All-orders Skyrmions

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We consider a special class of Skyrme-like Lagrangians which include higher-order terms in the derivatives of the pion field but leave the degree of the chiral angle equation at 2. Explicit Lagrangians are constructed up to order 24. They are found to be in agreement with a previous conjecture regarding the general form of the static energy density to all orders for the hedgehog solution. In addition, the static energy density gets zero contribution from Lagrangians of order 10, 14, 18, and 22, suggesting that this result extends to all order $4k+2$ for $k\geq 2$. We then proceed to prove both conjectures.

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I. INTRODUCTION

Despite the successes of the Skyrme model [1,2], it is overshadowed by the arbitrariness in the choice of a stabilizing term proposed by Skyrme and also in the choice of the chiral-symmetry-breaking term, for that matter [3]. Clearly, the Skyrme term is not the only expression of order 4 in the derivatives of the pion field [4], but more important, all higher-order terms have been entirely omitted in this model. Physically, this approximation is valid only in the small-momentum (or large-distance) region since an effective theory of QCD should include these terms in principle [5,6]. On the other hand, adding higher-order terms arbitrarily would make it virtually impossible to get any serious predictions for two reasons: The number of the parameters fixed by the experimental measurements must remain small; otherwise, the model loses any predictive power and higher orders imply in general a higher degree for the chiral angle differential equation, which eventually becomes intractable.

Recently, we proposed [7] a systematic approach for the introduction of higher orders. We built an effective Lagrangian order by order such that the differential equation for the chiral angle of the hedgehog solution remains of degree 2. The explicit construction was performed up to order 12 in derivatives of the pion field. These calculations showed a definite pattern in the expression for the static energy density, which led to the conjecture that it could be extended to all orders and the distinct possibility that terms of all orders could be summed. This defines a special class of all-orders effective Lagrangian and a whole new class of Skyrmions [8,3].

Jackson, Weiss, and Wirzba [9] generalized this pattern to a hedgehog solution on the hypersphere $S^3(L)$ [10,11] and went on to prove the conjecture for a less restricted case, the so-called "weak conjecture." In the next section, we pursue the rigid construction of Ref. [7] and compute the energy density up to order 24. These explicit calculations show that the conjecture still holds up to this order and reveals a second pattern: The energy density is zero for terms with 10, 14, 18, and 22 derivatives of the pion field. It also exhibits a much richer structure, which will turn out to be instrumental in the formal proof of the conjecture in Ref. [7] (Sec. III) and in the generalization of the second pattern for all $4k+2$ orders with $k\geq 2$ (Sec. IV). Finally, we write the general expressions for this special class of all-orders Skyrmions and offer a discussion of the results.

II. MORE TERMS TO THE SKYRME LAGRANGIAN

The Lagrangian proposed by Skyrme [1] is the sum of two terms:

$$\mathcal{L}_S=\mathcal{L}_1+\mathcal{L}_2\ ,$$

with

$$\mathcal{L}_1=c_1\mathrm{Tr}L_\mu L^\mu,\quad \mathcal{L}_2=c_2\mathrm{Tr}[L_\mu,L^\nu][L_\nu,L^\mu]\ ,\tag{2.1}$$

where $L_\mu=U^\dagger\partial_\mu U$, U being an SU(2) matrix. The first term $\mathcal{L}_1$ becomes the Lagrangian for the nonlinear σ model if one chooses to substitute the degrees of freedom in U by σ and π fields by writing $U=(\sigma+i\tau\cdot\pi)/f_\pi$. The second term $\mathcal{L}_2$ contains higher-order derivatives in the pion field and can account for nucleon-nucleon interactions via pion exchange. $\mathcal{L}_2$ was originally added by Skyrme to allow for solitonic solutions.

