Anomalies in the Fujikawa method using parameter-dependent regulators

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We propose an extended definition of the regularized Jacobian which allows the calculation of anomalies using parameter-dependent regulators in the Fujikawa approach. This extension incorporates the basic Green's function of the problem in the regularized Jacobian, allowing us to interpret a specific regularization procedure as a way of selecting the finite part of the Green's function, in complete analogy with what is done at the level of the effective action. In this way we are able to consider the effect of counterterms in the regularized Jacobian in order to relate different regularization procedures. We also discuss the ambiguities that arise in our prescription due to some freedom in the place where we can insert the regulator, using charge-conjugation invariance as a guiding principle. The method is applied to the case of vector and axial-vector anomalies in two- and four-dimensional quantum electrodynamics. In the first situation we recover the standard family of anomalies calculated by the point-splitting regularization prescription. We also study in detail an alternative choice in the position of the regulator and we calculate explicitly all the currents that generate the families of anomalies that we are considering. Next we extend the calculation to four dimensions, using the same prescriptions as before, and we compare the results with those obtained from the point-splitting calculation, which we also perform in the case of the vector anomaly. A discussion of the relation among the results obtained by different regularization prescriptions is given in terms of the allowed counterterms in the regularized Jacobian, which are highly constrained by the requirement of charge-conjugation invariance.

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I. INTRODUCTION

Since their discovery in the late 1940's, anomalies have played an important role in the consistent construction of quantum field theories (QFT's) [1]. Nevertheless, their study based on the path-integral formulation of QFT's is rather recent and practically was started with the work of Fujikawa in 1979 [2]. Subsequently, most of the known results have been rederived using his method and a great deal of new discoveries have also been made.

In this paper we only consider anomalies arising from the fermionic sector of the theory, which means that all the bosonic fields will be considered as external ones. Anomalies arise when a symmetry present at the classical level can no longer be maintained after the theory is quantized. This is a consequence of the fact that QFT's need further specification: one must provide a prescription for regularizing them. Then, it might happen that no regulator exists which preserves all the original classical symmetries.

The basic ingredients of the Fujikawa method are as follows. (i) The independence of the effective bosonic action of the theory upon arbitrary changes of the fermionic integration variables. This naturally introduces into the method the corresponding Jacobian associated with such transformation of integration variables which ensures that the path integral remains, in fact, unchanged. (ii) The use of the specific local transformation of the fermionic variables which generate the corresponding Noether currents related to the symmetry under consideration in the classical action. These two basic points allow Fujikawa to relate the mean value of the divergence of the symmetry current with the change of the corresponding Jacobian to first order of the transformation parameters.

Our work has been motivated by the present status of the discussion of the one-parameter family of vector and axial-vector anomalies in two- and four-dimensional QED, from the point of view of the Fujikawa method. The existence of such a family in two dimensions has been previously established, for example, by point-splitting calculations [3,4] and is given by

$$\partial_\mu J_V^\mu = \frac{e}{2\pi}(1-a)\partial^\mu A_\mu \ , \tag{1.1a}$$

$$\partial_\mu J_A^\mu = \frac{e}{2\pi}(1+a)\epsilon^{\mu\nu}\partial_\mu A_\nu \ , \tag{1.1b}$$

in Euclidean space, where $J_V^\mu = \mathrm{Tr}(\gamma^\mu S)$ and $J_A^\mu = \mathrm{Tr}(\gamma^\mu\gamma_5 S)$ are the respective currents. Here, S is a shorthand notation for the Green's function of the operator $(i\not\partial - e\not A)$, regularized in the standard manner of the point-splitting method. Equations (1.1) incorporate the basic feature which characterizes any true anomaly: no choice of parameters (a in this case) can make both currents simultaneously conserved. In this case the vector current is coupled to the photon field so that the physical requirement of charge conservation forces us to choose $a=1$ and, consequently, the axial-vector anomaly appears. The family (1.1) has been previously obtained using the Fujikawa approach by heavily relying upon very specific properties of the two-dimensional Dirac matrices [4,5].

The main issue to be discussed in this paper is the possibility of obtaining correct results in the path-integral calculation of anomalies by using parameter-dependent regulators. Our point of view here is that the anomaly is completely given in terms of the Jacobian which only needs to be regularized appropriately. This can be done by extending the Fujikawa results for the calculation of the relevant Jacobians [6,7]. In particular, we shall be concerned with regulators constructed from the operator $\not{D}_a=\not{\partial}+iea\not{A}$ where the parameter a is real to ensure Hermiticity in Euclidean space.

The freedom introduced by parameter-dependent regularization has proved to be very useful in the consistent definition of models lacking a gauge symmetry principle as in the case of the chiral Schwinger model, for example [8]. Recently, the authors of Ref. [9] have also considered the problem of obtaining the family of anomalies (1.1) in the path-integral formulation of QFT. Following a different route than us, they expand the action in terms of the eigenfunctions of $\not{D}_a$, derive the corresponding Ward identities for the currents, and subsequently regularize such expressions. Their results, which only include the two-dimensional case, coincide with those of Refs. [6,7]. A detailed comparison of their method with the one discussed in this work, especially in the four-dimensional case, would be of much interest, and it is deferred for future work.

This paper is organized as follows. Section II contains a review of the proposed generalization of the regularized Jacobian, which naturally includes the possibility of using parameter-dependent regulators. In Sec. III we discuss two kinds of ambiguities present in our method: the first one has to do with the freedom of selecting the position of the regulator to define our regularized Jacobian and the second one is analogous to the freedom that appears at the level of the effective action, where the possibility of adding counterterms which are local in the external field and its derivatives allows for the possibility of relating two different regularization prescriptions. Great emphasis is given to charge-conjugation invariance in elucidating these questions. In this section we also calculate the regularized currents that give rise to each family of anomalies that we obtain. Since our prescription can be directly generalized to higher dimensions, we have performed the analogous calculations for four dimensions in Sec. IV, where two families of anomalies are obtained. The relations among them and also with the standard family of anomalies obtained by the point-splitting prescription are discussed. Finally, there are two Appendixes: Appendix A deals with the point-splitting calculation of the vector anomaly in four dimensions and Appendix B contains some details of the calculations included in Sec. IV.

II. EXTENDED DEFINITION OF THE REGULATED JACOBIAN

Quantized Dirac fermions coupled to a background electromagnetic field in Euclidean space are described by the generating functional

$$Z(A)=\int \mathcal{D}\bar{\chi}\mathcal{D}\chi \exp\left[\int (dx)\bar{\chi}i\not{D}\chi\right], \tag{2.1}$$

where $\not{D}=\gamma^\mu D_\mu=\gamma^\mu(\partial_\mu+ieA_\mu)$. Our γ matrices are anti-Hermitian satisfying $\{\gamma^\mu,\gamma^\nu\}=2g^{\mu\nu}$ with $g^{\mu\nu}=-\mathrm{diag}I$. The Dirac operator is Hermitian and we pass from Minkowski to Euclidean space using the conventions of Ref. [2].

In order to obtain the Ward identities related to a symmetry of the classical Lagrangian appearing in (2.1), we follow the standard steps. Let us consider the fermionic part of such an infinitesimal symmetry transformation which we write as

$$\chi=[1+K(\eta)]\psi,\quad \bar{\chi}=\bar{\psi}[1+L(\eta)], \tag{2.2}$$

where K and L are operators depending on infinitesimal local parameters which we call $\eta=\eta(x)$ in compact notation. The transformations (2.2) are such that they leave the Lagrangian invariant when the parameters are independent of position. Now we use the transformations (2.2) as a linear change of variables in the generating functional obtaining

$$Z(A)=\mathcal{J}(\eta)\int \mathcal{D}\bar{\psi}\mathcal{D}\psi \exp\left[\int (dx)\bar{\psi}i(\not{D}+\not{D}K+L\not{D})\psi\right], \tag{2.3}$$

where we have kept in the action only terms to first order in the parameters, and $\mathcal{J}(\eta)$ stands for the Jacobian of the transformation. The explicit calculation of the terms in the exponential leads to the identification of the currents in the form

$$Z(A)=\mathcal{J}(\eta)\int \mathcal{D}\bar{\psi}\mathcal{D}\psi \exp\left[\int (dx)\bar{\psi}i\not{D}\psi - \int (dx)j^\mu(x)\partial_\mu\eta(x)\right]. \tag{2.4}$$

The Ward identities are obtained by functionally differentiating both sides of (2.4) with respect to $\eta(x)$. The result for $Z^{-1}[\delta Z/\delta\eta(x)]=0$ is

$$0=\partial^\mu J_\mu(x)+\left.\frac{\delta \ln\mathcal{J}(\eta)}{\delta\eta(x)}\right|_{\eta=0}, \tag{2.5}$$

where $J_\mu(x)=\langle j_\mu(x)\rangle$ is the usual average value of $j_\mu(x)$. We have gone through the derivation of the well-known expression (2.5) in order to introduce our notation and conventions.

