Dynamical violation of parity and chiral symmetry in three-dimensional four-Fermi theory

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We study dynamical symmetry breaking in (2+1)-dimensional quantum electrodynamics with four-Fermi couplings. Using the large-N expansion and an effective potential method we show that the four-Fermi couplings can be relevant when the coupling constants are near certain critical points. We show that the four-Fermi interactions can break either parity and time-reversal symmetries or a certain discrete flavor symmetry and analyze how the symmetry-breaking pattern is modified by electromagnetic interactions. We discuss the critical behavior of the theory in the large-N expansion in both the leading and next-to-leading orders.
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I. INTRODUCTION

Dynamical symmetry breaking plays a central role in many current areas of physics research [1–3]. Beginning with the work of Nambu and Jona-Lasinio [1], it has been investigated as a mechanism for generating the fermion mass spectrum in elementary-particle physics. The breaking of the approximate chiral symmetry of QCD is now widely accepted as a realization of this phenomenon [2]. It is also proposed as a mechanism for symmetry breaking in the electroweak sector of the standard model [3]. Among its appealing features are the fact that is requires fewer *ad hoc* parameters and fewer elementary fields than the alternative spontaneous symmetry breaking with elementary Higgs fields.

A systematic analysis of dynamical symmetry breaking in realistic field theories requires nonperturbative methods. Commonly used are numerical lattice simulations, analytic approximation schemes such as the study of Dyson-Schwinger equations in the quenched, planar limit, and variational methods. Another useful laboratory where much as been learned is the solution of models in lower dimensions, particularly in $1+1$ dimensions where there are numerous solvable interacting quantum field theories with discrete symmetries broken by composite order parameters. In this paper we shall study the dynamical breaking of parity and time reversal and a certain discrete flavor (chiral) symmetry in a strongly coupled $(2+1)$-dimensional field theory using a conventional effective potential approach and the large-N approximation. Our method is especially suited to field theories with four-Fermi interactions where the order parameter is the vacuum expectation value of a local composite operator.

Perhaps the classic example of dynamical symmetry breaking in $1+1$ dimensions occurs in the Gross-Neveu model [4,5], which is strictly renormalizable and exhibits asymptotic freedom, dimensional transmutation, and dynamical chiral-symmetry breaking through the generation of fermion mass. It was conjectured long ago [5] that the generalization of the Gross-Neveu model to $2+1$ dimensions, although not renormalizable in an expansion in powers of the four-Fermi coupling, would be renormalizable in the large-N expansion and would exhibit some of the features of the $(1+1)$-dimensional model such as dynamical symmetry breaking. This possibility has recently been discussed by several authors [6–11]. We shall elaborate on a recent investigation [11] of dynamical breaking of parity and time-reversal invariance and also certain discrete flavor symmetries in the $(2+1)$-dimensional Gross-Neveu model with additional U(1) gauge fields.

We shall consider a continuum field theory with two kinds of four-Fermi couplings as well as a U(1) gauge coupling:

$$S=\int d^3x\left[\sum_{a=1}^{N} A\bar{\psi}_a i\gamma_\mu\partial\mu\psi_a+\frac{\lambda A^2}{2N\Lambda}\left[\sum_{a=1}^{N}\bar{\psi}_a\psi_a\right]^2+\frac{\kappa A^2}{2N\Lambda}\left[\sum_{a=1}^{N}\bar{\psi}_a\tau^3\psi_a\right]^2+A\sum_{a=1}^{N}\bar{\psi}_a\gamma_\mu e\mathcal{A}_\mu\psi_a+\frac{N}{4\Lambda}\mathcal{F}_{\mu\nu}\mathcal{F}_{\mu\nu}\right], \tag{1.1}$$

with N flavors of complex four-component fermions and a U(1) gauge field $\mathcal{A}_\mu$ with $\mathcal{F}_{\mu\nu}=\partial_\mu\mathcal{A}_\nu-\partial_\nu\mathcal{A}_\mu$ in three-dimensional Euclidean space. Λ is the ultraviolet cutoff and A, λ, κ, and e are dimensionless, cutoff-dependent constants. $A^{1/2}$ is the fermion wave-function renormalization parameter, λ and κ are the bare four-Fermi vertices and e is the bare electric charge. These constants have been scaled by the appropriate powers of the cutoff to make them dimensionless, i.e., so that the only dimensional parameter in the model is the ultraviolet cutoff.

The γ matrices are

$$\gamma_\mu=\begin{bmatrix}\sigma_\mu & 0\\ 0 & \sigma_\mu\end{bmatrix},\quad \tau^3=\begin{bmatrix}\mathcal{I} & 0\\ 0 & -\mathcal{I}\end{bmatrix}, \tag{1.2}$$

where σ_μ, $\mu=1,2,3$ are the Pauli matrices and $\mathcal{I}$ is the 2×2 unit matrix. (Notation and definitions of symbols for Euclidean fermions is discussed in detail in Appendix A.)

Three kinds of mass operator are important in this model: a scalar fermion mass $\bar\psi\tau^3\psi$, a pseudoscalar fermion mass $\bar\psi\psi$, and a Chern-Simons topological mass term for the photon field $\epsilon^{\mu\nu\lambda}\mathcal{A}_\mu\partial_\nu\mathcal{A}_\lambda$. The latter changes by an exact derivative under a gauge transformation and therefore has a gauge-invariant spacetime integral [12,13]. These mass terms are distinguished by the way in which they transform under discrete symmetries.

The action (1.1) has $U(N)\times U(N)$ symmetry. Equation (1.1) also has a discrete Z_2 symmetry:

$$Z_2:\psi\to\tau^1\psi,\quad Z_2:\bar\psi\to\bar\psi\tau^1,\quad Z_2:\mathcal{A}_\mu\to\mathcal{A}_\mu\ , \tag{1.3}$$

where

$$\tau^1=\begin{bmatrix}0 & \mathcal{I}\\ \mathcal{I} & 0\end{bmatrix}\ . \tag{1.4}$$

Under this transformation the scalar fermion mass operator $\bar\psi\tau^3\psi$ changes sign,

$$Z_2:\bar\psi\tau^3\psi\to-\bar\psi\tau^3\psi\ . \tag{1.5}$$

and the mass operators $\bar\psi\psi$ and $\epsilon^{\mu\nu\lambda}\mathcal{A}_\mu\partial_\nu\mathcal{A}_\lambda$ are invariant:

$$Z_2:\bar\psi\psi\to\bar\psi\psi,\quad Z_2:\epsilon^{\mu\nu\lambda}\mathcal{A}_\mu\partial_\nu\mathcal{A}_\lambda\to\epsilon^{\mu\nu\lambda}\mathcal{A}_\mu\partial_\nu\mathcal{A}_\lambda\ . \tag{1.6}$$

Since $\bar\psi\tau^3\psi$ is not invariant under Z_2, a nonvanishing expectation value $\langle\bar\psi\tau^3\psi\rangle\neq0$ would indicate that the vacuum of the theory is not symmetric. This expectation value is therefore an order parameter for Z_2 symmetry breaking.

The action (1.1) is also invariant under the Euclidean three-dimensional parity

$$P:\psi(x_1,x_2,x_3)\to\gamma^2\tau^1\psi(x_1,-x_2,x_3)\ , \tag{1.7}$$

$$P:\bar\psi(x_1,x_2,x_3)\to-\bar\psi(x_1,-x_2,x_3)\gamma^2\tau^1\ , \tag{1.8}$$

$$P:(\mathcal{A}_1,\mathcal{A}_2,\mathcal{A}_3)(x_1,x_2,x_3)\to(\mathcal{A}_1,-\mathcal{A}_2,\mathcal{A}_3)(x_1,-x_2,x_3)\ . \tag{1.9}$$

The mass operator $\bar\psi\tau^3\psi$ transforms like a scalar,

$$P:\int d^3x\ \bar\psi\tau^3\psi\to\int d^3x\ \bar\psi\tau^3\psi\ , \tag{1.10}$$

and $\bar\psi\psi$ like a pseudoscalar:

$$P:\int d^3x\ \bar\psi\psi\to-\int d^3x\ \bar\psi\psi\ . \tag{1.11}$$

The topological mass term is also parity odd:

$$P:\int d^3x\ \epsilon^{\mu\nu\lambda}\mathcal{A}_\mu\partial_\nu\mathcal{A}_\lambda\to-\int d^3x\ \epsilon^{\mu\nu\lambda}\mathcal{A}_\mu\partial_\nu\mathcal{A}_\lambda\ . \tag{1.12}$$

Either of the vacuum expectation values $\langle\bar\psi\psi\rangle$ or $\langle\int\epsilon^{\mu\nu\lambda}\mathcal{A}_\mu\partial_\mu\mathcal{A}_\lambda\rangle$ are order parameters for parity breaking. If one of these operators has an expectation value, radiative corrections can induce an expectation value for the other one [14–17]. In general this occurs when the physical fermion has a pseudoscalar mass larger in magnitude than the scalar mass [11]. Then the physical fermion and the physical photon would have parity-violating masses and the Coulomb interaction would be short ranged.

The Z_2 and parity symmetries forbid the appearance of a bare mass for either the fermions or the photon in the action (1.1). If the physical fermion spectrum is to have a mass gap at least one of these symmetries must be broken dynamically. Furthermore, the photon can have a topological mass only if parity is broken. In the following we shall seek solutions of the model (1.1) which break either Z_2 or parity or both through the generation of fermion masses. We also find that, with the particular choice of four-Fermi couplings taken in (1.1), either parity or chiral Z_2 symmetry can be broken, but there is no solution where both are simultaneously broken. We shall discuss the effects of choosing other four-Fermi couplings in Sec. II.