Using the hedgehog ansatz $U(r)=\exp[i\tau\cdot\hat{r}F(r)]$, where F is the chiral angle, the Lagrangian $\mathcal{L}_1$ becomes

$$\mathcal{L}_1\equiv c_1\mathrm{Tr}L_\mu L^\mu=2c_1[(N-1)a+b]\ ,\tag{2.2}$$

with

$$a\equiv\frac{\sin^2F}{r^2}\quad\text{and}\quad b\equiv F'^2\ .$$

The number of spatial dimensions, $N=3$, coincides with the number of SU(2) generators. Although it is quartic in the derivatives of the pion field, $\mathcal{L}_2$ adds only a quadratic contribution in F' to the Lagrangian for the static hedgehog solution:

$$\begin{aligned}\mathcal{L}_2&\equiv c_2\mathrm{Tr}[L_\mu,L^\nu][L_\nu,L^\mu]\\&=8c_2(N-1)a[(N-2)a+2b]\ .\end{aligned}\tag{2.3}$$

As mentioned above, the Skyrme model can only be considered as a prototype of an effective theory of QCD. *A priori,* there is no physical grounds for excluding higher-order derivatives in the pion field from the effective Lagrangian. Indeed, a large-N_c analysis suggests that bosonization of QCD would most likely involve an infinite number of mesons. And if this is the case, then taking the appropriate decoupling limits (or large-mass limits) for higher-spin mesons leads to an all-orders Lagrangian for pions. From the point of view of QCD perturbation theory, one can also expect such terms in hadron interactions since they are connected to higher-twist effects. One example of higher-order terms is the piece proposed by Jackson *et al.* [12]:

$$\mathcal{L}_J = c_J \mathrm{Tr}(B^\mu B_\mu) \text{ with } B_\mu = \epsilon_{\mu\nu\rho\sigma} L^\nu L^\rho L^\sigma \ . \quad (2.4)$$

On the other hand, several attempts were made to incorporate vector mesons in the Skyrme picture [4]. These procedures are characterized by the addition of a piece to the Lagrangian which describes free vector mesons, typically of the form of an SU(2) gauge Lagrangian $\mathrm{Tr}(F_{\mu\nu}F^{\mu\nu})$ and the substitution of the derivative by a covariant derivative to account for scalar-vector interactions. In the large-mass limit of the vector mesons, they decouple and an effective self-interaction for scalar mesons is induced as $F_{\mu\nu} \to f_{\mu\nu} \equiv [L_\mu, L_\nu]$.

Following this approach, we have proposed [7] to represent contributions of order $2n$ in the derivative of the pion field in terms of the trace of a product of n $f_{\mu\nu}$'s. Such terms would presumably come from gauge-invariant quantities involving a similar expression with n field strengths $F_{\mu\nu}$ and describing higher-spin mesons, e.g.,

$$\mathrm{Tr}(F^\nu_\mu F^\rho_\nu \cdots F^\nu_\xi) \to \mathrm{Tr}(f^\nu_\mu f^\rho_\nu \cdots f^\nu_\xi) \ .$$

Note that $\mathrm{Tr}(f^\nu_\mu f^\rho_\nu f^\nu_\rho)$ is equivalent to the Jackson term of (2.4) for the hedgehog solution.

Digressing for a moment, let us consider the behavior of such terms under rotations. Following the usual approach of Adkins, Nappi, and Witten [2] and introducing time dependence through a solution of the form $U_t = A(t)U(r)A^\dagger(t)$ with $A(t) = \exp(i\tau\cdot\omega t)$, we find that the Skyrme action can be written

$$S = S_0 + \mathcal{J}\int dt\, \mathrm{Tr}(\dot{A}\dot{A}^\dagger) + O(A^3) \ ,$$

where S_0 is the action for the static solution and $\mathcal{J}$ is the moment of inertia. The action is at most second order in the time derivative, and since $\dot{A} = i\tau\cdot\omega A$, the second term is proportional to $\omega^i\omega^j \mathrm{Tr}(\tau_i\tau_j)$. But physically the Skyrme term can be associated with a vector object (ρ meson). Furthermore, the Jackson term, or the equivalent term $\mathrm{Tr}(f^\nu_\mu f^\rho_\nu f^\nu_\rho)$, which also represents a vector object (ω meson), is again only quadratic in time derivatives [13]. As one increases the number n of f^ν_μ's, the action gets contributions with four (n=4,5), six (n=6,7), eight (n=8,9), etc., time derivatives which presumably can be associated with spin 2,3,4,... objects, respectively. In other words, adding terms of order n should, in principle, be related to an interaction involving the exchange of a meson, or a combination of mesons, with spin equal to the integer part of $n/2$ or lower.