Now we concentrate on the Jacobian $\mathcal{J}(\eta)$ which is usually calculated by defining the path-integral measure in (2.1) via the expansion of the fermionic fields in terms of an appropriate set of complete basis functions and subsequently using the transformation (2.2) to calculate the changes in $\mathcal{D}\bar{\chi}$ and $\mathcal{D}\chi$ separately [2]. Because we are interested in studying the parameter dependence of the vector and axial-vector anomalies, we could perfectly well take our basis $\{\phi_n^a\}$ as the eigenvectors of the operator $\not{D}_a = \not{\partial} + iea\not{A}$, for the part of the measure corresponding to $\mathcal{D}\chi$. Following the standard steps and regulating the divergent trace by inserting the factor $\exp[-(\lambda_n^a/M^2)]$, with λ_n^a being the eigenvalues of $\not{D}_a$, we obtain

$$\mathcal{J}_\psi(\eta) = 1 - \mathrm{Tr}[K \exp(-\not{D}_a^2/M^2)] \equiv 1 - \mathrm{Tr}(KR_1) \ , \quad (2.6)$$

for the contribution to the Jacobian $\mathcal{J}(\eta)$ arising from the change of variables (2.2) in the corresponding measure. The symbol Tr denotes the trace over Dirac (labeled by tr) as well as spacetime indices. The contribution to the Jacobian of the remaining variables can be calculated in an analogous way. In principle, we could use a different regulator R_2, as is done, for example, in the case of non-Hermitian regulators where $\not{D}\not{D}^\dagger$ and $\not{D}^\dagger\not{D}$ are, respectively, used. The final result for the Jacobian is

$$\mathcal{J}(\eta) = \mathcal{J}_{\bar{\psi}}\mathcal{J}_\psi(\eta) = 1 - \mathrm{Tr}(KR_1 + LR_2) \ . \quad (2.7)$$

Some comments regarding expression (2.7) for the particular case of the vector $[K = -L = i\alpha(x)]$ transformations are now in order. In the first place we remark that (2.7) is indeed able to produce a nonzero vector anomaly as appropriate to the family (1.1). Nevertheless, one would be faced with the rather embarassing situation of obtaining a conserved vector current in all cases where $R_1 = R_2$ with a parameter $a \neq 1$. This would mean that gauge invariance is maintained in spite of regulating the theory in a non-gauge-covariant way ($a \neq 1$). The case $a = 0$ is even more dramatic because this regulator will also give a zero value for the axial-vector anomaly [10]. The previous comments point to the fact that, when we want to regularize a theory in a parameter-dependent way, a more careful definition of the Jacobian is required in order that the results obtained make sense. This is exactly the point of the extension of the regularization procedure previously presented in Refs. [6,7]. The basic idea in such references was to start from the definition

$$\int \mathcal{D}\bar{\psi}\mathcal{D}\psi \exp\left[\int (dx)\bar{\psi}P\psi\right] \equiv \det(P) \ , \quad (2.8)$$

together with the fact that the Jacobian is precisely the extra factor that produces a coordinate-independent integral. Thus, using (2.8) and setting (2.1) equal to (2.3) we obtain

$$\mathcal{J}(\eta) = \frac{\det(\not{D})}{\det(\not{D} + \not{D}K + L\not{D})} \ , \quad (2.9)$$

which can be rewritten as

$$\mathcal{J}(\eta) = \det[1 - \not{D}^{-1}(\not{D}K + L\not{D})]$$
$$= 1 - \mathrm{Tr}[\not{D}^{-1}(\not{D}K + L\not{D})] \ , \quad (2.10)$$

where $\not{D}^{-1}$ denotes the Green's function of the operator and we recall that K, L are first-order quantities in the infinitesimal parameters. Up to now all manipulations starting from (2.1) have been only formal and in order to properly define expression (2.10) it is necessary to adopt a suitable regularization scheme. The prescription

$$\ln\mathcal{J}_R(\eta) = -\mathrm{Tr}[\not{D}^{-1}(\not{D}K + L\not{D})R] \equiv T \ , \quad (2.11)$$

was proposed in Ref. [6]. Now we make some remarks upon (2.11). The first thing we notice is that $\mathcal{J}_R(\eta)$ does not coincide, in general, with the standard calculation where the dependence upon $\not{D}$ does not appear. One can recover from (2.11) the current expression for the Jacobian

$$\ln\mathcal{J}(\eta) = -\mathrm{Tr}[(K + L)R] \ , \quad (2.12)$$

which corresponds to the use of Eq. (2.7) with one regulator $R = R_1 = R_2$, only if the further condition $[R, \not{D}] = 0$ is imposed. This essentially means that $R = f(\not{D})$, which says that in using (2.12) we are also forced to set $a = 1$. In other words, we now understand that expression (2.12) is a good starting point to calculate the vector and axial-vector anomalies only when the gauge-covariant regulating operator $\not{D}$ is used, instead of an arbitrary regulator as might appear to be the case from the expression (2.7). The basic difference between the standard Fujikawa expression (2.12) and the one proposed in (2.11), when considered as a regulated definition of the naive form (2.10), is that, in the former case, the cyclic property of the trace is used directly in the unregulated expression, which is subsequently regularized, while in the latter situation one first regulates (2.10) and then uses all properties of matrix algebra. In order for both expressions (2.7) and (2.11) to be equal, it is necessary that $R_2 = \not{D}R_1\not{D}^{-1}$ in the former. This property is relevant to the calculation of anomalies in systems having non-Hermitian Dirac operators [4,11]. The particular choice $R_1 = \exp(-\not{D}^\dagger\not{D}/M^2)$ in our general expression (2.11) together with

$$R_2 = \not{D}\exp(-\not{D}^\dagger\not{D}/M^2)\not{D}^{-1} = \exp(-\not{D}\not{D}^\dagger/M^2) \ ,$$

explains the specific choice of regularization used in those references.

Finally we give a short description of the calculation of the family (1.1) in two dimensions using the regulating operator $\not{D}_a = \not{\partial} + iea\not{A}$. Our conventions are $\gamma^\mu\gamma^\nu = g^{\mu\nu} + \epsilon^{\mu\nu}\gamma_5$ with $\epsilon^{14} = -i$ and $\gamma_5^2 = +1$. To this end, it is more convenient to rewrite the corresponding expressions for T in (2.11) in the following way:

$$T^V = i\,\mathrm{Tr}(\not{D}^{-1}\alpha[\not{D}, R]) \ , \quad (2.13a)$$

$$T^A = -i\,\mathrm{Tr}(2\beta\gamma_5 R + \not{D}^{-1}\beta\gamma_5[\not{D}, R]) \ . \quad (2.13b)$$

Here the superscript V refers to the vector case, while A labels the axial-vector case. Again we see from (2.13) that only in the case $[\not{D}, R] = 0$ are the usual expressions for both anomalies recovered.

The first term in (2.13b) corresponds to the standard Fujikawa calculation with $A_\mu \to aA_\mu$ and reduces to

$$2i\,\mathrm{Tr}(\beta\gamma_5 R) = \frac{ea}{\pi}\int (dx)\beta(x)\epsilon^{\mu\nu}\partial_\mu A_\nu \ . \quad (2.14)$$

The remaining pieces are incorporated in the expression

$$X=\mathrm{Tr}(\not{D}^{-1}Q[\not{D},R]) \ , \qquad (2.15)$$

where Q stands either for $i\alpha I$ or $-i\beta\gamma_5$. The commutator is now rewritten as $[\not{D}=\not{D}_a+(1-a)ie\not{A}]$

$$[\not{D},R]=ie(1-a)[\not{A},R] \qquad (2.16)$$

and a plane-wave basis is used to calculate the trace. Operating the regulator upon the corresponding plane waves produces the usual shift $\not{D}_a \rightarrow \not{D}_a - i\not{k}$ giving

$$X=ie(1-a)\int \frac{(dk)}{(2\pi)^2}(dx)(dy)\exp[ik(x-y)]\exp(k^2/M^2)\mathrm{tr}\{G(x,y)Q(y)[\exp(i\{\not{k},\not{D}_a\}-\not{D}_a^2)/M^2, \not{A}(y)]\} \ , \qquad (2.17)$$

where $G(x,y)\equiv\langle x|\not{D}^{-1}|y\rangle$. The finite contributions to X come from the free Green's function together with the leading power of k/M in the expansion of the exponential in the commutator of (2.17). After rescaling $k \rightarrow Mp$ and taking the limit $M^2 \rightarrow \infty$ we are left with

$$X=\frac{ie(1-a)}{4\pi}\int (dx)\mathrm{tr}[\gamma^\mu Q(x)\gamma^\nu]\partial_\mu A_\nu(x) \qquad (2.18)$$

from where the vector and axial-vector contributions are easily recovered. The final answer for the anomalies correspond exactly to the family (1.1).