The model (1.1) is not ultraviolet renormalizable in a perturbative expansion in the coupling constants λ/Λ and κ/Λ, and formally, the four-Fermi operators are irrelevant. Furthermore, perturbation theory in the dimensional electromagnetic coupling constant $e\sqrt{\Lambda}$ is super-renormalizable in the ultraviolet but has infrared divergences when the fermions are massless [12]. However, it has long been argued that the four-Fermi couplings are renormalizable in the large-N expansion [5–11,18,19]. There is a recent rigorous construction of a model with one four-Fermi coupling ($\lambda=0=e$) in the context of the large-N expansion [9]. There it was shown to have a strong-coupling fixed point where the four-Fermi coupling is a relevant operator, i.e., generates nontrivial behavior of the S matrix. It has also been argued that a four-Fermi interaction of the kind considered here can drive dynamical symmetry breaking and dynamical fermion mass generation if the coupling is sufficiently strong [8,10,11].

In the large-N expansion the electromagnetic coupling is also ultraviolet renormalizable and its infrared behavior is improved [12,13,17–19]. It has been shown that for sufficiently strong coupling (sufficiently large $1/N$) the electromagnetic interaction can break the Z_2 symmetry [18–27]. Also, a preliminary investigation showed that the electromagnetic interactions oppose the spontaneous breaking of parity [25].

In this paper we shall elaborate on the phase diagram of the model (1.1) in the three-parameter coupling-constant space of $\lambda,\kappa,1/N$.

In Sec. II we describe an effective potential approach to dynamical symmetry breaking. We find the large-N limit of the phase diagram and show that P and Z_2 can be broken for sufficiently strong coupling.

In Sec. III we examine the renormalization of the model and next-to-leading-order corrections to the effective potential from both four-Fermi and U(1) gauge interactions. We discuss the critical behavior of the second-order phase transition. We believe that the next-to-leading-order computation both demonstrates the renormalizability of the four-Fermi theory and provides an aid in estimating the radius of convergence of the large-N expansion. There is a previous result on the renormalization in large-N with a single four-Fermi coupling and without further coupling to electromagnetic interactions [28]. There it is asserted that an additive renormalization of the auxiliary scalar fields of the model is necessary. Here we show that this is not necessary and only conventional multiplicative renormalization is needed to make the critical theory finite.

Section IV contains a discussion, and some useful formulas are summarized in the Appendix.

II. DYNAMICAL SYMMETRY BREAKING

To set up the $1/N$ expansion we introduce the Lagrange multiplier fields ϕ and χ and to rewrite the continuum action (1.1) as

$$S=\int d^2x \left[A\bar{\psi}i\gamma_\mu\partial_\mu\psi+iB\phi\bar{\psi}\psi+iC\chi\bar{\psi}\tau^3\psi + \frac{NB^2\Lambda}{2A^2\lambda}\phi^2+\frac{NC^2\Lambda}{2A^2\kappa}\chi^2+A\bar{\psi}\gamma_\mu e\mathcal{A}_\mu\psi + \frac{N}{4\Lambda}\mathcal{F}_{\mu\nu}\mathcal{F}_{\mu\nu}\right] . \tag{2.1}$$

Solving the equations of motion for the scalar fields ϕ and χ and substituting the solutions into (2.1) yields the four-Fermi interaction terms in the action (1.1). The cutoff-dependent renormalization constants A,B,C and the cutoff-dependent coupling constants λ,κ in (2.1) are to be chosen so that, in each order of the $1/N$ expansion, the correlation functions remain finite as the ultraviolet cutoff Λ is taken to infinity.

The partition function is given by

$$Z[J_1,J_2]=\int d\phi\, d\chi\, d\mathcal{A}_\mu d\psi\, d\bar{\psi}\exp\left[-S-\int(J_1\phi+J_2\chi)\right] \tag{2.2}$$

and the thermodynamic potential is

$$W[J_1,J_2]=-\ln Z[J_1,J_2] . \tag{2.3}$$

The expectation value of the scalar fields are given by

$$\langle\phi\rangle=\frac{\delta}{\delta J_1}W[J_1,J_2]\equiv m_1 , \tag{2.4}$$

$$\langle\chi\rangle=\frac{\delta}{\delta J_2}W[J_1,J_2]\equiv m_2 . \tag{2.5}$$

The free energy is obtained by the Legendre transformation

$$\Gamma[m_1,m_2]=W[J_1,J_2]-\int m_1J_1-\int m_2J_2 \tag{2.6}$$

and has the derivatives

$$\frac{\delta}{\delta m_1}\Gamma[m_1,m_2]=-J_1 , \tag{2.7}$$

$$\frac{\delta}{\delta m_2}\Gamma[m_1,m_2]=-J_2 . \tag{2.8}$$

We shall consider the equilibrium theory where $J_1=J_2=0$. This requires that we minimize the free energy with respect to m_1,m_2.

The functional integral

$$\Gamma=-\ln\int d\phi\, d\chi\, d\mathcal{A}_\mu d\psi\, d\bar{\psi}\exp\left[-S+\int(\phi-m_1)\frac{\delta}{\delta m_1}\Gamma+\int(\chi-m_2)\frac{\delta}{\delta m_2}\Gamma\right] \tag{2.9}$$

gives a self-consistent equation for Γ. It can be simplified by translating the integration variables $\phi\to\phi+m_1,\chi\to\chi+m_2$. With the definition

$$\Gamma[m_1,m_2]=\int\left[\frac{NB^2\Lambda}{2\lambda A^2}m_1^2+\frac{NC^2\Lambda}{2\kappa A^2}m_2^2\right]+\tilde{\Gamma}[m_1,m_2] , \tag{2.10}$$

this gives

$$\tilde{\Gamma}=-\ln\int d\phi\, d\chi\, d\mathcal{A}_\mu d\psi\, d\bar{\psi}\exp\left\{-\int\left[A\bar{\psi}\left[i\gamma_\mu\partial_\mu+i\frac{B}{A}m_1+i\frac{C}{A}m_2\tau^3\right]\psi+iB\phi\bar{\psi}\psi+iC\chi\bar{\psi}\tau^3\psi + \frac{N}{4\Lambda}\mathcal{F}_{\mu\nu}\mathcal{F}_{\mu\nu}+A\bar{\psi}\gamma_\mu e\mathcal{A}_\mu\psi+\frac{NB^2\Lambda}{2\lambda A^2}\phi^2+\frac{NC^2\Lambda}{2\kappa A^2}\chi^2-\phi\frac{\delta}{\delta m_1}\tilde{\Gamma}-\chi\frac{\delta}{\delta m_2}\tilde{\Gamma}\right]\right\} . \tag{2.11}$$

This is a self-consistent equation which determines $\tilde{\Gamma}$ as a functional of m_1 and m_2. Γ is the free energy which is to be minimized in order to find the values of m_1 and m_2 at equilibrium. It is also the sum of all one-(ϕ, χ)-particle-irreducible Feynman diagrams with arbitrary numbers of classical fields $\langle\phi\rangle=m_1$ and $\langle\chi\rangle=m_2$ attached to external lines. If we use the functions $(B/A)m_1+(C/A)m_2\tau$ as the fermion mass, Γ is the sum of all irreducible, connected vacuum bubble diagrams.

Feynman rules for the $1/N$ expansion are defined as follows.

(i) We first consider the Feynman rules for the coupling-constant expansion in powers of λ,κ with propagators and vertices depicted in Fig. 1.

(ii) We then find the scalar field inverse propagators to leading order in the $1/N$ expansion. To do this we must compute the fermion loop diagrams depicted in Fig. 2 to get the one-loop self-energies of the scalars, which is the only contribution to their proper self-energy part which is of order N. All other corrections are of order N^0 or of higher order in $1/N$. Using the function

$$f(p,m)=\left[\frac{\Lambda}{\pi^2}-\frac{|m|}{2\pi}-\frac{p^2+4m^2}{4\pi p}\arctan\frac{p}{2|m|}\right], \tag{2.12}$$

the inverse scalar propagator is

$$\Delta^{-1}(k)=N\begin{bmatrix}\frac{B^2\Lambda}{\lambda A^2}-\frac{B^2}{A^2}D_{\phi\phi} & -\frac{BC}{A^2}D_{\phi\chi}\\ -\frac{BC}{A^2}D_{\chi\phi} & \frac{C^2\Lambda}{\kappa A^2}-\frac{C^2}{A^2}D_{\chi\chi}\end{bmatrix}, \tag{2.13}$$

where

$$D_{\phi\phi}=D_{\chi\chi}=f\left[p,\frac{B}{A}m_1+\frac{C}{A}m_2\right]+f\left[p,\frac{B}{A}m_1-\frac{C}{A}m_2\right], \tag{2.14}$$

$$D_{\chi\phi}=D_{\phi\chi}=f\left[p,\frac{B}{A}m_1+\frac{C}{A}m_2\right]-f\left[p,\frac{B}{A}m_1-\frac{C}{A}m_2\right] \tag{2.15}$$

are the components of the one-loop proper self-energy function.

