Inflation in an exponential-potential scalar field model

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A spatially flat cosmological scalar field (Φ) model with the scalar field potential $\propto \exp(-\Phi/\sqrt{p})$, $p>1$, provides a simple class of inflationary cosmologies (which includes the usual exponential expansion inflation) that may be used as an analytical testing ground to help understand the predictions of the inflation model of the very early Universe. We divide the evolution of this model into three distinct epochs: scalar-field dominance and conventional radiation and baryon dominance; in each epoch we only account for irregularities in the dominant form of matter. We present closed-form solutions of the (synchronous gauge) relativistic linear perturbation equations that govern the evolution of inhomogeneities. These classical solutions, augmented with quantum-mechanically motivated initial conditions and joining conditions to match the expressions for the irregularities at the scalar-field–radiation and radiation-baryon transitions, are used to estimate the large-time form of the spectrum of energy-density irregularities, of the local departure velocity from homogeneous expansion, of large-scale fluctuations in the microwave background temperature, and of the gravitational-wave energy density. The inflation epoch results agree with those found from a purely quantum-mechanical analysis. Depending on the value of p this model can have more large-scale power than the usual scale-invariant spectrum (at the expense of less small-scale power) and would seem to be marginally better at forming large-scale structure than the canonical model; however, the decrease in small-scale power serves to exacerbate the problem of late galaxy formation. As the model approaches the exponential expansion inflation limit, the power spectrum tends towards the scale-invariant form, although, in this limit the numerical prefactor diverges. We find that transverse peculiar velocity perturbations are not generated. Normalizing by fitting to the observed large-scale departure velocity, we find that models which stop inflating around 10^7-10^{16} GeV are not obviously observationally inconsistent.

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I. INTRODUCTION AND SUMMARY

The simplest model for the formation of observed structure in the Universe (galaxies, clusters of galaxies, etc.) makes use of the characteristic ability of gravitational instability to amplify small initial spatial inhomogeneities in the mass distribution to form larger-scale objects. In one version of this scenario the small initial spatial inhomogeneities are assumed to result from quantum-mechanical fluctuations during a very early period of inflation. A currently popular example of this scenario is what is known as the biased, adiabatic, scale-invariant, cold nonluminous matter [hereafter, cold dark matter (CDM), i.e., extremely weakly interacting matter with almost no primeval thermal velocity] scheme [1].

The scale-invariant CDM scenario, which assumes a scale-invariant spectrum for the adiabatic energy-density fluctuations [2] (such a spectrum is present in a number of particle-physics-inspired inflation models of the very early Universe), has been analyzed in some detail. There seem to be two problems with this scenario [3]: if the spatial momentum-space power spectrum of energy-density fluctuations $P(k,t)$ (which is $\propto k$ in this scenario) is normalized by fitting to the observed large-scale fluctuations in galaxy number counts (on a scale of order 10–30 Mpc), there is relatively little power on small scales, which means that galaxies form late; furthermore, with this normalization, the model predicts that aggregations of mass are anticorrelated on scales larger than $\sim 50-100$ Mpc (which also does not seem to agree with the observations).

Although there have been some attempts to find inflation models which have more power on large scales [4], we believe that the more pressing issue is to find models which enhance the power spectrum on small scales, so that galaxies can form at a sufficiently early epoch. (The models discussed in this paper do not have enhanced small-scale power.) However, in models in which galaxies form before a redshift of a few it would be very difficult to reconcile the low dynamical estimates of the mean mass density with the negligibly small space curvature preferred by inflation via the construct of balancing $(1-\Omega_{\rm dyn})$ against some form of CDM since gravitational instability, given time, ensures that all forms of pressureless matter cluster democratically. [The density parameter Ω is the ratio of the mean mass density to the Einstein–de Sitter value, $\Omega(t)=8\pi G\rho/(3H^2)$, where G is Newton's gravitational constant, H is Hubble's parameter, and ρ the relevant energy density. $\Omega_{\rm dyn}$ denotes the dynamical estimate of the value of the density parameter, on scales up to ~ 10 Mpc.] To alleviate this problem one might assume that $(1-\Omega_{\rm dyn})$ is balanced by some kind of nonluminous, nearly homogeneous, energy density, for example, a cosmological constant [5] or a cosmological-

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"constant"-like scalar field [6,7].

The scale-invariant power spectrum of adiabatic fluctuations in the standard exponential expansion inflation model of the very early Universe is a consequence of the form of the potential, $V(\Phi)$, of the inflaton field, Φ— an exceedingly flat inflaton potential results in an exponentially expanding scale factor, $a(t) \propto \exp(Ht)$, and a scalar field two-point function that varies logarithmically with scale [8], which results in a scale-invariant power spectrum [9]. (The belief that the scale-invariant spectrum is a consequence of the time translation symmetry of de Sitter spacetime is not correct. This is because the solution, of the relativistic linear perturbation theory equations, corresponding to this symmetry is a decaying, gauge-dependent, solution.) One way of modifying the power spectrum would be to consider a different form for the inflaton potential [10]. There are a number of possible forms; in this paper we consider a potential of the form $V(\Phi) \propto \exp(-\Phi/\sqrt{p})$, $p > 0$, which results in a scale factor $a(t) \propto t^p$ (for a spatially flat cosmological model); if $p > 1$ this model describes an inflating cosmology (i.e., the time derivative of the scale factor increases with time) [11], with an energy-density power spectrum that can have enhanced large-scale power compared to the scale-invariant spectrum. In the limit $p \to \infty$ this model reduces to the usual model of exponential expansion inflation. This class of power-law inflation models was proposed and examined by Lucchin and Matarrese [12,13,7].

The main purpose of this paper is to study this simple example of an inflation modified hot big-bang cosmology in some detail, so as to develop some understanding of the observable predictions of inflation. We shall model the evolution of the Universe by dividing it into three distinct epochs. During the earliest, scalar-field-dominated, epoch the scalar field energy density powers the inflationary expansion of the Universe. This model for the very early Universe is then patched on to a conventional hot big-bang model with a radiation-dominated epoch and a baryon-dominated, present, epoch.

Even though a specific particle-physics-based scalar field inflation model of the very early Universe is complete (in the sense that it has a given scalar field potential and reheating temperature), we believe that the vast number of such models makes it more worthwhile to focus on a class of macrophysical inflation models, rather than to concentrate on a particular microphysics-based model, and to study the constraints observations impose on the reheating temperature and the scalar field potential during inflation. [Of course, if one then wants to use the models, with parameters that lie in the observationally desired range, to examine further observational consequences of inflation, one is open to the criticism that these models are "fine-tuned"—unfortunately, in the absence of a theory of the very early Universe (as opposed to a model), this is the best that can be done.] In other words, we wish to ask the question: Given the present observational constraints on the large-scale structure of the Universe, can one usefully constrain the energy scale at which inflation must have ended (i.e., the particle-physics energy scale which might be responsible for the large-scale structure of the Universe)? This question can only be posed in the context of a particular model of inflation. For reasons which we shall discuss below, we believe that the macrophysical model of inflation we examine is of sufficient (large time cosmological) generality that the answer to this question might not be totally irrelevant. Interestingly, a preliminary comparison to a fair fraction of the large-scale observational data suggests the rather weak bound that models which stop inflating around $\sim 10^7-10^{16}$ GeV are not obviously inconsistent with the data. We emphasize that these models do not seem to require a small dimensionless parameter (of order 10^{-10}), that some other inflation models seem to require, to not disagree with the observations. (We note that models at the lower end of this energy range will be more accessible to the tests of high-energy experimental physics; they will, however, probably require a new baryosynthesis subscenario.) A more careful comparison will probably reduce this spread in energy—the main purpose of this paper is not to draw detailed quantitative conclusions, rather, we wish to develop a framework which in the future might provide the appropriate setting for such an analysis.

Perhaps the most striking feature of inflation is its ability to enormously expand inflation epoch length scales on which quantum mechanics rules, to length scales that are now of cosmological significance. This makes it possible for inflation epoch zero-point quantum fluctuations to transmute to large-scale irregularities in an otherwise homogeneous large time cosmological model, and to maybe be responsible for the observed large-scale structure of the Universe. We are particularly interested in the behavior, in linear perturbation theory, of energy-density irregularities in this simple inflation model. We use the methods of Ref. [14] to derive closed-form solutions of the (synchronous gauge) equations of relativistic linear perturbation theory that govern the evolution of spatial irregularities. These general solutions depend on constants of integration; to determine the constants of integration in the inflation era expressions we use quantum-mechanically motivated initial conditions [8]. (We shall see, in Sec. III C, that these initial conditions only determine the constants of integration for gauge-invariant solutions; for instance, they seem to leave undetermined the constant of integration which corresponds to the gauge-dependent, time translation invariance solution.) These explicit solutions allow one to use this simple model to compare different methods of estimating the inflation epoch spectrum of spatial irregularities. An analysis of the inflation epoch of this model using the Dirac-Wheeler-DeWitt formalism of Ref. [9] is described elsewhere [15]. The expressions for the inflation epoch perturbations found here agree with those found from a purely quantum-mechanical analysis in Ref. [15].

Given the explicit form of the scalar field and gravitational irregularities during inflation, we may use the results of Refs. [14,16] to derive expressions for the perturbations in the large time (radiation-fluid- and matter-fluid-dominated) universe by joining the expressions for the perturbations at the transitions. We assume that the transitions (from the scalar-field-dominated epoch to the radiation-dominated epoch and from the radiation-dominated epoch to the matter-dominated epoch) are instantaneous and that the matter stress tensor is always dominated by one type of matter—the scalar field in the scalar-field-dominated epoch, radiation in the radiation-dominated epoch and baryons in the matter-dominated epoch (so that the total present density parameter $\Omega_{\rm now}=1$)—i.e., in each epoch we only account for perturbations in the dominant form of matter.

We model the transitions by requiring that the equation of state (and hence the pressure) change instantaneously and discontinuously at the spatial hypersurface on which the local (total) energy density reaches an appropriate critical value (the transitions, after all, are governed by local physics). This spatially homogeneous local energy-density hypersurface differs from the corresponding constant time synchronous gauge hypersurface only by terms that are of first- (or higher-) order in perturbation theory [16]. This means that we may match the scale factor and the homogeneous "background" part of the energy density on the constant time spatial hypersurface. The joining conditions for the constants of integration in the expressions for the perturbations in the large-time universe are derived by requiring that the linearized equations of covariant conservation of stress-energy and Einstein's equations do not become singular on the spatially homogeneous local energy-density transition hypersurfaces [16]. These joining conditions differ from two earlier sets of joining conditions (which differ from each other) [16].

Given the approximate nature of our model for the transitions (especially the reheating transition), it is important to be able to identify those parts of the final results which do not depend sensitively on the joining conditions used. In a preliminary attempt at studying this issue, we have, in Ref. [17], derived joining conditions for a transition at a spatially homogeneous local scalar field spatial hypersurface and used these joining conditions in the analysis of an inflation model—a discussion of the dependence of the final results on changes in the joining condition prescription is presented in Refs. [17,18].

The energy-density hypersurface joining conditions may be used to determine the constants of integration in the expressions for the fractional energy density, peculiar velocity and metric perturbations in the radiation- and matter-dominated epochs. Since this model does not include the physics which describes the coupling between radiation and matter (as well as other small scale processes), the expressions we have derived are only valid on fairly large scales. We, therefore, focus on perturbations on scales larger than the Hubble radius during the transition from scalar field dominance to radiation dominance and the transition from radiation dominance to matter dominance. These large-time expressions for the perturbations determine, among other things, the baryon-dominated epoch form of the power spectrum of energy-density irregularities. We find that the adiabatic mode, for perturbations that reentered the Hubble radius in the baryon-dominated epoch, has a power spectrum that is given by

$$P(k,t)\equiv\langle\delta_B(k,t)\delta_B(-k,t)\rangle = \frac{16\pi}{m_P^2}k^{2(1-\nu)}|A|^2a^2 , \tag{1.1}$$

where δ_B denotes the fractional energy-density perturbation in the baryon-dominated epoch, k is the magnitude of the spatial momentum of the mode under consideration, $\nu=(2+q)/[2(2-q)]$, $q=2/p$, m_P is the Planck mass and the coefficient in this expression is

$$|A|=\frac{3}{20}\left[\frac{3\pi}{\sqrt{2}q}\right]^{1/2}(2-q)^{\nu+1/2}\frac{\csc(\nu\pi)}{\Gamma(-\nu)} \times\left[\frac{\lambda_R}{k}\right]^{7/2}\left[\frac{\tilde{\lambda}_B}{k}\right]^{3/4} \times[a(t_{R\Phi})H(t_{R\Phi})]^{\nu+3/2}[a(t_{BR})H(t_{BR})]^{-3/4} \tag{1.2}$$

where $t_{R\Phi}$ and t_{BR} are the synchronous gauge (transition) times of equal scalar field and radiation energy density and equal radiation and baryon energy density and λ_R and $\tilde{\lambda}_B$ are characteristic wave numbers, at these transitions, defined below Eqs. (4.34) and (4.43) below. We note that for the allowed values of $q\in[0,2]$ (for which the model of the early Universe inflates) the power spectrum index $n\equiv 2(1-\nu)$ ranges from 1 to $-\infty$. The time dependence of this power spectrum is of the standard form, the spatial momentum dependence agrees with the results of Ref. [12], while the numerical prefactor seems to disagree with what was found in Ref. [12].

In the limit $q=2\epsilon^2\sim 0$ this model reduces to the step function potential exponential expansion model studied in Ref. [9]. In this limit the power spectrum reduces to the standard scale-invariant form [2]

$$P(k,t)=\frac{16\pi}{m_P^2}k|A|^2a^2 \tag{1.3}$$

[it should now be clear that our neglect of the effects arising from the coupling of radiation to matter means that this equation is only valid for large-scale perturbations (see, for instance, Ref. [19]); without including this physics we cannot estimate the ("galaxy formation") epoch when small scales ($\sim$ galactic scales) go nonlinear], where the coefficient is now given by

$$|A|=\frac{1}{\epsilon}\frac{1}{20}\left[\frac{3}{\sqrt{2}}\right]^{3/2}\left[\frac{\lambda_R}{k}\right]^{7/2}\left[\frac{\tilde{\lambda}_B}{k}\right]^{3/4} \times[a(t_{R\Phi})H(t_{R\Phi})]^2[a(t_{BR})H(t_{BR})]^{-3/4} . \tag{1.4}$$

In the exponential expansion limit the power spectrum diverges like ϵ^{-2}. (This divergence seems to be unrelated to the infrared divergence of (massless, or almost so, minimally coupled, scalar) field theory in de Sitter spacetime, studied in Ref. [8], which is responsible for the spatial momentum dependence ($\propto k$) of Eq. (1.3).) The physical origin of this divergence lies in a property of the quasi–de Sitter spacetime that results in this limit: the quasi–de Sitter background energy density evolves so slowly that the spatially homogeneous local energy-density hypersurfaces are exceedingly displaced from the corresponding synchronous gauge constant time hypersurfaces; i.e., one must wait for a substantial time before the change in ρ_b is large enough to compensate for δ_Φ, even if δ_Φ obeys the standard assumptions of linear perturbation theory. This would not be a problem if one stayed permanently in the de Sitter model, but the Universe must eventually become radiation dominated and the transition must take place simultaneously on a spatially homogeneous local energy-density hypersurface (since it is governed by local physics). In the radiation-dominated epoch the spatially homogeneous local energy-density hypersurfaces are not anomalously displaced from the corresponding constant time synchronous gauge hypersurfaces; since we have matched the homogeneous background fields on a constant time hypersurface and need to match the perturbations on a spatially homogeneous local energy-density hypersurface and since the local energy-density hypersurface is anomalously displaced in the de Sitter epoch, we are forced to choose δ_R, during radiation dominance, to be very large to achieve the needed matching. It might, therefore, seem that, independent of the detailed form of the joining conditions used at the transition, this (almost geometric) property of de Sitter spacetime always results in an exceedingly large power spectrum in the baryon-dominated epoch (this is true provided the scalar-field–radiation transition occurs rapidly on the relevant Hubble time scale [18,20]). Our analysis of the transition breaks down when ϵ is very small because we make the assumption that the local energy-density hypersurface and the corresponding constant time synchronous gauge hypersurface are "close," which is no longer valid (this does not signal a breakdown of the standard linear perturbation theory assumptions on the de Sitter side of the transition hypersurface)—however, for ϵ small, but not exceedingly so, our analysis holds and one sees the proto-divergence begin to develop.

Although the simplified treatment of the reheating transition presented here does not preclude the possibility that a more complete analysis might alter the detailed form of our (exponential expansion inflation limit) conclusions, we would be surprised if the relevant small-scale physics is found to significantly influence the large-scale form of the power spectrum. As we will discuss in more detail below, this divergence only rules out an exact de Sitter model; a naive analysis of the final results for an $\epsilon \sim 10^{-8}$ (or even smaller) shows no obvious inconsistency with the observations.

At this point, it is perhaps appropriate to note that, at first sight, the final result of a substantial fraction of previous analyses of density irregularities arising from de Sitter inflation [21–24] differs from this result (this will be elaborated on in Sec. V B and in Refs. [18,20]). [We note that a general formula presented at an intermediate state of some of these analyses, when applied to the model studied here, does contain this divergence.] More specifically, in earlier analyses the formula which is used to claim that inflation requires a very small ("fine-tuned") microphysical coupling constant if it is to not disagree with the observations also suggests that the large-time perturbations arising from exact de Sitter inflation vanish. (We emphasize that here "fine-tuned" means that the observationally desired value of the coupling constant conflicts with the value suggested by the microphysics of the model; this should not be confused with the "fine-tuning" problem of the electroweak model [25], which is something entirely different.) This difference is the result of the difference between the treatment of reheating used here and that used in earlier analyses; since the scalar field model studied in the earlier analyses differs from the model we have examined here, we postpone a comparison to Refs. [26,18,20] (where we consider the model used in earlier analyses).

As a spinoff of our analysis of inhomogeneities, we find that transverse peculiar velocity perturbations are not generated in this class of scalar field inflationary models. This means that such models of inflation are not the appropriate description of the very early Universe in the primeval turbulence scheme for structure formation. It is unclear, at present, whether this effect may be used to observationally distinguish the scalar field inflation mechanism for generating the progenitors of large-scale structure from other mechanisms. (This seems unlikely since other mechanisms for producing these progenitors will probably only generate decaying transverse peculiar velocity perturbations which would seem to be extremely difficult to detect observationally.)

We also study gravitational-wave perturbations. One characterization of these perturbations is the spatial momentum-space two-point function; we find, for perturbations that reentered the Hubble radius in the baryon dominated epoch,

$$\langle h_i^{(B)}(k) h_i^{(B)}(-k) \rangle = \frac{16\pi}{m_P^2} k^{-2(\nu+4)} a^{-3} |D|^2 \left\{ 1 - \cos\left[\frac{4k}{aH}\right] - \frac{4k}{aH}\sin\left[\frac{4k}{aH}\right] + 4\left[\frac{k}{aH}\right]^2 \left[1+\cos\left[\frac{4k}{aH}\right]\right]\right\}, \tag{1.5}$$

where there is no implied summation over the index i which can correspond to either the $+$ or $\times$ transverse polarization of the graviton ($h_+ = h_{22} - h_{33}$ and $h_\times = h_{23} + h_{32}$) and the numerical prefactor is given by

$$|D| = \sqrt{\pi}\, 3 (\tfrac{3}{2})^{5/4} (2-q)^{\nu+1/2} \frac{\csc(\nu\pi)}{\Gamma(-\nu)} \left[\frac{\tilde{\lambda}_R}{k}\right]^{3/2} \left[\frac{\tilde{\lambda}_B}{k}\right]^{-15/4} [a(t_{R\Phi})H(t_{R\Phi})]^{\nu+3/2} [a(t_{BR})H(t_{BR})]^{1/4}, \tag{1.6}$$

where $\tilde{\lambda}_R$ is a characteristic wave number at the scalar-field–radiation transition and is related to λ_R through $\tilde{\lambda}_R = \sqrt{3}\lambda_R$. We find that the two-point function becomes time independent for long-wavelength gravitational waves (and in this limit is proportional to $k^{-2(\nu+1)}$, which reduces to k^{-3} in the exponential expansion limit of the model for the very early Universe), while on small scales it oscillates about a nonzero mean with amplitude $\propto k^{-2(\nu+3)} a^{-2}$. The energy density carried by the gravitational-wave perturbations is

$$\varepsilon_i(k,t)=\frac{m_P^2}{32\pi}\left[\langle \dot{h}_i(k,t)\dot{h}_i(-k,t)\rangle+\frac{k^2}{a^2}\langle h_i(k,t)h_i(-k,t)\rangle\right], \tag{1.7}$$

where an overdot denotes a derivative with respect to t and again there is no implied summation over i. Dimensionally, this expression plays the role of $\rho_b[\langle\delta(k)\delta(-k)\rangle]^{1/2}$ in the ideal fluid model. When applied to the model studied here we find that this becomes

$$\varepsilon_i(k)=k^{-2(\nu+4)}a^{-6}|F|^2\left\{9+16\left[\frac{k}{aH}\right]^2+32\left[\frac{k}{aH}\right]^4 - \left[9-56\left[\frac{k}{aH}\right]^2\right]\cos\left[\frac{4k}{aH}\right]-\left[36\frac{k}{aH}-32\left[\frac{k}{aH}\right]^3\right]\sin\left[\frac{4k}{aH}\right]\right\}, \tag{1.8}$$

where $|F|$ is defined through

$$|F|=\sqrt{\pi}\frac{9}{2}\left[\frac{3}{2}\right]^{3/4}(2-q)^{\nu+1/2}\frac{\csc(\nu\pi)}{\Gamma(-\nu)}\left[\frac{\tilde{\lambda}_R}{k}\right]^{3/2}\left[\frac{\tilde{\lambda}_B}{k}\right]^{-21/4}[a(t_{R\Phi})H(t_{R\Phi})]^{\nu+3/2}[a(t_{BR})H(t_{BR})]^{1/4}. \tag{1.9}$$

In the long-wavelength limit the energy-density spectrum of gravitational waves is proportional to $k^{-2\nu}a^{-2}$ (which reduces to $k^{-1}a^{-2}$ in the de Sitter limit of the inflation model). In the short-wavelength limit we find that the dominant part of the energy-density spectrum does not oscillate and scales as $k^{-2(\nu+2)}a^{-4}$. We emphasize that both the spatial momentum dependence and the scale factor dependence of the energy-density spectrum differ from that of the graviton two-point function.

To compare theory with observation we have to use the same coordinate system for the theoretical expressions and the observational data. Although this issue does not yet seem to have been addressed in any detail in the relativistic perturbation formalism, conventional wisdom suggests that theoretical expressions in the Newtonian approximation (the limit in which the peculiar velocity is small and one focuses on scales small compared to the Hubble scale) may be directly contrasted with observational data. We, therefore, choose to transform the theoretical expressions to the instantaneously Newtonian synchronous coordinate system. This is the coordinate system in which, at the epoch when the observation was made, the time derivative of the trace of the metric perturbation is removed (and on scales much less than the Hubble radius the expressions for perturbations in these coordinates reduce to those which may be derived from the corresponding Newtonian perturbation equations). Furthermore, the limited resolution of an observation of a given quantity means that it only senses a coarse-grain average value for this position space quantity; we account for this effect by using a window function to suppress high spatial momentum modes in the Fourier transform (which takes the momentum-space expression derived for this quantity) back to position space.

Two quantities of particular interest are the large-time, baryon-dominated epoch, forms of the mean-square measure of the fractional mass distribution and the mean-square measure of the (longitudinal) local departure velocity from homogeneous expansion. The mean-square measure of the fractional mass distribution is defined by

$$\left\langle\left[\frac{\delta\hat{M}}{\hat{M}}(t_N|R)\right]^2\right\rangle=4\int_{-\infty}^{\infty}\frac{d^3k}{(2\pi)^3}\times\langle\hat{\delta}_B(\mathbf{k},t_N)\hat{\delta}_B(-\mathbf{k},t_N)\rangle e^{-|\mathbf{k}|^2R^2}, \tag{1.10}$$

where the carets denote that the instantaneously Newtonian synchronous coordinate system is being used, t_N is the time at which the observation was made, and R is the coordinate length scale which the observation was sensitive to (and we have used a Gaussian window function); a similar expression is used to define the mean-square measure of the local departure velocity from homogeneous expansion.

The exclusion, from this model, of the small-scale physics which describes the coupling of radiation to baryons requires that we normalize the large-time expressions, which describe the evolution of matter irregularities, by comparing the mean-square measure of the departure velocity from homogeneous expansion to that (tentatively) observed on intermediate ("great attractor") scales of ~ 60 Mpc [27]. (Linear perturbation theory should not be too inaccurate on these scales.) This results in a relation that is, for present purposes, best interpreted as determining the redshift of reheating (which is a "microphysical" parameter), $z_{R\Phi}$ in terms of the power-law index of the inflation epoch scalar field potential (another microphysical parameter), q.

We find at the present epoch, on scales small compared to the Hubble radius ($\mathcal{R}H_{\rm now}\ll 1$, where $H_{\rm now}$ is the present value of the Hubble parameter and $\mathcal{R}$ is a proper length scale), that the leading term in the fractional mass distribution is

$$\left\langle\left[\frac{\delta\hat{M}}{\hat{M}}(\mathcal{R})\right]^2\right\rangle=\frac{16\pi}{m_P^2}Q\Gamma\left[\frac{n}{2}+\frac{3}{2}\right](\mathcal{R}H_{\rm now})^{-(n+3)}, \tag{1.11}$$

where the factor Q is given by

$$Q=\frac{2^{5-2n}}{25\pi}\frac{(3-n)^{n-2}}{(1-n)}\frac{\csc^2(n\pi/2)}{\Gamma^2(n/2-1)}z_{R\Phi}^{5-n}z_{BR}^{(n-9)/2}H_{BR}^2 . \quad (1.12)$$

In this expression $z_{R\Phi}$ and z_{BR} (both $\gg 1$) are the redshifts of the scalar-field–radiation and radiation-baryon transitions and H_{BR} is the value of the Hubble parameter at the radiation-baryon transition. Equation (1.11) is only valid for $n>-3$ (because of an infrared divergence in the integrand of the transform of the momentum-space expression to position space). There is a qualitative change in the small-scale behavior of the mean-square measure of the (longitudinal) local departure velocity from homogeneous expansion when n passes through -1. We find

$$\langle[\delta\hat{V}^1(\mathcal{R})]^2\rangle=\frac{16\pi}{m_P^2}Q\frac{2^{(5-n)/2}}{(n+5)}\frac{3^{n+1}}{(-n-1)}\Delta_n \quad \text{for } -3\le n\le -1 \quad (1.13)$$

[where $\Delta_n=(-n-1)/4$ if $n=-1,-3$ and is unity otherwise], and

$$\langle[\delta\hat{V}^1(\mathcal{R})]^2\rangle=\frac{16\pi}{m_P^2}Q\Gamma\left[\frac{n}{2}+\frac{1}{2}\right]\Delta_n(\mathcal{R}H_{\rm now})^{-(n+1)} \quad \text{for } -1<n\le 1 \quad (1.14)$$

(where $\Delta_n=1/4$ if $n=1$ and is unity otherwise). The change in the behavior at $n=-1$ is well known; the main new feature here is that Eq. (1.13) is not cutoff dependent, unlike the corresponding expressions in the standard cosmological scenarios which must be cutoff in the infrared. In what follows we require $-3<n\le 1$, where the lower bound is a consequence of the infrared behavior of the momentum-space form of the mean-square fractional mass distribution and the upper bound is a consequence of the inflation model we have chosen.

To fix the normalization of the power spectrum we require that the expressions for the local departure velocity agree with the observations, which are conveniently summarized as

$$\langle[\delta\hat{V}^1(\mathcal{R})]^2\rangle=\gamma^2(\mathcal{R}H_{\rm now})^2 , \quad (1.15)$$

where the great attractor measurements (tentatively) suggest $\gamma\simeq 0.1$ on a scale $\mathcal{R}H_{\rm now}\simeq 2\times 10^{-2}$, Ref. [28]. This results in a relation that determines the redshift at the epoch of reheating, $z_{R\Phi}$, in terms of the spectral index n. With this relation, the amplitudes and spectral indices of all perturbations are fixed once n has been specified (in contradistinction to standard cosmological scenarios where, for instance, one must assume, as a separate initial condition at early time, the amplitude of gravitational-wave perturbations) and one may compare the resulting expressions (in an almost standard manner) to other observational data. Some of the results of a preliminary examination are as follows (the numerical values are meant to illustrate orders of magnitude and hence should not be taken too seriously).