Coming back to the construction of Lagrangians for a given n, we propose a number equal to the integer part of $n/2$ of independent Lagrangians containing $2n$ derivatives of the pion field in the form of a product of n f^ν_μ's. We begin with $\mathcal{L}_{na}$, which is the trace of an ordered product of f^ν_μ's. Then, in $\mathcal{L}_{nb}$, the first and third f^ν_μ's are regrouped in an anticommutator. Two more f^ν_μ's, the second and the fifth, are rearranged in an anticommutator in $\mathcal{L}_{nc}$, etc., until there remains either two (even n) or three (odd n). More explicitly,

$$\begin{aligned}
\mathcal{L}_{na} &= c_n \mathrm{Tr}(f^\nu_\mu f^\lambda_\nu f^\rho_\lambda f^\sigma_\rho f^\omega_\sigma f^\xi_\omega \cdots f^\mu_\eta) \ , \\
\mathcal{L}_{nb} &= c_n \mathrm{Tr}(\{f^\nu_\mu, f^\rho_\lambda\} f^\lambda_\nu f^\sigma_\rho f^\omega_\sigma f^\xi_\omega \cdots f^\mu_\eta) \ , \\
\mathcal{L}_{nc} &= c_n \mathrm{Tr}(\{f^\nu_\mu, f^\rho_\lambda\}\{f^\lambda_\nu, f^\omega_\sigma\} f^\sigma_\rho f^\xi_\omega \cdots f^\mu_\eta) \ , \quad (2.5) \\
\mathcal{L}_{nd} &= c_n \mathrm{Tr}(\{f^\nu_\mu, f^\rho_\lambda\}\{f^\lambda_\nu, f^\omega_\sigma\}\{f^\sigma_\rho, f^\eta_\xi\} \cdots f^\mu_\eta) \ , \\
&\vdots
\end{aligned}$$

$\mathcal{L}_n$, the linear combination free of $O(b^2)$ and higher-order terms, is normalized according to

$$\mathcal{L}_n = \mathcal{L}_{na} + c_{nb}\mathcal{L}_{nb} + c_{nc}\mathcal{L}_{nc} + c_{nd}\mathcal{L}_{nd} + \cdots \ . \quad (2.6)$$

This procedure ensures that all Lagrangians for a given order are linearly independent (the hedgehog ansatz is assumed).

This program was carried through up to $n=6$ in Ref. [7] with the result that the final form of the static energy density obeys the general expression

$$\mathcal{L}_n = \kappa_n a^{n-1}[(N-n)a + nb] \ , \quad (2.7)$$

for n=3,4,5,6 and where κ_n are constants. The results also indicated that, for the case $n=5$, the static energy density is simply zero.

This amazingly simple result motivated calculations to even higher n's with the aim to prove that there are no counterexamples of (2.7) at relatively low n's and to find if a zero energy density is unique to the case $n=5$. Starting at $n=7$, we get, according to (2.5),

$$\begin{aligned}
\mathcal{L}_{7a} &= -3584 c_7 a^4 b(a+2b)^2 \ , \\
\mathcal{L}_{7b} &= -1024 c_7 a^4 b(3a^2+4ab+8b^2) \ , \\
\mathcal{L}_{7c} &= -2048 c_7 a^4 b(a^2+2b^2) \ .
\end{aligned}$$

The appropriate linear combination of these objects gives

$$\mathcal{L}_7 = 384 c_7 a^6 (N-3)(-4a+7b) = 0 \ ,$$

since $N=3$. This is still in accordance with (2.7), but leads to zero as for $n=5$. Proceeding with the case $n=8$, we obtain

$$\begin{aligned}
\mathcal{L}_{8a} &= 1024 c_8 a^4 (a+2b)(a^3+6a^2b+28ab^2+8b^3) \ , \\
\mathcal{L}_{8b} &= 2048 c_8 a^4 (a+2b)(a^3+2a^2b+4ab^2+4b^3) \ , \\
\mathcal{L}_{8c} &= 4096 c_8 a^4 (a+2b)(a^3+2b^3) \ , \\
\mathcal{L}_{8d} &= 8192 c_8 a^4 (a^4+2b^4) \ .
\end{aligned}$$