(iii) We follow a similar procedure to find the inverse of the photon propagator to order N, i.e., we sum the loop diagrams shown in Fig. 3. The photon self-energy can be decomposed as

$$\Pi_{\mu\nu}(p)=(p^2\delta_{\mu\nu}-p_\mu p_\nu)\Pi_e(p^2)+\epsilon_{\mu\nu\lambda}p_\lambda\Pi_o(p^2) . \tag{2.16}$$

Here, Π_o is allowed only when parity is broken. The one-loop diagram leads to

$$\Pi_e(p^2)=\pi_e\left[p,\frac{B}{A}m_1+\frac{C}{A}m_2\right]+\pi_e\left[p,\frac{C}{A}m_2-\frac{B}{A}m_1\right], \tag{2.17}$$

$$\Pi_o(p^2)=\pi_o\left[p,\frac{B}{A}m_1+\frac{C}{A}m_2\right]-\pi_o\left[p,\frac{C}{A}m_2-\frac{B}{A}m_1\right], \tag{2.18}$$

where

$$\pi_e(p,m)=-e^2N\left[\frac{|m|}{4\pi|p|^2}+\frac{p^2-4m^2}{8\pi|p|^3}\arctan\frac{p}{2|m|}\right], \tag{2.19}$$

$$\pi_o(p,m)=\frac{e^2Nm}{2\pi|p|}\arctan\frac{p}{2|m|} . \tag{2.20}$$

The photon propagator to leading order in $1/N$ is the inverse of the matrix:

$$\Delta^{-1}_{\mu\nu}(p)=\frac{N}{\Lambda}(p^2\delta_{\mu\nu}-p_\mu p_\nu)-\Pi_{\mu\nu}(p)+\xi p_\mu p_\nu , \tag{2.21}$$

where we have used a relativistic R_ξ gauge fixing.

The $1/N$ resummed effective action would have a dynamically generated Chern-Simons term with a coefficient determined by the zero-momentum limit of the parity-odd part of the photon propagator:

$$\tfrac{1}{2}\Pi_0(0)=\frac{Ne^2}{8\pi}\left[\operatorname{sgn}\left[\frac{B}{A}m_1+\frac{C}{A}m_2\right]+\operatorname{sgn}\left[\frac{B}{A}m_1-\frac{C}{A}m_2\right]\right], \tag{2.22}$$

which vanishes when $|m_2|>|m_1|$ and is $\pm Ne^2/4\pi$ when $|m_2|<|m_1|$.

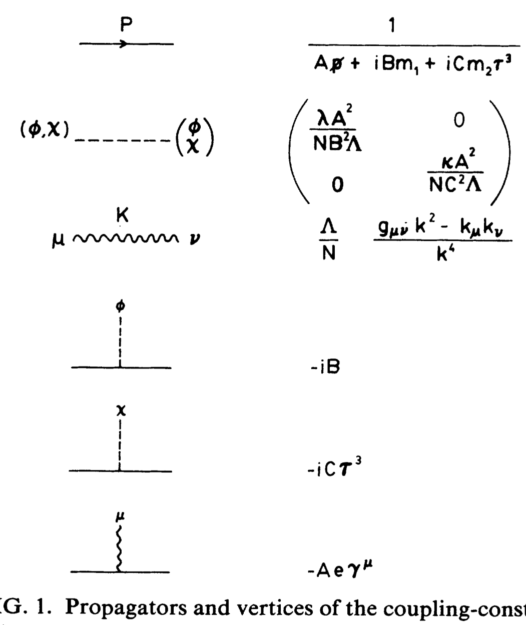

FIG. 1. Propagators and vertices of the coupling-constant expansion.

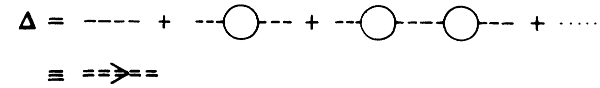

FIG. 2. The large-N resummed scalar propagator is the sum of all graphs of order N^{-1}.

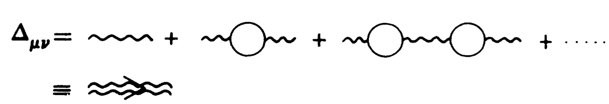

FIG. 3. The large-N resummed photon propagator is the sum of all graphs of order N^{-1}.

(iv) The Feynman rules, as summarized in Fig. 4, are the same as before with the exception that we use the large-N corrected scalar propagator $\Delta(p)$ and photon propagator $\Delta_{\mu\nu}(p)$, and we omit all Feynman graphs which have one-loop scalar or photon self-energy subgraphs.

With these rules, $\tilde{\Gamma}$ is given by the sum of the diagrams depicted in Fig. 5. Explicitly, we obtain

$$\frac{\Gamma[m_1,m_2]}{V}=\frac{N\Lambda B^2}{2\lambda A^2}m_1^2+\frac{N\Lambda C^2}{2\kappa A^2}m_2^2-N\int^{\Lambda}\frac{d^2k}{(2\pi)^3}\ln\det\left[\gamma_\mu k_\mu+i\frac{B}{A}m_1+i\frac{C}{A}m_2\tau^3\right]$$
$$+\frac{1}{2}\int^{\Lambda}\frac{d^3k}{(2\pi)^3}\ln\det\Delta^{-1}(k)+\frac{1}{2}\int^{\Lambda}\frac{d^3k}{(2\pi)^3}\ln\left[k^2\left[\frac{1}{\Lambda}-\Pi_e(k)\right]^2+\Pi_o(k)^2\right]\cdots , \tag{2.23}$$

where the contributions omitted are at least of order N^{-1}. The last two terms are the sum of ring diagrams shown in Fig. 5. Note that the electromagnetic coupling only influences the effective potential at next-to-leading order. The leading order is governed strictly by the four-Fermi interactions.

It is straightforward to integrate the leading order in $\tilde{\Gamma}$ to get the free energy

$$\frac{\Gamma[m_1,m_2]}{V}=\frac{N\Lambda b^2}{2\lambda a^2}m_1^2+\frac{N\Lambda c^2}{2\kappa a^2}m_2^2-\frac{N\Lambda}{\pi^2}\frac{b^2}{a^2}(m_1^2+m_2^2)+\frac{N}{6\pi}\frac{b^3}{a^3}(|m_1+m_2|^3+|m_1-m_2|^3)+\cdots , \tag{2.24}$$

where to leading order in $1/N$, we have taken $A=a$ and $B=b=C$, with a and b finite dimensionless constants. The cutoff dependence of λ and κ is yet to be determined. For now, it is sufficient to absorb the factors b/a into m_1 and m_2. Then, if we define the critical coupling constants

$$\lambda_c=\frac{\pi^2}{2}=\kappa_c , \tag{2.25}$$

the leading order in the large-N part of Γ can be written as

$$\frac{1}{V}\Gamma[m_1,m_2]=\frac{N\Lambda}{2}\left[\frac{1}{\lambda}-\frac{1}{\lambda_c}\right]m_1^2+\frac{N\Lambda}{2}\left[\frac{1}{\kappa}-\frac{1}{\kappa_c}\right]m_2^2+\frac{N}{6\pi}(|m_1+m_2|^3+|m_1-m_2|^3)+\cdots . \tag{2.26}$$

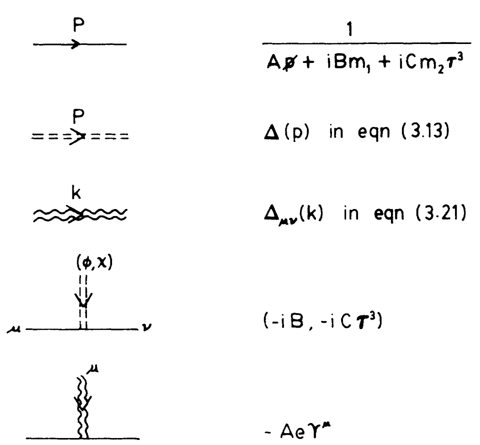

FIG. 4. Propagators and vertices of the large-N expansion. All graphs which contain as subgraphs those diagrams already included in the resummed scalar and photon propagators shown in Figs. 2 and 3 are to be omitted.

It is evident that if either of the four-Fermi couplings are sufficiently strong, either $\lambda>\lambda_c$ or $\kappa>\kappa_c$, the free energy is minimized by $m_1\neq 0$ or $m_2\neq 0$ and either the parity or Z_2 symmetry is broken dynamically. The large-N phase diagram is depicted in Fig. 6. The completely symmetric phase with $m_1=m_2=0$ is stable only in the region $\lambda<\lambda_c$ and $\kappa<\kappa_c$. When $\lambda>\lambda_c$ and also $\lambda>\kappa$ the stable minimum of (2.26) has $m_1\neq 0$ and $m_2=0$. When $\kappa>\kappa_c$ and $\kappa>\lambda$ the stable minimum is where $m_2\neq 0$ and $m_1=0$. There are no stable minima where both m_1 and m_2 are simultaneously nonzero. On the symmetry line $\lambda=\kappa>\lambda_c=\kappa_c$ the symmetry-breaking pattern is discontinuous and jumps from $m_1\neq 0$, $m_2=0$ for $\lambda>\kappa$ to $m_1=0$, $m_2\neq 0$ for $\kappa>\lambda$. At large N the electromagnetic interactions contribute to the effective action only at next-to-leading order. When $|m_1|>|m_2|$ the physical photon has a topological mass (2.22), and when $|m_1|<|m_2|$ the photon is massless. Here we have obtained this as a one-loop result. However, it is known that when the charged matter fields have a mass gap the topological mass term of the photon field receives quantum corrections only at one-loop order. Thus, although the topological photon mass is generic to a system where P is broken, when the P-conserving mass of the physical

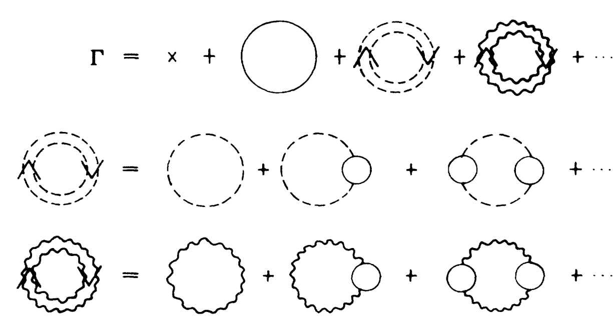

FIG. 5. Large-N expansion of the effective potential.