(i) The redshift of reheating, $z_{R\Phi}$, ranges from $\sim 3\times 10^{16}$ to $\sim 2\times 10^{29}$, where we have suppressed the dependence on $h(H_{\rm now}=100h\ {\rm km\,s^{-1}Mpc^{-1}})$ and assumed the standard hot big-bang value $z_{BR}=4\times 10^4h^2$, these numbers are correlated with the value of n—the lower end corresponds to $n\sim -3$, the upper to $n\sim 1$.

(ii) The corresponding energy scale at reheating, $E_{R\Phi}$, ranges from $\sim 7\times 10^3$ GeV ($n\sim -3$) to $\sim 3\times 10^{16}$ GeV ($n\sim 1$) (we have again chosen $h=1$).

(iii) The root-mean-square measure of the fractional mass distribution on the scale $\mathcal{R}H_{\rm now}\simeq 2\times 10^{-2}$ ranges from 10^{-2} (for $n\sim -5/2$) to 0.2 (for $n\sim 1$), the trend is not monotonic (in n).

(iv) One-third of the root-mean-square measure of the fractional mass distribution on the scale $\mathcal{R}H_{\rm now}\simeq 0.2$ ranges from $\sim(1-50)\times 10^{-4}$—this provides a very rough, order of magnitude, estimate of the fractional spatial anisotropy of the microwave background temperature resulting from the Sachs-Wolfe effect (we note that we have not bothered to account for the general-relativistic redshift correction, which should not alter the order of magnitude of these numbers).

These quantitative conclusions are preliminary; the important point to be noted is that there seems to be a large number of simple scalar field inflation models that are not obviously inconsistent with large-scale observational data. A qualitative feature of interest is that the adiabatic fluctuations in this model are not of the scale-invariant form characteristic of exponential expansion inflation; depending on the value of n the model has increased power on large scales (at the cost of reduced small-scale power) compared to the exponential expansion inflation model. This would lead to more large-scale structure than in the scale-invariant CDM mode, and might lead to a problem with the observational upper bound on the large-scale microwave background spatial anisotropy (our rough estimates tentatively suggest that we must have $n\gtrsim -1.5$ [29]). The reduction of small-scale power would seem to result in a late epoch of galaxy formation (which could also lead to observational problems), to quantify this would require a more complete treatment of the small-scale physics than we have attempted here.

In Sec. II we review the spatially flat exponential-potential scalar field inflation model [7]. In Sec. III we study the evolution of small spatial irregularities in the inflation epoch, present closed-form solutions of the equations of motion and use the remnants of general coordinate invariance in synchronous gauge to catalogue physical solutions. We than adapt the quantum-mechanical initial conditions of Ref. [8] to determine the constants of integration that appear in these solutions and present asymptotic approximations of our exact expressions valid at early times and at late times (in the scalar-field-dominated epoch). In Sec. IV we use the joining conditions of Ref. [16] and the results of Sec. III and Ref. [14] to determine expressions for small spatial irregularities in the radiation- and matter-dominated epochs.

Section V is the "applications" section. We first determine the form the constants of integration take for perturbations on scales that were larger than the Hubble scale during the transitions from scalar field dominance to radiation dominance and from radiation dominance to matter dominance. These expressions are used to derive the large time (baryon-dominated epoch) form of the peculiar velocity perturbation and the fractional energy-density irregularity power spectrum. The power spectrum is examined in the de Sitter (exponential expansion) inflation limit and is found to diverge. We then determine the baryon-dominated epoch form of the gravitational-wave perturbation two-point function, derive a measure of the energy density in these perturbations (which seems to be more general than that previously used) and apply it to this model of the very early Universe. To compare to observational data we transform the theoretical expressions which describe baryonic inhomogeneities to the instantaneously Newtonian synchronous coordinate system and derive mean-square measures of the fractional mass distribution and the local departure velocity from homogeneous expansion. The mean-square local departure velocity is compared to observational data on intermediate scales and we find that observationally allowed models are required to obey a relation between the redshift of reheating, $z_{R\Phi}$, and the index of the inflation epoch scalar field potential q. We conclude with a cursory comparison of the model to other large-scale observational data.

II. THE INFLATION MODEL

The exponential-potential scalar field model has been discussed in Ref. [7]; we collect some of the relevant results here.

The Einstein–scalar-field action is

$$S=\frac{m_P^2}{16\pi}\int dt\, d^3x\sqrt{-g}\,[-R+\tfrac{1}{2}g^{\mu\nu}\partial_\mu\Phi\partial_\nu\Phi-\tfrac{1}{2}V(\Phi)]\ , \tag{2.1}$$

where $m_P=G^{-1/2}$ is the Planck mass and the metric signature is $(+---)$. For a spatially homogeneous scalar field, Φ_0, the potential

$$V(\Phi_0)=\left[\frac{6-q}{3}\right]16\pi m_P^{-2}\rho_\Phi^{(0)}\times\exp\left[-\left[\frac{q}{2}\right]^{1/2}(\Phi_0-\Phi_0^{(0)})\right]\ , \tag{2.2}$$

causes the scalar field energy density to redshift as

$$\rho_\Phi=\rho_\Phi^{(0)}\left[\frac{a_0}{a}\right]^q\ ; \tag{2.3}$$

here $\Phi_0^{(0)}$ is the value of the scalar field when the scale factor $a=a_0$ and we have chosen the line element

$$ds^2=dt^2-a^2(t)(d\mathbf{x})^2\ ,$$

which describes homogeneous, isotropic, spatially flat cosmological models. The desired numerical values of q and $\rho_\Phi^{(0)}$ are to be determined from the observations.

With a potential of the form given by Eq. (2.2), the Einstein–scalar-field equations, for a spatially flat Friedmann-Lemaître-Robertson-Walker (FLRW) model, reduce to

$$\begin{aligned}&\ddot{y}+3\frac{\dot{a}}{a}\dot{y}-\frac{q}{4}\overline{H}^2e^{-y}=0\ ,\\&12\frac{\ddot{a}}{a}+\frac{4}{q}(\dot{y})^2-\overline{H}^2e^{-y}=0\ ,\end{aligned} \tag{2.4}$$

where an overdot denotes a derivative with respect to time, $y=\sqrt{(q/2)}(\Phi_0-\Phi_0^{(0)})$ and

$$\overline{H}^2=\left[\frac{6-q}{3}\right]16\pi m_P^{-2}\rho_\Phi^{(0)}\ .$$

A special solution (which is a time-dependent stable fixed point for the relevant range of q, [7]) of these equations is

$$\begin{aligned}&a_e(t)=a_0[1+M(t-t_0)]^{2/q}\ ,\\&y_e(t)=2\ln[1+M(t-t_0)]\ ,\end{aligned} \tag{2.5}$$

where $M=q\overline{H}/[2\sqrt{2(6-q)}]$. This solution describes an inflating universe if $q<2$.

When $q=2\epsilon^2\sim 0$ this model reduces to the step function potential model studied in Ref. [9]. In this limit the scalar field potential, Eq. (2.2), becomes

$$V(\Phi_0)=2\Lambda[1-\epsilon(\Phi_0-\Phi_0^{(0)})]\ , \tag{2.6}$$

where we have defined the cosmological constant $\Lambda=16\pi m_P^{-2}\rho_\Phi^{(0)}$, and the homogeneous solutions of the field equations reduce to (to lowest order in ϵ)

$$\begin{aligned}&a(t)=a_0\exp[H(t-t_0)]\ ,\\&\Phi_0=\Phi_0^{(0)}+2\epsilon H(t-t_0)\ ,\end{aligned} \tag{2.7}$$

where we have only retained the expanding solution and have defined the Hubble parameter $H=\sqrt{\Lambda/6}$. For later reference, we note that in this limit the background equation of state is given by

$$p_\Phi=-\left[1-\frac{2}{3}\epsilon^2\right]\rho_\Phi\ , \tag{2.8}$$

where the dominant first term in the parentheses corresponds to a cosmological-constant "fluid"; the time variation of the background energy density is extremely slow:

$$\dot{\rho}_\Phi=-2\epsilon^2H\rho_\Phi\ , \tag{2.9}$$

or

$$\rho_\Phi=\rho_\Phi^{(0)}\left[\frac{a_0}{a}\right]^{2\epsilon^2}\ . \tag{2.10}$$

III. INHOMOGENEITIES DURING INFLATION

In this section we study the evolution of inhomogeneities in the scalar field and in the spacetime metric for the exponential-potential inflation model, using the techniques developed in Ref. [14]. The synchronous gauge equations governing the evolution of inhomogeneities in a general Einstein-scalar field model have been derived in Sec. VII of Ref. [7]. These equations are

$$\ddot{\phi}+3\frac{\dot{a}}{a}\dot{\phi}-\frac{1}{a^2}\nabla^2\phi+\frac{1}{2}V''(\Phi_0)\phi=\frac{1}{2}\dot{h}\dot{\Phi}_0 , \quad (3.1)$$

$$\frac{1}{2}\ddot{h}+\frac{\dot{a}}{a}\dot{h}=\dot{\Phi}_0\dot{\phi}-\frac{1}{4}V'(\Phi_0)\phi , \quad (3.2)$$

$$\dot{h}_{,i}-\dot{h}_{ij,j}=\dot{\Phi}_0\partial_i\phi , \quad (3.3)$$

$$\frac{1}{a^2}(h_{ij,kk}+h_{,ij}-h_{ik,jk}-h_{jk,ik})-3\frac{\dot{a}}{a}\dot{h}_{ij} -\frac{\dot{a}}{a}\dot{h}\delta_{ij}-\ddot{h}_{ij}=\frac{1}{2}\delta_{ij}V'(\Phi_0)\phi ; \quad (3.4)$$

here spatial indices are raised and lowered with the metric $+\delta_{ij}$. To derive these equations we have linearized the metric about a spatially flat FLRW background, so the line element is given by

$$ds^2=dt^2-a^2(t)[\delta_{ij}-h_{ij}(t,\mathbf{x})]dx^i dx^j ,$$

where $a(t)$ is the FLRW scale factor, $h_{ij}(t,\mathbf{x})$ denotes the metric perturbations and h is the trace of the metric perturbations (and should not be confused with the parameter in the numerical expression for $H_{\rm now}$). We have also linearized the scalar field about a homogeneous background:

$$\Phi(t,\mathbf{x})=\Phi_0(t)+\phi(t,\mathbf{x}) .$$

We tabulate the solutions of Eqs. (3.1)–(3.4) in the following subsection.

A. Solutions

Since the analysis involved in integrating Eqs. (3.1)–(3.4) is very similar to that used to integrate the fluctuation equations in the ideal fluid model, Ref. [14], we omit technicalities and only list the solutions of the equations. We Fourier expand the fluctuations ϕ and h_{ij}:

$$\phi(t,\mathbf{x})=\int\frac{d^3k}{(2\pi)^3}[\phi(t,\mathbf{k})e^{i\mathbf{k}\cdot\mathbf{x}}+\phi^*(t,\mathbf{k})e^{-i\mathbf{k}\cdot\mathbf{x}}] ,$$

$$h_{ij}(t,\mathbf{x})=\int\frac{d^3k}{(2\pi)^3}[h_{ij}(t,\mathbf{k})e^{i\mathbf{k}\cdot\mathbf{x}}+h^*_{ij}(t,\mathbf{k})e^{-i\mathbf{k}\cdot\mathbf{x}}] ,$$

where $\mathbf{k}$ is the spatial coordinate momentum, $k_i\in[-\infty,\infty]$, $\mathbf{k}\cdot\mathbf{x}=\delta_{ij}k_ix_j$ and $\phi(t,\mathbf{k})=\phi^*(t,-\mathbf{k})$. It is convenient to change from the independent variable t (time) to x, where

$$e^x=t-t_0+m^{-1} . \quad (3.5)$$

The wave equations for the fluctuations ϕ and h, Eqs. (3.1) and (3.2), in the exponential potential model, after the spatial Fourier transform, where $k=|k_ik_i|^{1/2}$ (in first-order perturbation theory Fourier modes with different wave vectors do not couple), are given by

$$\phi''(x)+\frac{6-q}{q}\phi'(x)+\left[\frac{2(6-q)}{q}+f(x)\right]\phi(x) =\left[\frac{2}{q}\right]^{1/2}h'(x) , \quad (3.6)$$

$$h''(x)+\frac{4-q}{q}h'(x) =\left[\frac{2}{q}\right]^{1/2}\left[4\phi'(x)+\frac{2(6-q)}{q}\phi(x)\right] , \quad (3.7)$$

where a prime denotes a derivative with respect to x and we have defined

$$f(x)=\frac{k^2}{a_0^2}M^{-4/q}e^{-2(2-q)x/q} . \quad (3.8)$$

Using Eq. (3.7), h' may be eliminated from Eq. (3.6). This results in

$$\phi'''+\frac{2(5-q)}{q}\phi''+\left[\frac{1}{q^2}(24-6q-q^2)+f(x)\right]\phi' +\left[\frac{2}{q^2}(6-q)(2-q)+f(x)\right]\phi=0 . \quad (3.9)$$

Since we have differentiated Eq. (3.6) to derive this equation, we must check whether all solutions of Eq. (3.9) satisfy Eq. (3.6).

The general solution of this equation is

$$\phi(x)=c_2e^{-x}+\left[\frac{q}{2-q}\right]\lambda^{-3/2}e^{-2x/q} \times[c_+G_1(x)+c_-G_2(x)] , \quad (3.10)$$

where we have defined

$$G_1(x)=H^{(1)}_{\nu-1}(\lambda e^{-(2-q)x/q})S_{3/2,\nu}(\lambda e^{-(2-q)x/q}) -(\nu+\tfrac{1}{2})H^{(1)}_{\nu}(\lambda e^{-(2-q)x/q}) \times S_{1/2,\nu-1}(\lambda e^{-(2-q)x/q}) ,$$

$$G_2(x)=H^{(2)}_{\nu-1}(\lambda e^{-(2-q)x/q})S_{3/2,\nu}(\lambda e^{-(2-q)x/q}) -(\nu+\tfrac{1}{2})H^{(2)}_{\nu}(\lambda e^{-(2-q)x/q}) \times S_{1/2,\nu-1}(\lambda e^{-(2-q)x/q})$$

(the normal modes have been chosen so that $G_2(x)=[G_1(x)]^*$). Here $H^{(i)}_{\mu}$ are Hankel functions and $S_{\mu,\nu}$ are Lommel functions, $c_2,c_\pm$ are $\mathbf{k}$-dependent constants of integration and

$$\nu=\frac{1}{2}\left[\frac{2+q}{2-q}\right], \quad \lambda=\frac{q}{2-q}\frac{k}{a_0}M^{-2/q} .$$

The term proportional to c_2 in Eq. (3.10) corresponds to the time-translation-invariant solution (Sec. III of Ref. [7]). The terms proportional to $c_\pm$ are linear combinations of "isocurvature" and adiabatic perturbations—as will become clear when we analyze the long-wavelength asymptotics of Eq. (3.10) [30].

We note, from Eq. (2.5), that $\dot{\Phi}_0\propto e^{-x}$ so the time-translation solution in Eq. (3.10) satisfies $\phi|_{c_2}\propto\dot{\Phi}_0$; furthermore, the $c_\pm$ terms in Eq. (3.10) do not seem to obey a similar equation. We shall see, in Sec. III B, that the time-translation solution is not gauge invariant; we shall also see, in Sec. III C, that it does not seem possible to find an initial condition which determines the corresponding constant of integration c_2. Although it might seem that some versions of the usual derivation of the energy-density irregularity power spectrum (in the exponential expansion inflation model) make use of the time-translation solution, when taken in the context of a careful definition of the hypersurfaces involved, these derivations are gauge invariant [31,18].

Using Eq. (3.10), we find that the general solution of Eq. (3.7) is given by

$$h(x)=c_1-3\left[\frac{2}{q}\right]^{1/2}c_2e^{-x}+e^{-(10-3q)x/(2q)}[c_+F_1(x)+c_-F_2(x)]+c_5e^{-(4-q)x/q}\,, \tag{3.11}$$

where c_1 and c_5 are **k**-dependent constants of integration (this is not the general solution when $q=4$),

$$F_1(x)=\left[\frac{2}{q}\right]^{1/2}\left\{\left[-\frac{3q}{2-q}\lambda^{-2}e^{2(2-q)x/q}-\frac{q}{2}\left[\frac{2-q}{4-q}\right]\right]\lambda^{1/2}e^{-(2-q)x/(2q)}G_1(x)+\frac{q}{2}\left[\frac{2-q}{4-q}\right]\lambda e^{-(2-q)x/q}H^{(1)}_{\nu-1}(\lambda e^{-(2-q)x/q})-\frac{q}{4-q}H^{(1)}_{\nu}(\lambda e^{-(2-q)x/q})-\frac{2q}{2-q}\left[\frac{6-q}{4-q}\right]\lambda^{-1}e^{(2-q)x/q}H^{(1)}_{\nu+1}(\lambda e^{-(2-q)x/q})\right\},$$

and $F_2(x)=[F_1(x)]^*$. The solution proportional to c_1 in Eq. (3.11) corresponds to the arbitrariness in rescaling a_0. The c_2 term is the solution corresponding to time-translation invariance while the $c_\pm$ terms are linear combinations of "isocurvature" and adiabatic perturbations. We shall see that c_5 must be related to c_2.

Focusing on a plane wave with propagation vector **k** along the x^1 axis and denoting the transverse directions by $x^I(I=2,3)$, we find that Eqs. (3.3) and (3.4) reduce to

$$h'_{22}(x)+h'_{33}(x)=\left[\frac{8}{q}\right]^{1/2}\phi(x)\,, \tag{3.12}$$

$$h'_{I1}(x)=0\,, \tag{3.13}$$

$$h''_{22}(x)+h''_{33}(x)+\frac{6-q}{q}[h'_{22}(x)+h'_{33}(x)]=\left[\frac{8}{q}\right]^{1/2}\left[\phi'(x)+\frac{6-q}{q}\phi(x)\right], \tag{3.14}$$

$$\frac{k^2}{a^2}[h_{22}(x)+h_{33}(x)]=-\frac{2}{q}e^{-2x}\left[2h'(x)+(2q)^{1/2}\phi'(x)-(6-q)\left[\frac{2}{q}\right]^{1/2}\phi(x)\right], \tag{3.15}$$

$$e^{2x}\frac{k^2}{a^2}h_+(x)+\frac{6-q}{q}h'_+(x)+h''_+(x)=0\,, \tag{3.16}$$

$$e^{2x}\frac{k^2}{a^2}h_\times(x)+\frac{6-q}{q}h'_\times(x)+h''_\times(x)=0\,, \tag{3.17}$$

$$h''_{I1}(x)+\frac{6-q}{q}h'_{I1}(x)=0\,, \tag{3.18}$$

here $h_+=h_{22}-h_{33}$ and $h_\times=h_{23}+h_{32}$ are the two physical degrees of freedom of the graviton and we have used $h_{ij}=h_{ji}$ and Eq. (3.2) to simplify some of these equations. Equations (3.14) and (3.15) describe the behavior of the induced perturbations in the curvature of spatial hypersurfaces, Eqs. (3.16) and (3.17) are the graviton equations of motion and Eq. (3.18) is the equation of motion for the h_{I1} components of the metric perturbations. Equations (3.12) and (3.13) are constraint equations.

The general solution of Eq. (3.14) is

$$h_{22}(x)+h_{33}(x)=-\left[\frac{8}{q}\right]^{1/2}c_2e^{-x}+e^{-(10-3q)x/(2q)}[c_+\tilde{F}_1(x)+c_-\tilde{F}_2(x)]+c_7+c_8e^{-(6-q)x/q}\,, \tag{3.19}$$

where c_7 and c_8 are **k**-dependent constants of integration (we shall see that they must vanish),

$$\tilde{F}_1(x)=-\left[\frac{q}{2-q}\right]\left[\frac{8}{q}\right]^{1/2}[\lambda^{-3/2}e^{3(2-q)x/(2q)}G_1(x)+\lambda^{-1}e^{(2-q)x/q}H^{(1)}_{\nu+1}(\lambda e^{-(2-q)x/q})]$$

and $\tilde{F}_2(x)=[\tilde{F}_1(x)]^*$. The c_2 term in Eq. (3.19) corresponds to the time-translation-invariant solution. The $c_\pm$ terms correspond to adiabatic fluctuations [we shall see when we analyze the long-wavelength asymptotics of Eq. (3.19) that one linear combination of the $c_\pm$ terms present in the scalar field solution, Eq. (3.10), is subdominant in the large-scale asymptotic expansion of the solution for $h_{22}+h_{33}$, Eq. (3.19)—this is the "isocurvature" or longitudinal peculiar velocity solution].

Using Eq. (3.19), we find, for the left-hand side of Eq. (3.12),

$$h'_{22}(x)+h'_{33}(x)=\left[\frac{8}{q}\right]^{1/2}c_2e^{-x}+\left[\frac{q}{2-q}\right]\left[\frac{8}{q}\right]^{1/2}\lambda^{-3/2}e^{-2x/q}[c_+G_1(x)+c_-G_2(x)]-\left[\frac{6-q}{q}\right]c_8e^{-(6-q)x/q}, \tag{3.20}$$

while from Eq. (3.10) we see that the right-hand side of Eq. (3.12) is given by

$$\left[\frac{8}{q}\right]^{1/2}\phi(x)=\left[\frac{8}{q}\right]^{1/2}c_2e^{-x}+\left[\frac{q}{2-q}\right]\left[\frac{8}{q}\right]^{1/2}\lambda^{-3/2}e^{-2x/q}[c_+G_1(x)+c_-G_2(x)]\ ; \tag{3.21}$$

to satisfy Eq. (3.12) we must require that c_8 vanish.

From Eq. (3.19) we find that the left-hand side of Eq. (3.15) reduces to

$$\begin{aligned}\frac{k^2}{a^2}[h_{22}(x)+h_{33}(x)]=&-\left[\frac{2-q}{q}\right]^2\left[\frac{8}{q}\right]^{1/2}\lambda^2c_2e^{-(4+q)x/q}-\left[\frac{2-q}{q}\right]\left[\frac{8}{q}\right]^{1/2}\lambda^{1/2}e^{-6x/q}[c_+G_1(x)+c_-G_2(x)]\\&-\left[\frac{2-q}{q}\right]\left[\frac{8}{q}\right]^{1/2}\lambda e^{-(14-q)x/(2q)}[c_+H^{(1)}_{\nu+1}(\lambda e^{-(2-q)x/q})+c_-H^{(2)}_{\nu+1}(\lambda e^{-(2-q)x/q})]\\&+\left[\frac{2-q}{q}\right]^2\lambda^2c_7e^{-4x/q}+\left[\frac{2-q}{q}\right]^2\lambda^2c_8e^{-(10-q)x/q},\end{aligned} \tag{3.22}$$

while the right-hand side of Eq. (3.15) is given by

$$\begin{aligned}\frac{-2}{q}e^{-2x}&\left[2h'(x)+(2q)^{1/2}\phi'(x)-(6-q)\left[\frac{2}{q}\right]^{1/2}\phi(x)\right]\\&=\frac{4}{q}\left[\frac{4-q}{q}\right]c_5e^{-(4+q)x/q}-\left[\frac{2-q}{q}\right]\left[\frac{8}{q}\right]^{1/2}\lambda^{1/2}e^{-6x/q}[c_+G_1(x)+c_-G_2(x)]\\&\quad-\left[\frac{2-q}{q}\right]\left[\frac{8}{q}\right]^{1/2}\lambda e^{-(14-q)x/(2q)}[c_+H^{(1)}_{\nu+1}(\lambda e^{-(2-q)x/q})+c_-H^{(2)}_{\nu+1}(\lambda e^{-(2-q)x/q})],\end{aligned} \tag{3.23}$$

where we have used Eqs. (3.10) and (3.11). To satisfy Eq. (3.15), we must require that c_7 and c_8 vanish and that c_5 be related to c_2 through

$$c_5=-\left[\frac{2}{q}\right]^{1/2}\frac{(2-q)^2}{2(4-q)}\lambda^2c_2\ . \tag{3.24}$$

Using Eq. (3.24) it may be established that Eqs. (3.10) and (3.11) satisfy Eqs. (3.6) and (3.7).

The solution of Eqs. (3.13) and (3.18) is given by

$$h_{I1}(x)=c_{I,1}\ , \tag{3.25}$$

where $c_{I,1}$ are **k**-dependent constants of integration. We shall see that the constants $c_{I,1}$ may be removed by a gauge transformation. As discussed in more detail below, we find, on comparing this equation to Eqs. (37) and (38) of Ref. [14], that these scalar field inflationary models will not generate transverse peculiar velocity perturbations in the large time universe. This will be true for all scalar-field-dominated inflation models, [32], so such models cannot be used to describe the very early Universe in primeval turbulence scenarios for structure formation.

The solutions of the graviton equations of motion, Eqs. (3.16) and (3.17), are given by

$$\begin{aligned}h_+(x)=e^{-(6-q)x/(2q)}&\\\times[c_{+1}&H^{(1)}_{\nu+1}(\lambda e^{-(2-q)x/q})\\&+c_{+2}H^{(2)}_{\nu+1}(\lambda e^{-(2-q)x/q})]\ ,\end{aligned} \tag{3.26}$$

$$\begin{aligned}h_\times(x)=e^{-(6-q)x/(2q)}&\\\times[c_{\times1}&H^{(1)}_{\nu+1}(\lambda e^{-(2-q)x/q})\\&+c_{\times2}H^{(2)}_{\nu+1}(\lambda e^{-(2-q)x/q})]\ ,\end{aligned} \tag{3.27}$$

where c_{+1}, c_{+2}, $c_{\times1}$, and $c_{\times2}$ are **k**-dependent constants of integration. These expressions coincide with the solutions for the gravitons in the ideal fluid model (Eqs. (39)

and (40) of Ref. [14]), up to an unimportant sign difference in the definition of λ, which arises from the slightly different conventions used in Ref. [14]. This is reasonable since gravitons are not sensitive to the nature of energy density driving the expansion—they respond to an ideal fluid or to a homogeneous scalar field, with the same background equation of state as that of the ideal fluid, in the same manner.

In summary, the solutions of Eqs. (3.1)–(3.4) are given by Eqs. (3.10), (3.11), (3.19), and (3.25)–(3.27) where c_7 and c_8 vanish and c_5 is related to c_2 through Eq. (3.24). This leaves ten **k**-dependent constants of integration (some of which describe gauge-dependent solutions) that must be determined from initial conditions. These are c_1 which describes the solution corresponding to the rescaling of a_0, c_2 which describes the time-translation-invariant solution, $c_\pm$ which describe adiabatic and "isocurvature" fluctuations, $c_{I,1}$ which describe the h_{I1} components of the metric perturbations, and c_{+1}, c_{+2}, $c_{\times 1}$, and $c_{\times 2}$ which describe the gravitons.

The metric perturbations $h_{22}+h_{33}$ measure perturbations in the curvature of spatial hypersurfaces. The Ricci tensor of the three-dimensional spatial hypersurface is given by

$$^{(3)}R_{ij}=\tfrac{1}{2}(h_{ij,kk}+h_{,ij}-h_{ik,kj}-h_{jk,ki}) , \tag{3.28}$$

so we have, for the induced spatial scalar curvature,

$$\begin{aligned} ^{(3)}R &= \frac{k^2}{a^2}[h_{22}(x)+h_{33}(x)] \\ &= -\frac{2}{q}e^{-2x}\left[2h'(x)+(2q)^{1/2}\phi'(x)\right. \\ &\qquad \left. -(6-q)\left[\frac{2}{q}\right]^{1/2}\phi(x)\right] , \end{aligned} \tag{3.29}$$

where we have made use of Eq. (3.15). Using Eq. (3.23) we find that this reduces to

$$\begin{aligned} ^{(3)}R = e^{-4x/q}\Bigg[&-\left[\frac{2-q}{q}\right]\left[\frac{8}{q}\right]^{1/2}\lambda^{1/2}e^{-2x/q}[c_+G_1(x)+c_-G_2(x)] \\ &-\left[\frac{2-q}{q}\right]\left[\frac{8}{q}\right]^{1/2}\lambda e^{-(6-q)x/(2q)} \\ &\times[c_+H^{(1)}_{\nu+1}(\lambda e^{-(2-q)x/q})+c_-H^{(2)}_{\nu+1}(\lambda e^{-(2-q)x/q})]+\frac{4}{q}\left[\frac{4-q}{q}\right]c_5e^{-x}\Bigg] . \end{aligned} \tag{3.30}$$

We shall see that the term proportional to c_5 in this equation is a gauge-dependent solution, while the linear combination of terms proportional to $c_\pm$ is, by definition, the adiabatic perturbation.