A Lagrangian free of $O(b^2)$ terms then must assume the form

$$\mathcal{L}_8=-2048c_8a^7(-5a+8b)\ ,$$

which is again in agreement with (2.7). The calculations were pushed up to order 24 ($n=12$) in the derivatives of the pion field. The details are shown in the Appendix. We only present their final forms here, which are

$$\begin{aligned}\mathcal{L}_9&=-11\,264c_9a^8(N-3)(-6a+9b)=0\ ,\\ \mathcal{L}_{10}&=-\frac{327\,680}{43}c_{10}a^9(-7a+10b)\ ,\\ \mathcal{L}_{11}&=162\,816c_{11}a^{10}(N-3)(-7a+11b)=0\ ,\\ \mathcal{L}_{12}&=-\frac{5\,767\,168}{147}c_{12}a^{11}(-9a+12b)\ .\end{aligned}\tag{2.8}$$

The regularity of these results suggests that some patterns could repeat to all orders. Here two patterns clearly arise and they can be summarized.

Conjecture A. Lagrangians constructed according to (2.5) and (2.6) lead to a static energy density of the form $\mathcal{L}_n \propto a^{n-1}[(N-n)a+nb]$ for all n. This possibility, evoked and used in Refs. [7, 8, and 3], is now verified up to $n=12$.

Conjecture B. For all odd $n \geq 5$, the energy densities are zero.

III. PROVING CONJECTURE A AND MORE

The explicit construction of the Lagrangians in Sec. II becomes more complex and time consuming as the order increases, and although we have adopted a systematic approach, the only recurring pattern seems to be in the unexpected final form (2.7). So, at first sight, the complexity of the calculations and the apparent lack of additional symmetry could prevent us from reaching a proof that the form (2.7) holds to all orders. But it turns out that such a symmetry exists and becomes manifest when we simply expand the static energy densities in terms of $b-a$ instead of b. Then we find that *every* Lagrangian in Sec. II and in the Appendix takes the form

$$\mathcal{L}_{n\xi}=\text{const}\times a^{n-1}[Na+n(b-a)+O((b-a)^2)]\ .\tag{3.1}$$

This observation is a key ingredient in the proof of both conjectures A and B and needs to be proven to all orders. It is also interesting to note that (3.1) applies to all Lagrangians above, not only those linear in b. The form of the Lagrangians in the previous section involves products of n $f_{\mu\nu}$'s ($\equiv[L_\mu,L_\nu]$). But the proof we shall provide here is much more general.

Consider a Lagrangian built from a product of L_μ's:

$$\mathcal{L}_n=c_n\text{Tr}(L_\mu L_\nu\cdots L^\mu\cdots L^\nu\cdots)\ .\tag{3.2}$$

Using the hedgehog ansatz, the expression in (3.2) can be separated into two pieces: a piece which contains all the dependence on the chiral angle F and a trace of a product of SU(2) generators [7]. Then $\mathcal{L}_n$ takes the general form

$$\mathcal{L}_n=(-)^n c_n m_{A\tilde{A}}m_{B\tilde{B}}\cdots m_{Z\tilde{Z}}T^{AB\cdots\tilde{A}\cdots\tilde{B}\cdots Z\tilde{Z}}\ ,\tag{3.3}$$

where the product of m's carries the F dependence according to

$$m_{PQ}\equiv\delta_{PQ}a-\hat{r}_P\hat{r}_Q(b-a)\ ,\tag{3.4}$$

with a and b defined as in (2.2) and $T^{A\tilde{A}B\tilde{B}\cdots Z\tilde{Z}}$ is a trace of Pauli τ matrices corresponding to (3.2) [e.g., $\mathcal{L}_1$ in (2.2) can be written $\mathcal{L}_1=-c_1m_{A\tilde{A}}\text{Tr}(\tau^A\tau^{\tilde{A}})$].