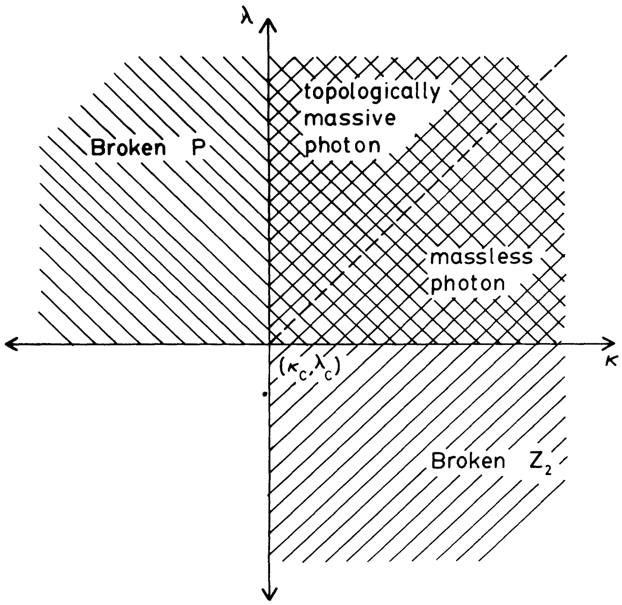

FIG. 6. The $N \to \infty$ limit of the phase diagram in the λ-κ plane. Z_2 is broken when $\lambda > \lambda_c$ and P is broken when $\kappa > \kappa_c$. When $\kappa > \lambda > \lambda_c$ the photon has a topological mass.

fermion is larger in magnitude than the P-violating mass of the physical fermion the photon remains massless.

The effective four-Fermi couplings are (see Fig. 7)

$$\lambda_{\text{eff}} = \frac{1}{\Lambda} \frac{\lambda \lambda_c}{\lambda_c - \lambda} , \tag{2.27}$$

$$\kappa_{\text{eff}} = \frac{1}{\Lambda} \frac{\kappa \kappa_c}{\kappa_c - \kappa} , \tag{2.28}$$

where we have used the cutoff Λ to restore the canonical dimension of the effective four-Fermi coupling. (At the critical points $\lambda = \lambda_c$ or $\kappa = \kappa_c$ the effective four-Fermi vertices do not have a finite local limit and in fact vary like an inverse power of momentum, $|\kappa|^{-1}$.) The effective couplings (2.32) and (2.33) are the zero-momentum limit of the sum of diagrams in Fig. 8. When $\lambda > \lambda_c$ or $\kappa > \kappa_c$ the effective four-Fermi interaction is attractive, consistent with formation of a fermion mass operator condensate. These interactions are finite only when the coupling constants are close to their critical values. In fact, if we fix λ_{eff} and κ_{eff} at finite constants of either sign then

$$\lambda = \frac{\lambda_c}{1 + \lambda_c / \Lambda \lambda_{\text{eff}}} \xrightarrow{\Lambda \to \infty} \lambda_c , \tag{2.29}$$

$$\kappa = \frac{\kappa_c}{1 + \kappa_c / \Lambda \kappa_{\text{eff}}} \xrightarrow{\Lambda \to \infty} \kappa_c , \tag{2.30}$$

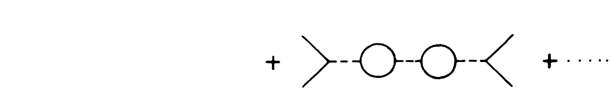

FIG. 7. The order-N^{-1} graphs contributing to the effective four-Fermi coupling.

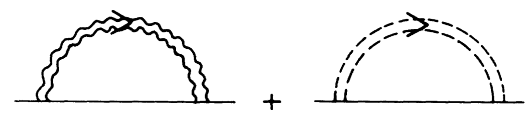

FIG. 8. Order N^{-1} corrections to the fermion propagator.

and $\lambda \to \lambda_c$ and $\kappa \to \kappa_c$ as $\Lambda \to \infty$. λ_c and κ_c are strong-coupling fixed points. If λ or κ deviate from these values the associated four-Fermi operator is irrelevant. If either P or Z_2 symmetry is broken the mass gap of both of the fermion species is infinite. If not, they are massless and the four-Fermi interactions are irrelevant and the physical properties of the system are governed by the Coulomb interactions.

Our analysis so far is specific to the four-Fermi interactions introduced in (1.1). Here, we note that to the leading order in large N our results apply to more general interactions. In fact, if we introduce a general four-Fermi interaction we have

$$S_{\text{int}} = \int d^3x \sum_{\delta=0}^{3} \frac{g_\delta}{2N\Lambda} (\bar{\psi} \tau_\delta \psi)^2 , \tag{2.31}$$

where $\tau_\delta = \{ \mathcal{I}, \tau_i \}$ are the unit and Pauli matrices. Also, we could introduce scalar fields m_δ so that (2.31) can be described by the action

$$S_{\text{int}} = \int d^3x \left[\frac{N\Lambda}{2g_\delta} m_\delta m_\delta + i m_\delta \bar{\psi} \tau_\delta \psi \right] , \tag{2.32}$$

and the leading order in large-N effective action would be

$$\frac{1}{V} \Gamma[m_\delta] = \frac{N\Lambda}{2g_\delta} m_\delta m_\delta - N \int^{\Lambda} \frac{d^3k}{(2\pi)^3} \ln \det(\gamma_\mu k_\mu + i m_\delta \tau_\delta) + \cdots \tag{2.33}$$

$$= \frac{N\Lambda}{2} \left[\frac{1}{g_\delta} - \frac{2}{g_c} \right] m_\delta^2 + \frac{N}{6\pi} [(|m_0| + |\mathbf{m}|)^3 + (|m_0| - |\mathbf{m}|)^3] + \cdots . \tag{2.34}$$

In the case where we have a full SU(2) chiral symmetry, $g_i = g$ for $i = 1,2,3$, and we get a situation very similar to that in (2.26) where now m_2 is identified with $|\mathbf{m}|$ and m_1 is identified with m_0. Also, either chiral SU(2) is broken for sufficiently large g or parity is broken for large enough $g_0 \equiv \lambda$, but the two symmetries are never simultaneously broken. When the chiral SU(2) is broken there will be massless composite Goldstone bosons resulting from the spontaneous breaking of the chiral symmetry. We could recover the model (1.1) by breaking the SU(2) symmetry explicitly, i.e., taking $g_1 = g_2 < g_c < g_3$. Then the Goldstone bosons acquire a mass proportional to $\Lambda(g_3 - g_1)$ which is large in the $\Lambda \to \infty$ limit.

The analysis in this section is arbitrarily accurate for sufficiently large N. To estimate corrections to the leading-order behavior we must compute to the next-to-leading order. This introduces logarithmic divergences and renormalization is required. A systematic procedure for renormalization of the theory described by (1.1) will be discussed in the next section.

III. RADIATIVE CORRECTIONS

In the previous two sections we established the existence at large N of critical couplings λ_c and κ_c for the model with action (2.1). In this section we shall examine the next-to-leading order in N corrections to that result. These are particularly interesting since this is where the first logarithmic divergences appear and where the renormalization constants A, B, and C are required to be cutoff dependent.