An alternative way of determining the spectrum of energy-density perturbations would be to evaluate the linearized stress tensor. From Eqs. (2.3) and (7.10) of Ref. [7] we have that the fractional perturbation in energy density corresponding to the scalar field perturbation ϕ is given by

$$\delta=\frac{\delta_{\rho\Phi}}{\rho\Phi}=2\frac{\dot\Phi_0\dot\phi+V'(\Phi_0)\phi/2}{(\dot\Phi_0)^2+V(\Phi_0)} , \tag{3.31}$$

or as a function of the variable x,

$$\delta(x)=\frac{1}{3}\left[\frac{q}{2}\right]^{3/2}\left[\phi'(x)-\frac{6-q}{q}\phi(x)\right] . \tag{3.32}$$

Using Eq. (3.10) this results in

$$\begin{aligned} \delta(x)=\frac{1}{3}\left[\frac{q}{2}\right]^{3/2}\Bigg[&-\frac{6}{q}c_2e^{-x}-\left[\frac{6}{2-q}\right]\lambda^{-3/2}e^{-2x/q}[c_+G_1(x)+c_-G_2(x)] \\ &+e^{-(10-3q)x/(2q)}[c_+H^{(1)}_\nu(\lambda e^{-(2-q)x/q})+c_-H^{(2)}_\nu(\lambda e^{-(2-q)x/q})]\Bigg] . \end{aligned} \tag{3.33}$$

We shall see that the term proportional to c_2 in this equation may be removed by a gauge transformation.

B. Remnants of general coordinate invariance

Consistent with the choice of synchronous gauge, one may still perform the following time-independent gauge transformations (i.e., the corresponding gauge parameters are time independent) on the metric perturbations without altering the physics:

$$\begin{aligned}
\delta h_{11}(x,k) &= -f^0(k)\frac{2(2-q)^2}{q(4-q)}\lambda^2 e^{-(4-q)x/q} \\
&\quad -2ikw_1(k)-f^0(k)\frac{4}{q}e^{-x} , \\
\delta h_{1I}(x,k) &= -ikw_I(k) , \\
\delta h_{IJ}(x,k) &= -f^0(k)\frac{4}{q}e^{-x}\delta_{IJ} , \\
\delta h_{+}(x,k) &= 0 = \delta h_{\times}(x,k) , \\
\delta h(x,k) &= -f^0(k)\frac{2(2-q)^2}{q(4-q)}\lambda^2 e^{-(4-q)x/q} \\
&\quad -2ikw_1(k)-f^0(k)\frac{12}{q}e^{-x} ,
\end{aligned} \tag{3.34}$$

where f^0 and w_i do not depend on the variable x. (We note that the two transverse degrees of freedom of the graviton, which are the only propagating modes, are gauge invariant.) From the definition of the stress tensor, Eqs. (7.7)–(7.9) of Ref. [7], and its transformation properties, we find

$$\begin{aligned}
\delta(\delta(x,k)) &= -2f^0(k)e^{-x} , \\
\delta\phi(x,k) &= f^0(k)\left[\frac{8}{q}\right]^{1/2} e^{-x} ,
\end{aligned} \tag{3.35}$$

where $\delta(x,k)$ is defined by Eq. (3.31) [we hope that the notation $\delta(\delta(x,k))$ does not lead to undue confusion].

It is easily verified that the choice $f^0(k)=-(c_2/2)(q/2)^{1/2}$ allows us to set to zero the terms proportional to c_2 and c_5 in Eqs. (3.10), (3.11) [here we use Eq. (3.24) to relate c_5 to c_2], (3.19), and (3.33), while the choice $w_1(k)=c_1/(2ik)$ allows us to remove the solution corresponding to the rescaling of a_0 from Eq. (3.11) and the choice $w_I(k)=c_{I,1}/(ik)$ allows us to set the h_{I1} part of the metric perturbations to zero. This leaves six physical solutions which are parametrized by the constants $c_\pm$, c_{+1}, c_{+2}, $c_{\times 1}$, and $c_{\times 2}$; these constants must be determined from initial conditions appropriate to the quantum mechanics of inflation [8,9].

It is interesting to contrast this analysis of the scalar field model with that of the ideal fluid model [14]. In the ideal fluid model one found that eight initial conditions were needed to determine the eight independent physical solutions which correspond to an adiabatic fluctuation, a longitudinal peculiar velocity perturbation (the "isocurvature" perturbation), two transverse peculiar velocity perturbations and four graviton solutions. In the scalar field model there are only six independent physical solutions, four corresponding to the gravitational-wave perturbations and two describing the scalar field perturbation. We note that in the scalar field model, the two scalar field solutions play the role of the adiabatic fluctuation and the longitudinal peculiar velocity perturbation in the ideal fluid model. Thus, on large scales (at late times) one would expect that the linear combination corresponding to the peculiar velocity perturbation will not perturb the curvature of spatial hypersurfaces [14].

It is also interesting to note (as we show below) that scalar-field-dominated inflation models provide initial conditions (at second Hubble radius crossing) that require that the transverse peculiar velocity perturbations in the ideal fluid (which is used to describe the large time universe) vanish.

The induced spatial curvature is a measure of the magnitude of adiabatic energy-density fluctuations; however, the expression in Eq. (3.29) is not gauge invariant. A gauge-invariant characterization of energy-density fluctuations [9,14] is

$$\Delta\rho = a^3\left[{}^{(3)}R + \frac{\dot a}{a}\frac{\partial}{\partial t}\left[\frac{3}{\nabla^2}h_{ij,ij}-h\right]\right] , \tag{3.36}$$

where $\nabla^2=\delta_{ij}\partial_i\partial_j$. Using Eqs. (3.11), (3.20), and (3.30) we find that this reduces to

$$\begin{aligned}
\Delta\rho = -\left[\frac{8}{q}\right]^{1/2} & a_0^3 M^{6/q} e^{(2-q)x/(2q)} \\
&\times[c_+ H_\nu^{(1)}(\lambda e^{-(2-q)x/q}) \\
&\quad + c_- H_\nu^{(2)}(\lambda e^{-(2-q)x/q})] ,
\end{aligned} \tag{3.37}$$

here we have set $c_7=0=c_8$. We note that the solution proportional to c_5 in Eq. (3.30), which may be set to zero by a gauge transformation, does not appear in Eq. (3.37).

C. Initial conditions and asymptotics

We have solved the equations of motion for small perturbations about a background FLRW–homogeneous-scalar-field cosmological solution for the exponential-potential scalar field model. To determine the constants of integration $c_\pm$, c_{+1}, c_{+2}, $c_{\times 1}$, and $c_{\times 2}$ (which correspond to gauge-invariant solutions) we study the inflation epoch expressions on small scales (or at early times, $k \gg \dot a$) and require that they reduce to the usual quantum-mechanical "vacuum state" form [8,9]. In this section we also derive asymptotic forms of the inflation epoch solutions valid at late times (on large scales)—in this limit we shall disregard some decaying (in time) terms for some of the fields.

It is useful to note that

$$\lambda e^{-(2-q)x/q} = \left[\frac{2}{2-q}\right]\frac{k}{aH} = -k\int^t \frac{dt'}{a(t')} = -k\tau , \tag{3.38}$$

where $H=(\dot a/a)$ is the Hubble parameter and conformal time τ is defined by

$$\tau=\int^t \frac{dt'}{a(t')} \ . \tag{3.39}$$

If we replace the constants of integration $c_\pm$ by the x-independent constants $\tilde{c}_\pm$ defined through

$$c_\pm=\tilde{c}_\pm \left[\frac{16\pi}{m_P^2}\right]^{1/2} \frac{k}{2} \left[\frac{q\pi}{2-q}\right]^{1/2} (a_0 M^{2/q})^{-5/2} \times e^{\pm i(\nu-1/2)\pi/2} \ , \tag{3.40}$$

then in the short-wavelength (early time) limit (i.e., for fluctuations well inside the Hubble radius for which $k \to \infty$) Eq. (3.10) reduces to

$$\phi(\mathbf{k},t)=\left[\frac{16\pi}{m_P^2}\right]^{1/2} \frac{1}{\sqrt{2ka^2}} \times \left[\tilde{c}_+ \exp\left[-ik\int^t \frac{dt'}{a(t')}\right] + \tilde{c}_- \exp\left[+ik\int^t \frac{dt'}{a(t')}\right]\right] . \tag{3.41}$$

We note that the prefactor on the right-hand side of this equation depends on the scale factor $a(t)$ in the manner expected for a relativistic scalar field. Even though ϕ is really a massive scalar field, for large enough spatial momentum its dispersion relation is that of a massless field, $\omega=|\mathbf{k}|$. The first factor in the prefactor is a consequence of the unconventional normalization of the scalar field action, Eq. (2.1). [In deriving Eq. (3.41) from Eq. (3.10) we have dropped the solution proportional to c_2; it is interesting to note that quantum mechanics does not seem to provide an initial condition to determine this constant of integration, which corresponds to a gauge-dependent solution—this also seems to be true for other gauge-dependent constants of integration.] In terms of the dimensionless scalar field $\chi(\mathbf{k},t)=a(t)\phi(\mathbf{k},t)$ Eq. (3.41) reduces to

$$\chi(\mathbf{k},\tau)=\left[\frac{16\pi}{m_P^2}\right]^{1/2} \frac{1}{\sqrt{2k}}(\tilde{c}_+ e^{-ik\tau}+\tilde{c}_- e^{+ik\tau}) \ , \tag{3.42}$$

where we have replaced t by conformal time τ. Since large spatial momentum modes sample regions of spacetime that are essentially flat (these momentum modes are indistinguishable from the corresponding momentum modes of a conformally coupled scalar field since, in this limit, the spatial gradient term in the scalar field action is far more important than the $R\phi^2$ conformal coupling term) the appropriate initial conditions (for the scalar field vacuum state) are derived by requiring that the momentum mode in Eq. (3.42) reduce to the corresponding expansion for the harmonic-oscillator "vacuum state" in a conformally flat spacetime [8,9]. This follows from the fact that a suitably rescaled, conformally coupled, scalar field, in a conformally flat spacetime, does not recognize as special the length scale set by the spacetime curvature [8]. Comparing to the Fourier expansion of Sec. III A we see that relativistic covariance requires that we choose

$$\tilde{c}_+=1 \quad \text{and} \quad \tilde{c}_-=0 \ , \tag{3.43}$$

where we have made the assumption that $\tilde{c}_\pm$ do not depend on k; this is equivalent to the initial condition of Ref. [8] for the scalar field ground-state wave functional.

Replacing the constants of integration c_{i1}, c_{i2} (where $i=+,\times$) by the x-independent constants $\tilde{c}_{i1}, \tilde{c}_{i2}$ where

$$\begin{aligned} c_{i1}&=\tilde{c}_{i1}\left[\frac{16\pi}{m_P^2}\right]^{1/2}\left[\frac{q\pi}{2-q}\right]^{1/2}(a_0 M^{2/q})^{-3/2} \times e^{+i(\nu+3/2)\pi/2} \ , \\ c_{i2}&=\tilde{c}_{i2}\left[\frac{16\pi}{m_P^2}\right]^{1/2}\left[\frac{q\pi}{2-q}\right]^{1/2}(a_0 M^{2/q})^{-3/2} \times e^{-i(\nu+3/2)\pi/2} \ , \end{aligned} \tag{3.44}$$

we find that at early times (on small scales) the expression for the graviton, Eqs. (3.26) and (3.27), become

$$h_i(\mathbf{k},t)=\left[\frac{16\pi}{m_P^2}\right]^{1/2} \frac{2}{\sqrt{2ka^2}} \times \left[\tilde{c}_{i1}\exp\left[-ik\int^t \frac{dt'}{a(t')}\right] + \tilde{c}_{i2}\exp\left[+ik\int^t \frac{dt'}{a(t')}\right]\right] ; \tag{3.45}$$

the appropriate "vacuum state" initial conditions are $\tilde{c}_{i1}=1$ and $\tilde{c}_{i2}=0$. The factor 2 in the numerator of the prefactor of Eq. (3.45) is a consequence of the extra factor of $\frac{1}{4}$ in the graviton kinetic term of the action, Eq. (2.1),

$$\frac{m_P^2}{16\pi}\frac{1}{4}[\tfrac{1}{2}a^3(\dot{h}_\times^2+\dot{h}_+^2)] \ ,$$

which should be contrasted with the scalar field kinetic term,

$$\frac{m_P^2}{16\pi}(\tfrac{1}{2}a^3\dot{\phi}^2) \ .$$

We note that the prefactor on the right-hand side of Eq. (3.45) depends on $a(t)$ in the expected manner.

Scalar field fluctuations outside the Hubble radius (i.e., in the long-wavelength or large-time limit when $k \ll \dot{a}$) are described by

$$\phi(\mathbf{k},x)=\tilde{c}_A k^{1-\nu} e^{-2(2-q)x/q} + \tilde{c}_I k^{1+\nu} e^{-(6-q)x/q} + \cdots \ , \tag{3.46}$$

where

$$\tilde{c}_A=\frac{\pi}{m_P}\left[\frac{q}{2-q}\right]^{3/2-\nu}(a_0 M^{2/q})^{\nu-5/2}\frac{2^{\nu+1}\csc(\nu\pi)}{\Gamma(1-\nu)(5/2-\nu)} \times(\tilde{c}_+ e^{i(\nu+1/2)\pi/2}+\tilde{c}_- e^{-i(\nu+1/2)\pi/2}) \ ,$$

$$\tilde{c}_I=\frac{\pi}{m_P}\left[\frac{q}{2-q}\right]^{3/2+\nu}(a_0 M^{2/q})^{-\nu-5/2}\frac{2^{-\nu+1}\csc(\nu\pi)}{\Gamma(1+\nu)(5/2+\nu)} \times(\tilde{c}_+ e^{-i(\nu+3/2)\pi/2}+\tilde{c}_- e^{i(\nu+3/2)\pi/2}) \ .$$

Again, we have dropped the term proportional to c_2 in Eq. (3.10) [there is, however, a term proportional to $k^{-3/2}e^{-x}$ in Eq. (3.46) which has the same time dependence as the c_2 term in Eq. (3.10) and is the slowest decaying term for some range of q]. It is pleasing to note that the time dependence of this result agrees with what was found in Sec. III of Ref. [7]. As discussed there, when $q<2$ the scalar field perturbation has no growing mode; the solution describes a stable fixed point—we emphasize, however, that scalar field irregularities, like most other perturbations, do evolve (in, for all practical purposes, a gauge-invariant fashion) outside the Hubble radius. The term proportional to $\tilde{c}_A$ is the adiabatic solution while that proportional to $\tilde{c}_I$ is the "isocurvature" solution—it does not contribute to spatial curvature perturbations on large scales (although it does perturb spatial curvature on small scales).

At large times the expressions for the graviton, Eqs. (3.26) and (3.27), reduce to

$$h_i(\mathbf{k},x)=\bar{c}_i^1 k^{-\nu-1}+\bar{c}_i^2 k^{\nu+1}e^{-(6-q)x/q}+\cdots\,, \tag{3.47}$$

where $i=+,\times$ and we have defined

$$\bar{c}_i^1=\frac{\pi}{m_P}2^{3+\nu}\left[\frac{q}{2-q}\right]^{-1/2-\nu}(a_0M^{2/q})^{\nu-1/2}\frac{\csc(\nu\pi)}{\Gamma(-\nu)}\times(\tilde{c}_{i1}e^{i(\nu-3/2)\pi/2}+\tilde{c}_{i2}e^{-i(\nu-3/2)\pi/2})\,,$$

$$\bar{c}_i^2=\frac{\pi}{m_P}2^{1-\nu}\left[\frac{q}{2-q}\right]^{3/2+\nu}(a_0M^{2/q})^{-\nu-5/2}\frac{\csc(\nu\pi)}{\Gamma(2+\nu)}\times(\tilde{c}_{i1}e^{-i(\nu+3/2)\pi/2}+\tilde{c}_{i2}e^{i(\nu+3/2)\pi/2})\,.$$

[The time independence of the leading term in Eq. (3.47) means that graviton fluctuations do not evolve outside the Hubble radius.] Comparing the two terms of Eq. (3.47) to those of Eq. (61) of Ref. [14] we see, as expected, that both the time (t) dependence and the spatial momentum dependence of these solutions are the same.

At early times we find that curvature of spatial hypersurfaces, Eq. (3.30), is given by (we have set $c_5=0$)

$${}^{(3)}R(\mathbf{k},t)=[a(t)]^{-(4+q)/2}\times\left[\hat{c}_+\exp\left[-ik\int^t\frac{dt'}{a(t')}\right]+\hat{c}_-\exp\left[+ik\int^t\frac{dt'}{a(t')}\right]\right]\,, \tag{3.48}$$

where we have defined

$$\hat{c}_\pm=2\tilde{c}_\pm\left[\frac{16\pi}{m_P^2}\right]^{1/2}\left[\frac{k}{q}\right]^{1/2}(a_0M^{2/q})^{q/2}e^{\pm i\pi/2}\,.$$

Expanding Eq. (3.30) for small k, i.e., for fluctuations at large times, we find

$${}^{(3)}R(\mathbf{k},x)=e^{-4x/q}[\hat{c}_A k^{1-\nu}+\hat{c}_2k^{3-\nu}e^{-2(2-q)x/q}+\hat{c}_Ik^{3+\nu}e^{-(6-q)x/q}+\cdots]\,, \tag{3.49}$$

where

$$\hat{c}_A=-\frac{\pi}{m_P}\left[\frac{q}{2-q}\right]^{-1/2-\nu}(a_0M^{2/q})^{\nu-5/2}\frac{2^{7/2+\nu}\csc(\nu\pi)}{\sqrt{q}\,\Gamma(-\nu)}(\tilde{c}_+e^{i(\nu+1/2)\pi/2}+\tilde{c}_-e^{-i(\nu+1/2)\pi/2})\,,$$

$$\hat{c}_2=\frac{\pi}{m_P}\left[\frac{q}{2-q}\right]^{5/2-\nu}(a_0M^{2/q})^{\nu-9/2}\frac{q^{-1/2}2^{3/2+\nu}\csc(\nu\pi)}{\Gamma(1-\nu)(5/2-\nu)}(\tilde{c}_+e^{i(\nu-3/2)\pi/2}+\tilde{c}_-e^{-i(\nu-3/2)\pi/2})\,,$$

$$\hat{c}_I=\frac{\pi}{m_P}\left[\frac{q}{2-q}\right]^{5/2+\nu}(a_0M^{2/q})^{-\nu-9/2}\frac{q^{-1/2}2^{3/2-\nu}\csc(\nu\pi)}{\Gamma(2+\nu)(5/2+\nu)}(\tilde{c}_+e^{-i(\nu-1/2)\pi/2}+\tilde{c}_-e^{i(\nu-1/2)\pi/2})\,.$$

Here, the dominant term (that proportional to $\hat{c}_A$) is the adiabatic solution, while the "isocurvature" or longitudinal peculiar velocity solution (the term proportional to $\hat{c}_I$) is subdominant. As with Eq. (3.46), we have dropped a term proportional to $k^{1/2}e^{-x}$ from within the square brackets; for some range of q this terms decays slower than the $\hat{c}_2$ and $\hat{c}_I$ terms in Eq. (3.49). From Eq. (3.49) we see that the leading term in the dimensionless quantity $a^2\,{}^{(3)}R$ is time independent—perturbations outside the Hubble radius in the curvature of the spatial hypersurface corresponding to adiabatic perturbations evolve in the expected fashion.

As noted in Sec. III of Ref. [7], in this model (for $q<2$) the scalar field perturbations, Eq. (3.46), do not grow even in the presence of nonzero space curvature fluctuations, Eq. (3.49). This should be contrasted with the behavior of the adiabatic solution in the ideal-fluid-dominated cosmological model where a nonzero space curvature perturbation is always accompanied by a growing fractional energy-density irregularity.

For small scale fluctuations, we find that δ, Eq. (3.33), is given by

$$\delta(\mathbf{k},t)=[a(t)]^{-(4-q)/2}\times\left[\check{c}_+\exp\left[-ik\int^t\frac{dt'}{a(t')}\right]+\check{c}_-\exp\left[+ik\int^t\frac{dt'}{a(t')}\right]\right]\,, \tag{3.50}$$

where we have set $c_2=0$ and

$$\check{c}_{\pm}=\tilde{c}_{\pm}\left[\frac{16\pi}{m_P^2}\right]^{1/2}\sqrt{qk}\,\frac{q}{12}(a_0M^{2/q})^{-q/2}e^{\mp i\pi/2}\,.$$

From Eq. (3.33) we find, in the large-time limit,

$$\delta(\mathbf{k},x)=\check{c}_A k^{1-\nu}e^{-2(2-q)x/q}+\check{c}_I k^{1+\nu}e^{-(6-q)x/q}+\cdots\,, \tag{3.51}$$

where we have defined

$$\check{c}_A=-\frac{1}{3}(10-3q)\left[\frac{q}{8}\right]^{1/2}\tilde{c}_A\,,$$

$$\check{c}_I=-\frac{1}{3}(6-q)\left[\frac{q}{2}\right]^{1/2}\tilde{c}_I\,,$$

and $\tilde{c}_A$ and $\tilde{c}_I$ are given below Eq. (3.46). Again, we have suppressed a term proportional to $k^{-3/2}e^{-x}$ in Eq. (3.51). It is pleasing to note that the spatial momentum dependence and the time dependence of the terms in Eq. (3.51) agree with those of Eq. (3.46). (We again emphasize that the fractional perturbation in the energy density does evolve outside the Hubble radius.) The term proportional to $\check{c}_A$ is the adiabatic solution (i.e., it depends on the coefficients $\tilde{c}_{\pm}$ in exactly the same way as the dominant term in ${}^{(3)}R$) while that proportional to $\check{c}_I$ is the "isocurvature" solution. As discussed above Eq. (3.50), even though there are nonzero space curvature perturbations, the fractional perturbation in the energy density, Eq. (3.51), decays in time.

It is interesting that higher-order terms in this asymptotic expansion (i.e., terms with higher powers of k in the numerator) grow slower—the next two terms are proportional to

$$k^{3-\nu}e^{-4(2-q)x/q} \quad\text{and}\quad k^{3+\nu}e^{-(10-3q)x/q}\,.$$

These terms are suppressed relative to those of Eq. (3.51) by a factor of $(k/aH)^2$; for inflation models aH is an increasing function of time. This should be compared to the ideal fluid model [14], where aH is a decreasing function of time and hence the higher-order terms in the large-scale asymptotic expansion grow with time (relative to the lower-order terms).

At early times, we find that the gauge-invariant measure of energy-density perturbations, $\Delta\rho$, Eq. (3.37), becomes

$$\Delta\rho(\mathbf{k},t)=[a(t)]^{(2-q)/2}\times\left[\check{c}_+\exp\left[-ik\int^t\frac{dt'}{a(t')}\right]+\check{c}_-\exp\left[+ik\int^t\frac{dt'}{a(t')}\right]\right]\,, \tag{3.52}$$

where

$$\check{c}_{\pm}=2\tilde{c}_{\pm}\left[\frac{16\pi}{m_P^2}\right]^{1/2}\left[\frac{k}{q}\right]^{1/2}(a_0M^{2/q})^{q/2}e^{\pm i\pi/2}\,.$$

While on large scales we find

$$\Delta\rho(\mathbf{k},x)=\check{c}_1k^{1-\nu}e^{2x/q}+\check{c}_2k^{1+\nu}e^{-x}+\cdots\,, \tag{3.53}$$

where we have defined

$$\check{c}_1=\frac{\pi}{m_P}\left[\frac{q}{2-q}\right]^{1/2-\nu}(a_0M^{2/q})^{1/2+\nu}\frac{2^{5/2+\nu}\csc(\nu\pi)}{\sqrt{q}\,\Gamma(1-\nu)}\times(\tilde{c}_+e^{i(\nu+1/2)\pi/2}+\tilde{c}_-e^{-i(\nu+1/2)\pi/2})\,,$$

$$\check{c}_2=\frac{\pi}{m_P}\left[\frac{q}{2-q}\right]^{1/2+\nu}(a_0M^{2/q})^{1/2-\nu}\frac{2^{5/2-\nu}\csc(\nu\pi)}{\sqrt{q}\,\Gamma(1+\nu)}\times(\tilde{c}_+e^{-i(\nu+3/2)\pi/2}+\tilde{c}_-e^{i(\nu+3/2)\pi/2})\,.$$

To compare the expressions derived here to those derived from a purely quantum-mechanical analysis in Ref. [15] it is most convenient to use the momentum-space two-point functions for the invariant measure of energy-density perturbations $\Delta\rho$ and for the graviton polarization h_+. From Eqs. (3.37), (3.40), and (3.43) we find

$$\langle\Delta\rho(k)\Delta\rho(-k)\rangle=\frac{16}{m_P^2}\frac{q\pi^2}{2-q}k^2aH\times H_\nu^{(1)}\{2k/[(2-q)aH]\}\times H_\nu^{(2)}\{2k/[(2-q)aH]\}\,, \tag{3.54}$$

which agrees with the corresponding expression, Eq. (4.31), given in Ref. [15]. While from Eqs. (3.26) and (3.44) we have

$$\langle h_+(k)h_+(-k)\rangle=\frac{16}{m_P^2}\frac{2\pi^2}{2-q}\frac{1}{Ha^3}\times H_{\nu+1}^{(1)}\{2k/[(2-q)aH]\}\times H_{\nu+1}^{(2)}\{2k/[(2-q)aH]\}\,, \tag{3.55}$$

which agrees with Eq. (4.25) of Ref. [15]. This also agrees with the expression for the graviton two-point function given in the first of Refs. [11] [see their Eqs. (2.4) and (2.6a) and the discussion below their Eq. (2.4)] up to unimportant phase factors and a relative factor of 2 (Eq. (3.55) seems to be larger than the expression given in Ref. [11]).

For $q=2\epsilon^2\sim0$ this model reduces to the step function potential model [Eqs. (2.6)–(2.10)] studied in Ref. [9]. In this limit we find that the inflation epoch power spectrum reduces to the scale-invariant form [2]

$$P_\Delta(k,t)=\langle\Delta_\Phi(k,t)\Delta_\Phi(-k,t)\rangle=\frac{8\pi}{m_P^2}\frac{\epsilon^2}{9}\frac{k}{a^4H^2}\,, \tag{3.56}$$

where Δ_Φ is defined in Eq. (4.23) below. [We note that the adiabatic terms in Δ_Φ and δ_Φ are related by $\delta_\Phi=\Delta_\Phi(10-3q)/(4-3q)$. This result is only valid for $0<q<2$—in particular, in the baryon-dominated model (for which $q=3$) we find $\delta_B=\Delta_B$, see the discussion in the paragraph below Eq. (4.46).] For completeness we note that the power spectrum, for perturbations on scales larger than the Hubble scale, is given by

$$P_\Delta(k,t)=\frac{4q}{9m_P^2}\Gamma^2(\nu)(2-q)^{2\nu-1}\frac{1}{a^3H}\left[\frac{k}{aH}\right]^{2(1-\nu)} . \tag{3.57}$$

For the exponential expansion model the inflation epoch graviton two-point function, Eq. (3.55), reduces to

$$\langle h_+(k)h_+(-k)\rangle=\frac{32\pi}{m_P^2}\frac{H^2}{k^3}\left[1+\left[\frac{k}{aH}\right]^2\right] \tag{3.58}$$

[inserting a factor of $16\pi/m_P^2$ in Eq. (5.18) of Ref. [8] we see, as expected in de Sitter spacetime, that the graviton two-point function is exactly a factor of 4 larger than the scalar field perturbation two-point function], while in the general q model, for perturbations on scales larger than the Hubble scale, we find that it is given by

$$\langle h_+(k)h_+(-k)\rangle=\frac{32\pi^2}{m_P^2}(2-q)^{2\nu+1}\frac{\csc^2(\nu\pi)}{\Gamma^2(-\nu)}\frac{1}{Ha^3}\times\left[\frac{k}{aH}\right]^{-2(\nu+1)} , \tag{3.59}$$

which is time independent [see Eq. (3.47)].

In the next section we shall use the explicit expressions for the inhomogeneities along with the inflation epoch constants of integration found here to derive the form of energy-density irregularities in the radiation- and matter-dominated epochs and to thereby determine, in Sec. V, the large-time power spectrum of energy-density irregularities.

