^2), \quad (36)$$

where

$$S \propto \int dt \left[-\frac{\dot{a}^2 a}{N} + Nka - N\frac{\Lambda}{3}a^3 - 6(\alpha + \beta) \left[\frac{\dot{N}^2 \dot{a}^2 a}{N^5} - \frac{2\dot{N}\ddot{a}a}{N^3} + \frac{\ddot{a}^2 a}{N^3} + \frac{\dot{a}^4}{N^3 a} + \frac{2k\dot{a}^2}{Na} + \frac{Nk^2}{a} \right] + \gamma \left[-\frac{\dot{a}^4}{3N^3 a} - \frac{2k\dot{a}^2}{Na} + \frac{Nk^2}{a} \right] \right] + O(\hbar^2), \quad (39)$$

where N is the lapse function and the gauge $N=1$ is chosen to recover the metric of (8). This is almost in the form of (13), on which the proof of Jaén, Llosa, and Molina [12] is based, with the differences that the mass term is not constant and that there is a nondynamical variable N , giving a first-order (gauge) constraint. A slightly generalized theorem that does apply to (39) is straightforwardly proven by the same method (for all times such that a , which acts as a nonconstant mass term, does not vanish [17]). This guarantees that all perturbatively expandable solutions to (9) are solutions to (36).

Next we examine the case of linearized gravity. Linearized gravitational fluctuations $h_{\mu\nu} = g_{\mu\nu} - g_{\mu\nu}^{(0)}$, where $g_{\mu\nu}^{(0)}$ is the de Sitter background metric, obey the classical field equations

$$0 + O(\hbar^2) = \delta(R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R + \Lambda g_{\mu\nu}) = -\frac{1}{2}\square h_{\mu\nu} - \frac{1}{2}\nabla_\mu \nabla_\nu h + \nabla_\alpha \nabla_{(\mu} h_{\nu)}^\alpha + \frac{1}{2}g_{\mu\nu}(\square h - \nabla_\alpha \nabla_\beta h^{\alpha\beta} + \Lambda h) - \Lambda h_{\mu\nu} \quad (40)$$

(with no gauge fixing), where all derivative operators are covariant with respect to the background metric [18]. At present, there exists no definitive proof of whether solutions of general relativity with a positive cosmological constant approach the de Sitter solution at late times (the “cosmic no-hair conjecture”). This, in some sense, would be a de Sitter analogue to the positive-energy theorem, by showing the stability of de Sitter space to (not necessarily small) perturbations. The behavior of linearized gravitational perturbations in background de Sitter space is known, however [19]. The helicity two perturbations

$$\tilde{\Lambda} \equiv \Lambda \left[1 - \gamma \frac{\Lambda}{3} \right]. \quad (37)$$

The corrections proportional to α and β both vanish [16]. The remaining equation of motion for the scale factor is of the same form as the classical case (30), with a (small) quantum correction to the cosmological constant proportional to Λ^2 . Thus all physical semiclassical solutions are the same as the classical solutions with $\Lambda \rightarrow \tilde{\Lambda}$:

$$a = \left[\frac{3}{\tilde{\Lambda}} \right]^{1/2} \cosh \left[\left[\frac{\tilde{\Lambda}}{3} \right]^{1/2} (t - t_0) \right]. \quad (38)$$

In particular, there are no de Sitter (or de Sitter-like) solutions for zero cosmological constant. Starobinsky inflation is not a physical solution.

The careful reader might be suspicious that other physical solutions might have been lost in the derivation of (36). That this is not so can be proven from the fact that (9) is derived from the action

freeze in (become constant) at late times. Locally, the freezing in occurs exponentially fast, and is pure gauge (though only locally). In this sense, at least for small amplitude perturbations, de Sitter space can be considered classically stable.

Before beginning the analysis of semiclassical corrections to gravitational waves on a de Sitter background, it is important to determine their regime of reliability. Semiclassical corrections (first order in \hbar) to perturbative gravity (first order in $h_{\mu\nu}$) are actually second order in perturbative parameters. We are only concerned with regimes where corrections second order in \hbar are too small to be considered, but we do wish to examine terms that are first order in both $h_{\mu\nu}$ and \hbar . If \hbar is sufficiently small that terms second order in \hbar can be neglected, but $h_{\mu\nu}$ is sufficiently large that terms first order in both $h_{\mu\nu}$ and \hbar are measurable, then effects second order in $h_{\mu\nu}$ (which we will not calculate) will dominate those effects that are about to calculate. Any predictions of the terms we will calculate must be taken in this context. There is one useful prediction we might extract. If the terms second order in $h_{\mu\nu}$ do not predict any instability of de Sitter space (in the sense defined above by the cosmological no-hair theorem), but the terms first order in both $h_{\mu\nu}$ and \hbar do, then the semiclassical instabilities might eventually dominate. What we will find, however, is that the semiclassical corrections to gravitational perturbations do not predict any instability, and therefore no predictions will be made at all.