A. Renormalization of the critical theory

It is most convenient to renormalize the critical theory with finite-momentum subtraction points. We shall do this by requiring the renormalization conditions that the inverse ϕ-ϕ and χ-χ correlation functions vanish at zero momentum,

$$\frac{1}{\lambda_{\text{eff}}^{\text{crit}}}=0=\frac{1}{\kappa_{\text{eff}}^{\text{crit}}} \tag{3.1}$$

(this defines the critical point), that the inverse fermion propagator is normalized as

$$S_{\text{crit}}^{-1}(p)=\gamma_\mu p_\mu \quad \text{when } p^2=\mu^2 , \tag{3.2}$$

and that the fermion-fermion-scalar vertices are

$$\tfrac{1}{4}\,\text{tr}\Gamma_{\psi\bar\psi\chi}^{\text{crit}}(p,-p,0)=-ib \quad \text{when } p^2=\mu^2 \tag{3.3}$$

and

$$\tfrac{1}{4}\,\text{tr}\tau^3\Gamma_{\psi\bar\psi\chi}^{\text{crit}}(p,-p,0)=-ic \quad \text{when } p^2=\mu^2 . \tag{3.4}$$

The bare (large-N resummed) propagators for the scalars and the photon (in the Landau gauge) are

$$\Delta_{\phi\phi}^{\text{crit}}(k)=\frac{A^2}{B^2N}\frac{4}{|k|} ,$$
$$\Delta_{\chi\chi}^{\text{crit}}(k)=\frac{A^2}{C^2N}\frac{4}{|k|} , \tag{3.5}$$
$$\Delta_{\chi\phi}^{\text{crit}}(k)=\Delta_{\phi\chi}^{\text{crit}}(k)=0 ,$$

$$\Delta_{\mu\nu}^{\text{crit}}(k)=\left[\delta_{\mu\nu}-\frac{k_\mu k_\nu}{k^2}\right]\frac{1}{N}\frac{1}{k^2/\Lambda+\frac{1}{8}e^2|k|} , \tag{3.6}$$

respectively. The critical bare Fermion propagator is

$$S_{\text{crit}}^{0}(p)=\frac{1}{A}\frac{1}{\gamma_\mu p_\mu} . \tag{3.7}$$

To compute the counterterms we shall use an ultraviolet cutoff Λ and the photon propagator

$$\tilde\Delta_{\mu\nu}^{\text{crit}}(k)=\left[\delta_{\mu\nu}-\frac{k_\mu k_\nu}{k_2}\right]\frac{8}{e^2N}\frac{1}{|k|} . \tag{3.8}$$

The renormalization counterterms computed with photon propagator (3.8) and with (3.6) differ at most by finite additive constants.

The large-N correction to the fermion propagator is

$$S_{\text{crit}}^{-1}(p)=A\gamma_\mu p_u+2\frac{A}{N}\int^\Lambda\frac{d^3k}{(2\pi)^3}\frac{1}{\gamma_\mu(p-k)_\mu}\frac{4}{|k|}-\frac{8A}{N}\int^\Lambda\frac{d^3k}{(2\pi)^3}\gamma_\mu\frac{1}{\gamma_\lambda(p-k)_\lambda}\gamma_\nu\left[\delta_{\mu\nu}-\frac{k_\mu k_\nu}{k^2}\right]\frac{1}{k} \tag{3.9}$$

$$=A\gamma_\mu p_\mu\left[1+\frac{4}{3\pi^2N}\left[\ln\frac{\Lambda}{p}+\frac{4}{3}\right]+\frac{8}{3\pi^2N}\left[\ln\frac{\Lambda}{p}-\frac{2}{3}\right]+O(N^{-2})\right] , \tag{3.10}$$

where the first contribution is from the fermion-scalar intermediate state and the second is from the fermion-photon intermediate state. Note that in the electromagnetic correction the dependence on e^2 cancels and in the scalar correction the dependence on B and C cancels. This determines A as

$$A=1-\frac{4}{3\pi^2N}\left[\ln\frac{\Lambda}{\mu}+\frac{4}{3}\right]-\frac{8}{3\pi^2N}\left[\ln\frac{\Lambda}{\mu}-\frac{2}{3}\right]+\cdots$$
$$=1-\frac{4}{\pi^2N}\ln\frac{\Lambda}{\mu}+\cdots , \tag{3.11}$$

which is approximately the power

$$A\approx\left[\frac{\mu}{\Lambda}\right]^{-4/\pi^2N}+\cdots \tag{3.12}$$

accurate to order N^{-1}. The renormalized fermion propagator is

$$S_{\text{crit}}\approx\frac{1}{\gamma_\mu p_\mu}\left[\frac{p}{\mu}\right]^{4/\pi^2N}+O(N^{-2}) . \tag{3.13}$$

The anomalous dimension of the fermion field operator is $D[\psi]=1+2/\pi^2N$.

The order-N^{-1} correction to the $\psi\bar\psi\phi$ vertex is given by the graphs in Fig. 9. The contribution of the one-loop graph in Fig. 9 to the $\psi\bar\psi\phi$ vertex function $\Gamma_{\psi\bar\psi\phi}^{\text{crit}}(p,-p,0)$ when the external ϕ line has zero momentum is

$$2i\frac{B}{N}\int^{\Lambda}\frac{d^3k}{(2\pi)^3}\frac{1}{(p-k)^2}\frac{4}{k}-i\frac{8B}{N}\int^{\Lambda}\frac{d^3k}{(2\pi)^3}\gamma_\mu\frac{1}{(p-k)^2}\gamma_\nu\left[\delta_{\mu\nu}-\frac{k_\mu k_\nu}{k^2}\right]\frac{1}{k} \quad (3.14)$$

$$=-iB\left[-\frac{4}{\pi^2N}\left[\ln\frac{\Lambda}{p}+1\right]+\frac{8}{N\pi^2}\left[\ln\frac{\Lambda}{p}+1\right]+\cdots\right], \quad (3.15)$$

where the first term is the contribution of the scalar-fermion loop and the second is the contribution of the photon-fermion loop. The contribution to $\mathrm{tr}\Gamma^{\mathrm{crit}}_{\psi\bar{\psi}\phi}(p,-p,0)$ of the two-loop graphs in Fig. 9 which have no internal photon lines vanishes upon taking the trace over Dirac matrices. This leaves the two-loop graphs with one or more internal photon lines. It is also easy to show that the graphs with one internal photon line vanish in the Landau gauge.

The integral corresponding to the graph with two internal photon lines is

$$64iB\int^{\Lambda}\frac{d^3l}{(2\pi)^3}\int^{\Lambda}\frac{d^3q}{(2\pi)^3}\,\mathrm{tr}\left[\frac{1}{\gamma\cdot q}\frac{1}{\gamma\cdot q}\gamma_\mu\frac{1}{\gamma\cdot(q-l)}\gamma_\nu\right]\left[\Delta_{\mu\rho}(l)\Delta_{\nu\sigma}(l)\tfrac{1}{4}\,\mathrm{tr}\left[\gamma_\rho\frac{1}{\gamma\cdot(p+l)}\gamma_\sigma\right]\right.$$
$$\left.+\Delta_{\mu\sigma}\Delta_{\nu\rho}(l)\tfrac{1}{4}\,\mathrm{tr}\left[\gamma_\rho\frac{1}{\gamma\cdot(p-l)}\gamma_\sigma\right]\right].$$

Note that this expression includes both the diagram shown in Fig. 9 and the contribution with crossed photon lines. Taking the trace over Dirac matrices yields the integral

$$256iB\int^{\Lambda}\frac{d^3l}{(2\pi)^3}\int^{\Lambda}\frac{d^3q}{(2\pi)^3}\left[\frac{-i\epsilon_{\mu\nu\lambda}(q-1)_\lambda}{q^2(q-l)^2}\right]\left[\Delta_{\mu\rho}(l)\Delta_{\nu\sigma}(l)\left[\frac{-i\epsilon_{\rho\sigma\kappa}(p+l)_\kappa}{(p+l)^2}\right]+\Delta_{\mu\sigma}\Delta_{\nu\rho}(l)\left[\frac{-i\epsilon_{\rho\sigma\kappa}(p-l)_\kappa}{(p-l)^2}\right]\right].$$

We first perform the loop integration over q. The result is

$$16iB\int^{\Lambda}\frac{d^3l}{(2\pi)^3}\epsilon_{\mu\nu\lambda}l_\lambda\frac{1}{l}$$
$$\times\left[\Delta_{\mu\rho}(l)\Delta_{\nu\sigma}(l)\left[\frac{\epsilon_{\rho\sigma\kappa}(p+l)_\kappa}{(p+l)^2}\right]\right.$$
$$\left.+\Delta_{\mu\sigma}\Delta_{\nu\rho}(l)\left[\frac{\epsilon_{\rho\sigma\kappa}(p-l)_\kappa}{(p-l)^2}\right]\right]. \quad (3.16)$$

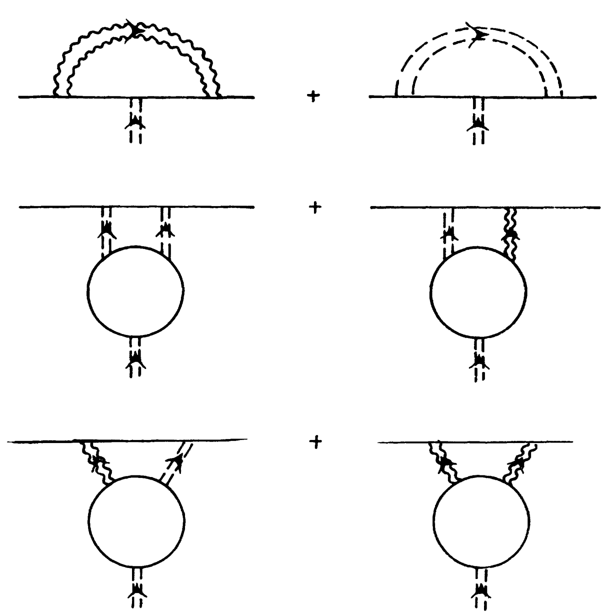

FIG. 9. Order N^{-1} corrections to the fermion-fermion-scalar vertex.

Contracting indices and performing the remaining integration yields

$$\frac{16iB}{\pi^2}\ln\frac{\Lambda}{p}. \quad (3.17)$$

The normalization condition (3.3) determines

$$B=b\left[1+\frac{4}{\pi^2N}\left[\ln\frac{\Lambda}{\mu}+1\right]+\frac{8}{\pi^2N}\left[\ln\frac{\Lambda}{\mu}-1\right]+\cdots\right]$$
$$\approx\left[\frac{\Lambda}{\mu}\right]^{12/\pi^2N}\left[1-\frac{4}{\pi^2N}+\cdots\right], \quad (3.18)$$

where the first term is due to four-Fermi interactions and the second term is due to electromagnetic interactions. Note that the magnitude of each contribution is characterized by $1/N$ rather than the four-Fermi or electromagnetic couplings.