IV. LARGE TIME: THEORETICAL CONSIDERATIONS

To be able to contrast the predictions of the exponential-potential inflation model with observations we have to evolve the expressions found in the previous section through the radiation-dominated epoch to the present, matter-dominated, epoch. We shall assume that the scalar field gets massive enough, at the end of inflation, so that it does not significantly influence the large-time evolution of the universe, and that the stress tensor for matter is always dominated by one kind of matter—the scalar field in the scalar-field-dominated epoch, the radiation fluid in the radiation-dominated epoch and the baryon fluid in the matter-dominated epoch (i.e., we only account for perturbations in the dominant form of matter). We shall also assume that the transition from the scalar-field-dominated epoch to the radiation-dominated epoch and the transition from the radiation-dominated epoch to the baryon-dominated epoch occur instantaneously, at different values of the synchronous gauge time (t) in different parts of space, when the local energy density drops to an appropriate critical value; i.e., we model the transitions by requiring that the equation of state (and hence the pressure) charge discontinuously at the spatial hypersurface on which the transition occurs.

That the value of the local energy density determines the hypersurface on which the transition occurs means that this hypersurface of simultaneity will differ from the corresponding synchronous gauge constant time hypersurface. Since the spatial gradients in the local energy density are of first order in the perturbations, the spatially homogeneous local energy-density transition hypersurface and the corresponding synchronous gauge constant time hypersurface only differ by terms that are of first order in the perturbations; i.e., these hypersurfaces coincide to zeroth order in the perturbations [16]. As a result we may match the scale factor and the homogeneous part of the energy density on the synchronous gauge constant time hypersurfaces corresponding to the transition from the scalar-field-dominated epoch to the radiation-dominated epoch (at redshift $z_{R\Phi}$) and the transition from the radiation-dominated epoch to the matter-dominated epoch (at redshift z_{BR}).

Implicit in our assumption that the transition from the scalar-field-dominated epoch to the radiation-dominated epoch occurs instantaneously on a spatially homogeneous energy-density hypersurface are two separate assumptions: that both the time scale on which the scalar field decays to radiation at the end of inflation and the time scale on which the radiation equilibrates (and hence can be treated as an ideal fluid) are much shorter than the time scale on which the perturbations evolve. These assumptions are, of course, slightly unphysical and in reality the scalar-field-radiation transition will be spread out over a time scale that is determined by the microphysics governing the decay of the scalar field to radiation as well as the microphysics governing the subsequent equilibration of radiation [18]; however, the more correct model (in which the time it takes to reheat the Universe is finite) is, at present, analytically intractable. Although the microphysics governing the time scale over which the equation of state changes during the radiation-baryon transition differs from that which governs the time scale for the scalar-field–radiation transition, we shall also model this second transition by a discontinuous change in the pressure. These small-scale approximations should not significantly affect the quantities which we wish to determine; i.e., the large-scale power spectrum of baryon energy-density irregularities and the large-scale power spectrum of gravitons in the large time, matter-dominated universe (the small-scale limit of our expressions are, of course, not physically relevant).

To derive the joining conditions that determine the constants of integration in the expressions for the perturbations during the radiation- and matter-dominated epochs (given in Ref. [14]) from the expressions for the perturbations in the scalar-field-dominated epoch (derived in the previous section) we require that the equations of covariant conservation of stress-energy and Einstein's equations do not become singular on the spatial hypersurfaces of spatially homogeneous local energy density on which the transitions occur [16]. The equations of motion we use are those of a general fluid model which allows for a spacetime-dependent "speed of sound" c_s, [16]. The equations of this model contain, as special cases, the scalar field model equations of motion, Eqs. (3.1)–(3.4), and the ideal fluid (constant c_s) model equations of motion, Eqs. (5)–(9) of Ref. [14] (see Ref. [16]); as a result, the joining conditions derived in this general model apply to more general transitions than those to which we apply them to in this paper.

Once the constants of integration in the expressions for the perturbations in the baryon-dominated epoch have been determined, current observations may be used to constrain the four free parameters of the model, $z_{R\Phi}, z_{BR}$, the present value of the Hubble parameter, $H_{\rm now}$, and the index of the scalar field potential, q. To focus on the characteristics of the inflation epoch we further simplify the model by requiring that the present Universe be baryon dominated and flat (so $\Omega_{\rm now}=1$), that the value of $H_{\rm now}=100h$ km s^{-1} Mpc^{-1} (this determines the present value of the baryon energy density up to a factor of h^2) and that the parameter z_{BR} has the conventional hot big-bang-value ($z_{BR}=4\times 10^4 h^2$)—this leaves the model with two free parameters $z_{R\Phi}$ and q (as well as h).

Determining the constants of integration. We first discuss the joining of the expressions for the spatially homogeneous part of the energy density and the scale factor at the transitions. Since the spatially homogeneous local energy-density transition hypersurface and the corresponding constant time synchronous gauge hypersurface coincide to lowest order in the perturbations, we match the scale factor and the spatially homogeneous part of the energy density on the appropriate constant time synchronous gauge hypersurface [16].

Matching the scale factor at $z_{R\Phi}$, the redshift of equal radiation and homogeneous scalar field energy density (or alternatively at the time $t_{R\Phi}$) we find, in the radiation-dominated epoch,

$$a(t)=a_1[1+M_R(t-t_{R\Phi})]^{1/2} , \tag{4.1}$$

where we have defined

$$a_1=a_0[1+M(t_{R\Phi}-t_0)]^{2/q} ,$$

and

$$M_R=2\left[\frac{8\pi}{3m_P^2}\rho_{bR}^{(1)}\right]^{1/2} .$$

The evolution of the scale factor in the scalar-field-dominated epoch is given in Eq. (2.5) and $\rho_{bR}^{(1)}$ is the value of the spatially homogeneous energy density (in radiation) at the scalar-field–radiation transition when $a=a_1$. Similarly we find, in the baryon-dominated epoch,

$$a(t)=a_2[1+M_B(t-t_{BR})]^{2/3} , \tag{4.2}$$

where t_{BR} is the time of equal baryon and radiation energy density,

$$a_2=a_0[1+M(t_{R\Phi}-t_0)]^{2/q}[1+M_R(t_{BR}-t_{R\Phi})]^{1/2}$$

and

$$M_B=\frac{3}{2}\left[\frac{8\pi}{3m_P^2}\rho_{bB}^{(2)}\right]^{1/2} ,$$

where $\rho_{bB}^{(2)}$ is the value of the energy density (in baryons) at the radiation-matter transition when $a=a_2$.

The redshift in the baryon-dominated epoch is given by

$$1+z=\frac{[1+M_B(t_{\rm now}-t_{BR})]^{2/3}}{[1+M_B(t-t_{BR})]^{2/3}} , \tag{4.3}$$

where $t_{\rm now}$ is the present value of t, while the redshift in the radiation-dominated epoch is

$$1+z=\frac{[1+M_R(t_{BR}-t_{R\Phi})]^{1/2}}{[1+M_R(t-t_{R\Phi})]^{1/2}} \times[1+M_B(t_{\rm now}-t_{BR})]^{2/3} , \tag{4.4}$$

and that in the scalar-field-dominated epoch is given by

$$1+z=\frac{[1+M(t_{R\Phi}-t_0)]^{2/q}}{[1+M(t-t_0)]^{2/q}}[1+M_R(t_{BR}-t_{R\Phi})]^{1/2} \times[1+M_B(t_{\rm now}-t_{BR})]^{2/3} . \tag{4.5}$$

From Eqs. (4.3)–(4.5) it is straightforward to get expressions relating z_{BR} to t_{BR} and $z_{R\Phi}$ to $t_{R\Phi}$.

Matching the scale factors at the transitions ensures that the homogeneous part of the energy density matches, i.e.,

$$\rho_{b\Phi}(t_{R\Phi})=\rho_{bR}(t_{R\Phi}) , \tag{4.6}$$

as well as

$$\rho_{bR}(t_{BR})=\rho_{bB}(t_{BR}) . \tag{4.7}$$

We note that having chosen to use observational evidence to fix z_{BR} (up to a factor of h^2) the numerical values in Eq. (4.7) are fixed, however, since we have not yet constrained $z_{R\Phi}$ (and hence the energy density in the scalar field just before reheating) the numerical values in Eq. (4.6) have not yet been fixed.

To determine the constants of integration in the expressions for the perturbations during the radiation- and matter-dominated epochs we require that the equations of covariant conservation of stress-energy and Einstein's equations on an appropriate spatially homogeneous local energy-density transition hypersurface are free of singularities [16]. (The equations we use are those of the spacetime-dependent "speed of sound" fluid model which is general enough to apply to all the types of matter that we need to consider, Sec. II of Ref. [16].) This allows us to determine a set of joining conditions, on the fields, at the transitions, Sec. III of Ref. [16].

For a transition at a corresponding synchronous gauge time t_{BR}, from a pretransition epoch dominated by a "fluid" denoted by R to a post-transition epoch dominated by a "fluid" denoted by B (where the coefficient in the equation of state and the "speed of sound" of the "fluids" are left unspecified) these joining conditions are

$$\delta_B(t_{BR},\mathbf{x})=\frac{\rho_{bB}(t_{BR})+p_{bB}(t_{BR})}{\rho_{bR}(t_{BR})+p_{bR}(t_{BR})}\delta_R(t_{BR},\mathbf{x}) , \tag{4.8}$$

$$[\rho_{bB}(t_{BR})+p_{bB}(t_{BR})]v_B^i(t_{BR},\mathbf{x})=[\rho_{bR}(t_{BR})+p_{bR}(t_{BR})]v_R^i(t_{BR},\mathbf{x})+\frac{1}{3}\frac{\rho_{bR}(t_{BR})}{a(t_{BR})H(t_{BR})}\frac{p_{bB}(t_{BR})-p_{bR}(t_{BR})}{\rho_{bR}(t_{BR})+p_{bR}(t_{BR})}\partial_i\delta_R(t_{BR},\mathbf{x}) , \tag{4.9}$$

$$h_{22}^{(B)}(t_{BR},\mathbf{k})+h_{33}^{(B)}(t_{BR},\mathbf{k})=h_{22}^{(R)}(t_{BR},\mathbf{k})+h_{33}^{(R)}(t_{BR},\mathbf{k}) , \tag{4.10}$$

$$h_{I1}^{(B)}(t_{BR},\mathbf{k})=h_{I1}^{(R)}(t_{BR},\mathbf{k}) , \tag{4.11}$$

$$\dot h_{22}^{(B)}(t_{BR},\mathbf{x})+\dot h_{33}^{(B)}(t_{BR},\mathbf{x})=\dot h_{22}^{(R)}(t_{BR},\mathbf{x})+\dot h_{33}^{(R)}(t_{BR},\mathbf{x})+2H(t_{BR})\frac{p_{bR}(t_{BR})-p_{bB}(t_{BR})}{\rho_{bR}(t_{BR})+p_{bR}(t_{BR})}\delta_R(t_{BR},\mathbf{x}) , \tag{4.12}$$

$$\dot h_{I1}^{(B)}(t_{BR},\mathbf{x})=\dot h_{I1}^{(R)}(t_{BR},\mathbf{x}) , \tag{4.13}$$

$$h^{(B)}(t_{BR},\mathbf{k})=h^{(R)}(t_{BR},\mathbf{k})-4\left[\frac{k}{a(t_{BR})H(t_{BR})}\right]^2\frac{p_{bR}(t_{BR})-p_{bB}(t_{BR})}{\rho_{bR}(t_{BR})+p_{bR}(t_{BR})}\times\frac{\rho_{bR}(t_{BR})}{\rho_{bR}(t_{BR})-3p_{bR}(t_{BR})}\frac{\rho_{bR}(t_{BR})}{\rho_{bB}(t_{BR})-3p_{bB}(t_{BR})}\delta_R(t_{BR},\mathbf{k}) , \tag{4.14}$$

$$\dot h^{(B)}(t_{BR},\mathbf{k})=\dot h^{(R)}(t_{BR},\mathbf{k})+3H(t_{BR})\frac{p_{bR}(t_{BR})-p_{bB}(t_{BR})}{\rho_{bR}(t_{BR})+p_{bR}(t_{BR})}\delta_R(t_{BR},\mathbf{k}) , \tag{4.15}$$

$$\begin{aligned}\dot\delta_B(t_{BR},\mathbf{k})=&\dot\delta_R(t_{BR},\mathbf{k})\\&-\frac{p_{bR}(t_{BR})-p_{bB}(t_{BR})}{\rho_{bR}(t_{BR})+p_{bR}(t_{BR})}\\&\times\left\{\frac{\rho_{bR}(t_{BR})+p_{bR}(t_{BR})}{2\rho_{bR}(t_{BR})}\dot h^{(R)}(t_{BR},\mathbf{k})\right.\\&\left.+H(t_{BR})\delta_R(t_{BR},\mathbf{k})\left[\frac{1}{3}\left[\frac{k}{a(t_{BR})H(t_{BR})}\right]^2+\frac{3}{2}\rho_{bR}^{-1}(t_{BR})[\rho_{bR}(t_{BR})+2p_{bR}(t_{BR})+p_{bB}(t_{BR})]\right]\right\}\\&-\frac{3H(t_{BR})\delta_R(t_{BR})}{\rho_{bR}(t_{BR})+p_{bR}(t_{BR})}\{c_{sB}^2[\rho_{bB}(t_{BR})+p_{bB}(t_{BR})]-c_{sR}^2[\rho_{bR}(t_{BR})+p_{bR}(t_{BR})]\} ,\end{aligned} \tag{4.16}$$

$$h_i^{(B)}(t_{BR},\mathbf{x})=h_i^{(R)}(t_{BR},\mathbf{x}) , \tag{4.17}$$

$$\dot h_i^{(B)}(t_{BR},\mathbf{x})=\dot h_i^{(R)}(t_{BR},\mathbf{x}) . \tag{4.18}$$

We first describe what these equations do (we shall assume some familiarity with the results and analysis of Ref. [14]). δ_B, v_B^1, and $h_{22}^{(B)}+h_{33}^{(B)}$ depend on three constants of integration, $c_2^{(B)}$ and $c_\pm^{(B)}$ which are determined by Eq. (4.8), by the $i=1$ part of Eqs. (4.9) and by Eq. (4.10). Equation (4.11) and the $i=I$ components of Eqs. (4.9) are four joining conditions for the four constants of integration, $c_{I,1}^{(B)}$ and $c_{I,2}^{(B)}$, in the solutions for v_B^I and $h_{I1}^{(B)}$. Equation (4.12) is a consistency equation that must be satisfied by the expressions for $c_\pm^{(B)}$ and $c_2^{(B)}$ derived from Eqs. (4.8)–(4.10)—this will provide a check on our algebraic manipulations. Equations (4.13) is another consistency condition—the expressions for $c_{I,2}^{(B)}$ must satisfy it. Equation (4.14) fixes the constant of integration $c_1^{(B)}$ in the solution for the trace of the metric perturbation, $h^{(B)}$ (it must be modified if either of the fluids is radiation, see the discussion below Eqs. (3.59) of Ref. [16], and is ignored in what follows since we do not have need for $c_1^{(B)}$). Equations (4.15) and (4.16) are further consistency equations. The four gravitational-wave perturbation constants of integration, $c_{i1}^{(B)}$ and $c_{i2}^{(B)}$ (where $i=+,\times$), are determined from Eqs. (4.17) and (4.18). The consistency conditions also follow from the perturbation equations of motion and the other joining conditions (see Sec. III of Ref. [16])—they contain no new information but serve to provide a check on our algebraic computations.

In the scalar-field-dominated epoch, the fractional energy-density δ_Φ is given by Eq. (3.31), while the transverse peculiar velocity perturbation is (Eq. (7.11) of Ref. [7])

$$v_\Phi^i=-\frac{m_P^2}{16\pi}\frac{\dot\Phi_0}{a(\rho_{b\Phi}+p_{b\Phi})}\partial_i\phi , \tag{4.19}$$

and the "speed of sound" $c_{s\Phi}$ may be determined from Eqs. (3.31) and (7.12) of Ref. [7]:

$$c_{s\Phi}^2\rho_{b\Phi}\delta_\Phi=\frac{m_P^2}{16\pi}[\dot{\Phi}_0\dot{\phi}-\tfrac{1}{2}V'(\Phi_0)\phi] \ . \tag{4.20}$$

We have shown, in Sec. IV of Ref. [16], that the constants of integration for the gauge-invariant solutions may be determined from gauge-invariant combinations of the above joining conditions. Along with the gauge-invariant joining conditions (4.17) and (4.18), which determine the four gauge-invariant constants of integration $c_{i1}^{(B)}$ and $c_{i2}^{(B)}$, we have

$$\Delta_B(t_{BR},\mathbf{k})=\Delta_R(t_{BR},\mathbf{k}) \ , \tag{4.21}$$

$$\frac{A_B(t_{BR},\mathbf{k})}{\rho_{bB}(t_{BR})+p_{bB}(t_{BR})}=\frac{A_R(t_{BR},\mathbf{k})}{\rho_{bR}(t_{BR})+p_{bR}(t_{BR})} \ , \tag{4.22}$$

where the gauge-invariant variables Δ and A are defined by

$$\Delta=\delta+3i\frac{aH}{k}\rho_b^{-1}(\rho_b+p_b)v^1 \ , \tag{4.23}$$

and

$$A=\delta-\frac{3}{4}\rho_b^{-1}(\rho_b+p_b)(h_{22}+h_{33}) \ , \tag{4.24}$$

where Δ is related to $\Delta\rho$ [Eq. (3.36)] through

$$\Delta\rho=-16\pi m_P^{-2}a^3\rho_b\Delta \ . \tag{4.25}$$

We choose to match the closed-form expressions for the irregularities (Sec. III and Appendices C and B of Ref. [14]) at $t_{R\Phi}$ and t_{BR} to determine the constants of integration and then, in Sec. V, focus on the approximate expressions valid for wavelengths that were large compared to the Hubble scale during the transitions—these are the fluctuations of interest in the large-time universe for which our expressions are physically relevant. Our expressions do not describe the physics of shorter-wavelength perturbations since we have not included the effects of reheating or of the coupling between radiation and matter. These effects cannot significantly affect the evolution of perturbations outside the Hubble radius and hence may be ignored if we only wish to determine the behavior of irregularities that first enter the Hubble radius during the matter-dominated epoch.

Using the joining conditions for $\delta,(\rho_b+p_b)v^1$ and $h_{22}+h_{33}$ [Eqs. (4.8)–(4.10)] at the scalar-field–radiation transition, we find, in the radiation-dominated epoch,

$$\delta_R(x)=c_2^{(R)}e^{-x}-2\lambda_R^{-5/2}e^{-x/2}[c_+^{(R)}G_1^R(x)+c_-^{(R)}G_2^R(x)] \ , \tag{4.26}$$

where

$$G_1^R(x)=\left[\frac{2}{\pi}\right]^{1/2}(i\lambda_R e^{x/2}-2-2i\lambda_R^{-1}e^{-x/2})\times\exp(i\lambda_R e^{x/2}) \tag{4.27}$$

[and $G_2^R=(G_1^R)^*$], as well as

$$v_R^1(x)=-i\lambda_R\frac{\sqrt{3}}{2}\{-\tfrac{1}{2}c_2^{(R)}e^{-x/2}+\lambda_R^{-5/2}[c_+^{(R)}\tilde{G}_1^R(x)+c_-^{(R)}\tilde{G}_2^R(x)]\} \ , \tag{4.28}$$

where we have defined

$$\tilde{G}_1^R(x)=-\left[\frac{2}{\pi}\right]^{1/2}(1+2i\lambda_R^{-1}e^{-x/2})\exp(i\lambda_R e^{x/2}) \tag{4.29}$$

[and $\tilde{G}_2^R=(\tilde{G}_1^R)^*$], and

$$h_{22}^{(R)}(x)+h_{33}^{(R)}(x)=c_2^{(R)}e^{-x}+2e^{3x/4}[c_+^{(R)}\tilde{F}_1^R(x)+c_-^{(R)}\tilde{F}_2^R(x)] \ , \tag{4.30}$$

where

$$\tilde{F}_1^R(x)=2i\left[\frac{2}{\pi}\right]^{1/2}\lambda_R^{-7/2}e^{-7x/4}\exp(i\lambda_R e^{x/2}) \tag{4.31}$$

[and $\tilde{F}_2^R=(\tilde{F}_1^R)^*$]. The integration constants in these formulas are given by

$$c_2^{(R)}=-c_2\left[\frac{q}{2}\right]^{1/2}-3\left[\frac{16\pi}{m_P^2}\right]^{1/2}k^{-3/2}\left[\frac{\sqrt{3}\pi}{q(2-q)}\right]^{1/2}\lambda_R^{1/2}e^{x_R/4}e^{i(\nu-1/2)\pi/2}$$
$$\times\left[\frac{2}{2-q}\lambda_R^2e^{x_R}\lambda^{-3/2}e^{3(2-q)\bar{x}_R/(2q)}G_1(\bar{x}_R)+\frac{1}{3}(2-q)H_\nu^{(1)}(\lambda e^{-(2-q)\bar{x}_R/q})\right.$$
$$\left.+\frac{2}{2-q}(2+\lambda_R^2e^{x_R})\lambda^{-1}e^{(2-q)\bar{x}_R/q}H_{\nu+1}^{(1)}(\lambda e^{-(2-q)\bar{x}_R/q})\right] \ , \tag{4.32}$$

$$c_+^{(R)} = -i\left[\frac{16\pi}{m_P^2}\right]^{1/2}\frac{\pi}{24}\frac{qk^{-3/2}}{(2-q)^{1/2}}\left[\frac{3}{q}\right]^{5/4}\lambda_R^4 e^{\bar{x}_R/4}e^{i(\nu-1/2)\pi/2}\exp(-i\lambda_R e^{x_R/2})$$
$$\times\left[\tfrac{1}{2}[2-q+\tfrac{1}{2}(4-q)i\lambda_R e^{x_R/2}]H_\nu^{(1)}(\lambda e^{-(2-q)\bar{x}_R/q}) + \frac{6}{2-q}(1+i\lambda_R e^{x_R/2})\lambda^{-1}e^{(2-q)\bar{x}_R/q}H_{\nu+1}^{(1)}(\lambda e^{-(2-q)\bar{x}_R/q})\right], \tag{4.33}$$

$$c_-^{(R)} = -i\left[\frac{16\pi}{m_P^2}\right]^{1/2}\frac{\pi}{24}\frac{qk^{-3/2}}{(2-q)^{1/2}}\left[\frac{3}{q}\right]^{5/4}\lambda_R^4 e^{\bar{x}_R/4}e^{i(\nu-1/2)\pi/2}\exp(i\lambda_R e^{x_R/2})$$
$$\times\left[\tfrac{1}{2}[-2+q+\tfrac{1}{2}(4-q)i\lambda_R e^{x_R/2}]H_\nu^{(1)}(\lambda e^{-(2-q)\bar{x}_R/q}) - \frac{6}{2-q}(1-i\lambda_R e^{x_R/2})\lambda^{-1}e^{(2-q)\bar{x}_R/q}H_{\nu+1}^{(1)}(\lambda e^{-(2-q)\bar{x}_R/q})\right]. \tag{4.34}$$

In these expressions c_2 is the constant of integration corresponding to the inflation epoch time translation solution, ν, λ, $a_0 M^{2/q}$, and the function $G_1(x)$ are given in Sec. III A, and we have defined

$$e^x = t - t_{R\Phi} + M_R^{-1}, \quad e^{x_R} = M_R^{-1},$$
$$e^{\bar{x}_R} = \frac{4}{q}e^{x_R}, \quad \lambda_R = \frac{2}{\sqrt{3}}\frac{k}{a_1}M_R^{-1/2};$$

we note that e^x jumps at the transition.

It may be verified, from the above equations, that the consistency conditions (4.12), (4.15), and (4.16), [33], are satisfied at the transition. It is pleasing to note that the constants of integration for the gauge-invariant terms $c_\pm^{(R)}$ do not depend on the inflationary epoch, gauge-dependent, time-translation solution constant of integration c_2. This is because the expressions for $c_\pm^{(R)}$, Eqs. (4.33) and (4.34), may also be derived by matching the gauge-invariant quantities Δ and $A/(\rho_b+p_b)$ at the transition, Eqs. (4.21) and (4.22). [Alternatively, the fact that $c_\pm^{(R)}$ do not depend on c_2 can be viewed as a rather nontrivial verification of our joining conditions, since one expects, on fairly general grounds, the coefficient of the large-time adiabatic solution to carry physical (i.e., gauge-invariant) information.] For completeness, we note that the gauge-invariant measure of energy-density perturbations, Δ_R, in the radiation-dominated epoch, is given by

$$\Delta_R = \left[\frac{8}{\pi}\right]^{1/2}\lambda_R^{-5/2}e^{-x/2}$$
$$\times[c_+^{(R)}(1-i\lambda_R e^{x/2})\exp(i\lambda_R e^{x/2}) + c_-^{(R)}(1+i\lambda_R e^{x/2})\exp(-i\lambda_R e^{x/2})]. \tag{4.35}$$

The last factor in Eq. (4.27) [as well as the similar factors in Eq. (4.35)] means that small-scale energy-density irregularities oscillate during the radiation-dominated epoch.

It is interesting to note that the gauge-dependent constants of integration corresponding to the time-translation solution in the scalar-field-dominated epoch, c_2, and in the radiation-dominated epoch, $c_2^{(R)}$ [Eq. (4.32)], cannot be simultaneously removed by a gauge transformation. The joining condition for the gauge parameter $f^0(k)$ (Eqs. (3.34) and (3.35) and Eqs. (C11)–(C13) of Ref. [14]) is consistent with Eq. (4.32); i.e., if we choose to remove c_2 in the inflationary epoch expressions then we automatically remove the c_2-dependent term in the radiation-dominated epoch expressions, Eq. (4.32)—a different choice for c_2 will result in a different value for $c_2^{(R)}$.

Similarly, one may match the trace of the metric perturbation at the scalar-field–radiation transition to determine the constant of integration $c_1^{(R)}$ in the radiation-dominated epoch. Since we do not require the explicit form of the trace of the metric perturbation h we shall not record $c_1^{(R)}$ here.

Using the joining conditions for $\delta, (\rho_b+p_b)v^1$ and $h_{22}+h_{33}$ at the radiation-baryon transition, we find, in the baryon-dominated epoch,

$$\delta_B(x) = c_2^{(B)}e^{-x} + c^{(B)}e^{2x/3}, \tag{4.36}$$

$$v_B^1(x) = i\frac{3}{8}\frac{k}{a_2}M_B^{-2/3}c_8^{(B)}e^{-2x/3}, \tag{4.37}$$

$$h_{22}^{(B)}(x) + h_{33}^{(B)}(x) = -\frac{40}{9}\frac{a_2^2}{k^2}M_B^{4/3}c^{(B)} + c_8^{(B)}e^{-x}, \tag{4.38}$$

where we have defined

$$e^x = t - t_{BR} + M_B^{-1}.$$

The constants of integration in these expressions are related to those in the radiation-dominated epoch through

$$c_2^{(B)} = c_2^{(R)} - 2\lambda_R^{-5/2}e^{\bar{x}_B/2}[c_+^{(R)}\bar{G}_1^R(\bar{x}_B) + c_-^{(R)}\bar{G}_2^R(\bar{x}_B)], \tag{4.39}$$

where we have defined

$$\bar{G}_1^R(\bar{x}_B)=\left[\frac{2}{\pi}\right]^{1/2}(-2i\lambda_R^{-1}e^{-\bar{x}_B/2}-2+i\lambda_R e^{\bar{x}_B/2}+\tfrac{4}{15}\lambda_R^2 e^{\bar{x}_B}+\tfrac{2}{15}i\lambda_R^3 e^{3\bar{x}_B/2})\exp(i\lambda_R e^{\bar{x}_B/2}) \tag{4.40}$$

[and $\bar{G}_2^R=(\bar{G}_1^R)^*$], and the constant of integration $c^{(B)}$, corresponding to a gauge-invariant solution, is given by

$$c^{(B)}=\frac{3}{5}\left[\frac{2}{\pi}\right]^{1/2}\tilde{\lambda}_B^{-1/2}e^{-x_B/12}\times\left[c_+^{(R)}\left[1+\frac{i}{2}\lambda_R e^{\bar{x}_B/2}\right]\exp(i\lambda_R e^{\bar{x}_B/2})+c_-^{(R)}\left[1-\frac{i}{2}\lambda_R e^{\bar{x}_B/2}\right]\exp(-i\lambda_R e^{\bar{x}_B/2})\right], \tag{4.41}$$

as well as

$$c_8^{(B)}=\tfrac{4}{3}c_2^{(R)}-\tfrac{16}{9}\lambda_R^{-5/2}e^{\bar{x}_B/2}\times[c_+^{(R)}\hat{G}_1^R(\bar{x}_B)+c_-^{(R)}\hat{G}_2^R(\bar{x}_B)], \tag{4.42}$$

where

$$\hat{G}_1^R(\bar{x}_B)=\left[\frac{2}{\pi}\right]^{1/2}\left[-\frac{i}{2}\lambda_R e^{\bar{x}_B/2}-1-3i\lambda_R^{-1}e^{-\bar{x}_B/2}\right]\times\exp(i\lambda_R e^{\bar{x}_B/2}) \tag{4.43}$$

[and $\hat{G}_2^R=(\hat{G}_2^R)^*$]. In these formulas the constants $c_2^{(R)}$ and $c_\pm^{(R)}$ are given by Eqs. (4.32)–(4.34) and we have defined

$$e^{x_B}=M_B^{-1},\quad e^{\bar{x}_B}=\tfrac{3}{4}e^{x_B},\quad \tilde{\lambda}_B=3k(a_2M_B^{2/3})^{-1}.$$

It is pleasing to note that the inflationary epoch, gauge-dependent, constant of integration c_2 does not appear in the expression for $c^{(B)}$ which corresponds to a gauge-invariant solution in the baryon-dominated epoch. It is straightforwardly verified that the consistency conditions (4.12), (4.15), and (4.16) are satisfied [33]. Contrasting Eqs. (4.32) and (4.39) we see that an appropriate choice for the value of inflation epoch constant of integration c_2, which corresponds to the gauge-dependent time-translation solution, may be used to remove one of either c_2, $c_2^{(R)}$, or $c_2^{(B)}$ [see the discussion in the paragraph following Eq. (4.35)].