The semiclassical field equation for linearized, gravitational perturbations on the semiclassical de Sitter background given by (38) is

$$\begin{aligned}
O(\hbar^2) &= \delta(R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} + \Lambda g_{\mu\nu} + \alpha^{(1)}H_{\mu\nu} + \beta^{(2)}H_{\mu\nu} + \gamma^{(3)}H_{\mu\nu}) \\
&= -\frac{1}{2}\square h_{\mu\nu} - \frac{1}{2}\nabla_\mu\nabla_\nu h + \nabla_\alpha\nabla_{(\mu}h_{\nu)}^\alpha + \frac{1}{2}g_{\mu\nu}(\square h - \nabla_\alpha\nabla_\beta h^{\alpha\beta} + \tilde{\Lambda}h) - \tilde{\Lambda}h_{\mu\nu} - (\alpha + \frac{1}{4}\beta)g_{\mu\nu}\square^2 h - \frac{1}{2}\beta\square^2 h_{\mu\nu} \\
&\quad + (\alpha + \frac{1}{2}\beta)\nabla_\mu\nabla_\nu\square h - (\alpha + \frac{2}{3}\beta + \frac{1}{6}\gamma)\tilde{\Lambda}g_{\mu\nu}\square h - (4\alpha + \frac{1}{3}\beta - \frac{1}{3}\gamma)\tilde{\Lambda}\square h_{\mu\nu} + (2\alpha + \beta)\tilde{\Lambda}\nabla_\mu\nabla_\nu h \\
&\quad - (\frac{2}{3}\alpha + \frac{1}{9}\beta + \frac{1}{9}\gamma)\tilde{\Lambda}^2 g_{\mu\nu}h + (\frac{8}{3}\alpha + \frac{4}{9}\beta + \frac{1}{9}\gamma)\tilde{\Lambda}^2 h_{\mu\nu} + (2\alpha + \frac{1}{2}\beta)g_{\mu\nu}\square\nabla^\alpha H_\alpha - (2\alpha + \beta)\nabla_\mu\nabla_\nu\nabla^\alpha H_\alpha \\
&\quad + \beta\square\nabla_{(\mu}H_{\nu)} - (2\alpha + \frac{1}{3}\beta - \frac{1}{3}\gamma)\tilde{\Lambda}g_{\mu\nu}\nabla^\alpha H_\alpha + (8\alpha + \frac{4}{3}\beta - \frac{2}{3}\gamma)\tilde{\Lambda}\nabla_{(\mu}H_{\nu)} , \tag{41}
\end{aligned}$$

where $H_\nu \equiv \nabla_\mu(h_\nu^\mu - \frac{1}{2}\delta_\nu^\mu)$ vanishes identically in harmonic gauge, and all sums over Greek indices are performed with respect to the background metric given by (8) and (38) [not (31)]. $\tilde{\Lambda}$ and Λ may be freely interchanged whenever preceded by a coefficient of order \hbar .

To find perturbatively expandable solutions to (41), the same strictly perturbative procedure as used above is applied. Multiply (41) by \hbar , take time derivatives as necessary, and substitute back. The resulting equation is considerably simpler:

$$\begin{aligned}
O(\hbar^2) &= -\frac{1}{2}\square h_{\mu\nu} - \frac{1}{2}\nabla_\mu\nabla_\nu h + \nabla_\alpha\nabla_{(\mu}h_{\nu)}^\alpha \\
&\quad + \frac{1}{2}g_{\mu\nu}(\square h - \nabla_\alpha\nabla_\beta h^{\alpha\beta} + \tilde{\Lambda}h) - \tilde{\Lambda}h_{\mu\nu} \\
&\quad - \frac{1}{3}\gamma\tilde{\Lambda}^2 h_{\mu\nu} . \tag{42}
\end{aligned}$$

Contributions proportional to α and β vanish. All higher-derivative corrections vanish. The only nonvanishing semiclassical correction is the last term. This is only a small correction of the same form as the preceding term, and it does not change the results of the stability of de Sitter space (from localized gravity modes).

V. CONCLUSION

The effective action of semiclassical gravity is derived as a perturbative approximation to the full (nonperturbative) effective action. It takes the form of a polynomial in \hbar , where terms higher order than \hbar are ignored. All information of nonperturbative and higher-order behavior has been discarded in this approximation, and this is inescapable in any derivation of the semiclassical field equations. Because of higher-derivative terms in the first-order corrections, however, there appear nonperturbative solutions to the semiclassical field equations. Since all nonperturbative information has already been lost, these nonperturbative solutions must be artifacts of the expansion, not indicative of new dynamics. By remaining within the perturbative formalism at every step of solving the field equations, all physical solutions are found and all nonperturbative pseudosolutions are avoided. This is achieved with mathematical rigor by treating the pertur-

bative expansion formally as an algebraic ring with zero divisors, and by remaining within this formalism until all (physical) solutions have been found.

A simple model demonstrated the appearance of nonperturbative pseudosolutions unrelated to the nonperturbative behavior of the full field equations. For consistency the pseudosolutions must be excluded. This excising of the pseudosolutions is especially significant for semiclassical gravity, where if the pseudosolutions were interpreted as physical, they would predict the instability of flat space.

At its weakest, the strictly perturbative method of solving higher-derivative expansions is a self-consistent method for removing ill-behaved (e.g., negative kinetic energy) solutions. At its most powerful, the method cuts through to the heart of the problem, bypassing the mathematical artifacts created by perturbatively expanding a higher-order action.

Starobinsky inflation, a class of de Sitter solutions to the semiclassical field equations driven solely by higher-order curvature effects, might potentially be useful in driving an inflationary epoch without requiring additional matter fields. Unfortunately, Starobinsky inflation was shown above to not be a physical prediction of self-consistent semiclassical gravity. Even if one were to assume that the nonperturbative solutions to semiclassical gravity, by chance or accident, were approximations to solutions of the full nonperturbative effective action, thus allowing Starobinsky inflation, then the prediction of instability of flat space would also follow. This is clearly in conflict with experiment.

Semiclassical corrections to linearized gravitational fluctuations on a de Sitter background were also examined, but no signs of instability were found.

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