III. REGULARIZATION AMBIGUITIES AND COUNTERTERMS IN TWO DIMENSIONS

Before going to the four-dimensional case, we would like to discuss how regularization ambiguities can be understood in the Fujikawa approach. We will focus on the simpler two-dimensional situation but many of our observations will be valid for the general case.

The prescription (2.11) possesses a further ambiguity that is produced by the choice of the position of the regulator R in Eq. (2.10). As we can easily see, there are nine regularized extensions of (2.10) after the cyclic property of the trace is used; four of them correspond to the expression (2.12) which we have already rejected because it produces a zero vector anomaly even though the regulator is not gauge covariant ($a\neq 1$). The remaining expressions are

$$T_1=-\mathrm{Tr}(\not{D}^{1}R\not{D}K+\not{D}^{1}L\not{D}R) \qquad (3.1)$$

together with

$$T_2=-\mathrm{Tr}(KR+\not{D}^{-1}L\not{D}R) \qquad (3.2a)$$

and

$$T_3=-\mathrm{Tr}(\not{D}^{-1}R\not{D}K+RL) \ . \qquad (3.2b)$$

These alternative possibilities of defining the regulated Jacobian remind us that in order to completely define a quantum theory we must incorporate the largest possible set of symmetries that can be compatible with the system. Up to now we have completely overlooked the discrete symmetries of quantum electrodynamics. In particular, let us consider the invariance under charge conjugation, defined by the transformation $\psi\rightarrow\mathcal{C}\bar{\psi}^t$, $\bar{\psi}\rightarrow-\psi^t\mathcal{C}^{-1}$ $A_\mu\rightarrow-A_\mu$, where the charge-conjugation matrix $\mathcal{C}$ satisfies $\mathcal{C}^{-1}\gamma_\mu\mathcal{C}=-\gamma_\mu^t$ and the superscript t means transposition in the space of Dirac matrices. In even dimensions $d=2n$ one has the transformation property $\mathcal{C}^{-1}\gamma_{d+1}\mathcal{C}=(-1)^n\gamma_{d+1}^t$ for the generalization γ_{d+1} of the matrix γ_5. The above properties imply that the vector and axial-vector currents transform as

$$j_V^\mu\rightarrow-j_V^\mu, \ \ j_A^\mu\rightarrow(-1)^n j_A^\mu \ , \qquad (3.3)$$

under charge conjugation. In order to maintain such a symmetry at the quantum level, the logarithm of the regulated Jacobian should transform accordingly. This requirement will allow us to select an adequate combination of the expressions (3.1) and (3.2) to define a conveniently regulated Jacobian. To this end we have to consider the transformation properties of the corresponding operators. Under charge conjugation they transform as

$$\not{D}\rightarrow\not{D}^T, \ \ R\rightarrow R^T \ , \qquad (3.4a)$$

$$K_V\rightarrow K_V^T \ \ (L_V=-K_V) \ , \qquad (3.4b)$$

$$K_A\rightarrow(-1)^n K_A^T \ \ (L_A=K_A) \ , \qquad (3.4c)$$

where the superscript T means the transposition operation that includes spacetime as well as Dirac indices. For the sake of definiteness, let us consider the vector case. Here we can easily verify that under charge conjugation $T_1^V\rightarrow-T_1^V$, while $T_2^V\leftrightarrow-T_3^V$. The analogous calculation for the axial-vector case leads to $T_1^A\rightarrow(-1)^nT_1^A$, $T_2^A\leftrightarrow(-1)^nT_3^A$. This means that, in principle, any appropriately weighted combination of T_1 and $(T_2+T_3)/2$ could be taken as a good candidate for a definition of the logarithm of the regulated Jacobian. We discuss the two separate cases in the following paragraphs.

In two dimensions, the calculation of the previous section showed that the terms that could possibly violate charge-conjugation invariance in T_2 vanished in the limit $M^2\rightarrow\infty$. As we shall see later, this is not the case in four dimensions.

Now let us consider the alternative prescription (3.1) which can be rewritten in the convenient way

$$T_1^V=X^V(A_\mu)-X^V(-A_\mu) \ , \qquad (3.5a)$$

$$T_1^A=-2i\,\mathrm{Tr}(\beta\gamma_5R)+X^A(A_\mu)+(-1)^nX^A(-A_\mu) \ , \qquad (3.5b)$$

where $X^V(A_\mu)$, $X^A(A_\mu)$ are defined in terms of Eq. (2.15) with the corresponding choices of Q. We have also made use of the charge-conjugated version of Eq. (2.15), which states that

$$X^{V}(-A_\mu)=-\mathrm{Tr}(Q\not{D}^{-1}[\not{D},R])\ , \tag{3.6a}$$

$$X^{A}(-A_\mu)=-(-1)^n\mathrm{Tr}(Q\not{D}^{-1}[\not{D},R])\ . \tag{3.6b}$$

In this way, the results of the calculation according to Eqs. (3.5) with the usual regulating operator $R=\exp[-(\not{D}_a/M)^2]$ are

$$\partial_\mu \tilde{J}^\mu_V=\frac{e}{\pi}(1-a)\partial^\mu A_\mu\ , \tag{3.7a}$$

$$\partial_\mu \tilde{J}^\mu_A=\frac{e}{\pi}\epsilon_{\mu\nu}\partial^\mu A^\nu\ . \tag{3.7b}$$

It is a rather surprising result of this choice of regularization that the axial-vector anomaly turns out to be independent of the parameter a and also that the physical value is obtained. These features will show up again in the four-dimensional calculation, and deserve further discussion.

Our next task is to understand the family of anomalies (1.1) and (3.7) in terms of possible counterterms which would, in principle, allow for the possibility of relating different regularization prescriptions. Usually, local counterterms, which are polynomials in the external field and its derivatives, appear as a freedom in the effective action of the theory. They account for the different possibilities of extracting the finite part of an otherwise divergent quantity. In the construction of the anomalies that we have presented here we do not have at our disposal such an effective action from where the currents are calculated as functional derivatives with respect to the external fields.

In the present calculation, the expression for $\ln\mathcal{J}(\alpha)$ given by any regularized extension of Eq. (2.10) plays the role corresponding to the change of the effective action and each of these expressions, for the vector and axial-vector cases, are also defined up to local polynomials in the external field A_μ which arise from the need to regularize the short-distance behavior of the Green's function $G(x,x)=\langle x|\not{D}^{-1}|x\rangle$ that appears in our definition of the Jacobian. These ambiguities arising from the Green's function induce some freedom in the calculation of the divergence of the respective currents, and should allow us to connect one regularization scheme with another.

In the two-dimensional case, the Green's function G is defined up to the local polynomial $\delta G=\rho\gamma^\mu A_\mu(x)$ [12], which induces the counterterms

$$\begin{aligned}\Delta\mathcal{A}^{V}&=2\rho\partial^\mu A_\mu\ ,\\ \Delta\mathcal{A}^{A}&=-2\rho\epsilon^{\mu\nu}\partial_\mu A_\nu\ ,\end{aligned} \tag{3.8}$$

as the freedom in the explicit form of the anomalies. Here our notation is $\mathcal{A}=\partial_\mu J^\mu$. In particular, this means that the members of the family of anomalies (1.1) can be reshuffled among themselves. This property is related to the question is it possible to obtain the physical value of our family of anomalies by the following method? (i) First regularize using whatever convenient value of the parameter a, say $a=0$, for example, (ii) then change the value of the vector anomaly to zero (physical requirement) by choosing a suitable counterterm, and, finally, (iii) using the previously determined counterterm, obtain the physical value for the axial anomaly. One can generalize this sequence in order to move from one value of the parameter in the family to any other and we will refer to this as the group property of the family. In fact, the family (1.1) possesses such a group property because the anomalies corresponding to any two different parametrizations labeled by a and a' are related by counterterms of the type (3.8) with parameter $\rho=(-e/4\pi)(a'-a)$. In this way, starting from any a it is possible to move the vector anomaly to zero (which is the physical value) by adequately choosing $\rho=(-e/4\pi)(1-a)$. Then, the corresponding counterterm in the axial-vector anomaly will indeed change its value to the physical one. Notice that this is an alternative procedure to the direct choice $a=1$ in Eqs. (1.1) and provides a consistency check on the regularization prescription (2.11).