In the baryon-dominated epoch the gauge-invariant measure of energy-density perturbations, Δ_B, is given by

$$\Delta_B=(c_2^{(B)}-\tfrac{3}{4}c_8^{(B)})e^{-x}+c^{(B)}e^{2x/3}, \tag{4.44}$$

where

$$c_2^{(B)}-\tfrac{3}{4}c_8^{(B)}=\tfrac{8}{3}\lambda_R^{-5/2}e^{\bar{x}_B/2}[c_+^{(R)}\check{G}_1^R(\bar{x}_B)+c_-^{(R)}\check{G}_2^R(\bar{x}_B)] \tag{4.45}$$

(which, as expected, does not depend on c_2), and we have defined

$$\check{G}_1^R(\bar{x}_B)=\left[\frac{2}{\pi}\right]^{1/2}\left[1-i\lambda_R e^{\bar{x}_B/2}-\frac{1}{5}\lambda_R^2 e^{\bar{x}_B}-\frac{i}{10}\lambda_R^3 e^{3\bar{x}_B/2}\right]\exp(i\lambda_R e^{\bar{x}_B/2}) \tag{4.46}$$

and $\check{G}_2^R=(\check{G}_1^R)^*$. Equations (4.41) and (4.45) may also be derived by matching Δ and $A/(\rho_b+p_b)$ at the transition. We note that unlike Eq. (4.35), Δ_B in the matter-dominated epoch, Eq. (4.44), does not oscillate inside the Hubble radius.

It is pleasing to note that the growing mode in Δ_B [Eq. (4.44)] and δ_B [Eq. (4.36)] have exactly the same numerical coefficient; only the coefficient of the decaying mode in δ_B is gauge dependent—this is the reason why, the comments of some proponents of the gauge-invariant formalism notwithstanding, synchronous gauge computations of the large-time behavior of energy-density irregularities are physically correct (i.e., gauge invariant).

We now consider the large-time transverse peculiar velocity perturbations. The relevant joining conditions are Eqs. (4.9) and (4.11). From Eq. (4.19) we see that the transverse peculiar velocity perturbation v_Φ^I vanishes in the scalar-field-dominated epoch (this is true no matter what the form of the scalar field potential); Eq. (4.9) then implies $v_R^I=0=v_B^I$ or $c_{I,2}^{(R)}=0=c_{I,1}^{(B)}$ (where we have used Eqs. (C9) and (B16) of Ref. [14]). Since h_{I1} is time-independent during inflation [Eq. (3.25)], Eq. (4.11) implies $h_{I1}^{(R)}=h_{I1}=h_{I1}^{(B)}$ or $c_{I,1}^{(R)}=c_{I,1}=c_{I,1}^{(B)}$ (from Eqs. (C8) and (B15) of Ref. [14]). [It is obvious that these expressions for the constants of integration satisfy the consistency condition (4.13).] This result means that such scalar field models of inflation do not generate transverse peculiar velocity perturbations, $v^I=0$, and so are not the appropriate description of the very early Universe in the primeval turbulence scenario for structure formation.

The joining conditions for the graviton degrees of freedom at the transition are that h_i and $\dot{h}_i$ (where $i=+,\times$) should match, Eqs. (4.17) and (4.18). From Eqs. (39) and (40) of Ref. [14] and Eqs. (3.26), (3.27), and (3.44) we find, in the radiation-dominated epoch,

$$h_i^{(R)}=e^{-x/4}[c_{i1}^{(R)}H_{-1/2}^{(1)}(\tilde{\lambda}_R e^{x/2})+c_{i2}^{(R)}H_{-1/2}^{(2)}(\tilde{\lambda}_R e^{x/2})], \tag{4.47}$$

where $\tilde{\lambda}_R=\sqrt{3}\lambda_R$ and

$$c_{i1}^{(R)}=\frac{\pi}{8}\left[\frac{16\pi}{m_P^2}\right]^{1/2}\left[\frac{2\pi}{2-q}\right]^{1/2}k^{-3/2}\tilde{\lambda}_R^{5/2}e^{x_R/2}e^{i(\nu-3/2)\pi/2}$$
$$\times[H_{-3/2}^{(2)}(\tilde{\lambda}_R e^{x_R/2})H_{\nu+1}^{(1)}(\lambda e^{-(2-q)\bar{x}_R/q})+H_{-1/2}^{(2)}(\tilde{\lambda}_R e^{x_R/2})H_{\nu}^{(1)}(\lambda e^{-(2-q)\bar{x}_R/q})]\ ,$$
$$c_{i2}^{(R)}=-\frac{\pi}{8}\left[\frac{16\pi}{m_P^2}\right]^{1/2}\left[\frac{2\pi}{2-q}\right]^{1/2}k^{-3/2}\tilde{\lambda}_R^{5/2}e^{x_R/2}e^{i(\nu-3/2)\pi/2}$$
$$\times[H_{-3/2}^{(1)}(\tilde{\lambda}_R e^{x_R/2})H_{\nu+1}^{(1)}(\lambda e^{-(2-q)\bar{x}_R/q})+H_{-1/2}^{(1)}(\tilde{\lambda}_R e^{x_R/2})H_{\nu}^{(1)}(\lambda e^{-(2-q)\bar{x}_R/q})]\ . \tag{4.48}$$

In the matter dominated epoch we find

$$h_i^{(B)}=e^{-x/2}[c_{i1}^{(B)}H_{-3/2}^{(1)}(\tilde{\lambda}_B e^{x/3})+c_{i2}^{(B)}H_{-3/2}^{(2)}(\tilde{\lambda}_B e^{x/3})]\ , \tag{4.49}$$

where we have defined

$$c_{i1}^{(B)}=i\frac{\pi}{4}\left[\frac{2}{\sqrt{3}}\right]^{1/2}\tilde{\lambda}_B e^{7x_B/12}\{c_{i1}^{(R)}[H_{-5/2}^{(2)}(\tilde{\lambda}_B e^{x_B/3})H_{-1/2}^{(1)}(\tilde{\lambda}_R e^{\bar{x}_B/2})-H_{-3/2}^{(2)}(\tilde{\lambda}_B e^{x_B/3})H_{-3/2}^{(1)}(\tilde{\lambda}_R e^{\bar{x}_B/2})]$$
$$+c_{i2}^{(R)}[H_{-5/2}^{(2)}(\tilde{\lambda}_B e^{x_B/3})H_{-1/2}^{(2)}(\tilde{\lambda}_R e^{\bar{x}_B/2})-H_{-3/2}^{(2)}(\tilde{\lambda}_B e^{x_B/3})H_{-3/2}^{(2)}(\tilde{\lambda}_R e^{\bar{x}_B/2})]\}\ ,$$
$$c_{i2}^{(B)}=-i\frac{\pi}{4}\left[\frac{2}{\sqrt{3}}\right]^{1/2}\tilde{\lambda}_B e^{7x_B/12}\{c_{i1}^{(R)}[H_{-5/2}^{(1)}(\tilde{\lambda}_B e^{x_B/3})H_{-1/2}^{(1)}(\tilde{\lambda}_R e^{\bar{x}_B/2})-H_{-3/2}^{(1)}(\tilde{\lambda}_B e^{x_B/3})H_{-3/2}^{(1)}(\tilde{\lambda}_R e^{\bar{x}_B/2})]$$
$$+c_{i2}^{(R)}[H_{-5/2}^{(1)}(\tilde{\lambda}_B e^{x_B/3})H_{-1/2}^{(2)}(\tilde{\lambda}_R e^{\bar{x}_B/2})-H_{-3/2}^{(1)}(\tilde{\lambda}_B e^{x_B/3})H_{-3/2}^{(2)}(\tilde{\lambda}_R e^{\bar{x}_B/2})]\}\ . \tag{4.50}$$

Although the expressions in Eqs. (4.47)–(4.50) can be written in a slightly simpler form (Hankel functions of half-integral order are a product of an exponential and a polynomial) we shall not bother to do so, at this stage, since the resulting expressions are not much more illuminating. We note that both in the radiation-dominated epoch, Eq. (4.47), and in the baryon-dominated epoch, Eq. (4.49), graviton perturbations oscillate inside the Hubble radius. It is also interesting to note that in the scalar-field-dominated epoch, on small scales $h_i \propto \exp[-i(k\tau-\mathbf{k}\cdot\mathbf{x})]$ (where τ is conformal time) while in both the radiation-dominated and matter-dominated epochs h_i has a contribution proportional to $\exp[-i(k\tau-\mathbf{k}\cdot\mathbf{x})]$ as well as a term proportional to $\exp[i(k\tau+\mathbf{k}\cdot\mathbf{x})]$; in a matter- or radiation-dominated universe the graviton "vacuum" state initial condition would have required that the second type of term be absent. If we discard these terms in our model we would find that we could not require that both h_i and $\dot{h}_i$ match at the transition.

V. LARGE-TIME EXPRESSIONS FOR LARGE-SCALE INHOMOGENEITIES

In this section we evaluate the power spectrum of energy-density irregularities, the spectrum of local departure velocity from homogeneous expansion and the gravitational-wave energy-density spectrum in the large-time, baryon-dominated, universe. The microphysics underlying this inflation model is, at present, unknown, rather, the normalization of the power spectrum will be determined from observational data. We choose to determine the normalization of the power spectrum from observational data because the scalar field inflation picture of the very early Universe (like all other scenarios) is, at present, incomplete—the reheating temperature and the inflation epoch scalar field potential differ from model to model and experimental high-energy physics has not yet distinguished a preferred standard model. Even though a given, microphysics-based, inflation model has a fixed reheating temperature and scalar field potential, the vast number of grand unified, higher-dimensional and superstring-inspired inflation models makes it more worthwhile to work with a class of macrophysical models rather than to concentrate on any single model that is motivated by a particular choice of short-distance philosophy.

The model also does not include the physics which describes the coupling of radiation to matter so the fractional energy-density irregularity power spectrum we derive is not valid on scales on which this coupling is important—it is only correct on large scales ($\gtrsim$ few tens of megaparsecs). We are, hence, forced to fix the normalization of the power spectrum at an unconventionally large scale. On these ("great attractor") scales, the relevant observed quantity is the local departure velocity from homogeneous expansion [28]. Requiring that the theoretical expression agree with the (tentative) observational data results in a relation between the two free parameters of the model: $z_{R\Phi}$, the redshift at the epoch of reheating, and q, the index of the inflation epoch scalar field potential (as well as h).

Prior to comparing theory to observation we must ensure that the theoretical expressions and the corresponding observational measurement are recorded in the same coordinate system. We transform the theoretical expressions to the coordinate system in which the time derivative of the trace of the metric perturbation has been removed on a specified spatial hypersurface on which the observation was made; this is the instantaneously Newtonian synchronous coordinate system. Furthermore, we may only compare a position space course-grain average of the theoretical expression to the observational measurement.

A. Large-scale approximation

The characteristics of the inflation epoch are most easily examined if we focus on perturbations on scales larger than the Hubble radius during the transitions from scalar field dominance to radiation dominance and from radiation dominance to baryon dominance. These perturbations are unaffected by small scale processes such as entropy production and baryosynthesis at reheating and the interaction of radiaton with matter—they are truly primeval. (Since our expressions for the perturbations do not include the effects of small-scale processes they are only physically relevant for such large-scale perturbations.) The large-scale perturbations are the leading asymptotic terms of the expressions derived in the previous sections and in Ref. [14] when $k/[a(t_{R\Phi})H(t_{R\Phi})]$ and $k/[a(t_{BR})H(t_{BR})]$ are small.

For perturbations that were outside the Hubble radius during the scalar-field–radiation transition, the constants which characterize the energy-density and longitudinal peculiar velocity perturbations in the radiation-dominated epoch, Eqs. (4.32)–(4.34), reduce to

$$c_2^{(R)}=-c_2\left[\frac{q}{2}\right]^{1/2}-6\left[\frac{16\pi}{m_P^2}\right]^{1/2}\left[\frac{\pi}{q}\right]^{1/2}(2-q)^{\nu+1/2}\frac{\csc(\nu\pi)}{\Gamma(-\nu)}e^{i(\nu+1/2)\pi/2}k^{-3/2}\left[\frac{k}{a(t_{R\Phi})H(t_{R\Phi})}\right]^{-\nu-3/2} \tag{5.1}$$

and

$$c_\pm^{(R)}=\pm\left[\frac{16\pi}{m_P^2}\right]^{1/2}\frac{3}{8}\pi\left[\frac{2}{q}\right]^{1/2}(2-q)^{\nu+1/2}\frac{\csc(\nu\pi)}{\Gamma(-\nu)}e^{i(\nu-1/2)\pi/2}\left[\frac{\lambda_R}{k}\right]^{7/2}k^2\left[\frac{k}{a(t_{R\Phi})H(t_{R\Phi})}\right]^{-\nu-3/2}; \tag{5.2}$$

in these equations the constant c_2 corresponds to the inflation epoch time-translation invariant solution and ν is defined in Sec. III A. (We have only recorded the leading terms in these asymptotic expansions.) In the matter-dominated era we find that Eqs. (4.39), (4.41), and (4.42) reduce to, for perturbations that will first reenter the Hubble radius in the baryon-dominated epoch,

$$c_2^{(B)}=-c_2\left[\frac{q}{2}\right]^{1/2}-i\left[\frac{16\pi}{m_P^2}\right]^{1/2}\frac{(2-q)}{\sqrt{q\pi}}\Gamma\left[\frac{5}{4}-\frac{\nu}{2}\right]\Gamma\left[\frac{5}{4}+\frac{\nu}{2}\right]k^{-3/2}$$
$$-\frac{1}{20}\left[\frac{16\pi}{m_P^2}\right]^{1/2}\left[\frac{\pi}{q}\right]^{1/2}(2-q)^{\nu+1/2}\frac{\csc(\nu\pi)}{\Gamma(-\nu)}e^{i(\nu+1/2)\pi/2}k^{-3/2}\left[\frac{k}{a(t_{R\Phi})H(t_{R\Phi})}\right]^{-\nu-3/2}\left[\frac{k}{a(t_{BR})H(t_{BR})}\right]^4, \tag{5.3}$$

$$c^{(B)}=\frac{9}{20}\left[\frac{16\pi}{m_P^2}\right]^{1/2}\left[\frac{3\pi}{\sqrt{2}q}\right]^{1/2}(2-q)^{\nu+1/2}\frac{\csc(\nu\pi)}{\Gamma(-\nu)}e^{i(\nu+1/2)\pi/2}$$
$$\times\left[\frac{\lambda_R}{k}\right]^{7/2}\left[\frac{\tilde{\lambda}_B}{k}\right]^{-1/4}k^{7/4}\left[\frac{k}{a(t_{R\Phi})H(t_{R\Phi})}\right]^{-\nu-3/2}\left[\frac{k}{a(t_{BR})H(t_{BR})}\right]^{3/4}, \tag{5.4}$$

$$c_8^{(B)}=-c_2\frac{4}{3}\left[\frac{q}{2}\right]^{1/2}-i\frac{4}{3}\left[\frac{16\pi}{m_P^2}\right]^{1/2}\frac{(2-q)}{\sqrt{q\pi}}\Gamma\left[\frac{5}{4}-\frac{\nu}{2}\right]\Gamma\left[\frac{5}{4}+\frac{\nu}{2}\right]k^{-3/2}$$
$$-\frac{7}{81}\left[\frac{16\pi}{m_P^2}\right]^{1/2}\left[\frac{\pi}{q}\right]^{1/2}(2-q)^{\nu+1/2}\frac{\csc(\nu\pi)}{\Gamma(-\nu)}e^{i(\nu+1/2)\pi/2}k^{-3/2}\left[\frac{k}{a(t_{R\Phi})H(t_{R\Phi})}\right]^{-\nu-3/2}\left[\frac{k}{a(t_{BR})H(t_{BR})}\right]^4. \tag{5.5}$$

In deriving Eqs. (5.3) and (5.5) we have made use of

$$\frac{k}{a(t_{BR})H(t_{BR})}\gg\frac{k}{a(t_{R\Phi})H(t_{R\Phi})}.$$

For future reference we note that the combination $c_2^{(B)}-(3/4)c_8^{(B)}$, which is a constant of integration for a gauge-invariant solution [Eq. (4.45)], is given by

$$c_2^{(B)} - \frac{3}{4} c_8^{(B)} = \frac{2}{135} \left[\frac{16\pi}{m_P^2} \right]^{1/2} \left[\frac{\pi}{q} \right]^{1/2} (2-q)^{\nu+1/2} \frac{\csc(\nu\pi)}{\Gamma(-\nu)} e^{i(\nu+1/2)\pi/2} \times k^{-3/2} \left[\frac{k}{a(t_{R\Phi})H(t_{R\Phi})} \right]^{-\nu-3/2} \left[\frac{k}{a(t_{BR})H(t_{BR})} \right]^4 . \tag{5.6}$$

To derive some of Eqs. (5.3)–(5.6) one needs to retain subleading terms in Eqs. (5.1) and (5.2); in some cases it is necessary, because of cancellations, to go to the fourth or fifth order in the asymptotic expansion.

For perturbations that were outside the Hubble radius during the scalar-field–radiation transition, the constants which characterize the graviton in the radiation-dominated era, Eqs. (4.48), become

$$c_{i(\frac{1}{2})}^{(R)} = \pm \left[\frac{16\pi}{m_P^2} \right]^{1/2} \frac{\pi}{4} (2-q)^{\nu+1/2} \frac{\csc(\nu\pi)}{\Gamma(-\nu)} e^{i(\nu+3/2)\pi/2} \left[\frac{\tilde{\lambda}_R}{k} \right]^{3/2} \left[\frac{k}{a(t_{R\Phi})H(t_{R\Phi})} \right]^{-\nu-3/2} , \tag{5.7}$$

while for perturbations that reenter the Hubble radius in the matter-dominated epoch we find that Eqs. (4.50) reduce to

$$c_{i(\frac{1}{2})}^{(B)} = \pm \left[\frac{16\pi}{m_P^2} \right]^{1/2} \left[\frac{3}{8} \right]^{3/4} \pi (2-q)^{\nu+1/2} \frac{\csc(\nu\pi)}{\Gamma(-\nu)} e^{i(\nu-1/2)\pi/2} \times \left[\frac{\tilde{\lambda}_R}{k} \right]^{3/2} \left[\frac{\tilde{\lambda}_B}{k} \right]^{-3/4} k^{-3/4} \left[\frac{k}{a(t_{R\Phi})H(t_{R\Phi})} \right]^{-\nu-3/2} \left[\frac{k}{a(t_{BR})H(t_{BR})} \right]^{-1/4} . \tag{5.8}$$

We shall use these results, in the next two subsections, to evaluate the large-time, baryon-dominated epoch, form of the peculiar velocity perturbation, the power spectrum of energy-density irregularities and the energy-density spectrum of gravitational-wave perturbations that reenter the Hubble radius in the baryon-dominated epoch. It is straightforward to derive the corresponding formulas for large-scale perturbations in the radiation-dominated epoch—we shall not record these expressions here.

B. Peculiar velocity perturbations and the power spectrum of energy-density irregularities

Transverse peculiar velocity perturbations vanish [see the discussion in the paragraph above Eq. (4.47)]. We note that these are gauge-invariant perturbations. In the baryon-dominated epoch the longitudinal peculiar velocity perturbation (which transforms under the remnants of general coordinate invariance in synchronous gauge) decays linearly with the scale factor, Eq. (4.37).

$$v_B^1 = i\tfrac{3}{8} k c_8^{(B)} a^{-1} , \tag{5.9}$$

where $c_8^{(B)}$ is given by Eq. (5.5).

There are two ways of characterizing large-time energy-density inhomogeneities—δ_B [Eq. (4.36)] and Δ_B [Eq. (4.44)]. Δ_B is invariant to the remnants of general coordinate invariance in synchronous gauge while δ_B is most straightforwardly related to the large-scale anisotropy of the cosmic microwave background temperature. We note that both Δ_B and δ_B have two terms one of which varies as t^{-1} while the other grows as $t^{2/3}$. (As previously discussed, the growing modes in Δ_B and δ_B are identical—δ_B does carry gauge-invariant information.)

To determine the corresponding power spectra we need to evaluate $\langle \delta_B(k)\delta_B(-k) \rangle$ and $\langle \Delta_B(k)\Delta_B(-k) \rangle$. We find, in the baryon-dominated epoch,

$$P_\Delta(k,t) \equiv \langle \Delta_B(k,t)\Delta_B(-k,t) \rangle = \frac{16\pi}{m_P^2} k^{2(1-\nu)} (|A|^2 a^2 + |B_\Delta| a^{-1/2} + |C_\Delta|^2 a^{-3}) , \tag{5.10}$$

where $|A|, |B_\Delta|$, and $|C_\Delta|$ correspond to gauge-invariant solutions (for which expressions are given below). The term proportional to $|A|^2$ in this equation is the growing adiabatic mode (and has the expected dependence on the scale factor). Since we have used a quantum-mechanical Fourier expansion to transform to spatial momentum space (Sec. III A) this definition of the power spectrum differs, by a factor of 4, from what is conventional in cosmology. In particular, the mass autocorrelation function is

$$\xi(|\mathbf{x}-\mathbf{y}|,t) \equiv \langle \delta_B(\mathbf{x},t)\delta_B(\mathbf{y},t) \rangle = \int_{-\infty}^{\infty} \frac{d^3k}{(2\pi)^3} 4 \langle \delta_B(\mathbf{k},t)\delta_B(-\mathbf{k},t) \rangle e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{y})} .$$

We note that the numerical coefficient of the decaying term in δ_B depends on c_2, the constant of integration for the inflation epoch, gauge-dependent, time-translation solution [see Eqs. (4.32), (4.39), and (5.3)]. If we use the remnants of general coordinate invariance in synchronous gauge, during inflation, to choose c_2 such that $c_2^{(B)}$ vanishes, then the only term in $P_\delta(k)\equiv\langle\delta_B(k)\delta_B(-k)\rangle$ is the term proportional to $|A|^2$ in Eq. (5.10). [This argument differs from the more familiar one used to discard these terms in δ_B, i.e., that the other terms in P_δ will, given time, soon be overshadowed by the growing adiabatic term.] In this gauge $c_8^{(B)}$ is defined by Eq. (5.6) where $c_2^{(B)}=0$ and then the coefficient in Eq. (5.9) is fixed with $kc_8^{(B)}\propto k^{2-\nu}$.

The coefficients in Eq. (5.10) are given by

$$|A|=\frac{3}{20}\left[\frac{3\pi}{\sqrt{2}q}\right]^{1/2}(2-q)^{\nu+1/2}\frac{\csc(\nu\pi)}{\Gamma(-\nu)}\times\left[\frac{\lambda_R}{k}\right]^{7/2}\left[\frac{\tilde{\lambda}_B}{k}\right]^{3/4}\times[a(t_{R\Phi})H(t_{R\Phi})]^{\nu+3/2}[a(t_{BR})H(t_{BR})]^{-3/4}\,, \tag{5.11}$$

$$|B_\Delta|=\frac{1}{25}\frac{\pi}{(2)^{1/4}}\frac{1}{q}(2-q)^{2\nu+1}\frac{\csc^2(\nu\pi)}{\Gamma^2(-\nu)}\times\left[\frac{\lambda_R}{k}\right]^{7/2}\left[\frac{\tilde{\lambda}_B}{k}\right]^{-3/4}\times[a(t_{R\Phi})H(t_{R\Phi})]^{2\nu+3}[a(t_{BR})H(t_{BR})]^{-19/4}\,, \tag{5.12}$$

$$|C_\Delta|=\frac{2\sqrt{3}}{45}\left[\frac{\pi}{q}\right]^{1/2}(2-q)^{\nu+1/2}\frac{\csc(\nu\pi)}{\Gamma(-\nu)}\left[\frac{\tilde{\lambda}_B}{k}\right]^{-3/2}\times[a(t_{R\Phi})H(t_{R\Phi})]^{\nu+3/2}[a(t_{BR})H(t_{BR})]^{-4}\,. \tag{5.13}$$

It is interesting to note that the spatial momentum dependence of all three terms in the gauge-invariant power spectrum, Eq. (5.10), is the same. The allowed range of q is from 0 to 2 and the corresponding range of the power spectrum index $n=2(1-\nu)$ is from 1 to $-\infty$.

In the limit $q=2\epsilon^2\rightarrow0$ the inflation model reduces to the usual exponential expansion model [Eqs. (2.6)–(2.10)]; in this limit the power spectrum of energy-density irregularities reduces to

$$P_\Delta(k,t)=\frac{16\pi}{m_P^2}k(|A|^2a^2+|B_\Delta|a^{-1/2}+|C_\Delta|^2a^{-3})\,, \tag{5.14}$$

where the coefficients are now given by

$$|A|=\frac{1}{\epsilon}\frac{1}{20}\left[\frac{3}{\sqrt{2}}\right]^{3/2}\left[\frac{\lambda_R}{k}\right]^{7/2}\left[\frac{\tilde{\lambda}_B}{k}\right]^{3/4}\times[a(t_{R\Phi})H(t_{R\Phi})]^2[a(t_{BR})H(t_{BR})]^{-3/4}\,, \tag{5.15}$$

$$|B_\Delta|=\frac{1}{\epsilon^2}\frac{1}{50(2)^{1/4}}\left[\frac{\lambda_R}{k}\right]^{7/2}\left[\frac{\tilde{\lambda}_B}{k}\right]^{-3/4}\times[a(t_{R\Phi})H(t_{R\Phi})]^4[a(t_{BR})H(t_{BR})]^{-19/4}\,, \tag{5.16}$$

$$|C_\Delta|=\frac{1}{\epsilon}\frac{1}{15}\left[\frac{2}{3}\right]^{1/2}\left[\frac{\tilde{\lambda}_B}{k}\right]^{-3/2}\times[a(t_{R\Phi})H(t_{R\Phi})]^2[a(t_{BR})H(t_{BR})]^{-4}\,. \tag{5.17}$$

We note that the spatial momentum dependence of the dominant adiabatic part of the power spectrum [the term proportional to $|A|^2$ in Eq. (5.14)] is of the standard scale-invariant form [2]. Since we have not included the effects arising from the coupling of matter to radiation, this expression does not correctly describe the short-wavelength form of the power spectrum, Eq. (6) of Ref. [19]. It is interesting to note that in this limit the power spectrum diverges like ϵ^{-2}; as far as we are aware this effect has not previously been discussed.