Expanding the product of m's in (3.3) according to (3.4) and neglecting for the moment the contributions of $O((b-a)^2)$ and higher, $\mathcal{L}_n$ can be cast in the form

$$\begin{aligned}\mathcal{L}_n=(-)^n c_n a^{n-1}(d_{A\tilde{A}}T^{A\tilde{A}BB\cdots ZZ}+d_{B\tilde{B}}T^{AAB\tilde{B}}+\cdots)\\ +O((b-a)^2)\ ,\end{aligned}\tag{3.5}$$

where d is defined as

$$d_{PQ}\equiv\frac{\delta_{PQ}}{n}a+\hat{r}_P\hat{r}_Q(b-a)\ .$$

Finally, summing over repeated indices, it is easy to see that the traces in (3.5) must be proportional to the Kronecker δ (e.g., $T^{A\tilde{A}BB\cdots ZZ}=\text{const}\times\delta_{A\tilde{A}}$). Substituting the traces by the appropriate δ, one gets the final expression

$$\mathcal{L}_n=(-)^n c_n\text{const}\times[Na+n(b-a)]+O((b-a)^2)\ .\tag{3.6}$$

The result (3.6) also holds for any Lagrangian of order $2n$ *in derivatives constructed from* $2n$ *currents* L_μ*'s* since such Lagrangians can be expressed as linear combinations of $\mathcal{L}_n$'s with the form (3.2). Moreover, all the Lagrangians mentioned in Sec. II and any Lagrangian constructed with $f_{\mu\nu}$ commutators fall into this class. Therefore, (3.6) implies that conjecture A is true.

It is also easy to see that a Lagrangian which can be expressed as a product of a number p of lower-order Lagrangians described in (3.6) also obeys the same relation. This is readily shown from the general expression

$$\mathcal{L}_n^p=c_n\mathcal{L}_\alpha\mathcal{L}_\beta\cdots\mathcal{L}_\omega\ ,\tag{3.7}$$

where $\alpha,\beta,\ldots,\omega$ are positive integers such that $\alpha+\beta+\cdots+\omega=n$. Then each piece on the right-hand side (RHS) can be written according to (3.6), and their product has the simple form

$$\mathcal{L}_n^p=\text{const}\times N^{p-1}a^{n-1}[Na+n(b-a)+O((b-a)^2)]\ .\tag{3.8}$$

Accordingly, finding a static energy density linear in b, as in Sec. II, can be achieved with Lagrangians of the form (3.7). Jackson, Weiss, and Wirzba in Ref. [9] rely on such a construction to propose the formula

$$\mathcal{E}_{n+1}=-\mathcal{E}_n\mathcal{E}_1+\mathcal{E}_{n-1}\mathcal{E}_2-\tfrac{1}{3}\mathcal{E}_{n-2}\mathcal{E}_3\ ,\tag{3.9}$$

where $\mathcal{E}_n\equiv a^{n-1}[3a+n(b-a)]$, and write a similar expression for $\mathcal{L}_n$. But since this induction relation involves $\mathcal{E}_1$ (or $\mathcal{L}_1$), the proof cannot apply to conjectures A and B where Lagrangians are by assumption of the form (2.5), i.e., polynomials in $f_{\mu\nu}$ commutators. When this last requirement is relaxed, it is referred to as the *weak conjecture* in Ref. [9].

There are also some limitations on how this induction relation can be modified to prove conjectures A and B. One starts from Lagrangians leading to $\mathcal{L}_n = c_n \text{const} \times a^{n-1}[3a + n(b-a)]$, but in general, the constant can be zero as is the case for $n=5,7,9,11$ in Sec. II. Nonetheless, a formula similar to (3.9) can be found for n even ≥ 6:

$$\mathcal{L}_{n+2} = -\mathcal{L}_n \mathcal{L}_2 + \mathcal{L}_{n-2}\mathcal{L}_4 - \tfrac{1}{3}\mathcal{L}_{n-4}\mathcal{L}_6 ,$$

where none of the Lagrangians $\mathcal{L}_2$, $\mathcal{L}_4$, and $\mathcal{L}_6$ leads to a zero energy density. The result in (3.6) is, however, much more general and, of course, provides a proof of conjecture A for all n. It will also be instrumental for the proof of conjecture B.