The renormalized vertex function is

$$\Gamma^{\mathrm{crit}}_{\psi\bar{\psi}\phi}(p,-p,0)=-ib\left[1-\frac{12}{\pi^2N}\ln\frac{\mu}{p}+\cdots\right]$$
$$\approx-ib\left[\frac{\mu}{p}\right]^{-12/\pi^2N}. \quad (3.19)$$

A similar computation leads to the leading corrections to the $\psi\bar{\psi}\chi$ vertex. The resulting cutoff dependence of the coefficient C is

$$C=c\left[1+\frac{4}{\pi^2N}\left[\ln\frac{\Lambda}{\nu}+1\right]\right.$$
$$\left.+\frac{8}{\pi^2N}\left[\ln\frac{\Lambda}{\mu}-1\right]+\cdots\right]=\frac{c}{b}B, \quad (3.20)$$

and the renormalized vertex function is

$$\Gamma^{\text{crit}}_{\psi\bar\psi\chi}(p,-p,0)=-ic\tau^3\left[1-\frac{12}{\pi^2N}\ln\frac{\mu}{p}+\cdots\right]$$
$$\approx -ic\tau^3\left[\frac{\mu}{p}\right]^{-12/\pi^2N} . \tag{3.21}$$

Note that the strength of the interactions of the fermions with a zero-momentum scalar is small when the momentum of the fermion is small; i.e., the interaction is infrared-free. However, unlike the case of a coupling-constant expansion, it is the value of N which controls the validity of the approximations, and at least to the order that we have considered, N is scale independent. The correction to leading-order behavior here is small if N is sufficiently large. Here the next-to-leading order is smaller than the leading order if $N \gtrsim 2$.

A change in the arbitrary parameter μ which appears in renormalized quantities can always be canceled by a change in the parameters b and c or the wave-function normalization of the fermion.

B. The effective potential 1: Pure four-Fermi theory

We cannot compute the integrals in the penultimate and the last terms of (2.23) explicitly. However, we can obtain a Taylor expansion of these quantities, and therefore also the effective potential to second order in the parameters m_1 and m_2.

We begin by considering the pure four-Fermi theory in the absence of electromagnetic interactions. Note that the leading-order calculation in Sec. II did not use the electromagnetic interactions. The renormalization constants for the pure four-Fermi theory [from Eq. (3.11)] are

$$A_{4f}=\left[\frac{\Lambda}{\mu}\right]^{-4/3\pi^2N}\left[1-\frac{16}{9\pi^2N}+\cdots\right] , \tag{3.22}$$

$$B_{4f}=C_{4f}=\left[\frac{\Lambda}{\mu}\right]^{4/\pi^2N}\left[1+\frac{4}{\pi^2N}+\cdots\right]b \tag{3.23}$$

(accurate to order N^{-1}), where we have set $b=c$.

Since there are massless fermions when $m_1=\pm m_2$ we do not expect the effective potential to be analytic there. This is already apparent in the cubic terms in (2.26) which depend on the absolute values $|m_1+m_2|$ and $|m_1-m_2|$ and are therefore not three times differentiable at $m_1=\pm m_2$. To order N^0 the effective potential is twice differentiable at $m_1 \pm m_2$, and therefore the quadratic terms in m_1 and m_2 can be isolated.

Taylor expansion of the penultimate integral in (2.23) gives

$$\frac{1}{2}\int^{\Lambda}\frac{d^3k}{(2\pi)^3}\ln\det\Delta^{-1}(k)=\text{const}+\left[\frac{B^2}{A^2}\frac{m_1^2}{2}+\frac{C^2}{A^2}\frac{m_2^2}{2}\right]\frac{2\Lambda}{\pi^2}\left[2+4\left[\frac{1}{\lambda}-\frac{1}{\lambda_c}\right]\ln 4\left[\frac{1}{\lambda}-\frac{1}{\lambda_c}\right]+4\left[\frac{1}{\kappa}-\frac{1}{\kappa_c}\right]\ln 4\left[\frac{1}{\kappa}-\frac{1}{\kappa_c}\right]\right]+\cdots \tag{3.24}$$

and combining with the leading-order result gives

$$\frac{\Gamma}{V}=\text{const}+\left\{\left[\frac{1}{\lambda}-\frac{1}{\lambda_c}\right]\left[1+\frac{8}{\pi^2N}\ln 4\left[\frac{1}{\lambda}-\frac{1}{\lambda_c}\right]\right]+\frac{8}{\pi^2N}\left[\frac{1}{\kappa}-\frac{1}{\kappa_c}\right]\ln 4\left[\frac{1}{\kappa}-\frac{1}{\kappa_c}\right]\right\}\frac{N\Lambda B^2}{A^2}\frac{m_1^2}{2}$$
$$+\left\{\left[\frac{1}{\kappa}-\frac{1}{\kappa_2}\right]\left[1+\frac{8}{\pi^2N}\ln 4\left[\frac{1}{\kappa}-\frac{1}{\kappa_c}\right]\right]+\frac{8}{\pi^2N}\left[\frac{1}{\lambda}-\frac{1}{\lambda_c}\right]\ln 4\left[\frac{1}{\lambda}-\frac{1}{\lambda_c}\right]\right\}\frac{N\Lambda C^2}{A^2}\frac{m_2^2}{2}+\cdots , \tag{3.25}$$

where we have included an order-N^{-1} correction to the critical couplings:

$$\frac{1}{\lambda_c}=\frac{1}{\kappa_c}=\frac{2}{\pi^2}\left[1-\frac{2}{N}\right] . \tag{3.26}$$

All corrections to (3.25) either vanish faster than m_1^2, m_2^2 as $m_1^2, m_2^2 \to 0$ or are at least of order N^{-1}.

Thus, the inverse correlation lengths for the renormalized scalar fields are

$$\xi_1=\left\{\left[\frac{1}{\lambda}-\frac{1}{\lambda_c}\right]\left[1+\frac{8}{\pi^2N}\ln 4\left[\frac{1}{\lambda}-\frac{1}{\lambda_c}\right]\right]+\frac{8}{\pi^2N}\left[\frac{1}{\kappa}-\frac{1}{\kappa_c}\right]\ln 4\left[\frac{1}{\kappa}-\frac{1}{\kappa_c}\right]\right\}\frac{\Lambda B^2}{A^2} \tag{3.27}$$

and

$$\xi_2=\left\{\left[\frac{1}{\kappa}-\frac{1}{\kappa_c}\right]\left[1+\frac{8}{\pi^2N}\ln 4\left[\frac{1}{\kappa}-\frac{1}{\kappa_c}\right]\right]+\frac{8}{\pi^2N}\left[\frac{1}{\lambda}-\frac{1}{\lambda_c}\right]\ln 4\left[\frac{1}{\lambda}-\frac{1}{\lambda_c}\right]\right\}\frac{\Lambda C^2}{A^2} \tag{3.28}$$

for ϕ and for χ, respectively. In the λ-κ plane the line $\xi_1[\lambda,\kappa]=0$ gives a line of second-order critical points of the parity-breaking phase transition, and $\xi_2[\lambda,\kappa]=0$ gives a line of second-order critical points where Z_2 symmetry is broken. The parameters ξ_1 and ξ_2 are the renormalized coefficients of the quadratic terms in scalar fields in the effective action. These can be made finite by tuning λ and κ sufficiently close to the zeros of the functions

$\xi_1[\lambda,\kappa]$ and/or $\xi_2[\lambda,\kappa]$. Since these functions have a simultaneous zero, it is possible to make both ξ_1 and ξ_2 finite in the continuum limit.

Equations (3.27) and (3.28) can be inverted to obtain

$$\frac{1}{\lambda}=\frac{1}{\lambda_c}+\frac{A^2\xi_1}{B^2\Lambda}\left[1-\frac{8}{\pi^2 N}\ln\frac{4\xi_1}{\Lambda}\right]-\frac{8}{\pi^2 N}\frac{A^2\xi_2^2}{C^2\Lambda}\ln\frac{4\xi_2}{\Lambda}\,, \tag{3.29}$$

$$\frac{1}{\kappa}=\frac{1}{\kappa_c}+\frac{A^2\xi_2^2}{C^2\Lambda}\left[1-\frac{8}{\pi^2 N}\ln\frac{4A^2\xi_2}{C^2\Lambda}\right]-\frac{8}{\pi^2 N}\frac{A^2\xi_1}{B^2\Lambda}\ln\frac{4A^2\xi_1}{B^2\Lambda}\,, \tag{3.30}$$

which show that the linear running of λ and κ to λ_c and κ_c, respectively, is corrected by logarithms at next-to-leading order. We can derive renormalization-group equations by considering the logarithmic derivatives of (3.29) and (3.30) with respect to the cutoff, holding the low-energy, physical parameters of the model b,μ,ξ_1, and ξ_2 fixed. The result is

$$\Lambda\frac{\partial}{\partial\Lambda}\left[\frac{1}{\lambda}+\frac{1}{\kappa}-\frac{2}{\lambda_c}\right]_{b,\mu,\xi_1,\xi_2}=-\left[1+\frac{16}{3\pi^2 N}\right]\left[\frac{1}{\lambda}+\frac{1}{\kappa}-\frac{2}{\lambda_c}\right], \tag{3.31}$$

$$\Lambda\frac{\partial}{\partial\Lambda}\left[\frac{1}{\lambda}-\frac{1}{\kappa}\right]_{b,\mu,\xi_1,\xi_2}=-\left[1+\frac{32}{3\pi^2 N}\right]\left[\frac{1}{\lambda}-\frac{1}{\kappa}\right]. \tag{3.32}$$