The physical origin of this divergence lies in a rather interesting property of the quasi–de Sitter spacetime that results in this limit—even though δ_Φ is small (i.e., even though the standard assumptions of linear perturbation theory are valid) the spatially homogeneous local energy-density hypersurfaces in this spacetime are exceedingly displaced from the corresponding synchronous gauge constant time hypersurfaces. This is because the background evolution is so slow, Eq. (2.9), that one must go to a synchronous gauge hypersurface of substantially different time before the change in $\rho_{b\Phi}$ is large enough to compenste for δ_Φ. It is straightforward to quantify this; in the relevant limit Eq. (3.33) reduces to

$$\delta_\Phi(k,t)=-\epsilon^3c_2H-i\epsilon\left[\frac{16\pi}{m_P^2}\right]^{1/2}\frac{k^{-3/2}}{3\sqrt{2}}\times H\left[3-3i\frac{k}{aH}+\left[\frac{k}{aH}\right]^2\right]\times\exp\left[i\frac{k}{aH}\right]\,. \tag{5.18}$$

This expression may also be derived by solving the linear perturbation equations, in the background described by Eqs. (2.6) and (2.7), to zeroth order in ϵ (i.e., we work to zeroth order in ϵ in the first-order perturbation equations for the fundamental fields ϕ and h_{ij}, while we work to first nontrivial order in ϵ in the homogeneous background equations and in the definition of derived fields such as δ_Φ); with this method of derivation the first term in Eq. (5.18) would not be seen. Now from the discussion around Eqs. (3.3)–(3.7) of Ref. [16] we have that the spatial momentum-space form of the temporal displacement of the spatially homogeneous local energy-density hypersurface from the corresponding constant time (t_Φ) synchronous gauge hypersurface is given by, in the scalar-field-dominated epoch,

$$\Delta t(t_\Phi,\mathbf{k})=-\frac{\rho_{b\Phi}(t_\Phi)}{\dot{\rho}_{b\Phi}(t_\Phi)}\delta_\Phi(t_\Phi,\mathbf{k})\,. \tag{5.19}$$

From Eqs. (2.9) and (5.18) this results in

$$\Delta t(t_\Phi,\mathbf{k}) = -\epsilon\frac{c_2}{2} - \frac{i}{\epsilon}\left[\frac{16\pi}{m_P^2}\right]^{1/2}\frac{(2k)^{-3/2}}{3} \times\left[3-3i\frac{k}{aH}+\left[\frac{k}{aH}\right]^2\right] \times\exp\left[i\frac{k}{aH}\right], \quad (5.20)$$

where all time-dependent quantities on the right-hand side are evaluated at $t=t_\Phi$. The first term on the right-hand side corresponds to the time-translation solution (it is, as expected, time independent) while the second term corresponds to all other solutions (and in particular the adiabatic solution); the first term, for a sufficiently sensible, fixed, value of c_2 (which is arbitrary), does not diverge, while the second term diverges linearly for small ϵ. This means that the spatial hypersurface on which, in particular, the adiabatic mode is spatially homogeneous is not "close" to the corresponding synchronous gauge constant time hypersurface. Now the reheating phase transition must occur on a spatially homogeneous local energy-density hypersurface (because it is governed by local physics) and in the radiation-dominated epoch the spatially homogeneous local energy-density hypersurfaces and the corresponding synchronous gauge constant time hypersurfaces are "close" (i.e., for reasonable values of δ_R); we have joined the background fields on a constant time hypersurface, to be able to join the perturbations we need to get the two corresponding spatially homogeneous local energy-density hypersurfaces (the scalar-field-dominated epoch one and the radiation-dominated epoch one) to line up—this requires an unreasonably large value of δ_R (for sufficiently small ϵ).

For small enough ϵ the assumption that the transition spatial hypersurface is close to the constant time spatial hypersurface breaks down and invalidates our results—however for ϵ small (but not excessively so) they are relevant and hence the "divergence" in Eqs. (5.14)–(5.17) is physically meaningful. [It is unclear, at present, how to deal with the exact de Sitter case. It is possible that it behaves sensibly (although we suspect that it does not); one might conceive of approaching it from another direction in the space of potentials, i.e., instead of Eq. (2.6) we might construct another potential, which in the limit of a small parameter vanishing approaches a constant value. In any case it is unlikely, for the physical purposes of inflation, that an exact de Sitter spacetime is of much relevance. It is also possible that this divergence disappears when one correctly accounts for the effects of the perturbations on the evolution of the background.]

We now consider whether this divergence might have been anticipated from the general expression derived in previous analyses (for instance, a representative sample, which is by no means complete, includes Refs. [21–24]) of density irregularities arising from inflation. From Eqs. (5.10) and (5.14)–(5.17) we see that as we increase q from $2\epsilon^2$ (or as we increase ϵ^2 from 0) the numerical prefactor becomes smaller and the power spectrum index, $n=1-2\epsilon^2$, decreases from unity; i.e., a small numerical prefactor is accompanied by a deviation from a scale-invariant power spectrum and a deviation away from an exceedingly flat potential (and exponential expansion inflation). The intuition developed from some of the results of the analyses of Refs. [21–24] suggests the opposite conclusion. In particular, they consider a model with an inflation epoch scalar field potential of the form $V(\Phi)\simeq V_0-\lambda f(\Phi)$, where V_0 is large and dominates during inflation and the small second term causes the scalar field to slowly roll during inflation. λ is a coupling constant (which microphysics suggests is of order unity) that governs deviations away from the exact de Sitter solution and $f(\Phi)$ is a polynomial in Φ that might include logarithmic terms (see, for instance, Eq. (4) of the first of Refs. [23] where $f\propto\Phi^4$ or Eqs. (3.7)–(3.9) of Ref. [24]). [In the exponential expansion inflation limit of our model the step function potential, Eq. (2.6), is $V(\Phi)=V_0-\epsilon 2\Lambda(\Phi-\Phi_0^{(0)})$.] These analyses typically conclude that at second Hubble radius crossing $P(k)=\langle\delta(k)\delta(-k)\rangle\propto\lambda$ (see, for instance, Eq. (22) of the first of Refs. [23], the equation in the first column on page 297 of Ref. [21], Eq. (18) of Ref. [22] or Eq. (3.17) of Ref [24]); a more careful analysis of this result suggests that if one wished to make the numerical value of $P(k)$ small one should consider microphysical theories in which $\lambda \ll 1$ (this is the "fine-tuning" problem that the inflation scenario is supposed to have, i.e., density perturbations at second Hubble radius crossing do not violate the bounds on large-scale spatial anisotropy of the cosmic microwave background temperature only if this coupling constant is exceedingly small during the inflation epoch). The limit $\lambda\to 0$ corresponds to an exceedingly flat potential and hence exponential expansion inflation; this result of these references suggests that the numerical value of $P(k)$ decreases as the deviation from exponential expansion inflation decreases, which differs from our conclusion that the numerical value of $P(k)$ decreases as the deviation from exponential expansion inflation increases.

At this stage, it is perhaps appropriate to note that a general formula presented in earlier analyses, when applied to the model studied here, suggests a result somewhat similar to what we have found. In our notation their expression for the power spectrum at a second Hubble radius crossing is given by

$$P(k)=\langle\delta(k)\delta(-k)\rangle\propto\frac{H^2}{\dot\Phi_0^2}k^4\langle\phi(k)\phi(-k)\rangle ,$$

where the time-dependent quantities on the far right of this equation are evaluated at first Hubble radius crossing (see, for instance, Eq. (16) of the first of Refs. [23] and Eq. (7.8) of the second reference or Eq. (2.37c) of Ref. [24]). It is straightforward to establish, in de Sitter spacetime, that H is independent of ϵ and $\dot\Phi_0^2\propto(\rho_{b\Phi}+p_{b\Phi})\propto\epsilon^2H^2$ (i.e., the kinetic energy of the scalar field is negligible compared to the potential energy during the "slow-rolling" epoch of the exponential expansion inflation scenario). In this limit the expression for the scalar field perturbation, Eq. (3.10), reduces to

$$\phi(k,t)=\epsilon^2 c_2 H+\left[\frac{16\pi}{m_P^2}\right]^{1/2}\frac{k^{-3/2}}{\sqrt{2}}H\left[i+\frac{k}{aH}\right]\times\exp\left[i\frac{k}{aH}\right].$$

The first term is the gauge-dependent time-translation-invariance solution, which vanishes when $\epsilon=0$ (for a reasonable, fixed, choice for the value of the constant of integration c_2 which does not seem to be determined by the quantum mechanics of inflation), the second term includes the adiabatic solution and exactly agrees with the expression given in Eq. (5.18) of Ref. [8] (which was derived using the convention $16\pi/m_P^2=1$). If one ignores the time-translation solution, Sec. III of Ref. [18], then $\langle\phi(k)\phi(-k)\rangle$ is independent of ϵ. This results in $P(k)\propto\epsilon^{-2}$ which agrees with the ϵ dependence of Eqs. (5.14)–(5.17). That this effect is not present in the final result of Refs. [21–24] is a consequence of the different treatment of the reheating transition used in these analyses; in these analyses reheating takes a time comparable to the Hubble time at this epoch, while in our analysis reheating is much more rapid (this is discussed in Refs. [18,20]).

Finally, we note that the analysis of the Dimopoulos-Raby geometric hierarchy super-symmetric grand-unified-theory model presented in Sec. III B of Ref. [24] also illustrates the effect of the de Sitter spacetime divergence. In this model the deviation of the scalar field potential from that corresponding to a constant cosmological constant is governed by their parameter c_2 [Eq. (3.19) of Ref. [24]]; Eq. (3.23) of this reference suggests $P(k)\propto(c_2)^{-2}$ so the power spectrum increases as the deviation from exponential expansion inflation becomes smaller.

To elaborate on the physical implications of this divergence we must first phenomenologically relate the epoch of reheating to the scalar field exponential-potential index q by making use of observational data. We shall return to this later on in this section.

C. Gravitational-wave energy-density spectrum

In the matter-dominated epoch, the momentum-space graviton two-point function, corresponding to the polarization state $i=+$ or $\times$, for perturbations that reenter the Hubble radius in this epoch, is given by

$$\langle h_i^{(B)}(k)h_i^{(B)}(-k)\rangle=\frac{16\pi}{m_P^2}k^{-2(\nu+4)}a^{-3}|D|^2\left\{1-\cos\left[\frac{4k}{aH}\right]-\frac{4k}{aH}\sin\left[\frac{4k}{aH}\right]+4\left[\frac{k}{aH}\right]^2\left[1+\cos\left[\frac{4k}{aH}\right]\right]\right\}. \quad (5.21)$$

(it is interesting to note that the factor in curly brackets is a universal function of k/aH, i.e., unlike the prefactor it is not an explicit function of q), where we have used Eqs. (4.49) and (5.8) and there is no implied summation over the index i on the left hand side of this equation (as is the case in all other equations, involving gravitons, in this subsection); the numerical coefficient $|D|$ is

$$|D|=\sqrt{\pi}3\left[\frac{3}{2}\right]^{5/4}(2-q)^{\nu+1/2}\frac{\csc(\nu\pi)}{\Gamma(-\nu)}\times\left[\frac{\tilde{\lambda}_R}{k}\right]^{3/2}\left[\frac{\tilde{\lambda}_B}{k}\right]^{-15/4}\times[a(t_{R\Phi})H(t_{R\Phi})]^{\nu+3/2}[a(t_{BR})H(t_{BR})]^{1/4}. \quad (5.22)$$

The time dependence and spatial momentum dependence of Eq. (5.21) (including the term in curly brackets) agrees with the corresponding baryon-dominated epoch expression which may be derived from the results of the first of Refs. [11] [see their Eqs. (2.10) and (2.11)]. On large scales Eq. (5.21) reduces to the time-independent expression

$$\langle h_i^{(B)}(k)h_i^{(B)}(-k)\rangle=\frac{16\pi}{m_P^2}k^{-2(\nu+1)}|E|^2, \quad (5.23)$$

where the numerical coefficient is now given by

$$|E|=\frac{1}{3}\sqrt{\pi}\left[\frac{3}{2}\right]^{3/4}(2-q)^{\nu+1/2}\frac{\csc(\nu\pi)}{\Gamma(-\nu)}\times\left[\frac{\tilde{\lambda}_R}{k}\right]^{3/2}\left[\frac{\tilde{\lambda}_B}{k}\right]^{3/4}\times[a(t_{R\Phi})H(t_{R\Phi})]^{\nu+3/2}[a(t_{BR})H(t_{BR})]^{1/4}. \quad (5.24)$$

On small scales it is clear that Eq. (5.21) oscillates (the factor $\{1+\cos[4k/(aH)]\}$ in the last term in the curly brackets oscillates between 0 and 1, the square root of this expression is $\propto\cos[2k/(aH)]$ so, as expected, the amplitude oscillates about zero mean); in this limit the factor in front of the oscillatory term is proportional to $k^{-2(\nu+3)}a^{-2}$. (We note that it is only on these small scales that the perturbations we have been calling gravitational-wave perturbations behave like "true" waves.)

In the de Sitter inflation limit ($q=2\epsilon^2\sim 0$) of the very early Universe model we find that Eq. (5.21) becomes

$$\langle h_i^{(B)}(k)h_i^{(B)}(-k)\rangle=\frac{16\pi}{m_P^2}k^{-9}a^{-3}|D_0|^2\left\{1-\cos\left[\frac{4k}{aH}\right]-\frac{4k}{aH}\sin\left[\frac{4k}{aH}\right]+4\left[\frac{k}{aH}\right]^2\left[1+\cos\left[\frac{4k}{aH}\right]\right]\right\}, \quad (5.25)$$

where the coefficient now takes the form

$$|D_0|=3\left[\frac{3}{2}\right]^{5/4}\left[\frac{\tilde{\lambda}_R}{k}\right]^{3/2}\left[\frac{\tilde{\lambda}_B}{k}\right]^{-15/4}\times[a(t_{R\Phi})H(t_{R\Phi})]^2[a(t_{BR})H(t_{BR})]^{1/4}; \quad (5.26)$$

we note that unlike the energy-density irregularity two-point function, this expression does not diverge in the de Sitter limit. The time dependence of Eq. (5.25) agrees with the baryon-dominated epoch results of Ref. [34] [see the last of their Eqs. (5) and the equation following it], however, although both the k^{-9} factor and the factor in curly brackets in Eq. (5.25) appear in the corresponding expression derived from the results of Ref. [34] their expression seems to have an extra, spatial momentum-dependent, factor $\sin^2(k\tau_s+\varphi)$, where τ_s is the value of conformal time at reheating and φ is a phase, which does not appear in Eq. (5.25); furthermore, the numerical prefactors seem to differ. For long-wavelength perturbations Eq. (5.25) reduces to the time-independent expression

$$\langle h_i^{(B)}(k)h_i^{(B)}(-k)\rangle=\frac{16\pi}{m_P^2}k^{-3}|E_0|^2, \quad (5.27)$$

where $|E_0|$ is given by

$$|E_0|=\frac{1}{3}\left[\frac{3}{2}\right]^{3/4}\left[\frac{\tilde{\lambda}_R}{k}\right]^{3/2}\left[\frac{\tilde{\lambda}_B}{k}\right]^{3/4}\times[a(t_{R\Phi})H(t_{R\Phi})]^2[a(t_{BR})H(t_{BR})]^{1/4}. \quad (5.28)$$

Small scale perturbations oscillate, Eq. (5.25), and the prefactor is proportional to $k^{-7}a^{-2}$.

We now turn to deriving the energy-density spectrum of gravitational-wave perturbations: $\varepsilon_i(k,t)$, where $i=+,\times$. Using the techniques described in Refs. [14,15] one may expand the action (2.1) to quadratic order in the perturbations. This expanded action may be used to derive the corresponding quadratic order Hamiltonian; rewriting this Hamiltonian in terms of the fields and their velocities we find, for a gravitational wave of momentum k propagating in the x^1 direction with polarization i, that the energy-density spectrum $\varepsilon_i(k,t)$ is given by

$$\varepsilon_i(k,t)=\frac{m_P^2}{32\pi}\left[\langle\dot{h}_i(k,t)\dot{h}_i(-k,t)\rangle+\frac{k^2}{a^2}\langle h_i(k,t)h_i(-k,t)\rangle\right]. \quad (5.29)$$

The expectation value of the gravitational-wave part of the quadratic order Hamiltonian is related to $\varepsilon_i(k,t)$ through

$$\langle H_2(t)\rangle=\int_{-\infty}^{\infty}\frac{d^3k}{(2\pi)^3}a^3(t)[\varepsilon_+(k,t)+\varepsilon_\times(k,t)];$$

the square of ε_i plays a role somewhat similar to that of $\rho_b^2\langle\delta(k)\delta(-k)\rangle$ in the case of fluid perturbations. As far as we are aware, this characterization of gravitational-wave perturbations has not previously been used; in the high spatial frequency (WKB) limit ε_i is presumably related to the expressions for the (time-time component of the) effective gravitational-wave stress tensor given in Refs. [35]. It would be intersting to more closely examine the relation between the expressions given in Refs. [35] and those derived, for gravitational-wave perturbations, from the relevant parts of the expectation value of the generators of time translation (the Hamiltonian) and space translation (the momenta).

Using Eqs. (4.49), (5.8), and (5.21) we find that the energy-density spectrum for gravitational-wave perturbations in this model is given by

$$\varepsilon_i(k)=k^{-2(\nu+4)}a^{-6}|F|^2\left\{9+16\left[\frac{k}{aH}\right]^2+32\left[\frac{k}{aH}\right]^4-\left[9-56\left[\frac{k}{aH}\right]^2\right]\cos\left[\frac{4k}{aH}\right]-\left[36\frac{k}{aH}-32\left[\frac{k}{aH}\right]^3\right]\sin\left[\frac{4k}{aH}\right]\right\}, \quad (5.30)$$

where

$$|F|=\sqrt{\pi}\frac{9}{2}\left[\frac{3}{2}\right]^{3/4}(2-q)^{\nu+1/2}\frac{\csc(\nu\pi)}{\Gamma(-\nu)}\times\left[\frac{\tilde{\lambda}_R}{k}\right]^{3/2}\left[\frac{\tilde{\lambda}_B}{k}\right]^{-21/4}\times[a(t_{R\Phi})H(t_{R\Phi})]^{\nu+3/2}[a(t_{BR})H(t_{BR})]^{1/4}. \quad (5.31)$$

For long-wavelength perturbations this reduces to

$$\varepsilon_i(k,t)=k^{-2\nu}a^{-2}|G|^2, \quad (5.32)$$

where the numerical prefactor $|G|$ is given by

$$|G|=\frac{\sqrt{\pi}}{3\sqrt{2}}\left[\frac{3}{2}\right]^{3/4}(2-q)^{\nu+1/2}\frac{\csc(\nu\pi)}{\Gamma(-\nu)}\times\left[\frac{\tilde{\lambda}_R}{k}\right]^{3/2}\left[\frac{\tilde{\lambda}_B}{k}\right]^{3/4}\times[a(t_{R\Phi})H(t_{R\Phi})]^{\nu+3/2}[a(t_{BR})H(t_{BR})]^{1/4}; \quad (5.33)$$

we note that the long-wavelength contribution to the graviton energy density, Eq (5.32), depends on spatial momenta and the scale factor in a manner that differs from that of the two-point function, Eq. (5.23) [in fact, in this limit $\varepsilon_i(k)=(m_P^2/16\pi)(k/a)^2\langle h_i(k)h_i(-k)\rangle/2$]. As far as we are aware, there has been no previous estimate of the energy-density spectrum of gravitational-wave perturbations arising from inflation in the very early Universe. It is interesting to note that the a^{-2} dependence of Eq. (5.32) is not the appropriate scale factor dependence of the energy density of a (massless) relativistic field; this is not unexpected since these perturbations are "true" gravitational waves only on small scales. For short-wavelength perturbations the dominant part of Eq. (5.30) does not oscillate (this is also not unexpected since the energy density of gravitational-wave perturbations must contribute to the expansion of the Universe) and is proportional to $k^{-2(\nu+2)}a^{-4}$, which is the expected dependence on the scale factor. In the de Sitter limit of the model for the very early Universe Eq. (5.32) reduces to

$$\varepsilon_i(k,t)=k^{-1}a^{-2}|G_0|^2 , \tag{5.34}$$

where

$$|G_0|=\frac{1}{3\sqrt{2}}\left[\frac{3}{2}\right]^{3/4}\left[\frac{\tilde{\lambda}_R}{k}\right]^{3/2}\left[\frac{\tilde{\lambda}_B}{k}\right]^{3/4} \times[a(t_{R\Phi})H(t_{R\Phi})]^2[a(t_{BR})H(t_{BR})]^{1/4} , \tag{5.35}$$

while on small scales the energy density is proportional to $k^{-5}a^{-4}$; again these expressions do not diverge.

We plan to eventually use these results to examine the constraints that the observational limits on gravitational waves impose on this model of the very early Universe.

D. Predictions and observational constraints

Theoretical cosmology has been partially hamstrung by the difficulties involved in determining a reliable observational estimate for the density parameter on scales greater than a few tens of megaparsecs (a recent application of the number count of conserved objects versus redshift test [36] holds promise, although it is still in the early stages of development). Since it is unlikely that a reliable large-scale estimate will be available before the turn of the millennium, we have decided to assume that $\Omega_{\rm now}=1$ (which is the preferred, but not the only, value suggested by the inflation scenario). We shall also use the conventional factor h to account for the uncertainty in the Hubble parameter. With these caveats the large-scale features of the scalar field inflation model are governed by the values of the parameters $z_{R\Phi}$ and q.

The normalization of the linear perturbation theory power spectrum may be determined by a best fit to the observed mass autocorrelation function on "intergalactic" scales (~ 10 Mpc) and the observed peculiar velocity perturbation (which is still tentative) on "great attractor" scales (~ 60 Mpc). This would require an expression for the power spectrum that is also valid on "intergalactic" scales and needs a more complete treatment than we have attempted here—we are, hence, forced to determine the normalization of the power spectrum by normalizing the theoretical expression for the peculiar velocity perturbation on scales on which the coupling between radiation and matter can be ignored (for instance, "great attractor" scales, Ref. [28]), [27]. Fixing this normalization results in a relation that determines $z_{R\Phi}$ in terms of q (or vice versa, which would be more convenient if one had an experimentally preferred model for baryosynthesis)—the models are then classified by the vaue of q.

There are then at least two "independent" observational upper bounds (for which our linear perturbation theory expressions are valid) that may be used to constrain q—observational constraints on large-scale spatial anisotropy in the cosmic microwave background temperature constrain the large-scale energy-density irregularity power spectrum and the large-scale gravitational-wave energy-density spectrum (we shall not discuss this constraint here). Observational constraints on the small-scale microwave background anisotropy will also constrain q; however, in this case one has to correctly account for more small-scale physics than has been done here.

A detailed analysis of the linear perturbation theory predictions for these observational tests will require a more complete analysis of the small-scale form of the power spectrum of energy-density irregularities. It will also require some consideration of the observational methods used to make the measurements, as well as, perhaps, a more correct treatment of the radiation-baryon transition. Such an analysis certainly deserves consideration and we hope to return to it elsewhere; in what follows we shall present preliminary, order of magnitude, estimates of what the model predicts and how it is constrained by the observational upper bounds.

To compare with observational data we must first transform our expressions to the coordinate system used in the observations. Since the peculiar velocity is small, on the (small) scales of interest, conventional wisdom suggests the use of the coordinates used in the Newtonian approximation of the relativistic theory (Sec. 84 of Ref. [37])—the instantaneously Newtonian (or locally Minkowski) synchronous coordinate system. These are synchronous coordinates, $\hat{x}^\mu=(\hat{t},\hat{x}^i)$, in which the time derivative of the trace of the metric perturbation, $\hat{\partial}_0\hat{h}(\hat{x})$, vanishes on a spatial hypersurface at the time $\hat{t}=\hat{t}_N$ when the observation was carried out. These coordinates are related to the synchronous coordinates we have mostly worked with so far through the equations

$$\hat{t}=t-\Delta t(t_N,\mathbf{x}),\quad \hat{x}^i=x^i-f^i(t,\mathbf{x}) , \tag{5.36}$$

where $\hat{t}_N=t_N-\Delta t(t_N)$ and we shall determine Δt and f^i below. (The following manipulations are very similar to those of Sec. III of Ref. [16] so we shall omit technical details which may be found there; $\Delta t(t_N)$ and $f^i(t)$ are, of course, not to be confused with $\Delta t(t_{BR})$ and $f^i(t)$ defined there.) To ensure that the $\hat{x}$ coordinates are synchronous, we must require

$$f^i(t,\mathbf{x})=\partial_i\Delta t(t_N,\mathbf{x})\int^t\frac{dt'}{a^2(t')}+\omega_i(\mathbf{x}) ; \tag{5.37}$$

we shall set $w_i=0$ in what follows.

Requiring that $\hat{\partial}_0\hat{h}$ vanish, we find

$$\Delta t(t_N)=-\frac{1}{9}H^{-2}\left[1+\frac{2}{9}\left[\frac{k}{aH}\right]^2\right]^{-1}\dot{h}^{(B)}(t_N)\ , \tag{5.38}$$

where all time-dependent quantities on the right-hand side of this equation are evaluated at $t=t_N$. In these coordinates, the fractional energy-density perturbation $\hat{\delta}_B$ and the longitudinal peculiar velocity perturbation $\hat{v}_B^1$ are given by

$$\hat{\delta}_B(\hat{t}_N)=\delta_B(t_N)-3H(t_N)\Delta t(t_N)\ , \tag{5.39}$$

$$\hat{v}_B^1(\hat{t}_N)=v_B^1(t_N)-i\frac{k}{a(t_N)}\Delta t(t_N)\ . \tag{5.40}$$

These expressions obey the Newtonian equation $\hat{\partial}_0\hat{\delta}_B=-\hat{\partial}_i\hat{v}_B^i/\hat{a}$—hence the name "Newtonian" coordinates. It is straightforward to verify that the other "Newtonian" equation, $\hat{\partial}_0^2\hat{\delta}_B+2\hat{H}\hat{\partial}_0\hat{\delta}_B=(4\pi/m_P^2)\hat{\rho}_{bB}\hat{\delta}_B$ also holds in these coordinates.

Using the results of Sec. IV as well as those of Appendix B of Ref. [14], Eqs. (5.39) and (5.40) result in

$$\hat{\delta}_B(k,t_N)=(1+18\tilde{\lambda}_B^{-2}e^{-2x_N/3})^{-1}\times[(1+30\lambda_B^{-2}e^{-2x_N/3})c^{(B)}e^{2x_N/3}+(c_2^{(B)}-\tfrac{3}{4}c_8^{(B)})e^{-x_N}]\ , \tag{5.41}$$

$$\hat{v}_B^1(k,t_N)=(1+18\tilde{\lambda}_B^{-2}e^{-2x_N/3})^{-1}2i\tilde{\lambda}_B^{-1}\times[c^{(B)}e^{x_N/3}-\tfrac{3}{2}(c_2^{(B)}-\tfrac{3}{4}c_8^{(B)})e^{-4x_N/3}]\ , \tag{5.42}$$

where

$$e^{x_N}=\frac{1}{3\sqrt{3}}\left[\frac{\tilde{\lambda}_B}{k}\right]^{3/2}a^{3/2}(t_N)\ ,$$

and all the other symbols in Eqs. (5.41) and (5.42) have been defined in Sec. IV. It is pleasing to note that in the limit when $k\gg aH$ the dominant terms in Eq. (5.41) are proportional to $t^{2/3}$ and t^{-1}, which agrees with the results of the Newtonian analysis, Eq. (11.7) of Ref. [37]. In this limit the dominant terms in Eq. (5.42) are proportional to $t^{1/3}$ and $t^{-4/3}$, which agrees with Eq. (14.7) of Ref. [37].