IV. ODD ORDERS: CONJECTURE B

The second conjecture involves Lagrangians with an odd number of $f_{\mu\nu}$ commutators. For odd $n \geq 5$ all Lagrangians at most linear in b must yield zero static energy densities. Again, any attempt to provide a proof to all odd n seems destined to failure because of the rapidly increasing complexity of the problem. But much like for the proof of conjecture A, the explicit calculations of Sec. II do provide a hint to the wanted procedure. Quite unexpectedly, none of the static energy densities for n odd (including $n=3$) have a contribution $O(a^n)$. Since all Lagrangians also obey (2.7), then $\text{const} \times a^n(3-n)=0$. This is certainly true for $n=3$ for any value of the constant. However, for $n \neq 3$, it is required that the constant$=0$ or $\mathcal{L}_n=0$ for n odd. The remainder of the proof consists in establishing that there are no $O(a^n)$ contributions to the Lagrangians for all odd n.

The term of order $O(a^n)$ in the Lagrangian $\mathcal{L}_{n\xi}$ is easy to isolate from (3.3):

$$\mathcal{L}_{n\xi} = \text{const} \times a^n \Delta_{A\tilde{A}} \Delta_{B\tilde{B}} \cdots \Delta_{Z\tilde{Z}} \epsilon^{\tilde{A}B\alpha} \epsilon^{\tilde{B}C\beta} \cdots \epsilon^{\tilde{Z}A\omega} \times \text{Tr}(\{\tau^\alpha, \tau^\beta, .., \tau^\omega\}_{\text{perm}}) + O(b) , \tag{4.1}$$

where, for brevity, we wrote $\Delta_{pq} = \delta_{pq} - \hat{r}_p \hat{r}_q$. The presence of antisymmetric tensors ϵ is directly attributable to the fact that the Lagrangian is a polynomial of $f_{\mu\nu}$ commutators. The remaining trace can be any permutation of an odd number of Pauli τ matrices.

However, the trace of an odd number of τ matrices *always* generates at least one more antisymmetric tensor ϵ_{ijk}, where i,j,k take values in the set $\alpha, \beta, .., \omega$. Any such contribution contracts three other ϵ tensors explicitly written in (4.1). Without loss of generality, we could write (4.1) as

$$\mathcal{L}_{n\xi} = \text{const} \times a^n \Delta_{A\tilde{A}} \Delta_{B\tilde{B}} \Delta_{C\tilde{C}} \epsilon^{\tilde{A}Bi} \epsilon^{\tilde{B}Cj} \epsilon^{\tilde{C}Ak} \epsilon^{ijk} + O(a^{n-1}b) , \tag{4.2}$$

where all other summations over spatial Lorentz indices are factored in the constant. The term of $O(a^n)$ is easy to compute using elementary algebra:

$$\mathcal{L}_{n\xi} = \text{const} \times a^n (N-3)(N-1) + O(a^{n-1}b) . \tag{4.3}$$

Since the number of spatial dimension N is 3, there are no $O(a^n)$ contribution to $\mathcal{L}_{n\xi}$. Furthermore, $\mathcal{L}_{n\xi}$ also obeys (2.7), which implies that conjecture B is justified.

V. ALL-ORDERS SKYRMIONS

In summary, the effective Lagrangians described above define a special class of Skyrmions. They involve all orders in the derivatives of the pion field, but their energy densities are only second order in the derivative of the chiral angle F. Summing to all orders, the mass of the static solution takes the general form

$$\begin{aligned} M_s &= 8\pi c_1 \int r^2 dr \sum_{m=1}^{\infty} h_m a^{m-1}[Na + m(b-a)] \\ &= 8\pi c_1 \int r^2 dr [N\chi(a) + (b-a)\chi'(a)] , \end{aligned} \tag{5.1}$$

where $\chi(x) = \sum_{m=1}^{\infty} h_m x^m$ and $\chi' = d\chi(x)/dx$. Some of the properties of χ were carefully enumerated in Ref. [9]. Reflecting the property described in conjecture B, we add one more constraint to this set: namely,

$$h_m = 0 ,$$

for any odd $m \geq 5$ or, equivalently,

$$\chi^{(5)}(x) + \chi^{(5)}(-x) = 0 \quad \text{with} \quad \chi^{(5)}(x) = \frac{d^5\chi(x)}{dx^5} .$$