Now we consider the family of anomalies (3.7) which is obtained from the regularization prescription (3.1). In this case the only way to achieve the physical result is to choose $a=1$ in (3.7a). Any effort to set $\mathcal{A}^{V}=0$ by adding a counterterm would spoil the correct result (3.7b) for $\mathcal{A}^{A}$. In addition, it is not possible to relate both families (1.1) and (3.7) by means of the allowed counterterms given in Eq. (3.8).

In order to understand better the differences that have arisen between the regularization procedures (3.1) and (3.2), we would need to go one step behind and try to identify the corresponding regularized currents which produce the divergences that are directly calculated in the Fujikawa approach. The identification of the corresponding regularized currents in the Fujikawa method is not a direct operation since we do not have at our disposal the effective action of the system; instead we depend on the logarithm of the Jacobian. Thus, the currents may be identified from their divergence, which introduces a further indeterminacy in the problem.

The regularization prescription (2.11) considers the combination $(\not{D}K+L\not{D})$ as a unity. This is very convenient for our purposes because this term leads to $i\gamma^\mu\partial_\mu\alpha(x)$ or $i\gamma^\mu\gamma_5\partial_\mu\beta(x)$ in the vector and axial-vector cases, respectively. Then, from expressions (2.5) we can identify the regularized current as

$$J^\mu(y)\equiv-\lim_{M\to\infty}\mathrm{tr}\int(dx)\frac{dk}{(2\pi)^2}e^{-ikx}G(x,y)r^\mu R_y e^{iky}\ , \tag{3.9}$$

where $r^\mu_V=i\gamma^\mu$, $r^\mu_A=i\gamma^\mu\gamma_5$, respectively, and R_y is such that $\langle y|R=R_y\langle y|$. The above identification is such that $\int(dy)J^\mu(y)\partial_\mu\eta(y)$ exactly reproduces the regularized expression for the integrated anomaly obtained from Eq. (2.11) or (3.2a). Notice also that the naive limit of (3.9) is the formal expression $J^\mu(y)=\mathrm{tr}[G(y,y)r^\mu]$ as it should. Expression (3.9) is valid for arbitrary dimensions and can be exactly calculated in the two-dimensional case giving

$$\begin{aligned}J^\mu_V&=\frac{e}{\pi}\left[\frac{\partial^\mu\partial^\alpha}{\partial^2}-\left[\frac{1+a}{2}\right]g^{\mu\alpha}\right]A_\alpha\ ,\\ J^\mu_A&=-\epsilon^{\mu\nu}J_{V\nu}\end{aligned} \tag{3.10}$$

in the Euclidean case [8]. It is a simple matter to show that the currents (3.10), which coincide with those obtained in the point-splitting calculation, reproduce indeed the family of anomalies (1.1).

Now let us consider the prescription (3.1), which can be rewritten as

$$\mathrm{Tr}(\not{D}^{-1}R\not{D}K+\not{D}^{-1}L\not{D}R) =\mathrm{Tr}(\not{D}^{-1}[R,\not{D}]K)+\mathrm{Tr}[\not{D}^{-1}(L\not{D}+\not{D}K)R] \quad (3.11)$$

in terms of (3.2a). We observe that the difficulty in identifying the corresponding currents that are responsible for the family (3.7) arises from the first term in the right-hand side of (3.11) which depends on $\eta(x)$ instead of $\partial_\mu\eta(x)$. One can force this dependence at the expense of introducing an extra nonlocal contribution to the current, either in the form

$$-\int dy\, dx\, \partial^\mu J(y)\Delta_0(y,x)\partial_\mu\eta(x)\ , \quad (3.12a)$$

or

$$+\int dy\, dx\, J(y)S_0(y,x)\gamma_\mu\partial^\mu\eta(x)\ , \quad (3.12b)$$

where $J(y)$ denotes the remaining pieces of the above-mentioned first term, and Δ_0, and S_0 are the Green's functions of the operators ∂^2 and $\not{\partial}$, respectively. Then we have at least two possible currents which we can associate to the regularization prescription (3.1). They are

$${}_i\tilde{J}^\mu={}_iJ^\mu+J^\mu \ (i=1,2) \quad (3.13)$$

with

$${}_1J_A^\mu(y)=\lim_{M^2\to\infty}\mathrm{tr}\int\frac{dk}{(2\pi)^2}dx\,dx' \times e^{ikx}G(x,x')[R_{x'},\not{D}_{x'}] \times e^{ikx'}i\gamma_5\Delta_0(x'-y)\overleftarrow{\partial}{}_y^\mu \quad (3.14a)$$

or

$${}_2J_A^\mu(y)=-\lim_{M^2\to\infty}\mathrm{tr}\int\frac{dk}{(2\pi)^2}dx\,dx' \times e^{-ikx}G(x,x')[R_{x'},\not{D}_{x'}] \times i\gamma_5S_0(x'-y)\gamma^\mu\ . \quad (3.14b)$$

The corresponding vector currents are obtained by making the replacement $\gamma_5\to1$. The results of the calculation are

$$\begin{aligned}{}_1J_V^\mu&=\frac{e(1-a)}{2\pi}\frac{1}{\partial^2}\partial^\mu\partial^\alpha A_\alpha\ ,\\ {}_1J_A^\mu&=\frac{e(1-a)}{2\pi}\frac{1}{\partial^2}\partial^\mu\epsilon^{\rho\sigma}\partial_\rho A_\sigma\ ,\end{aligned} \quad (3.15)$$

together with

$$\begin{aligned}{}_2J_V^\mu&=\frac{e}{\pi}\left[\frac{1-a}{2}\right]\left[2\frac{1}{\partial^2}\partial^\mu\partial^\alpha A_\alpha-A^\mu\right]\ ,\\ {}_2J_A^\mu&=-\epsilon^{\mu\nu}{}_2J_{V\nu}\ .\end{aligned} \quad (3.16)$$

Both currents ${}_i\tilde{J}^\mu$ defined in (3.13) lead to the family of anomalies (3.7), but differ from the standard ones (3.10) by extra nonlocal terms due to the presence of the first term of the right-hand side of Eq. (3.11) and also are not gauge invariant. We expected that the charge-conjugation-covariant prescription would be naturally singled out as the correct one. Nevertheless, this was not the case and even worse, we found that the results obtained with this prescription do not have the group property and also they are not related by allowed counterterms to the other prescriptions [point-splitting or (2.11)]. These properties can be traced back to the fact that prescription (3.1), though covariant under charge conjugation, does not incorporate $(L\not{D}+\not{D}K)$ as a unity but has an extra contribution which is responsible for the additional nonlocal terms. This result favors prescription (3.2) as the correct one.

IV. THE FOUR-DIMENSIONAL CASE

The anomaly calculations in four dimensions are completely analogous to those described in the previous section and most of the details are presented in the appendices because of the more complicated nature of the algebra that it is involved. Again, our starting point is the one-parameter-dependent point-splitting calculation of the vector and axial-vector anomalies. The latter case can be found in the literature [13], so that we just quote the Euclidean result which is

$$\partial_\mu J_A^\mu=-\frac{(1+a)ie^2}{2(4\pi)^2}\epsilon^{\mu\nu\alpha\beta}F_{\mu\nu}F_{\alpha\beta} \quad (4.1a)$$

in our conventions. We were not able to find the corresponding calculation for the divergence of the vector current with arbitrary a, which is then performed in Appendix A. The result is

$$\partial_\mu J_V^\mu=-\frac{e(1-a)}{(4\pi)^2}\lim_{M_0\to\infty}\partial_\mu[-2M_0^2A^\mu+\tfrac{1}{6}\partial_\nu\partial^\nu A^\mu -\tfrac{2}{3}e^2(1-a)^2A_\nu A^\nu A^\mu]\ . \quad (4.1b)$$

The expected null result obtained in the gauge-invariant calculation ($a=1$) was previously verified explicitly in Ref. [14]. Let us remark that the values for the anomalies in the point-splitting calculation must be covariant under charge conjugation, which amounts to the transformation properties $J_V^\mu\to-J_V^\mu$, $J_A^\mu\to J_A^\mu$, in four dimensions. This fact can be directly verified from expression (A3) for the regularized current, taking into account the transformation properties of the Green's function together with the prescription that the short-distance limit must be taken symmetrically. Equations (4.1) constitute a family of vector and axial-vector anomalies in four dimensions which is the analogue of the one given in Eqs. (1.1) for the two-dimensional case.