There are lines of fixed points where $\lambda=\kappa$ and where $1/\lambda+1/\kappa=(4/\pi^2)(1-2/N+\cdots)$. Explicit running of the coupling constants can be found by solving these equations as

$$\left[\frac{1}{\lambda}-\frac{1}{\kappa}\right]=\frac{1}{b^2}\left[\frac{\mu}{\Lambda}\right]^{1+32/3\pi^2 N}\left[\frac{\xi_1}{\mu}-\frac{\xi_2}{\mu}\right], \tag{3.33}$$

$$\left[\frac{1}{\lambda}+\frac{1}{\kappa}-\frac{2}{\lambda_c}\right]=\left[\frac{\mu}{\Lambda}\right]^{1+16/3\pi^2 N}\left[\left[\frac{\xi_1}{\mu}\right]^{1-16/\pi^2 N}+\left[\frac{\xi_2}{\mu}\right]^{1-16/\pi^2 N}\right]. \tag{3.34}$$

Equations (3.33) and (3.34) indicate that λ flows toward κ and that both λ and κ flow toward $\lambda_c=\kappa_c$ as we take the continuum limit $\Lambda/\mu\to\infty$.

The logarithms in (3.28) can also be interpreted as a manifestation of nontrivial scaling behavior. It is straightforward to deduce critical exponents associated with this scaling. If we set $\kappa=\lambda$ and approach the critical point (λ_c,κ_c) in the κ-λ plane along the diagonal from the symmetric phase, we get

$$\xi_1=\xi_2=\left[\frac{1}{\lambda}-\frac{1}{\lambda_c}\right]\left[1+\frac{16}{\pi^2 N}\ln 4\left[\frac{1}{\lambda}-\frac{1}{\lambda_c}\right]\right]\frac{N\Lambda B^2}{A^2} \tag{3.35}$$

$$\approx\frac{1}{4}\left[4\left[\frac{1}{\lambda}-\frac{1}{\lambda_c}\right]\right]^{1+16/\pi^2 N}\frac{N\Lambda B^2}{A^2}\,. \tag{3.36}$$

Here, the critical exponent which governs how fast the correlation lengths of the scalar fields vanish as $\lambda\to\lambda_c$ is $1+16/\pi^2 N$. In general this scaling behavior depends on the direction from which we approach the critical line. For example, if instead we set $\kappa=\kappa_c$ and consider $\xi_1[\lambda,\kappa_c]$, the same critical exponent is $1+8/\pi^2 N$.

We can also expand the effective potential due to the four-Fermi interactions to third order. The result for the third-order term is

$$\frac{4}{3\pi^3}\ln\left[16\left[\frac{1}{\lambda}-\frac{1}{\lambda_c}\right]\left[\frac{1}{\kappa}-\frac{1}{\kappa_c}\right]\right]\times\left[\left|\frac{B}{A}m_1+\frac{C}{A}m_2\right|^3+\left|\frac{B}{A}m_1-\frac{C}{A}m_2\right|^3\right], \tag{3.37}$$

and the leading- [from Eq. (2.24)] and next-to-leading-order cubic terms combined are

$$\approx\frac{N}{6\pi}\bar{b}^3\left[\frac{\Lambda}{\mu}\right]^{16/\pi^2 N}\left[16\left[\frac{1}{\lambda}-\frac{1}{\lambda_c}\right]\left[\frac{1}{\kappa}-\frac{1}{\kappa_c}\right]\right]^{8/N\pi^2}\times(|m_1+m_2|^3+|m_1-m_2|^3) \tag{3.38}$$

$$=\frac{N}{6\pi}\bar{b}^3\left[\frac{16\Lambda^2}{\mu^2}\left[\frac{1}{\lambda}-\frac{1}{\lambda_c}\right]\left[\frac{1}{\kappa}-\frac{1}{\kappa_c}\right]\right]^{8/N\pi^2}\times(|m_1+m_2|^3+|m_1-m_2|^3) \tag{3.39}$$

$$\approx\frac{N}{6\pi}\bar{b}^3\left[\frac{16\xi_1\xi_2}{\mu^2}\right]^{8/N\pi^2}(|m_1+m_2|^3+|m_1-m_2|^3)\,, \tag{3.40}$$

where we have used (3.22) and (3.23) for A, B, and C and $\bar{b}^3=b^3(1+20/3\pi^2 N+\cdots)$. The result is finite in the limit of the infinite ultraviolet cutoff $\Lambda\to\infty$ since $\xi_1\approx\Lambda(1/\lambda-1/\lambda_c)$ and $\xi_2\approx\Lambda(1/\kappa-1/\kappa_c)$ are finite in the critical regime. This confirms that the effective potential to order N^0 is made finite by the counterterms given in Eqs. (3.22) and (3.23) and supports the renormalizability of the full theory.

The sum of (3.25) and (3.40) gives the effective potential to cubic order and to order N^0. Corrections to this formula either vanish faster than cubic order as m_1 and m_2 vanish or are at least of order N^{-1}. The remaining corrections at order N^0 are also finite as $\Lambda\to\infty$. Thus, with the cutoff dependence (3.22) and (3.23) and with λ and κ tuned so that ξ_1 and ξ_2 in (3.27) and (3.28) are finite, the effective potential is ultraviolet finite to both leading- and next-to-leading order in large-N.

The critical coupling in (3.26) gets larger as N get smaller and goes to infinity when $N \approx 2$. This indicates that, beyond the leading order in large N, the quantum fluctuations due to the four-Fermi interaction resist breaking of both parity and chiral Z_2 symmetry. As we shall see in the following, this is particularly important in the latter case where the next-to-leading-order effect of the four-Fermi interactions can cancel the leading-order tendency of the electromagnetic interactions to break chiral symmetry.

C. The effective potential 2: Electromagnetic radiative corrections

To order N^0 the quadratic term in the effective potential contained in the last, electromagnetic term in (2.23) can be obtained as

$$\frac{1}{2}\int^{\Lambda}\frac{d^3k}{(2\pi)^3}\ln\left[k^2\left[\frac{1}{\Lambda}-\Pi_e(k)\right]^2+\Pi_o(k)^2\right]$$
$$=-\frac{1}{2\pi^2}\Lambda e^2\ln\left[1+\frac{8}{e^2}\right]\left[\frac{B^2}{2A^2}m_1^2+\frac{C^2}{2A^2}m_2^2\right]+\frac{8\Lambda}{\pi^2}\frac{1}{1+8/e^2}\frac{C^2}{2A^2}m_1^2+\cdots \quad (3.41)$$

The first term arises from the dependence of Π_e on the induced fermion masses and does not distinguish between parity and chiral Z_2 symmetry breaking. The second term comes from Π_o and can be regarded as arising from the induced topological photon mass, which is present when parity is spontaneously broken. Note that it only depends on the parity-odd mass m_1. The expression (3.41) contains no cutoff-dependent logarithms, and therefore this contribution to the effective potential modifies the critical values of the coupling constants, rather than the critical exponents. The modified critical couplings are

$$\frac{1}{\lambda_c}=\frac{2}{\pi^2}\left[1-\frac{2}{N}+\frac{e^2}{4N}\ln\left[1+\frac{8}{e^2}\right]-\frac{4}{N}\frac{1}{1+8/e^2}\right], \quad (3.42)$$

$$\frac{1}{\kappa_c}=\frac{2}{\pi^2}\left[1-\frac{2}{N}+\frac{e^2}{4N}\ln\left[1+\frac{8}{e^2}\right]\right], \quad (3.43)$$

or, if we invert (3.42) and (3.43) by assuming that N is very large,

$$\lambda_c=\frac{\pi^2}{2}\left[1+\frac{2}{N}-\frac{e^2}{4N}\ln\left[1+\frac{8}{e^2}\right]+\frac{4}{N}\frac{1}{1+8/e^2}+\cdots\right], \quad (3.44)$$

$$\kappa_c=\frac{\pi^2}{2}\left[1+\frac{2}{N}-\frac{e^2}{2N}-\frac{e^2}{4N}\ln\left[1+\frac{8}{e^2}\right]+\cdots\right]. \quad (3.45)$$

Note that the electromagnetic contributions decrease the critical coupling κ_c and therefore favor Z_2 symmetry breaking. However, the electromagnetic terms in (3.44) are always positive and therefore increase λ_c. Thus the electromagnetic corrections resist the dynamical breaking of parity. This is consistent with the results of Ref. [25].

The function $\frac{1}{4}e^2\ln(1+8/e^2)$ in κ_c is a monotonically increasing function of e^2 with maximum 2 at $e^2=\infty$. Thus, when $e^2=\infty$ the electromagnetic interactions balance the tendency of next-to-leading-order four-Fermi interactions to increase the critical couplings. The electromagnetic correction is the same magnitude as the leading-order result when $N\approx 2$.