The differential equation for the chiral angle then reads

$$\begin{aligned} 0 &= \sum_{m=1}^{\infty} m h_m \left[\frac{\sin^2 F}{r^2}\right]^{m-1} \left[F'' + (N+1-2m)\frac{F'}{r} + (m-1)F'^2 \frac{\cos F}{\sin F} - (N-m)\frac{\sin F \cos F}{r^2}\right] \\ &= \chi'\left[\frac{\sin^2 F}{r^2}\right]\left[F'' + (N-1)\frac{F'}{r} - N\frac{\sin F \cos F}{r^2}\right] + \frac{\sin^2 F}{r^2}\chi''\left[\frac{\sin^2 F}{r^2}\right]\left[-2\frac{F'}{r} + F'^2\frac{\cos F}{\sin F} + \frac{\sin F \cos F}{r^2}\right] . \end{aligned} \tag{5.2}$$

Following the procedure in Ref. [2], the moment of inertia $\mathcal{I}$ is then simply

$$\begin{aligned} \mathcal{I} &= \frac{16\pi c_1}{3} \int r^4 dr \sum_{m=1}^{\infty} m h_m a^{m-1}[(3-m)a + (m-1)b] \\ &= \frac{16\pi c_1}{3} \int r^4 dr\, a[2\chi'(a) + (b-a)\chi''(a)] , \end{aligned} \tag{5.3}$$

and the axial-vector coupling constant

$$g_A=\frac{16\pi c_1}{3a}\int r^2dr\sum_{m=1}^{\infty}mh_m(-a)^{m-1}(\{\alpha[(3-m)a+(m-1)b]+\beta a\})$$
$$=\frac{16\pi c_1}{3a}\int r^2dr\,a\{[(2\alpha+\beta)\chi'(-a)+(b-a)\alpha\chi''(-a)]\}\ , \qquad (5.4)$$

where $\alpha=\sin 2F/2r$ and $\beta=F'$. These expressions provide most of the information out of the Skyrme model and of the special class of the all-orders effective Lagrangian. Some examples have been considered in Refs. [8], [3], and [9], demonstrating that soliton solutions clearly exist for this class of model.

The construction of these models relies on two main assumptions: (a) The degree of the differential equation of the chiral angle F is 2, and (b) the effective Lagrangian (except for $\mathcal{L}_1$) is built from products of the commutators $[L_\mu,L_\nu]$. The relations from Secs. III and IV can still be used to a certain extent if one or two of these assumptions are relaxed. We can identify at least two cases where important information can still be extracted from either relation. The first case comes when we relax (a) completely and impose no constraint on the degree of the differential equations. Clearly, contributions to the static energy densities obey the pattern $a^{n-1}[(N-n)a+nb]+O(b^2)$. On the other hand, the results of Sec. IV are not as restrictive and only imply that terms of order $O(a^n)$ are absent for odd $n\geq 5$. We could also consider a second example where condition (b), this time, is weakened to require that the Lagrangian be constructed out of L_μ currents alone. This only affects the conclusions regarding odd n, which do not apply anymore. Higher orders are still described by the pattern (2.7).

Finally, we stress again the importance of including higher-order terms in the construction of a QCD effective Lagrangian. Their presence is essential. They correspond physically to interactions which involve higher-spin mesons (or higher-twist effects). They should also modify considerably the small-distance (or large-momentum) behavior of hadrons. These two features are reminiscent of the Regge trajectories and the region of validity of the parton picture, respectively. At this point, however, we can only hope that a Skyrme-like Lagrangian of this type could link such diverse concepts.