Next we turn to the calculation of anomalies in the Fujikawa approach using the extended definition for the regulated Jacobian introduced previously in Eqs. (2.13) and (3.2a) with the regulating operator $R=\exp[-(\not{D}_a/M)^2]$. The calculation according to the prescription (2.13) is summarized in Appendix B and leads to the following family of anomalies in their Euclidean version:

$$\partial_\mu J_V^\mu = \frac{(1-a)e}{(4\pi)^2} \lim_{M\to\infty} \partial_\mu(-2M^2 A^\mu + 2iae A_\nu F^{\mu\nu} + \tfrac{2}{3}\{\partial^\mu\partial^\nu A_\nu - ie(1-a)[A_\nu \partial^\mu A^\nu + \partial_\nu(A^\mu A^\nu)] - e^2(1-a)^2 A_\nu A^\nu A^\mu\}) , \tag{4.2a}$$

$$\partial_\mu J_A^\mu = \frac{(1+a^2)ie^2}{2(4\pi)^2} \epsilon^{\mu\nu\alpha\beta} F_{\mu\nu} F_{\alpha\beta} . \tag{4.2b}$$

Let us remark that, in an abuse of notation, we will be using the same symbols for the currents in different regularization schemes, even though we know from the previous section that, in general, such currents are expected to be different.

The four-dimensional calculation shows explicitly that the prescription (3.2a) is not covariant under charge conjugation. Consequently, one should consider the average $\frac{1}{2}(T_2+T_3)$ to complete the regularization procedure. Nevertheless, the axial anomaly obtained from T_2 transforms correctly under this symmetry. Contrary to the two-dimensional case, the anomaly calculation according to the prescription (3.2), in its symmetrized form, does not directly reproduce the point-splitting results (4.1). Later in this section we will explain how results (4.1) and (4.2) can be related via counterterms in the corresponding Jacobians.

Equation (4.2b) shows that the parameter-dependent coefficient of the axial anomaly is $(1+a^2)/2$ instead of the value $(1+a)/2$ obtained in the point-splitting calculation. We remind the reader that a must be a real number in order for $\not{D}_a$ to be Hermitian. The above result may lead to the uncomfortable feeling that the axial anomaly could not be made zero in principle, even though this is not the physical situation. Relaxing the condition of a to be real does not allow us to choose $a=i$ in order to set (4.2b) equal to zero, because then $\not{D}_a$ would not be Hermitian and the whole calculation according to Eq. (3.2a) would be wrong. The possibility of considering complex values of the parameter can be realized by using the regulators $R_1 = \exp(-\not{D}_a^\dagger \not{D}_a/M^2)$ and $R_2 = \exp(-\not{D}_a \not{D}_a^\dagger/M^2)$ in Eq. (2.7). This calculation gives the factor $[a^2 + aa^* + 4a^{*2} + 6(a^*-a) + 6]/12$, which reduces to $(1+a^2)/2$ when $a=a^*$, and which can be made zero for the choice $a = -2 \pm i\sqrt{30}/2$ that, anyway, is different from the naive choice $a=i$.

Now we consider the anomaly calculation according to prescription (3.1) which is invariant under charge conjugation. Some details of the calculation are presented at the end of the Appendix B and here we only write the results, which are

$$\partial_\mu J_V^\mu = \frac{2(1-a)e}{(4\pi)^2} \times \lim_{M\to\infty} \partial_\mu\{-2M^2 A^\mu + \tfrac{2}{3}[\partial^\mu\partial^\nu A_\nu - e^2(1-a)^2 A_\nu A^\nu A^\mu]\} , \tag{4.3a}$$

$$\partial_\mu J_A^\mu = \frac{ie^2}{(4\pi)^2} \epsilon^{\mu\nu\alpha\beta} F_{\mu\nu} F_{\alpha\beta} . \tag{4.3b}$$

In complete analogy with the two-dimensional case, the axial anomaly is again independent of the parameter a and apparently has the correct physical value even before imposing any requirement about gauge invariance.

Having presented three alternative calculations of the vector and axial-vector anomalies in four dimensions, we would like to discuss now some relations among such results. In this section we will not intend to identify the currents which lead to the corresponding anomalies, mainly because of the calculational difficulties in this case.

The first thing we notice in all the results obtained for the vector anomaly is that our regularization prescriptions have not been completely successful because the terms linear in A_μ with no derivatives diverge. This means that we still have to identify the finite part of this contribution to the Jacobian. After this is done we realize that such finite pieces are defined up to a local counterterm $\int dy\, \alpha(y)\partial_\mu A^\mu(y)$. We can think of this term as arising from an effective ambiguity in the short-distance behavior of the four-dimensional Green's function given by $\delta G \sim \gamma^\mu A_\mu$. This counterterm does not contribute to the axial-vector anomaly because $\mathrm{Tr}(\gamma^\mu\gamma^\nu\gamma_5)=0$ in four dimensions. Now we consider the possible counterterms that we can have in the corresponding Jacobians arising directly from the identification of the short-distance behavior of the Green's function. In four dimensions, the subtraction of such infinities leads to an arbitrariness in the Green's function determined by the following local polynomials in A_μ and its derivatives:

$$\delta_1 G \sim \not{\partial}\partial_\mu A^\mu, \quad \delta_2 G \sim \partial_\mu \partial^\mu \not{A} , \tag{4.4}$$

$$\begin{aligned} &\delta_3 G \sim \gamma^\mu\gamma^\nu\gamma^\rho A_\mu \partial_\nu A_\rho , \\ &\delta_4 G = \not{A}\, \partial_\mu A^\mu , \\ &\delta_5 G = \not{\partial} A^2 , \\ &\delta_6 G = A^\mu \partial_\mu \not{A} , \end{aligned} \tag{4.5}$$

$$\delta_7 G \sim A^2 \not{A} . \tag{4.6}$$

Such terms arise when the divergences of $G(x,x)$ are labeled by powers of A_μ and correspond to all possible terms of dimension $[m^3]$ that can be constructed with the γ matrices, A_μ, and its derivatives. The vector anomaly would get contributions from any one of the terms above and those in Eq. (4.5) will induce terms violating charge-conjugation invariance. On the other hand, the axial anomaly will get a contribution only from the first expression in Eq. (4.5). We then see that, in four dimensions (and, in general, in $4n$ dimensions) charge-conjugation invariance puts severe constraints upon the possibility of relating the results for the anomalies obtained via different regularization prescriptions, or results with different parameters within a specific regularization.

Let us consider the point-splitting calculation, for example. Suppose that we want to change the axial anomaly with parameter a to a corresponding one with parameter a'. This could only be done with a counterterm induced by the first local polynomial in (4.5) which, nevertheless, would generate terms that violate charge-conjugation invariance in the vector anomaly. This is a simple way of realizing that the family of anomalies (4.1) in four dimensions does not possess the group property exhibited by the analogous two-dimensional family. Thus, the physical result in the four-dimensional family (4.1) can only be achieved by demanding gauge invariance choosing $a=1$.

Now let us compare the point-splitting calculation (4.1) with the one using the regularization prescription (3.2a). The latter calculation does not produce a result covariant under charge conjugation for the vector anomaly and the counterterm arising from the first local polynomial in (4.5) needed to change the factor $(1+a^2)$ into $(1+a)$ in the axial anomaly, gives contributions of similar type to those terms already present in (4.2a). Unfortunately, the elimination of these charge-conjugation-violating terms in the vector anomaly does not constitute any check on our calculation. This is because we have at our disposal three parameters, arising from the last three local polynomials in (4.5), to cancel a given linear combination of the terms $A^\mu\partial_\nu A^\nu$, $A^\nu\partial^\mu A_\nu$, $A^\nu\partial_\nu A^\mu$ that appear inside the total derivative term in Eq. (4.2a). This can always be done and we do not write the results for the required coefficients because they are not very illuminating. The only remaining difference between the results (4.1) and (4.2) for the vector anomaly is the coefficient in the term linear in A_μ with three derivatives. Again, one result can be transformed into the other by means of the counterterm arising from any of the local polynomials in (4.4).