From Eq. (5.41) one finds that the (spatial momentum space) fractional energy-density power spectrum, for perturbations that reentered the Hubble radius in the baryon-dominated epoch, is

$$\hat{P}_\delta(k,t_N)\equiv\langle\hat{\delta}_B(k,t_N)\hat{\delta}_B(-k,t_N)\rangle=\frac{16\pi}{m_P^2}k^{2(1-\nu)}\left[1+\frac{9}{2}\left[\frac{a(t_N)H(t_N)}{k}\right]^2\right]^{-2}\times\left\{|A|^2\left[1+\frac{15}{2}\left[\frac{a(t_N)H(t_N)}{k}\right]^2\right]^2a^2(t_N)+|B_\Delta|\left[1+\frac{15}{2}\left[\frac{a(t_N)H(t_N)}{k}\right]^2\right]a^{-1/2}(t_N)+|C_\Delta|^2a^{-3}(t_N)\right\}\ , \tag{5.43}$$

where we have used Eqs. (5.4) and (5.6) and the numerical prefactors $|A|$, $|B_\Delta|$, and $|C_\Delta|$ are defined by Eqs. (5.11)–(5.13). Similarly, the longitudinal peculiar velocity perturbation two-point function is given by

$$\langle\hat{v}_B^1(k,t_N)\hat{v}_B^1(-k,t_N)\rangle=\frac{16\pi}{m_P^2}k^{-2\nu}\left[1+\frac{9}{2}\left[\frac{a(t_N)H(t_N)}{k}\right]^2\right]^{-2}\times[|A_v|^2a(t_N)+|B_v|a^{-3/2}(t_N)+|C_v|^2a^{-4}(t_N)]\ , \tag{5.44}$$

where $|A_v|$, $|B_v|$, and $|C_v|$ are defined by

$$|A_v|=2\sqrt{3}\left[\frac{\tilde{\lambda}_B}{k}\right]^{-3/2}|A|\ , \tag{5.45}$$

$$|B_v|=-18\left[\frac{\tilde{\lambda}_B}{k}\right]^{-3}|B_\Delta|\ , \tag{5.46}$$

$$|C_v|=3\sqrt{3}\left[\frac{\tilde{\lambda}_B}{k}\right]^{-3/2}|C_\Delta|\ . \tag{5.47}$$

In terms of the power spectrum index $n\equiv2(1-v)$ [see the discussion below Eq. (5.13)] the power of spatial momentum in the prefactor of Eq. (5.43) is n (which is allowed to range from $-\infty$ to 1 in this inflation model of the very early Universe) and that in the perfactor of Eq. (5.44) is $n-2$ (which ranges from $-\infty$ to -1 in this model).

Note that in terms of the proper wave number, $\kappa=k/a(t_N)$, the factor, in Eqs. (5.43) and (5.44), $a(t_N)H(t_N)/k=H(t_N)/\kappa$. To determine the normalization of the power spectrum we are interested in the behavior of the power spectrum and the peculiar velocity perturbation two-point function on scales much smaller than the Hubble scale—for a momentum mode with a wavelength corresponding to such a scale the $H(t_N)/\kappa$ terms are not significant; however, they contribute a correction $\gtrsim 50\%$ to the "adiabatic" term in $[\langle\hat{\delta}_B(k)\hat{\delta}_B(-k)\rangle]^{1/2}$ [the term proportional to $|A|^2a^2(t_N)$ in Eq. (5.43)] for a momentum mode with a wavelength of the size of the Hubble scale and so are presumably of interest for an analysis of the large-scale microwave background anisotropy. In any case, even through we shall eventually be mostly interested in perturbations on scales well inside the Hubble radius, this Newtonian approximation is most properly made in position space rather than in momentum space.

To contrast the theory with observational data we need to derive position space expressions which are related to the fractional energy-density and longitudinal peculiar velocity two-point functions. These position space expressions should only sense a coarse-grain average of the corresponding quantity, so large spatial momentum modes cannot contribute significantly to them—mathematically this is accomplished by using a window function to suppress these contributions.

A measure of the departure away from homogeneity of the mass distribution (the fractional mass distribution) on a coordinate length scale $\hat{R}$, about a point $\hat{x}^i_0$, is

$$\frac{\delta\hat{M}}{\hat{M}}(\hat{t}_N|\hat{R})\equiv\frac{\int_{-\infty}^{\infty}[\hat{\rho}(\hat{x}^i,\hat{t}_N)-\hat{\rho}_b(\hat{t}_N)]W(|\hat{x}^i_0-\hat{x}^i|,\hat{R})\hat{a}^3(\hat{t}_N)d^3\hat{x}}{\int_{-\infty}^{\infty}\hat{\rho}_b(\hat{t}_N)W(|\hat{x}^i_0-\hat{x}^i|,\hat{R})\hat{a}^3(\hat{t}_N)d^3\hat{x}}\,, \tag{5.48}$$

where W is a window function (see Sec. 26 of Ref. [37]) whose precise form is determined by the manner in which the observation of the fractional mass distribution was performed. For our purposes it suffices to adopt a normalized Gaussian window,

$$W(|\hat{x}^i_0-\hat{x}^i|,\hat{R})=(2\pi\hat{R}^2)^{-3/2}\exp[-|\hat{x}^i_0-\hat{x}^i|^2/(2\hat{R}^2)]\,, \tag{5.49}$$

in which case Eq. (5.48) reduces to

$$\frac{\delta\hat{M}}{\hat{M}}(t_N|R)=\int_{-\infty}^{\infty}\hat{\delta}_B(\mathbf{x},t_N)W(|\mathbf{x}_0-\mathbf{x}|,R)d^3x\,, \tag{5.50}$$

where we have neglected the difference between the $\hat{x}$ and x coordinates since Eq. (5.50) is a first-order expression and the difference in coordinate systems will only contribute to this equation in the next order in perturbation theory (here R should not be confused with the subscript for radiation). Fourier expanding $\hat{\delta}_B$ and performing the spatial integration we find that the mean-square measure of the mass distribution is given by

$$\left\langle\left[\frac{\delta\hat{M}}{\hat{M}}(t_N|R)\right]^2\right\rangle=4\int_{-\infty}^{\infty}\frac{d^3k}{(2\pi)^3}\langle\hat{\delta}_B(\mathbf{k},t_N)\hat{\delta}_B(-\mathbf{k},t_N)\rangle e^{-|\mathbf{k}|^2R^2}\,, \tag{5.51}$$

where, as explained above, the factor of 4 on the right-hand side of this equation is the result of our Fourier expansion conventions. We emphasize that $\hat{\delta}_B$ is the quantity that determines the fractional mass distribution—one must not use the gauge-invariant fractional energy-density perturbation here.

A measure of the (longitudinal) local departure velocity from homogeneous expansion on a coordinate length scale $\hat{R}$, about a point $\hat{x}^i_0$, is

$$\delta\hat{V}^1(\hat{t}_N|\hat{R})=\frac{\int_{-\infty}^{\infty}\hat{v}^1_B(\hat{x}^i,\hat{t}_N)W(|\hat{x}^i_0-\hat{x}^i|,\hat{R})d^3\hat{x}}{\int_{-\infty}^{\infty}W(|\hat{x}^i_0-\hat{x}^i|,\hat{R})d^3\hat{x}}\,; \tag{5.52}$$

manipulations similar to those above, with a Gaussian window function, lead to the mean-square measure

$$\langle[\delta\hat{V}^1(t_N|R)]^2\rangle=4\int_{-\infty}^{\infty}\frac{d^3k}{(2\pi)^3}\langle\hat{v}^1_B(\mathbf{k},t_N)\hat{v}^1_B(-\mathbf{k},t_N)\rangle\times e^{-|\mathbf{k}|^2R^2}\,, \tag{5.53}$$

where we have again replaced $\hat{x}$ coordinates by x coordinates. We note that the momentum-space Gaussian window in this equation and in Eq. (5.51) suppresses large spatial momentum modes—these are the modes for which our analysis is incomplete (because of the neglect of the coupling between radiation and matter).

From Eq. (5.44) we find, in the exponential potential inflation model of the very early universe, that Eq. (5.53) reduces to

$$\begin{aligned}\langle[\delta\hat{V}^1(z_N|\mathcal{R})]^2\rangle=\frac{16\pi}{m_P^2}&[|\bar{A}_v|^2(1+z_N)^{(n-1)/2}+|\bar{B}_v|(1+z_N)^{2+n/2}+|\bar{C}_v|^2(1+z_N)^{(9+n)/2}]\\&\times\sum_{l=0}^{\infty}(l+1)\Bigg[(-\tfrac{9}{2})^l(\mathcal{R}H_N)^{2l-n-1}\Gamma(n/2+1/2-l)\\&\qquad-2(l+1)\sum_{j=0}^{\infty}\frac{(-1)^{j+l}(9/2)^{n/2+1/2+j}(\mathcal{R}H_N)^{2j}}{j![(n/2+3/2+j)^2-(l+1)^2]}\Bigg]\,,\end{aligned} \tag{5.54}$$

where z_N and H_N are the redshift and Hubble parameter at time $t=t_N$, the proper length scale $\mathcal{R}\equiv a(t_N)R$, and we have defined

$$|\bar{A}_\nu|=\frac{(2-q)^{\nu+1/2}}{5\sqrt{\pi q}}\frac{\csc(\nu\pi)}{\Gamma(-\nu)}(1+z_{R\Phi})^{2-\nu}H_{R\Phi}^{\nu-1/4}\times(1+z_{BR})^{3(\nu-3/2)/2}H_{BR}^{5/4-\nu}\,, \tag{5.55}$$

$$|\bar{B}_\nu|=-\frac{(2-q)^{2\nu+1}}{75\pi q}\frac{\csc^2(\nu\pi)}{\Gamma^2(-\nu)}(1+z_{R\Phi})^{1/2-2\nu}H_{R\Phi}^{2\nu+5/4}\times(1+z_{BR})^{3\nu-7/2}H_{BR}^{3/4-2\nu}\,, \tag{5.56}$$

$$|\bar{C}_\nu|=\frac{(2-q)^{\nu+1/2}}{30\sqrt{\pi q}}\frac{\csc(\nu\pi)}{\Gamma(-\nu)}(1+z_{R\Phi})^{-\nu-3/2}H_{R\Phi}^{\nu+3/2}\times(1+z_{BR})^{(3\nu-5/2)/2}H_{BR}^{-1/2-\nu}\,, \tag{5.57}$$

where $H_{R\Phi}$ and H_{BR} are the values of the Hubble parameter at the scalar-field–radiation and radiation-baryon transitions and $z_{R\Phi}$ and z_{BR} are the corresponding values of the redshift. The expression on the right-hand side of Eq. (5.54), which is derived in the Appendix, is formally valid only for $\nu<7/2$ (i.e., $n>-5$) because of an infrared divergence in the integrand, although nothing drastic seems to happen (to the right-hand side) if n does not obey this bound. We note that even for $n>-5$ there are values of n (for instance, -3, -1, 1) for which individual terms on the right-hand side are singular (these exceptional cases are examined more carefully below). For future reference, we note that from Eqs. (5.55)–(5.57) and the relation $(H_{R\Phi}/H_{BR})^{1/2}=(1+z_{R\Phi})/(1+z_{BR})$ we find

$$-\frac{1}{3}\frac{|\bar{A}_\nu|^2}{|\bar{B}_\nu|}=\frac{1}{6}\frac{|\bar{A}_\nu|}{|\bar{C}_\nu|}=(1+z_{BR})^{5/2}\,, \tag{5.58}$$

or, with the value we have assumed, for z_{BR},

$$-\frac{1}{3}\frac{|\bar{A}_\nu|^2}{|\bar{B}_\nu|}=\frac{1}{6}\frac{|\bar{A}_\nu|}{|\bar{C}_\nu|}\simeq 3\times10^{11}h^5\,; \tag{5.59}$$

this relation will allow us to simplify the large-time form of the local departure velocity, Eq. (5.54).

In a manner similar to the derivation of the mean-square measure of the local departure velocity, we find, from Eqs. (5.43) and (5.51), that the mean-square measure of the fractional mass distribution is given by

$$\left\langle\left[\frac{\delta\hat{M}}{\hat{M}}(z_N|\mathcal{R})\right]^2\right\rangle=\frac{16\pi}{m_P^2}\sum_{l=0}^{\infty}(-1)^l(l+1)\left\{\left[\frac{9}{2}\right]^l(\mathcal{R}H_N)^{2l-n-3}\left[|W_1|\Gamma\left[\frac{n}{2}+\frac{3}{2}-l\right]+|W_2|(\mathcal{R}H_N)^2\Gamma\left[\frac{n}{2}+\frac{1}{2}-l\right]+|W_3|(\mathcal{R}H_N)^4\Gamma\left[\frac{n}{2}-\frac{1}{2}-l\right]\right]\right.$$
$$+2(l+1)\sum_{j=0}^{\infty}\frac{(-1)^{j+1}}{j!}\left[\frac{9}{2}\right]^{n/2+3/2+j}\times(\mathcal{R}H_N)^{2j}\{|W_1|[(n/2+5/2+j)^2-(l+1)^2]^{-1}+\tfrac{2}{9}|W_2|[(n/2+3/2+j)^2-(l+1)^2]^{-1}$$
$$\left.+\tfrac{4}{81}|W_3|[(n/2+1/2+j)^2-(l+1)^2]^{-1}\}\right\}\,, \tag{5.60}$$

where

$$|W_1|=|\bar{A}_\nu|^2(1+z_N)^{1/2-\nu}-\frac{2}{3}|\bar{B}_\nu|(1+z_N)^{3-\nu}+\frac{4}{9}|\bar{C}_\nu|^2(1+z_N)^{11/2-\nu}\,, \tag{5.61}$$

$$|W_2|=15|\bar{A}_\nu|^2(1+z_N)^{1/2-\nu}-5|\bar{B}_\nu|(1+z_N)^{3-\nu}\,, \tag{5.62}$$

$$|W_3|=\frac{225}{4}|\bar{A}_\nu|^2(1+z_N)^{1/2-\nu}\,, \tag{5.63}$$

and $|\bar{A}_\nu|$, $|\bar{B}_\nu|$, and $|\bar{C}_\nu|$ have been defined in Eqs. (5.55)–(5.57). To derive Eq. (5.60) we must require $\nu<5/2$ (or $n>-3$) because of an infrared divergence in the integrand; this is a stronger constraint than that which follows from the derivation of Eq. (5.54). [It remains to be seen whether this divergence just means that we need to find a more careful way of defining the relevant position space expressions (in the $n<-3$ models); physically, these models have a lot of power on large scales and (aside from probably being observationally ruled out) should, perhaps, make sense.] In what follows we impose $-3<n\le1$, where the upper limit is a consequence of requiring that this particular class of models of the very early Universe inflate. We suspect that there is another class of inflation models which are not restricted

to $n \le 1-$ we hope to return to this elsewhere.

To determine the normalization of the power spectrum we normalize the mean-square measure of the local departure velocity, Eq. (5.54), to observational data on "great attractor" scales (~ 60 Mpc), Refs. [28]. The redshift at which the observations are made is $z_N \simeq 0$ and the "great attractor" scale corresponds to $\mathcal{R}H_N \ll 1$, so we only retain the leading term in Eq. (5.54). [As a consequence of Eq. (5.59) we may ignore $|\bar{B}_v|$ and $|\bar{C}_v|^2$ compared to $|\bar{A}_v|^2$.] There is a qualitative change in the asymptotic behavior of $\langle [\delta\hat{V}^1]^2 \rangle$ when n passes through -1; we find, on "small" scales and at the present epoch,

$$\langle [\delta\hat{V}^1(\mathcal{R})]^2 \rangle = \frac{16\pi}{m_P^2} Q \frac{2^{(5-n)/2}}{n+5} \frac{3^{n+1}}{-n-1} \Delta_n \quad \text{for } -3 \le n \le -1 \qquad (5.64)$$

[where $\Delta_n = (-n-1)/4$ if $n=-1, -3$ (this is discussed below) and unity otherwise] and

$$\langle [\delta\hat{V}^1(\mathcal{R})]^2 \rangle = \frac{16\pi}{m_P^2} Q \Gamma\left[\frac{n}{2}+\frac{1}{2}\right] \Delta_n (\mathcal{R}H_{\text{now}})^{-(n+1)} \quad \text{for } -1 < n \le 1 \qquad (5.65)$$

[where $\Delta_n = 1/4$ if $n=1$ (this is also discussed below) and unity otherwise], which is singular on very small scales. We note that our analysis ignores two major effects which will certainly change the very-small-scale behavior of $\langle (\delta\hat{V}^1)^2 \rangle$ (and could conceivably render it nonsingular)—the coupling of radiation to matter and the nonlinear effects associated with the late stage of development of irregularities. The change in the asymptotic behavior at $n=-1$ is well known; the main new feature is that, unlike in the standard scenario where the correlation function (5.44) does not have the extra k-dependent factor in the denominator (i.e., its k dependence is just k^{n-2}) and the infrared divergence must be cutoff, the expression here is not cutoff dependent. The coefficient Q in Eqs. (5.64) and (5.65) is

$$Q = \frac{2^{5-2n}}{25\pi} \frac{(3-n)^{n-2}}{1-n} \frac{\csc^2(n\pi/2)}{\Gamma^2(n/2-1)} z_{R\Phi}^{5-n} z_{BR}^{(n-9)/2} H_{BR}^2 , \qquad (5.66)$$

where we have made use of $z_{R\Phi} \gg 1$, $z_{BR} \gg 1$ and the relation $H_{R\Phi} = H_{BR}(z_{R\Phi}/z_{BR})^2$.

For the exceptional values of $n(=-3,-1,1)$ of interest, it can be shown that the term in large parentheses in Eq. (5.54) reduces to

$$\sum_{j=0}^{\infty} \frac{(-1)^{j+l}}{j!} \frac{(9/2)^{n/2+1/2+j}}{n/2+5/2+l+j} (\mathcal{R}H_N)^{2j} + \delta_{n,1}(-1)^l (\mathcal{R}H_N)^{-(n+1)} \exp[-\tfrac{9}{2}(\mathcal{R}H_N)^2] , \qquad (5.67)$$

where $\delta_{n,1}$ is unity for $n=1$ and vanishes for $n \neq 1$; in this form none of the terms in Eq. (5.54) are singular for the range of n of interest. This expression results in the correction factor Δ_n in Eq. (5.64) when $n=-3$ or -1. For $n=1$ it is important to remember that the ϵ^{-2} divergence is in the factor in front of the series in Eq. (5.54); i.e., the Fourier transform is not divergent at $n=1$, which is the value that must be used in Eq. (5.67). This results in the correction factor Δ_n in Eq. (5.65).

On small scales and at the present epoch the mean-square measure of the fractional mass distribution, Eq. (5.60), reduces to

$$\left\langle \left[\frac{\delta\hat{M}}{\hat{M}}(\mathcal{R})\right]^2 \right\rangle = \frac{16\pi}{m_P^2} Q \Gamma\left[\frac{n}{2}+\frac{3}{3}\right] (\mathcal{R}H_{\text{now}})^{-(n+3)} ; \qquad (5.68)$$

it is pleasing to note that this scale dependence agrees with Eq. (26.5) of Ref. [37]. This equation [like Eq. (5.65)] is singular on very small scales—similar caveats also hold here.

Parenthetically, we note that if we retain only the adiabatic term in Eq. (5.10), we find, in the original synchronous coordinates, that the present fractional mass distribution agrees with Eq. (5.68), i.e.,

$$\left\langle \left[\frac{\delta M}{M}(\mathcal{R})\right]^2 \right\rangle = \left\langle \left[\frac{\delta\hat{M}}{\hat{M}}(\mathcal{R})\right]^2 \right\rangle \Bigg|_{\mathcal{R}H_{\text{now}} \ll 1} . \qquad (5.69)$$

This is expected since, inside the Hubble radius, the adiabatic mode is invariant to the remnants of general coordinate invariance in synchronous gauge. In these coordinates and with the further choice $c_2^{(B)}=0$ [to fix the remnants of general coordinate invariance in synchronous gauge, see the discussion in the paragraph above Eq. (5.11)] we find that the mean-square measure of the local departure velocity from homogeneous expansion is given by

$$\langle [\delta V^1(\mathcal{R})]^2 \rangle = \frac{16\pi}{m_P^2} Q_V \Gamma\left[\frac{n}{2}+\frac{5}{2}\right] (\mathcal{R}H_{\text{now}})^{-(n+5)} \qquad (5.70)$$

[which differs from both Eqs. (5.64) and (5.65)], where

$$Q_V = \tfrac{4}{81} |\bar{C}_v|^2 , \qquad (5.71)$$

and $|\bar{C}_v|$ is defined by Eq. (5.57).

It is conventional to express the observed local departure velocity from homogeneous expansion in terms of the corresponding Hubble velocity:

$$\langle [\delta\hat{V}^1(\mathcal{R})]^2 \rangle = \gamma^2 (\mathcal{R}H_{\text{now}})^2 , \qquad (5.72)$$

where the great attractor observations (tentatively) suggest $\gamma \simeq 0.1$ on scales $\mathcal{R}H_{\text{now}} \simeq 2\times 10^{-2}$, Refs. [28].

Combining Eqs. (5.64)–(5.66) and (5.72) we find

$$z_{R\Phi}^{n-5} = \frac{3^n 2^{5(5-n)/2}}{25\pi} \frac{(n-2)^2(3-n)^{n-2}}{(n+5)(n^2-1)} \Gamma^2\left[1-\frac{n}{2}\right] \times \frac{\Delta_n}{\gamma^2} \frac{\rho_{\text{now}}}{m_P^4} z_{BR}^{(n-3)/2} (\mathcal{R}H_{\text{now}})^{-2} \quad \text{for } -3 \le n \le -1 , \qquad (5.73)$$

and

$$z_{R\Phi}^{n-5}=\frac{2^{2(5-n)}}{75\pi}\frac{(n-2)^2(3-n)^{n-2}}{1-n}\Gamma\left[\frac{n}{2}+\frac{1}{2}\right]\Gamma^2\left[1-\frac{n}{2}\right]\times\frac{\Delta_n}{\gamma^2}\frac{\rho_{\rm now}}{m_P^4}z_{BR}^{(n-3)/2}(\mathcal{R}H_{\rm now})^{-(n+3)}\quad\text{for } -1<n\le 1\ , \tag{5.74}$$

where $\rho_{\rm now}$ is the present homogeneous background energy density in baryons (with $\Omega_{\rm now}=1$) and $m_P{}^{-4}\rho_{\rm now}\simeq 3.6\times 10^{-123}h^2$. Using the numerical values chosen above, these expressions reduce to

$$z_{R\Phi}\simeq 75\sqrt{2}(3-n)\left[\frac{(3-n)^3(2-n)^2}{(n+5)(n^2-1)}\Gamma^2\left[1-\frac{n}{2}\right]\Delta_n\times 1.1\times 10^{-112}\right]^{1/(n-5)}h^{(n-1)/(n-5)}\quad\text{for } -3\le n\le -1\ , \tag{5.75}$$

and

$$z_{R\Phi}\simeq 2.5\times 10^3(3-n)\times\left[\frac{(3-n)^3(2-n)^2}{1-n}\Gamma\left[\frac{n}{2}+\frac{1}{2}\right]\Gamma^2\left[1-\frac{n}{2}\right]\Delta_n\times 2.4\times 10^{-105}\right]^{1/(n-5)}h^{(n-1)/(n-5)}\quad\text{for } -1<n\le 1\ . \tag{5.76}$$

In the limit when this model of the early Universe reduces to the exponential expansion inflation model ($n=1-2\epsilon^2$) Eq. (5.76) reduces to

$$z_{R\Phi}\simeq 5.4\times 10^{29}\sqrt{\epsilon}h^{\epsilon^2/2}\ , \tag{5.77}$$

we note that this number is not unreasonable even for fairly small ϵ. To get a rough idea of the range of numerical values, we note that Eqs. (5.75) and (5.76) imply

$$z_{R\Phi}\simeq\begin{cases}3.8\times 10^{26}h^{1/9} & (n=1/2)\ ,\\ 2.2\times 10^{24}h^{1/5} & (n=0)\ ,\\ 2.9\times 10^{22}h^{3/11} & (n=-1/2)\ ,\\ 1.2\times 10^{21}h^{1/3} & (n=-1)\ ,\\ 3.5\times 10^{19}h^{5/13} & (n=-3/2)\ ,\\ 2.4\times 10^{18}h^{3/7} & (n=-2)\ ,\\ 2.3\times 10^{17}h^{7/15} & (n=-5/2)\ ,\\ 3.1\times 10^{16}h^{1/2} & (n=-3)\ ,\end{cases} \tag{5.78}$$

where the first four expressions have been derived from Eq. (5.76) and the last four from Eq. (5.75).

The characteristic features of the trend in Eqs. (5.77) and (5.78) seem to be a consequence of two interesting phenomena: the stability of the model of the very early Universe to small spatial irregularities and the anomalous displacement, in de Sitter spacetime, of the spatially homogeneous local energy-density hypersurfaces relative to the constant time hypersurfaces. We have shown, in Sec. III of Ref. [7] and in Sec. III C here, that small spatial irregularities decay, as a power of time with exponent $-4/(1-n)$, even though there are nonzero spatial curvature perturbations; we note that these perturbations decay the slowest for $n=-3$ and the fastest for $n=1-2\epsilon^2$. If we require that the power spectrum has a fixed amplitude, on a given scale in the large-time universe, then as we raise n from -3 the model would have to compensate for the faster decay at larger n by spending more time in the radiation-dominated epoch (relative to a model with a lower value of n) to take advantage of the fact that perturbations grow on scales larger than the Hubble scale in this epoch. However, when we raise n towards 1 the geometric property of anomalous hypersurface displacement in de Sitter spacetime starts becoming important—since this results in a very large value of the fractional energy-density perturbation after the scalar-field–radiation transition the model cannot afford to spend too much time in the radiation-dominated epoch (when n is near 1) if the power spectrum in the large-time universe is to have a fixed amplitude—this is the reason why $z_{R\Phi}$ in Eq. (5.77) is $\propto\sqrt{\epsilon}$.

The energy scale at reheating may be conveniently expressed, in units of the Planck mass, as

$$\left[\frac{\rho_{R\Phi}}{m_P^4}\right]^{1/4}\simeq 1.7\times 10^{-32}z_{R\Phi}\ , \tag{5.79}$$

where $\rho_{R\Phi}$ is the value of the homogeneous background energy density at the scalar-field–radiation transition and $z_{R\Phi}$ is given by either Eq. (5.75) or Eq. (5.76). In more conventional units the energy scale at the scalar-field–radiation transition, $E_{R\Phi}=(\rho_{R\Phi})^{1/4}$ is

$$E_{R\Phi}\simeq 2.1\times 10^{-13}z_{R\Phi}\ \text{GeV}\ . \tag{5.80}$$

In the exponential expansion inflation model this expression reduces to

$$E_{R\Phi}\simeq 1.1\times 10^{17}\sqrt{\epsilon}h^{\epsilon^2/2}\ \text{GeV}\ , \tag{5.81}$$

which, as expected, is not unreasonably small, even for small ϵ. As n is varied, the numerical range of this energy scale is

$$E_{R\Phi}\ (\text{in GeV})\simeq\begin{cases}8.0\times 10^{13}h^{1/9} & (n=1/2)\ ,\\ 4.6\times 10^{11}h^{1/5} & (n=0)\ ,\\ 6.1\times 10^{9}h^{3/11} & (n=-1/2)\ ,\\ 2.5\times 10^{8}h^{1/3} & (n=-1)\ ,\\ 7.4\times 10^{6}h^{5/13} & (n=-3/2)\ ,\\ 5.0\times 10^{5}h^{3/7} & (n=-2)\ ,\\ 4.8\times 10^{4}h^{7/15} & (n=-5/2)\ ,\\ 6.5\times 10^{3}h^{1/2} & (n=-3)\ .\end{cases} \tag{5.82}$$

Having related the redshift of the scalar-field–radiation transition to the power spectrum index, we now turn to estimating the value of other observable quantities. The following analysis is, of course, almost standard; the main new feature is that, unlike in the standard cosmological scenarios, the amplitude and spectral index of all the perturbations are now determined once the value of n is specified (in the standard scenarios the amplitude of gravitational-wave perturbations are not necessarily fixed once the corresponding quantity for energy-density irregularities has been fixed.

Given an estimate of the mean-square local departure velocity from homogeneous expansion on a scale $\mathcal{R}H_{\rm now} \ll 1$, Eq. (5.72), the mean-square measure of the fractional mass distribution, Eq. (5.68), on the same scale, is

$$\left\langle \left[\frac{\delta \hat{M}}{\hat{M}}(\mathcal{R}) \right]^2 \right\rangle = \frac{2^{(n-5)/2}}{3^{n+1}}(-n-1)(n+5) \times \Gamma\left[\frac{n}{2}+\frac{3}{2}\right] \frac{\gamma^2}{\Delta_n}(\mathcal{R}H_{\rm now})^{-(n+1)} \quad \text{for } -3 < n \le -1 \tag{5.83}$$

[$n=-3$ is no longer allowed because the fractional mass distribution transform diverges in the infrared], and

$$\left\langle \left[\frac{\delta \hat{M}}{\hat{M}}(\mathcal{R}) \right]^2 \right\rangle = \frac{n+1}{2}\frac{\gamma^2}{\Delta_n} \quad \text{for } -1 < n \le 1 \ . \tag{5.84}$$

On scales $\mathcal{R}H_{\rm now} \simeq 2\times 10^{-2}$ with $\gamma \simeq 0.1$ these equations result in

$$\left\langle \left[\frac{\delta \hat{M}}{\hat{M}} \right]^2 \right\rangle^{1/2} \simeq \begin{cases} 0.2 \ (n=1) \ , \\ 8.6\times 10^{-2} \ (n=1/2) \ , \\ 7.1\times 10^{-2} \ (n=0) \ , \\ 5.0\times 10^{-2} \ (n=-1/2) \ , \\ 0.14 \ (n=-1) \ , \\ 2.3\times 10^{-2} \ (n=-3/2) \ , \\ 1.7\times 10^{-2} \ (n=-2) \ , \\ 1.2\times 10^{-2} \ (n=-5/2) \ ; \end{cases} \tag{5.85}$$

the mathematical reason for the rather large changes at $n=-1,1$ is the factor Δ_n in Eqs. (5.83) and (5.84), the physics behind them is not yet as obvious.