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APPENDIX

The static energy densities for the Lagrangians (2.5) with $n=9,10,11,12$ (assuming the hedgehog ansatz) are as follows. For $n=9$,

$$\mathcal{L}_{9a}=-6144c_9a^5b(3a^3+18a^2b+40ab^2+24b^3)\ ,$$
$$\mathcal{L}_{9b}=-4096c_9a^5b(a+2b)(5a^2+4ab+12b^2)\ ,$$
$$\mathcal{L}_{9c}=-8192c_9a^5b(3a^3+2a^2b+2ab^2+8b^3)\ ,$$
$$\mathcal{L}_{9d}=-16\,384c_9a^5b(a^3+2b^3)\ .$$

For $n=10$,

$$\mathcal{L}_{10a}=4096c_{10}a^5(a+2b)^2(a^3+6a^2b+42ab^2+8b^3)\ ,$$
$$\mathcal{L}_{10b}=8192c_{10}a^5(a^5+6a^4b+22a^3b^2+36a^2b^3+48ab^2+16b^5)\ ,$$
$$\mathcal{L}_{10c}=16\,384c_{10}a^5(a^5+4a^4b+6a^3b^2+2a^2b^3+12ab^4+8b^5)\ ,$$
$$\mathcal{L}_{10d}=32\,768c_{10}a^5(a+2b)(a^4+2b^4)\ ,$$
$$\mathcal{L}_{10e}=65\,536c_{10}a^5(a^5+2b^5)\ .$$

For $n=11$,

$$\mathcal{L}_{11a}=-90\,112c_{11}a^6b(a+2b)(a^3+6a^2b+16ab^2+8b^3)\ ,$$
$$\mathcal{L}_{11b}=-16\,384c_{11}a^6b(a+2b)(7a^3+18a^2b+2ab^2+32b^3)\ ,$$
$$\mathcal{L}_{11c}=-32\,768c_{11}a^6b(a+2b)(5a^3+2a^2b+2ab^2+12b^3)\ ,$$
$$\mathcal{L}_{11d}=-65\,536c_{11}a^6b(3a^4+2a^3b+2ab^3+8b^4)\ ,$$
$$\mathcal{L}_{11e}=-131\,072c_{11}a^6b(a^4+2b^4)\ .$$

For $n=12$,

$$\mathcal{L}_{12a}=16\,384c_{12}a^6(a^6+12a^5b+108a^4b^2+448a^3b^3+840a^2b^4+576ab^5+64b^6)\ ,$$
$$\mathcal{L}_{12b}=32\,768c_{12}a^6(a+2b)(a^5+6a^4b+32a^3b^2+44a^2b^3+72ab^4+16b^5)\ ,$$
$$\mathcal{L}_{12c}=65\,536c_{12}a^6(a^6+6a^5b+20a^4b^2+18a^3b^3+20a^2b^4+48ab^5+16b^6)\ ,$$
$$\mathcal{L}_{12d}=131\,072c_{12}a^6(a^6+4a^5b+6a^4b^2+2a^2b^4+12ab^5+8b^6)\ ,$$
$$\mathcal{L}_{12e}=262\,144c_{12}a^6(a+2b)(a^5+2b^5)\ ,$$
$$\mathcal{L}_{12f}=524\,288c_{12}a^6(a^6+2b^6)\ .$$

[1] T. H. R. Skyrme, Proc. R. Soc. London **A247**, 260 (1958); **A262**, 236 (1959).
[2] G. S. Adkins, C. R. Nappi, and E. Witten, Nucl. Phys. **B228**, 552 (1983).
[3] L. Marleau, Phys. Rev. D **43**, 885 (1991).
[4] I. Zahed and G. E. Brown, Phys. Rep. **142**, 1 (1986).
[5] G. 't Hooft, Nucl. Phys. **B72**, 461 (1974).
[6] E. Witten, Nucl. Phys. **B160**, 57 (1979).
[7] L. Marleau, Phys. Lett. B **235**, 141 (1990).
[8] S. Dubé and L. Marleau, Phys. Rev. D **41**, 1606 (1990).
[9] A. D. Jackson, C. Weiss, and A. Wirzba, Nucl. Phys. **A529**, 741 (1991).
[10] N. S. Manton, Commun. Math. Phys. **111**, 469 (1987).
[11] A. D. Jackson, C. Weiss, A. Wirzba, and A. Linde, Nucl. Phys. **A494**, 523 (1989).
[12] A. Jackson, A. D. Jackson, A. S. Goldhaber, G. S. Brown, and L. C. Castillo, Phys. Lett. **154B**, 101 (1985).
[13] A. Jackson, A. D. Jackson, and V. Pasquier, Nucl. Phys. **A432**, 567 (1985).