Finally, we comment on the family (4.3) obtained through the charge-conjugation-invariant prescription (3.1). We obtain results similar to the two-dimensional case: the axial-vector anomaly turns out to be independent of the parameter a and also with the correct physical value. Nevertheless, the vector anomaly is nonzero and parameter dependent. Because of charge-conjugation invariance, this family is not related by counterterms to any of the previous ones and does not possess the group property either. Even though we did not calculate the currents that produce this family of anomalies, we conjecture that, in analogy with the two-dimensional case, the regularization prescription (3.1) induces extra nonlocal terms in these currents that account for these rather bizarre results.

We conclude with the following remarks. (i) Our generalization (2.11) of the regularized Fujikawa Jacobian has allowed us to understand the regularization ambiguities appearing due to local terms in the effective action as ambiguities arising from local counterterms in the Green's function appearing in the Jacobian. The ability of using such counterterms is highly restricted by charge-conjugation invariance, as is clear in the four-dimensional case. (ii) The nice group property exhibited in the two-dimensional case by the family of anomalies given in Eqs. (1) is not extended to four dimensions, essentially due to charge-conjugation invariance. (iii) The alternative possibility of placing the regulating operator according to Eq. (3.1), though covariant under charge conjugation *ab initio*, has to be rejected because it induces extra nonlocal terms in the currents.

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APPENDIX A: THE POINT-SPLITTING METHOD

In this appendix we present the calculation of the vector anomaly using the point-splitting method in Minkowski space. The vector current is given by

$$j_V^\mu(x|\epsilon|a)=\bar{\psi}(x+\epsilon/2)\gamma^\mu\psi(x-\epsilon/2)\times\exp\left[-iea\int_{x-\epsilon/2}^{x+\epsilon/2}A_\mu dx^\mu\right], \qquad (A1)$$

where ϵ is a small spacelike interval, and a is a parameter which should be set equal to one in order to have gauge invariance; i.e., for $a=1$ the anomaly must vanish.

The divergence of $j_V^\mu(x|\epsilon|a)$ is

$$\partial_\mu j_V^\mu(x|\epsilon|a)=iej_V^\mu(x|\epsilon|a)\times\left[A_\mu(x+\epsilon/2)-A_\mu(x-\epsilon/2)-a\partial_\mu\int_{x-\epsilon/2}^{x+\epsilon/2}A^\alpha(y)dy_\alpha\right]. \qquad (A2)$$

To obtain the anomaly of the vector current we take first the vacuum expectation value of (A2) and then the symmetric limit $\epsilon\to 0$, which leads to

$$H=\partial_\mu J_V^\mu=e\lim_{\epsilon\to 0}\mathrm{tr}\gamma^\mu G(x-\epsilon/2,x+\epsilon/2)\times\left[A_\mu(x+\epsilon/2)-A_\mu(x-\epsilon/2)-a\partial_\mu\int_{x-\epsilon/2}^{x+\epsilon/2}A^\alpha(y)dy_\alpha\right]\times\exp\left[-iae\int_{x-\epsilon/2}^{x+\epsilon/2}A^\alpha(y)dy_\alpha\right]. \qquad (A3)$$

In this expression $G(x-\epsilon/2,x+\epsilon/2)$ is the Green's function of the fermion in the external field A_μ. The first thing that we would like to check, before making any calculation, is whether Eq. (A3) transforms correctly under charge conjugation. Under the corresponding transformations the right hand side of (A3) changes into

$$H \to -e\lim_{\epsilon\to 0}\mathrm{tr}\gamma^\mu G(x+\epsilon/2,x-\epsilon/2)\times\left[-A_\mu(x+\epsilon/2)+A_\mu(x-\epsilon/2)+a\partial_\mu\int_{x-\epsilon/2}^{x+\epsilon/2}A^\alpha(y)dy_\alpha\right]\times\exp\left[+iae\int_{x-\epsilon/2}^{x+\epsilon/2}A^\alpha(y)dy_\alpha\right]. \quad \text{(A4)}$$

Taking into account that the limit $\epsilon\to 0$ is taken symmetrically, it is possible to interchange ϵ by $-\epsilon$, so that H transforms as $H\to -H$. This is in agreement with the transformation property of the vector current $j_V^\mu \to -j_V^\mu$ under charge conjugation. This result implies that the only terms that can contribute to the anomaly of the vector current are odd in the electromagnetic potential A_μ.

To calculate the divergence (A3) we expand the fields to $O(\epsilon^3)$ and solve the integrals to the same order. The result is

$$H=\partial_\mu J_V^\mu = e\lim_{\epsilon\to 0}\mathrm{tr}\gamma^\mu G(x-\epsilon/2,x+\epsilon/2)\left[(\epsilon^\alpha - iae\epsilon^\alpha\epsilon^\sigma A_\sigma)(\partial_\alpha A_\mu - a\partial_\mu A_\alpha)+\frac{\epsilon^\alpha\epsilon^\sigma\epsilon^\rho}{2}[\tfrac{1}{12}\partial_\rho\partial_\sigma\partial_\alpha A_\mu-(a/12)\partial_\rho\partial_\sigma\partial_\mu A_\alpha - (ae)^2A_\sigma A_\rho(\partial_\alpha A_\mu - a\partial_\mu A_\alpha)]\right]. \quad \text{(A5)}$$

Substituting the expansion of $G(x,y)$, we obtain contributions to the anomaly of zero, first, and second order in the external field. We denote this power series by

$$H=H_0+H_1+H_2+\cdots ,$$

where H_0 corresponds to the free term of the Green's function, and it is given by

$$H_0=e\lim_{\epsilon\to 0}\mathrm{tr}\int\frac{dp}{(2\pi)^4}\exp(ip\epsilon)\gamma^\mu\frac{\not p}{p^2}\left[(\epsilon^\alpha - iae\epsilon^\alpha\epsilon^\sigma A_\sigma)(\partial_\alpha A_\mu - a\partial_\mu A_\alpha)+\frac{\epsilon^\alpha\epsilon^\sigma\epsilon^\rho}{2}[\tfrac{1}{12}\partial_\rho\partial_\sigma\partial_\alpha A_\mu-(a/12)\partial_\rho\partial_\sigma\partial_\mu A_\alpha - (ae)^2A_\sigma A_\rho(\partial_\alpha A_\mu - a\partial_\mu A_\alpha)]\right]. \quad \text{(A6)}$$

After taking the trace it is easy to see that (A6) diverges in the limit $\epsilon\to 0$. To avoid this we introduce a Pauli-Villars regulator with mass M_0. Calculating the integral and taking the limit $\epsilon\to 0$, we obtain

$$H_0=-\frac{2e(1-a)}{(4\pi)^2}\lim_{M_0\to\infty}\partial_\mu[-M_0^2A^\mu+\tfrac{1}{12}\partial_\alpha\partial^\alpha A^\mu - \tfrac{1}{3}(ae)^2A_\alpha A^\alpha A^\mu]. \quad \text{(A7)}$$

The first term in the right-hand side of (A7) diverges in the limit $M_0\to\infty$, but this divergence can be eliminated introducing a mass renormalization.

The first-order term in the external field of the Green's function contributes with

$$H_1=e^2\lim_{\epsilon\to 0}\mathrm{tr}\int dz\frac{dk\,dp}{(2\pi)^8}\exp[-ik(x-\epsilon/2-z)]\exp[-ip(z-x-\epsilon/2)]\times\gamma^\mu S(k)\not A(z)S(p)[(\epsilon^\alpha - iae\epsilon^\alpha\epsilon^\sigma A_\sigma)(\partial_\alpha A_\mu - a\partial_\mu A_\alpha)]. \quad \text{(A8)}$$

The result of calculating this expression is

$$H_1=-\frac{4e^3a(1-a)}{3(4\pi)^2}\partial_\mu(A^\mu A^\nu A_\nu). \quad \text{(A9)}$$

In the contribution of second order in A_μ in the Green's function, the only term that gives a nonvanishing result is

$$H_2=e^3\,\mathrm{tr}\int dz\,dw\frac{dp\,dq\,dk}{(2\pi)^{12}}\exp[iz(p-q)]\exp[ix(k-p)]\exp[iw(q-k)]\times\exp[(i\epsilon/2)(k+p)]\gamma^\mu S(p)\not A(z)S(q)\not A(w)S(k)\epsilon^\alpha(\partial_\alpha A_\mu - a\partial_\mu A_\alpha). \quad \text{(A10)}$$

After calculating the integrals and taking the trace we obtain

$$H_2=\frac{2e^3(1-a)}{3(4\pi)^2}\partial_\mu(A_\nu A^\nu A^\mu)\ . \tag{A11}$$

Adding all the contributions of the divergence of the vector current we are led to

$$\partial_\mu J_V^\nu=-\frac{e(1-a)}{(4\pi)^2}\lim_{M_0\to\infty}\partial_\mu[-2M_0^2A^\mu+\tfrac{1}{6}\partial_\nu\partial^\nu A^\mu -\tfrac{2}{3}e^2(1-a)^2A_\nu A^\nu A^\mu]\ . \tag{A12}$$

This result shows clearly that, when $a=1$, i.e., the case of a gauge-invariant current, the vector anomaly vanishes. Finally, we remark that our conventions are such that the same expression (A12) is valid in the Euclidean formulation.