The fractional mass distribution on a scale $\tilde{\mathcal{R}}H_{\rm now}$ is related to that on the scale $\mathcal{R}H_{\rm now}$ on which the local departure velocity is measured through

$$\left\langle \left[\frac{\delta \hat{M}}{\hat{M}}(\tilde{\mathcal{R}}) \right]^2 \right\rangle^{1/2} = \left\langle \left[\frac{\delta \hat{M}}{\hat{M}}(\mathcal{R}) \right]^2 \right\rangle^{1/2} \left[\frac{\mathcal{R}H_{\rm now}}{\tilde{\mathcal{R}}H_{\rm now}} \right]^{(n+3)/2} , \tag{5.86}$$

where the fractional mass distribution on the scale $\mathcal{R}H_{\rm now}$ is given by either Eq. (5.83) or (5.84). To get a very rough estimate of the fractional mass distribution on smaller scales we consider a scale $\tilde{\mathcal{R}}H_{\rm now} \simeq 10^{-2}$ (the neglect of the coupling between radiation and matter means that these numbers should not be taken too seriously); we find

$$\left\langle \left[\frac{\delta \hat{M}}{\hat{M}} \right]^2 \right\rangle^{1/2} \simeq \begin{cases} 0.80 \ (n=1) \ , \\ 0.29 \ (n=1/2) \ , \\ 0.20 \ (n=0) \ , \\ 0.12 \ (n=-1/2) \ , \\ 0.28 \ (n=-1) \ , \\ 0.039 \ (n=-3/2) \ , \\ 0.024 \ (n=-2) \ , \\ 0.014 \ (n=-5/2) \ ; \end{cases} \tag{5.87}$$

although the numbers are quite crude, they do illustrate the problem (at small scales) with shifting the power to large wavelengths—this effect will probably result in a later epoch of galaxy formation in models with a smaller value of n. To get a rough estimate of the behavior of the fractional mass distribution on larger scales we consider a scale $\mathcal{R}H_{\rm now} \simeq 0.2$ [this number was chosen as a compromise since we expect the subleading term in the series (5.60), on this scale, to modify the numerical values below only by $\simeq 5-10\%$]. On this scale, we find

$$\frac{1}{3}\left\langle \left[\frac{\delta \hat{M}}{\hat{M}} \right]^2 \right\rangle^{1/2} \simeq \begin{cases} 6.6\times 10^{-4} \ (n=1) \ , \\ 5.1\times 10^{-4} \ (n=1/2) \ , \\ 7.5\times 10^{-4} \ (n=0) \ , \\ 9.4\times 10^{-4} \ (n=-1/2) \ , \\ 4.7\times 10^{-3} \ (n=-1) \ , \\ 1.3\times 10^{-3} \ (n=-3/2) \ , \\ 1.8\times 10^{-3} \ (n=-2) \ , \\ 2.2\times 10^{-3} \ (n=-5/2) \end{cases} \tag{5.88}$$

(here we have ignored the general-relativistic redshift correction, which should not change the order of magnitude of these numbers). These numbers give a very rough, order of magnitude, estimate of $\delta T/T$ (the fraction perturbation in the microwave background temperature) on large scales. Except for the anomalous jump at $n=-1$ the general trend is what one would expect. These numbers also suggest that the models with $n \lesssim -1$ are probably inconsistent with the observational constraints on large-scale $\delta T/T$ [29]. A more accurate estimate will require, at the very least, retaining higher-order terms in Eq. (5.60). One might also need to examine more carefully the gauge in which the fractional mass distribution is related to $\delta T/T$ (since the major contributor is the adiabatic mode, one might expect this to not be a very significant effect, however, the issue of gauge choice is complicated by the fact that the perturbations one is interested in are on scales $\mathcal{R}H_{\rm now} \sim 1$).

The difference between the functional dependence on $\mathcal{R}H_{\rm now}$ of Eqs. (5.64) and (5.65) might prove useful in constraining the power spectrum index (at least as far as this model is concerned since the difference is model dependent, although one might hope not sensitively so, i.e., it might be a rather generic feature of the behavior of large-time velocity perturbations in the inflation modified gravitational instability scenario). If the observed fractional departure velocity scales as

$$\left\langle \left[\frac{\delta \hat{V}^1}{\hat{V}_H} \right]^2 \right\rangle^{1/2} \propto (\mathcal{R} H_{\rm now})^{-1} , \quad (5.89)$$

where $\hat{V}_H$ is the Hubble velocity on the same scale, one might be lead to suspect $n \lesssim -1$; if the observed functional form is steeper,

$$\left\langle \left[\frac{\delta \hat{V}^1}{\hat{V}_H} \right]^2 \right\rangle^{1/2} \propto (\mathcal{R} H_{\rm now})^{-\beta} , \quad (5.90)$$

where $\beta > 1$ ($\beta = 2$ is the predicted value for the scale-invariant model), one might believe $n \gtrsim -1$ with $n = 2\beta - 3$. It is conceivable that a slight improvement on the present observational data might suffice for this test. One issue that might need to be examined is the precise relation between the longitudinal local departure velocity studied here and the observed local departure velocity, which is not necessarily longitudinal.

We defer an examination of the observational constraints on gravitational-wave perturbations to future work and close this section with a speculative remark. We note that in a cosmological model that is not based on a theory of the very early Universe (for instance, the canonical CDM model for which one assumes as initial conditions, at early time, the amplitude and spectral index of the energy-density irregularity power spectrum), the amplitude of the gravitational-wave energy-density spectrum must be assumed as an initial condition at early time. This is not necessary in the model studied here; the amplitude and spectral index of the gravitational-wave energy-density spectrum is determined, from the quantum mechanics of inflation, in terms of the same quantities which determine the power spectrum—$z_{R\Phi}$ and q. In particular, in Sec. V C we found that, on large scales, $\varepsilon_i \propto k^{-2\nu}$ [we note that the quantity directly related to the mean-square microwave background fractional temperature anisotropy is proportional to $k^2 \langle h_i(k) h_i(-k) \rangle$, which, on large scales, has the same spatial momentum dependence as $\varepsilon_i(k)$]; if the energy density of gravitational waves, $\rho_{\rm GW} \propto \int d^3k (\varepsilon_+ + \varepsilon_\times)$, is not to diverge in the infrared we must require $n > -1$. The ultraviolet behavior of the integrand does not result in a constraint on n since our expressions do not hold on small scales. If it is sensible to insist that the integrand in $\rho_{\rm GW}$ not diverge in the infrared (we note that the constraint $n > -3$ came from the similar requirement on $\langle (\delta \hat{M} / \hat{M})^2 \rangle$ in the infrared; however, this constraint seems to have more than a little support from the large-scale observational upper bounds on $\delta T / T$), then we are lead to suspect that $\langle (\delta \hat{V}^1 / \hat{V}_H)^2 \rangle^{1/2}$ must drop faster than $(\mathcal{R} H_{\rm now})^{-1}$, Eq. (5.90); if it is found to drop like $(\mathcal{R} H_{\rm now})^{-1}$ then this simple model of inflation is probably not the correct theory of the very early Universe (or, at the very least, one might either have to make sense of a gravitational-wave energy density that diverges in the infrared or speculate about what might have happened "before" inflation).

VI. CONCLUSION

We have constructed a simple, semirealistic, model of an inflation modified hot big-bang cosmology which may be used to derive, among other things, an expression for the (model-dependent) large-time, baryon-dominated epoch, form of the power spectrum of energy-density irregularities. Our model approximates the evolution of the Universe by dividing it into three distinct epochs: an early scalar-field–dominated inflation epoch, during which the expansion of the Universe is driven by a scalar field with an exponential potential; an intermediate, radiation-dominated, epoch; and the present, baryon-dominated ($\Omega_{\rm now} = 1$), epoch. In each of these epochs we have only accounted for perturbations in the dominant form of matter; also, the transitions between epochs have been approximated as instantaneous. Although we have argued that this approximate cosmological model suffices for the purpose of determining fairly accurate (or, at the very least, not grossly inaccurate) expressions for energy density (and gravitational-wave) irregularities in the large-time, baryon-dominated, epoch, we have not supported this belief with a quantitative underpinning. This issue certainly deserves further investigation—in particular, it would be interesting to know if it is possible to construct a simple extension of our model that includes the effects of reheating (and whether the effects associated with this transition significantly affect our conclusions). It would be difficult to overemphasize the need for a simple, semirealistic, inflation cosmological model that may be used to quantitatively analyze the entropy production and baryosynthesis subscenarios of canonical inflation.

We have presented closed-form solutions of the relativistic linear perturbation equations which govern the evolution of inhomogeneities in this simple class of inflation models. These closed-form expressions for the irregularities depend on constants of integration. To determine these constants of integration in the inflation epoch expressions we have adapted the, now standard, quantum-mechanical initial conditions of Refs. [8,9] for our classical analysis here. The inflation epoch expressions that follow from these initial conditions agree with those derived from a purely quantum-mechanical analysis [15]—this justifies the use of these initial conditions here. To determine the constants of integration in the large-time (radiation- and baryon-dominated epoch) expressions we have used joining conditions derived by requiring that the equations of motion for the inhomogeneities do not become singular at the transition hypersurfaces [16]. These joining conditions are necessary since we have only retained perturbations in the dominant component of the stress tensor in each epoch. Again, the issue of whether our single "fluid" instantaneous transition model is a good approximation to the real Universe could do with further study—we hope to eventually return to this elsewhere.

We have used these large-time expressions to derive the large-time form of the power spectrum of large-scale (i.e., an expression that is not valid on very small scales because of the neglect of the coupling between baryons and radiation) energy-density irregularities, the spectrum of the local departure velocity from homogeneous expansion and the energy-density spectrum of gravitational-wave perturbations. We have found that the energy-density perturbation power spectrum "diverges" in the limit in which the exponential-potential scalar field model reduces to the step function potential model (the exponential-expansion inflation limit). Although it is possible that a more complete treatment of the reheating transition or a careful accounting for the back reaction of the perturbations on the background will smooth the "divergence," we suspect that the change will not be qualitatively significant.

Since we have decided not to restrict ourselves to a particular microphysics-based model, we have had to normalize the power spectrum by comparing to observational data. (It is certainly of interest to attempt to find a consistent microphysical model that underlines this macrophysical model and to examine whether the numerical values of the microphysical parameters of the model that are observationally preferred conflict or agree with the range of numerical values that the underlying microphysics might suggest. Of course, the real test of the validity of this macrophysical model of inflation is observational.) Normalizing the power spectrum entails transforming the relevant theoretical expressions to the instantaneously Newtonian synchronous coordinate system "used" in the observations and comparing suitably coarse-grain averaged forms of these expressions to the corresponding observational measurements. [We note that although the use of gauge-invariant combinations of the dynamical variables tends to somewhat simplify the perturbation equations of motion, they do not facilitate the comparison to observational data (which is determined in a preferred gauge). In fact, as far as we are aware, the issue of comparing theory to observation has not yet been examined in the gauge-invariant formalism.] To normalize the power spectrum we have compared the derived mean-square measure of the local departure velocity from homogeneous expansion to that observationally determined on "great attractor" scales. This results in a relation that determines the redshift of reheating, $z_{R\Phi}$, in terms of the power-law index q of the inflation model scalar field potential.

We have found that as the fractional energy-density spectral index is raised from -3 the main effect is a rise in the redshift of reheating. When n approaches 1 (the scale-invariant value) the de Sitter spacetime "divergence" tends to lower $z_{R\Phi}$. The reason for this behavior was explained in Sec. V D. It is not yet clear if the $z_{R\Phi}-n$ relation is monotonic between -3 and ~ 1 or whether there is fine-structure superimposed on the general trend. The de Sitter spacetime "divergence" does not seem to be a particularly severe problem (except in the exact exponential expansion inflation limit, a limit in which our analysis is no longer valid and a limit which, in any case, is probably not very physically realistic). Our preliminary comparison to the observations suggests that models which stopped inflating in the range 10^7-10^{16} GeV (a given number is correlated with a fixed value of the spectral index n) are not obviously inconsistent.

Although we have focused on a macrophysical inflation model of the very early Universe, it might be argued that this large spread in energy scale at which inflation must end indicates that it is not difficult to construct models with small enough fractional energy-density perturbations at large times. One might even speculate that the main effects of the underlying microphysics would be to modify our simple model of the reheating transition; i.e., if the microphysics ensures sufficient inflation then the main qualitative effect of varying microphysical parameters (coupling constants) is to influence reheating dynamics and not the large-time amplitude of the fractional mass distribution (the microphysics would, of course, determine $z_{R\Phi}$).

Although the spread in the energy scale at which inflation must have ended, 10^7-10^{16} GeV, is large, the bounds seem to carry some nontrivial information. It is interesting to note that, in the context of this macrophysical model, it is probably not possible to associate inflation with the electroweak transition. The models at the lower end of this range, those which stop inflating late, will probably require a new mechanism for generating the observed baryon asymmetry. What, however, seems to be somewhat remarkable is the upper bound of $\sim 10^{16}$ GeV—if inflation, is responsible for the observed large-scale structure, then, in the context of this model, something must happen on energy scales ($\lesssim 10^{-3} m_P$) well below the Planck scale—this is very fortunate since it holds out hope that a model of the very early Universe, free of the complexities of quantum gravitation (and so a model that can be analyzed with the methods we have developed), might be able to explain the observed large-scale structure of the Universe. Rubakov, Sazhin, and Veryaskin [34] have also found a qualitatively similar result in a more restricted model (the scale-invariant model) from an analysis of the effects of gravitational-wave perturbations (which do not diverge in the exponential expansion inflation limit) on the microwave background spatial anisotropy (through a Sachs-Wolfe-like effect). (That the analysis of two "independent" observational tests suggest similar conclusions is, perhaps, not just a coincidence). Now that we have slightly more precise estimates for gravitational-wave perturbations we hope to be able to present a quantitatively more precise upper bound elsewhere.

The cosmological model we have considered here is, at large times, a spatially flat, baryon-dominated model with an adiabatic power spectrum $\propto k^n$ where $-3 < n \leq 1$ (in the original synchronous gauge coordinates; the power spectrum changes when transformed to instantaneously Newtonian coordinates). It might be argued that this model is sufficiently general yet simple enough to be used for a fairly reliable quantitative analysis of the scalar-field-dominated inflation scenario. (We suspect that there is an equally simple model which fills in, at least part of, the rest of the range of n of interest, $1 < n < 4$—we hope to return to this elsewhere.) The reason we believe that this model is (cosmologically) general enough is because it is the simplest (and probably unique, within a set of assumptions) scalar field model of the very early Universe that underlies part of the range of standard large-time adiabatic cosmological models on which attention has been focused. It is also more restrictive (and hence easier tested) than a standard cosmological model with the same power spectrum because the inflation epoch quantum-mechanical initial conditions make the expressions for the perturbations both more precise than in a conventional cosmological model and produce interrelations between the expressions for a variety of different perturbations (energy density, peculiar velocity and gravitational wave) which allows for the use of a variety of "independent" observational tests to attempt to coherently constrain the model.

If a power spectrum of the form that results in this model is thought not too inconsistent with the available observational data then a detailed comparison between the large-time theoretical expressions and the observations is called for. This is likely to be a more productive undertaking for such an inflation-based model (than for the case of a more phenomenological cosmological model). In recent years it has become clear that there are over half a dozen "independent" large-time observational tests that may be used to judge the validity of using a given inflation model to approximate the physics of the early Universe. In particular, the large-time fractional energy-density spectrum derived here should prove useful in determining an estimate for the spatial anisotropy in the temperature of the cosmic microwave background. This anisotropy, once detected, will eventually provide a fairly sensitive discriminator between various scenarios for structure formation and might provide the first observational information about very-short-distance physics. Current upper bounds on gravitational-wave perturbations, on both small scales and large scales, might also constrain the possible range of the relevant short-distance physics. We plan to return to the comparison of inflation theory with observational data in due course of time.

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APPENDIX: DERIVATION OF EQ. (5.54)

Here we consider the derivation of Eq. (5.54). From Eqs. (5.44) and (5.53) we have

$$\langle[\delta\hat{V}^1(t_N|R)]^2\rangle = Q\int_0^\infty dk\, k^{2(3-\nu)}\left[1+\frac{2}{9}\left[\frac{k}{a_N H_N}\right]^2\right]^{-2} \times e^{-k^2R^2}, \tag{A1}$$

where we have performed the momentum space angular integrations in Eq. (5.53), $a_N = a(t_N)$, $H_N = H(t_N)$ and

$$Q = \frac{16\pi}{m_P^2}\frac{8}{81}\frac{1}{\pi^2}(a_N H_N)^{-4} \times(|A_v|^2 a_N + |B_v| a_N^{-3/2} + |C_v|^2 a_N^{-4}) . \tag{A2}$$

We note that for large $k(\gg R^{-1})$ the exponential factor in the integrand in Eq. (A1) makes it sufficiently well behaved, while for nonsingular infrared behavior we must require $\nu < 7/2$. It is convenient to change variables, $k \to x$ where $x = k^2R^2$; this change, along with the definitions $\alpha = (2/9)(a_N R H_N)^{-2}$ and $\beta = 5/2-\nu$, results in

$$\langle[\delta\hat{V}^1(t_N|R)]^2\rangle = \frac{1}{2}QR^{2\nu-7}\int_0^\infty dx\, x^\beta(1+\alpha x)^{-2}e^{-x} . \tag{A3}$$

The integral in this equation may be reexpressed as

$$\int_0^{1/\alpha-\epsilon_1} dx\, x^\beta[1+\alpha x]^{-2}e^{-x} + \alpha^{-2}\int_{1/\alpha+\epsilon_2}^\infty dx\, x^{\beta-2} \times[1+1/(\alpha x)]^{-2}e^{-x} , \tag{A4}$$

where the infinitesimal positive convergence factors ϵ_1, ϵ_2 have been introduced to allow the binomial expansion of the terms in square parentheses, and it is understood that the equivalence is in the limit when they vanish. It is clear that $\alpha x < 1$ in the first integral while $1/(\alpha x) < 1$ in the second and so the standard assumption of the binomial expansion is satisfied. Expanding and interchanging the order of integration and summation (there are no divergences), we find Eq. (A4) becomes

$$\sum_{l=0}^\infty (-1)^l(l+1)\left[\alpha^l\int_0^{1/\alpha-\epsilon_1} dx\, x^{\beta+l}e^{-x} + \alpha^{-(l+2)}\int_{1/\alpha+\epsilon_2}^\infty dx\, x^{\beta-l-2}e^{-x}\right] . \tag{A5}$$

Integral representations of the incomplete gamma functions are (Ref. [38] Eqs. (6.5.2) and (6.5.3))

$$\gamma(b,u) = \int_0^u dx\, x^{b-1}e^{-x} \quad (\mathrm{Re}\, b > 0) , \tag{A6}$$

$$\Gamma(b,u) = \int_u^\infty dx\, x^{b-1}e^{-x} ; \tag{A7}$$

using these equations Eq. (A5) becomes

$$\sum_{l=0}^\infty (-1)^l(l+1)[\alpha^l\gamma(\beta+l+1, 1/\alpha-\epsilon_1) + \alpha^{-(l+2)}\Gamma(\beta-l-1, 1/\alpha+\epsilon_2)] . \tag{A8}$$

Furthermore

$$\Gamma(b,u) = \Gamma(b) - \gamma(b,\mu) \tag{A9}$$

(Eq. (6.5.3) of Ref. [38]), and

$$\gamma(b,u) = \sum_{j=0}^\infty \frac{(-1)^j}{j!}\frac{u^{b+j}}{(b+j)} \quad (|u| < \infty) \tag{A10}$$

(Eqs. (6.5.4) and (6.5.29) of Ref. [38]).

Using Eqs. (A9) and (A10) we find that Eq. (A8) reduces to

$$\sum_{l=0}^{\infty}(-l)^l(l+1)\left[\alpha^{-(l+2)}\Gamma(\beta-l-1)+\sum_{j=0}^{\infty}\frac{(-1)^{j+1}}{j!}\alpha^{-\beta-j-1}\left[\frac{2(l+1)}{[(\beta+j)^2-(l+1)^2]}+\alpha(\epsilon_1+\epsilon_2)\right]\right]\tag{A11}$$

where we have made use of the fact that $\alpha\epsilon_1 \ll 1$ and $\alpha\epsilon_2 \ll 1$; since the first term in the large parentheses does not vanish (for $j,l<\infty$) we may now set $\epsilon_1=0=\epsilon_2$. Equation (5.54) follows on combining Eqs. (A3) and (A11) with some straightforward algebra.

The derivation of Eq. (5.60) proceeds in a manner similar to that of Eq. (5.54) and will not be recorded here.

[1] G. R. Blumenthal, S. M. Faber, J. R. Primack, and M. J. Rees, Nature (London) **311**, 517 (1984); M. Davis, G. Efstathiou, C. S. Frenk, and S. D. M. White, Astrophys. J. **292**, 371 (1985), and references therein.

[2] E. R. Harrison, Phys. Rev. D **1**, 2726 (1970); P. J. E. Peebles and J. T. Yu, Astrophys. J **162**, 815 (1970); Ya. B. Zel'dovich, Mon. Not. R. Astron. Soc. **160**, 1P (1972).

[3] P. J. E. Peebles, in *The Structure of the Universe,* edited by J. Audouze (Reidel, Dordrecht, 1987).

[4] L. A. Kofman and A. D. Linde, Nucl. Phys. **B282**, 555 (1987); J. M. Bardeen, J. R. Bond, and G. Efstathiou, Astrophys. J. **321**, 28 (1987); J. M. Bardeen, J. R. Bond, and D. S. Salopek, CITA report, 1987 (unpublished); G. R. Blumenthal, A. Dekel, and J. R. Primack, Astrophys. J **326**, 539 (1988); J. R. Bond, D. S. Salopek, and J. M. Bardeen, CITA report, 1988 (unpublished).

[5] P. J. E. Peebles, Astrophys. J. **284**, 439 (1984).

[6] P. J. E. Peebles and B. Ratra, Astrophys. J. Lett. **325**, L17 (1988).

[7] B. Ratra and P. J. E. Peebles, Phys. Rev. D **37**, 3406 (1988).

[8] B. Ratra, Phys. Rev. D **31**, 1931 (1985).

[9] W. Fischler, B. Ratra, and L. Susskind, Nucl. Phys. **B259**, 730 (1985).

[10] An alternate scenario for the formation of large-scale structure is the baryonic isocurvature scheme, P. J. E. Peebles, Nature (London) **327**, 210 (1987); Astrophys. J. Lett. **315**, L73 (1987).

[11] The fact that the evolution $a(t)\propto t^p$ with $p>1$ describes an inflation cosmology was noted by L. F. Abbott and M. B. Wise, Nucl. Phys. **B244**, 541 (1984), although they did not specify the mechanism responsible for causing the Universe to inflate. They used the Sachs-Wolfe formalism [R. K. Sachs and A. M. Wolfe, Astrophys. J. **147**, 73 (1967)] to derive constraints on power-law inflation from a study of the microwave background spatial anisotropy due to gravitational-wave perturbations. They did not investigate the effects of energy-density perturbations on the microwave background radiation (this is what Sachs and Wolfe studied in their paper).

[12] F. Lucchin and S. Matarrese, Phys. Rev. D **32**, 1316 (1985).

[13] J. J. Halliwell, Phys. Lett. B **185**, 341 (1987); J. D. Barrow, *ibid.* **187**, 12 (1987); J. Yokoyama and K. Maeda, *ibid.* **207**, 31 (1988); M. Dennin, Princeton University Senior thesis, 1988.

[14] B. Ratra, Phys. Rev. D **38**, 2399 (1988).

[15] B. Ratra, Phys. Rev. D **40**, 3939 (1989).

[16] B. Ratra, Phys. Rev. D **43**, 3802 (1991).

[17] B. Ratra, Caltech Report No. GRP-229/CALT-68-1666, 1990 (unpublished).

[18] B. Ratra, Phys. Rev. D **44**, 365 (1991).

[19] P. J. E. Peebles, Astrophys. J. Lett. **263**, L1 (1982).

[20] B. Ratra, Phys. Lett. B **260**, 21 (1991).

[21] S. W. Hawking, Phys. Lett. **115B**, 295 (1982).

[22] A. A. Starobinsky, Phys. Lett. **117B**, 175 (1982).

[23] A. H. Guth and S.-Y. Pi, Phys. Rev. Lett. **49**, 1110 (1982); Phys. Rev. D **32**, 1899 (1985).

[24] J. M. Bardeen, P. J. Steinhardt, and M. S. Turner, Phys. Rev. D **28**, 679 (1983).

[25] L. Susskind, Phys. Rev. D **20**, 2619 (1979).

[26] B. Ratra, Caltech Report No. GRP-217/CALT-68-1594, 1989 (unpublished).

[27] J. Bardeen has pointed out that, for a more accurate estimate, it is preferable to (numerically) normalize the power spectrum by using the small-scale galaxy-galaxy correlation function data (which is more complete than the intermediate-scale "great attractor" data); however, for our preliminary order-of-magnitude estimates it suffices to (analytically) normalize the power spectrum on intermediate scales.

[28] C. A. Collins, R. D. Joseph, and N. A. Robertson, Nature (London) **320**, 506 (1986); A. Dressler, S. M. Faber, D. Burstein, R. L. Davies, D. Lynden-Bell, R. J. Terlevich, and G. Wegner, Astrophys. J. Lett. **313**, L37 (1987), and references therein.

[29] J. Bardeen has pointed out that more accurate numerical estimates (in which the power spectrum normalization is fixed on a smaller scale) suggest that n need be even closer to unity for the model to agree with observational data.

[30] By an "isocurvature" perturabtion we mean an inhomogeneity which, on large scales, does not perturb spatial curvature. In a model where the stress tensor is dominated by only one type of matter such a solution usually oscillates inside the Hubble radius and does perturb the curvature of spatial hypersurfaces on small scales.

[31] This was pointed out by J. Bardeen.

[32] J. Bardeen, Phys. Rev. D **22**, 1882 (1980), Sec. III B, has noted that scalar field transverse peculiar velocity perturbations are not generated because the time-space component of the scalar field stress-energy tensor is proportional to $\partial_i\phi$ which is curl-free and hence the vorticity tensor vanishes.

[33] To verify Eqs. (4.15) and (4.16) we need to use Eq. (C2) of Ref. [14]. The definition of the function F_1 given below Eq. (C2) is incorrect. It suffices, for our purposes, to note that $(h^{(R)})'=-\frac{3}{2}c_2^{(R)}e^{-x}(1+\frac{1}{2}\lambda_R^2e^x)+e^{3x/4}[c_+^{(R)}\overline{W}_1(x)+c_-^{(R)}\overline{W}_2(x)]$, where $\overline{W}_1(x)=-6(2/\pi)^{1/2}\lambda_R^{-7/2}e^{-7x/4}(i+\lambda_Re^{x/2})\exp(i\lambda_Re^{x/2})$ and $\overline{W}_2=(\overline{W}_1)^*$. This expression may be integrated to give the correct definition of F_1 in Eq. (C2) of Ref. [14].

[34] V. A. Rubakov, M. V. Sazhin, and A. V. Veryaskin, Phys. Lett. **115B**, 189 (1982).

[35] R. A. Isaacson, Phys. Rev. **166**, 1263 (1968); **166**, 1272

(1968); C. W. Misner, K. S. Thorne, and J. A. Wheeler, *Gravitation* (Freeman, San Francisco, 1973), Sec. 35.15.

[36] E. D. Loh and E. J. Spillar, Astrophys. J. **307**, L1 (1986); E. D. Loh, Phys. Rev. Lett. **57**, 2865 (1986).

[37] P. J. E. Peebles, *The Large-Scale Structure of the Universe* (Princeton University Press, Princeton, NJ, 1980).

[38] *Handbook of Mathematical Functions,* edited by M. Abramowitz and I. A. Stegun (Dover, New York, 1972).