In the absence of four-Fermi interactions, i.e., in three-dimensional quantum electrodynamics, it has been shown [22] by finding the solution of Dyson-Schwinger equations for the fermion mass operator in the large-N (or quenched planar) limit that there is a critical $N_{\rm crit}\approx 4$ such that for $N<N_{\rm crit}$ the electromagnetic interactions break chiral Z_2 symmetry and for $N>N_{\rm crit}$ the system is chirally symmetric. In the present case the next-to-leading-order effect of the four-Fermi coupling cancels the tendency of the electromagnetic interaction to reduce the critical coupling constant for chiral symmetry breaking. To the order that we have computed, it is not possible to conclude whether for small enough N the electromagnetic interactions renormalize κ_c to 0. It is necessary to estimate the relative magnitude of contributions from electromagnetic and four-Fermi interactions at next-to-next-to leading order in large N. Also, a computation to that order would help in obtaining a better estimate of the radius of convergence of the large-N expansion itself.

IV. DISCUSSION

The low-energy effective action of the four-Fermi theories near the critical points should contain the kinetic terms for the scalar fields ϕ and χ, which are stable if the corresponding parity and/or chiral symmetry is broken and which have masses $\sim 2m_1$ and $2m_2$, respectively.

Although the only effect of electromagnetic interactions in the effective potential to the order computed is to shift the critical four-Fermi couplings, they have a profound effect on the nature of the low-energy spectrum of the fermions. There, we distinguish three phases: *confining, Coulombic,* and *topologically massive.* When the fermions have a parity-invariant mass, or if parity breaking is weaker than chiral-symmetry breaking so that $|m_1|<|m_2|$, the photon is massless (there is no induced topological mass at one loop and a no-renormalization theorem [17] guarantees that it does not appear in higher-order corrections) and the Coulomb interaction is long ranged and logarithmic. The photon propagator has the low-momentum behavior

$$\Delta_{\mu\nu}(k)\sim\left[\delta_{\mu\nu}-\frac{k_\mu k_\mu}{k^2}\right]\frac{m}{k^2},$$

where m is proportional to a characteristic fermion mass. This gives a *confining* logarithmic Coulomb interaction between fermions and antifermions. In this case only neutral excitations should appear in the spectrum of finite-energy states of the model.

On the other hand, if all of the fermions are massless for N large enough that the large-N photon propagator is a good approximation,

$$\Delta_{\mu\nu}(k)\sim\left[\delta_{\mu\nu}-\frac{k_\mu k_\nu}{k^2}\right]\frac{1}{k},$$

the electromagnetic interaction is Coulombic ($\sim 1/|x-y|$) and the coupling strength is the dimensionless parameter $1/N$. This *Coulombic* phase should have asymptotic states with infrared-free fermions and antifermions as well as a propagating free photon similar to four-dimensional electrodynamics.

If parity breaking is stronger than chiral-symmetry breaking, i.e., if $|m_1|>|m_2|$, the low-energy effective action contains a Chern-Simons term for the photon with coefficient given by Eq. (2.22) and we obtain the *topologically massive* phase. The resulting topological mass of the photon screens the Coulomb interaction and the fermions are not confined. (This is also true for compact electrodynamics with topological mass [29].) In this phase the fermions have fractional statistics [30–34].

Four-Fermi interactions have recently been argued to play a role in the strong-coupling, chiral-symmetry-breaking phase of quantum electrodynamics in four dimensions [35]. There it has been shown that, because of the large anomalous dimension of the fermion mass operator, four-Fermi couplings can be relevant operators [36]. This idea has been used to construct a dynamical-symmetry-breaking scheme for the standard model where the Higgs boson is a t-$\bar{t}$ condensate. In the present case, near the critical points the effective dimension of the mass operator can be obtained from the large Euclidean momentum limit of the correlation functions:

$$\begin{bmatrix}\langle\bar\psi\psi,\bar\psi\psi\rangle & \langle\bar\psi\psi,\bar\psi\tau^3\psi\rangle\\ \langle\bar\psi\tau^3\psi,\bar\psi\psi\rangle & \langle\bar\psi\tau^3\psi,\bar\psi\tau^3\psi\rangle\end{bmatrix}\sim\frac{1}{N}\begin{bmatrix}\frac{1}{k} & 0\\ 0 & \frac{1}{k}\end{bmatrix}$$

in the large-k limit and $\dim[\bar\psi\psi]=1+O(N^{-1})$ and also $\dim[\bar\psi\tau^3\psi]=1+O(N^{-1})$.

APPENDIX: FERMIONS IN 2+1 DIMENSIONS

In this Appendix we shall fix the notation which is used in this paper and discuss the symmetries of the Dirac equation in 2+1 dimensions. Fermions in 2+1 dimensions are spinor representations of the Lorentz group SO(2,1). The minimal dimension of the Dirac matrices which satisfies the Clifford algebra

$$\{\gamma^\mu,\gamma^\nu\}=2g^{\mu\nu} \tag{A1}$$

[where the metric is taken as $g^{\mu\nu}=\mathrm{diag}\,(1,-1,-1)$] is 2×2. A particularly convenient choice is

$$\gamma^0=\sigma^2,\quad \gamma^1=i\sigma^3,\quad \gamma^2=i\sigma^1\,, \tag{A2}$$

where σ^i are the Pauli matrices. With this choice the Dirac operator

$$i\gamma_\mu\partial^\mu+m \tag{A3}$$

is real and the charge-conjugation symmetry of the action

$$S_0=\int d^3x\,\bar\psi(i\gamma^\mu\partial_\mu+m)\psi \tag{A4}$$

is

$$C:\psi\to\psi^{\dagger t},\quad C:\psi^\dagger\to\psi^t\,, \tag{A5}$$

where the dagger denotes Hermitian conjugation and t denotes the transpose. Making the substitution (A5) in the action (A4) and taking into account the anticommutativity of the spinor fields, we see that the action is self-conjugate.

It is possible to define a real spinor which obeys the constraint

$$\psi=\psi^{\dagger t}\,. \tag{A6}$$

The action with a real spinor differs from (A4) by a factor of 2:

$$S_0=\tfrac{1}{2}\int d^3x\,\bar\psi(i\gamma^\mu\partial_\mu+m)\psi\,, \tag{A7}$$

where now the fermion field obeys the constraint (A6). This minimal representation fermion field has the advantage of exhibiting the full flavor symmetry of the Dirac action. For example, if in (A7) there are $2N$ species of real fermions the explicit flavor symmetry would be O($2N$), whereas in its complex representation (A4) there would be a manifest U(N) [which is a subgroup of O($2N$)] symmetry. The full symmetry could only be seen by taking combinations of U(N) transformations and charge conjugations.

A disadvantage of using this parametrization of the fermions is that the analytic continuation to Euclidean space is more subtle. This is done by putting $x_0\to ix_3$ and defining the Euclidean space γ matrices as

$$\gamma_3^E=\gamma_0,\quad \gamma_1^E=i\gamma_1=\sigma^3,\quad \gamma_2^E=i\gamma_1=\sigma^1\,. \tag{A8}$$

We then define the Euclidean Fermi fields as

$$\psi_E=\psi(x_0\to ix_3),\quad \bar\psi_E=-i\bar\psi(x_0\to ix_3)\,, \tag{A9}$$

so that the Dirac action is

$$S=iS_E=i\int d^3x_E\bar\psi_E(i\gamma_\mu^E\partial_\mu+im)\psi_E \tag{A10}$$

and the path-integral representation of the partition function would be

$$Z=\int d\psi_E d\bar\psi_E e^{-S_E}=\det(i\gamma_\mu^E\partial_\mu+im)\,. \tag{A11}$$

In Euclidean space we can relabel the γ matrices as $\gamma_\mu^E=\sigma_\mu,\mu=1,2,3$, and, define charge conjunction as

$$\psi_E\to\sigma_2\bar\psi_E^t,\quad \bar\psi_E\to-\psi_E^t\sigma^2\,. \tag{A12}$$

The Euclidean action in (A10) is invariant under this transformation. Again, we can use this fact to decompose Euclidean fermions into their self-conjugate and anti-self-conjugate components. Self-conjugate Euclidean fermions obey the constraint

$$\psi_+=\sigma^2\bar\psi_+^t,\quad \bar\psi_-=-\psi_-^t\sigma^2\,, \tag{A13}$$

where

$$\psi_+\tfrac{1}{2}(\psi_E+\sigma^2\bar\psi_E^t),\quad \psi_-=\tfrac{1}{2}(\psi_e-\sigma^2\bar\psi_E^t)\,, \tag{A14}$$

and have action with an additional factor of $\frac{1}{2}$:

$$S_E=\int d^3x\, \tfrac{1}{2}\bar{\psi}_\pm(i\gamma_\mu^E\partial_\mu+im)\psi_\pm\ , \quad (A15)$$

$$Z=\int d\psi_\pm e^{-S_E}=[\det(i\gamma_\mu^E\partial_\mu+im)]^{1/2}\ . \quad (A16)$$

The Euclidean mass operator for complex fermions, $i\bar{\psi}\psi$, can be written in terms of real fermions as

$$i\bar{\psi}\psi=i(\psi_-^t\sigma^2\psi_-+\psi_+^t\sigma^2\psi_+)\ , \quad (A17)$$

which exhibits explicitly the O(2) [or O(2N) if there are N flavors of fermions] symmetry of the mass operator.

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