APPENDIX B: AXIAL AND VECTOR ANOMALIES IN FOUR DIMENSIONS

In this appendix we calculate the axial and vector anomalies in Euclidean space using the prescription given in Eq. (3.2a); i.e., we use expressions (2.13a) and (2.13b) with the regulator given by

$$R=\exp(-\not{D}_a^2/M^2)\ , \tag{B1}$$

where

$$\not{D}_a=\gamma^\mu(\partial_\mu+iea A_\mu)=\gamma^\mu D_{a\mu}\ ,$$
$$\not{D}_a^2=D_{a\mu}D_a^\mu+\frac{iae}{2}\gamma^\mu\gamma^\nu F_{\mu\nu}\ . \tag{B2}$$

Our conventions are as follows: $\gamma^\mu(\mu=1,2,3,4)$ are anti-Hermitian matrices satisfying $\{\gamma^\mu,\gamma^\nu\}=2g_{\mu\nu}$; $g_{\mu\nu}=\mathrm{diag}(----)$; $\gamma_5=\gamma^1\gamma^2\gamma^3\gamma^3\gamma^4$ with $\gamma_5^2=1$, and $\epsilon^{1234}=+1$.

We begin by calculating the first term in Eq. (2.13b). This contribution is equivalent to the Fujikawa computation of the axial anomaly with $A_\mu\to aA_\mu$. The result is

$$T_{A2}=\frac{ia^2e^2}{(4\pi)^2}\int dy\,\beta(y)\epsilon^{\mu\nu\rho\sigma}F_{\mu\nu}F_{\rho\sigma}\ . \tag{B3}$$

The second term in (2.13b) and the contribution (2.13a) to the vector anomaly can be written in similar form by introducing a matrix Q, with the value $Q_V=\alpha I$ for the vector anomaly and $Q_A=-\beta\gamma_5$ for the axial anomaly. In this way we have

$$T_1=i\,\mathrm{Tr}(\not{D}^{-1}Q[\not{D},R])\ . \tag{B4}$$

Calculating the trace over the spacetime and using Eq. (2.16), we are left with

$$\begin{aligned}T_1=-i(1-a)e\lim_{M\to\infty}\mathrm{tr}\int dx\,dy\frac{dk\,dq}{(2\pi)^8}\exp(-ikx)\exp[iy(k-q)]G(x,y)Q(y)\\ \times A_\rho(q)\exp(k^2/M^2)\left\{\gamma^\rho\exp\left[\frac{-1}{M^2}\left[D_{a\mu}D_a^\mu+2ik_\mu D_a^\mu+\frac{iae}{2}\gamma^\mu\gamma^\nu F_{\mu\nu}\right]\right]\right.\\ -\exp\left[\frac{-1}{M^2}\left[D_{a\mu}D_a^\mu-2iq_\mu D_a^\mu-q^2+2k_\mu q^\mu\right.\right.\\ \left.\left.\left.+2ik_\mu D_a^\mu+\frac{iae}{2}\gamma^\mu\gamma^\nu F_{\mu\nu}\right]\right]\gamma^\rho\right\}\ ,\end{aligned} \tag{B5}$$

where $G(x,y)=\langle x|\not{D}^{-1}|y\rangle$ with $\not{D}=\not{\partial}+ie\not{A}$. Rescaling $k\to Mp$, and expanding the exponentials up to terms that go like $1/M^3$, we obtain that the term inside the curly brackets in (B5) is

$$\begin{aligned}\{\ \}^\rho=\Big[&\frac{1}{M}2p_\mu q^\mu\gamma^\rho+\frac{1}{M^2}[iae(g^{\nu\rho}\gamma^\mu-g^{\rho\mu}\gamma^\nu)F_{\mu\nu}-(q^2+2iq_\mu D_a^\mu+2p_\mu p_\alpha q^\mu q^\alpha+4ip_\mu p_\alpha q^\alpha D_a^\mu)\gamma^\rho]\\ &+\frac{1}{M^3}\{aep_\alpha(g^{\nu\rho}\gamma^\mu-g^{\rho\mu}\gamma^\nu)(F_{\mu\nu}D_a^\alpha+D_a^\alpha F_{\mu\nu})-2q_\alpha p_\mu(D_a^\alpha D_a^\mu+D_a^\mu D_a^\alpha)\gamma^\rho\\ &+2iq^2p_\mu D_a^\mu\gamma^\rho-2p_\alpha q^\alpha[D_{a\mu}D_a^\mu+(iae/2)\gamma^\mu\gamma^\nu F_{\mu\nu}]\gamma^\rho+2p_\alpha q^\alpha(2iq_\mu D_a^\mu+q^2)\gamma^\rho\\ &+\tfrac{4}{3}p_\mu p_\nu p_\alpha(q^\mu q^\nu q^\alpha+3iq^\nu q^\alpha D_a^\mu-3q^\alpha D_a^\mu D_a^\nu)\gamma^\rho\}\Big]\ .\end{aligned} \tag{B6}$$

Now, to calculate (B5) we needed to expand the Green's function $G(x,y)$ in powers of the external field A_μ. For the first term in the expansion we have

$$T_{10}=(1-a)e\lim_{M\to\infty}\mathrm{tr}\int dy\frac{dp\,dq}{(2\pi)^8}\exp(-iqy)\frac{-M^4}{M\not{p}}Q(y)A_\rho(q)\exp(p^2)\{\ \}^\rho\ . \tag{B7}$$

Substituting (B6) into (B7) we can calculate the contribution to the axial anomaly with the result

$$T_{10A}=\frac{ia(1-a)e^2}{2(4\pi)^2}\int dy\,\beta(y)\epsilon^{\mu\nu\alpha\beta}F_{\mu\nu}F_{\alpha\beta}\ . \tag{B8}$$

The corresponding contribution to the vector anomaly involves a more tedious calculation since there is a large number of terms not vanishing under the trace. These are given by

$$\begin{aligned}T_{10V}=&-(1-a)e\lim_{M\to\infty}\mathrm{tr}\int dy\frac{dq\,dp}{(2\pi)^8}\exp(-iqy)\alpha(y)A_\rho(q)\exp(p^2)\\&\times\left\{2\frac{p_\alpha p_\mu}{p^2}M^2q^\mu\gamma^\alpha\gamma^\rho+\frac{\not p p_\alpha}{p^2}[ae(g^{\nu\rho}\gamma^\mu-g^{\mu\rho}\gamma^\nu)(F_{\mu\nu}D_a^\alpha+D_a^\alpha F_{\mu\nu})\right.\\&\qquad-2q_\mu(D_a^\mu D_a^\alpha+D_a^\alpha D_a^\mu)\gamma^\rho\\&\qquad+2iq^2D_a^\alpha\gamma^\rho-q^\alpha(2D_{a\mu}D_a^\mu+iae\gamma^\mu\gamma^\nu F_{\mu\nu})\gamma^\rho\\&\qquad+2q^\alpha(2iq_\mu D_a^\mu+q^2)\gamma^\rho]\\&\left.+\frac{4}{3}\frac{p_\eta p_\nu p_\alpha p_\mu}{p^2}(q^\mu q^\nu q^\alpha+3iq^\nu q^\alpha D_a^\mu-3q^\alpha D_a^\mu D_a^\nu)\gamma^\eta\gamma^\rho\right\}\ .\end{aligned} \tag{B9}$$

Tracing over the γ matrices and calculating one of the momentum integrals as an inverse Fourier transform, we obtain the result

$$\begin{aligned}T_{10V}=&\frac{-(1-a)e}{(4\pi)^2}\lim_{M\to\infty}\int dy\,\alpha(y)\{2M^2\partial_\mu A^\mu-2iaeA^\nu(F_{\mu\nu}D_a^\mu+D_a^\mu F_{\mu\nu})\\&-2iae\partial_\mu A_\nu F^{\mu\nu}-\tfrac{2}{3}[\partial_\mu A_\nu(D_a^\mu D_a^\nu+D_a^\nu D_a^\mu)+\partial_\mu\partial^\mu A_\nu D_a^\nu\\&+\partial_\mu A^\mu D_{a\nu}D_a^\nu+2\partial^\mu\partial^\nu A_\nu D_{a\mu}+\partial_\nu\partial^\nu\partial^\mu A_\mu]\}\ .\end{aligned} \tag{B10}$$

Reordering this expression, we finally obtain the zeroth-order contribution in the Green's function expansion to the vector anomaly:

$$\begin{aligned}T_{10V}=&\frac{(1-a)e}{(4\pi)^2}\lim_{M\to\infty}\int dy\,\alpha(y)\partial_\mu(-2M^2A^\mu+2iaeA_\nu F^{\mu\nu}\\&+\tfrac{2}{3}\{\partial^\mu\partial^\nu A_\nu+iae[A_\nu\partial^\mu A^\nu+\partial_\nu(A^\mu A^\nu)]-a^2e^2A_\nu A^\nu A^\mu\})\ .\end{aligned} \tag{B11}$$

The second contribution to the anomalies comes from the first-order term in the Green's function and it is

$$\begin{aligned}T_{11}=&(1-a)e^2\lim_{M\to\infty}\mathrm{tr}\int dz\,dy\frac{dk\,dq\,dt}{(2\pi)^{12}}\{\exp[iy(k+t-q)]\exp[-iz(k+t)]\\&\times S(-k)\not A(z)S(t)Q(y)A_\rho(q)\exp(k^2/M^2)\{\ \}^\rho\}\ ,\end{aligned} \tag{B12}$$

where $S(k)=1/\not k$ is the free propagator. Making a change of variables and rescaling in the usual way we can express (B12) in the form

$$T_{11}=(1-a)ie^2\lim_{M\to\infty}\mathrm{tr}\int dy\frac{dp\,dq\,dr}{(2\pi)^{12}}M^4\{\exp[-iy(q+r)]S(-Mp)\gamma^\alpha S(-Mp-r)Q(y)A_\alpha(r)A_\rho(q)\exp(p^2)\{\ \}^\rho\}\ . \tag{B13}$$

Introducing the expansion for the free propagator $S(-Mp-r)$ in powers of $1/M$ together with Eq. (B6), we get the result

$$T_{11}=(1-a)ie^2\mathrm{tr}\int dy\frac{dq\,dp\,dr}{(2\pi)^{12}}\exp[-iy(q+r)]A_\alpha(r)A_\rho(q)\exp(p^2)$$
$$\times\Bigg[2\gamma^\beta Q(y)[iae(g^{\nu\rho}\gamma^\mu-g^{\mu\rho}\gamma^\nu)F_{\mu\nu}-\gamma^\rho(2iq_\mu D_a^\mu+q^2)]\frac{p^\alpha p_\beta}{p^4}$$
$$-2\gamma^\beta Q(y)\gamma^\rho(4iq^\eta D_a^\mu+2q^\mu q^\eta)\frac{p^\alpha p_\beta p_\mu p_\eta}{p^4}$$
$$-\gamma^\alpha Q(y)[iae(g^{\nu\rho}\gamma^\mu-g^{\mu\rho}\gamma^\nu)F_{\mu\nu}-\gamma^\rho(2iq_\mu D_a^\mu+q^2)]\frac{1}{p^2}$$
$$+\gamma^\alpha Q(y)\gamma^\rho(4iq^\eta D_a^\mu+2q^\mu q^\eta)\frac{p_\mu p_\eta}{p^2}+2r_\eta q^\mu\gamma^\beta\gamma^\alpha\gamma^\eta Q(y)\gamma^\rho\frac{p_\beta p_\mu}{p^4}$$
$$-8r_\eta q^\mu\gamma^\beta Q(y)\gamma^\rho\frac{p^\alpha p^\eta p_\beta p_\mu}{p^6}+4r^\eta q^\mu\gamma^\alpha Q(y)\gamma^\rho\frac{p_\eta p_\mu}{p^4}\Bigg]\ . \tag{B14}$$

From this expression we obtain the contribution to the axial anomaly,

$$T_{11A}=\frac{i(1-a)e^2}{2(4\pi)^2}\int dy\,\beta(y)\epsilon^{\mu\nu\alpha\beta}F_{\mu\nu}F_{\alpha\beta}\ , \tag{B15}$$

whereas for the vector anomaly we have

$$T_{11V}=\frac{(1-a)e}{(4\pi)^2}\int dy\,\alpha(y)\tfrac{2}{3}\partial_\mu\{2ae^2A_\nu A^\nu A^\mu-ie[A_\nu\partial^\mu A^\nu+\partial_\nu(A^\mu A^\nu)]\}\ . \tag{B16}$$

The last contribution to the anomalies arises from the second-order term in the Green's function expansion, and it is given by

$$T_{12}=(1-a)e^3\lim_{M\to\infty}\mathrm{tr}\int dy\,dz\,dw\frac{dk\,dq\,dt\,dr}{(2\pi)^{16}}\exp(k^2/M^2)\exp[iy(k+t-q)]\exp[-iz(k+r)]$$
$$\times\exp[iw(r-t)]S(-k)A(z)S(r)A(w)S(t)Q(y)A_\rho(q)\{\ \}^\rho\ . \tag{B17}$$

If we again replace the free propagators, integrate over the position variables z and w, and take the inverse Fourier transforms, we obtain

$$T_{12}=\frac{(1-a)e^3}{12(4\pi)^2}\mathrm{tr}\int dy\ A_\eta A_\sigma\partial^\mu A_\rho(g_{\mu\nu}g_{\alpha\beta}+g_{\mu\alpha}g_{\nu\beta}+g_{\mu\beta}g_{\nu\alpha})\gamma^\nu\gamma^\eta\gamma^\alpha\gamma^\sigma\gamma^\beta Q(y)\gamma^\rho\ . \tag{B18}$$

From this expression it is easy to see that the contribution to the axial anomaly vanishes and that, for the vector anomaly, we have

$$T_{12V}=\frac{-(1-a)e^3}{(4\pi)^2}\int dy\,\alpha(y)\tfrac{2}{3}\partial_\mu(A_\nu A^\nu A^\mu)\ . \tag{B19}$$

The result of adding all the contributions to the axial anomaly is

$$T_A=\frac{(a^2+1)ie^2}{2(4\pi)^2}\int dy\,\beta(y)\epsilon^{\mu\nu\alpha\beta}F_{\mu\nu}F_{\alpha\beta}\ , \tag{B20}$$

whereas for the vector anomaly we have

$$T_V=\frac{(1-a)e}{(4\pi)^2}\lim_{M\to\infty}\int dy\,\alpha(y)\partial_\mu(-2M^2A^\mu+2iaeA_\nu F^{\mu\nu}+\tfrac{2}{3}\{\partial^\mu\partial^\nu A_\nu-ie(1-a)[A_\nu\partial^\mu A^\nu+\partial_\nu(A^\mu A^\nu)]$$
$$-e^2(1-a)^2A_\nu A^\nu A^\mu\})\ . \tag{B21}$$

Equation (B21) shows explicitly that the prescription (2.13a) does not transform correctly under charge conjugation, so that we now consider the prescription (3.1) which naturally does so. In order to calculate with this prescription we use Eq. (3.5). For the axial anomaly we have from Eqs. (B8) and (B15) that

$$X^A(A_\mu)=\frac{i(1-a^2)e^2}{(4\pi)^2}\int dy\,\beta(y)\epsilon^{\mu\nu\alpha\beta}F_{\mu\nu}F_{\alpha\beta}\ . \tag{B22}$$

Then, the axial anomaly in four dimensions calculated with the prescription (3.1) is

$$\partial^\mu J_\mu^A = \frac{ie^2}{(4\pi)^2}\epsilon^{\mu\nu\alpha\beta}F_{\mu\nu}F_{\alpha\beta} \ . \tag{B23}$$

For the vector anomaly, we have from the Eq. (B21), that

$$X^V(A_\mu) = T_V \tag{B24}$$

and then we obtain

$$\partial^\mu J_\mu^V = \frac{2(1-a)e}{(4\pi)^2}\partial_\mu \lim_{M\to\infty}\{-2M^2A^\mu + \tfrac{2}{3}[\partial^\mu\partial^\nu A_\nu - e^2(1-a)^2 A_\nu A^\nu A^\mu]\} \ , \tag{B25}$$

according to prescription (3.1).

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