Interpretation of the asymptotic S matrix for massless particles

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We discuss the existence and properties of the asymptotic S matrix in field theories with massless particles. We show for the classic case of the scattering of an electron in an external electromagnetic field that the S-matrix method yields the same results as the traditional cross-section method involving properly formed sums over physically degenerate initial as well as final states. In that case, we show that observables are independent of the particular choice of asymptotic Hamiltonian. We argue that the massless theory is unique and discuss its relation to the massless limit of the massive theory. Although the results do not depend on it, a novel aspect of our formalism is the interpretation of physical states involving massless particles as Fock states and the occurrence of physical transitions in Fock space.

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I. INTRODUCTION

The perturbative evaluation of transition amplitudes in gauge theories, such as QED or QCD, is plagued by divergences, mass singularities, associated with the massless particles that these theories contain. These singularities do not occur in off-shell Green's functions, but only in S-matrix elements or certain other quantities involving on-mass-shell particles such as form factors. Technically speaking, these divergences are an indication that the Feynman-Dyson S matrix between Fock-space states does not exist. Traditionally, the standard method of dealing with these divergences is to regulate them in some manner, usually by temporarily restoring a mass to all particles and summing up incoherent but physically indistinguishable cross sections, as in the classic Bloch-Nordsieck treatment of the soft-photon divergence or so-called "infrared catastrophe" [1,2]. Following Nelson [3], we shall refer to this procedure as the *cross-section method.* The mass divergences are conveniently divided into two classes: (1) *infrared* divergences, which deal with mass singularities associated with the emission of soft quanta, and (2) *collinear* divergences, related to the property that differing numbers of collinear, massless, on-shell particles can be degenerate in energy and momentum (and other good quantum numbers). It has been shown quite generally [4,5] that summing over physically degenerate states does indeed produce a cancellation of all mass divergences, both infrared and collinear.

There remain, however, unresolved issues concerning this procedure. First of all, for collinear singularities in the Abelian case and generally in the non-Abelian case, these cancellations require summing not only over indistinguishable final states, but also over degenerate initial states, a procedure that is somewhat controversial. The ability to entertain initial states corresponding to the superposition of states with arbitrary relative weights is fundamental to quantum mechanics, yet the sum over degenerate initial states requires a very special relationship among them in order to cancel mass singularities. So while some believe the cancellation would inevitably occur, others have their doubts. Lee and Nauenberg [5] themselves were somewhat equivocal on this point, indicating that their theorem implied the cancellation of singularities at least in the sense of an "ensemble average." Whether it would always occur in nature was left undecided. In this paper, among other things, we shall show that the requisite initial-state summation does inevitably occur in massless theories. Even if one is convinced of the cancellation, there is the question, for a particle of finite mass, how small must the mass be before it becomes negligible? We shall show that the relevant parameter is the ratio of mass to transverse momentum.

A second issue concerns whether the massless theory is unique and has a well-defined perturbation expansion, a point that was also raised in Ref. [5]. At first sight it would seem that the results do depend on the method by which mass singularities are regulated. It has been shown in special cases that dimensional regularization [6] also results in a cancellation of infrared divergences [7,8]. In certain cases those terms that diverge with an energy resolution ΔE or angular resolution $\delta\theta$ have been shown to be the same as when using a mass cutoff. However, there remain certain finite contributions that are *independent of* ΔE and $\delta\theta$, and it has not been shown that they are always independent of the method used to regulate the mass singularities. As we shall elaborate below, this has important physical consequences, yet is an issue that,

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until very recently [9], has received very little attention [7]. A closely related, physically relevant question is whether the high-energy limit of a massive theory coincides with the predictions of its massless counterpart. We shall show that results are indeed regulator independent and that observables are smooth (i.e., nonsingular) in the limit that any mass tends to zero.

With regard to the first point, the summation of degenerate initial states, there are persuasive arguments that the true asymptotic, "in" and "out" scattering states of quantum electrodynamics are asymptotic states that cannot be approximated as noninteracting particles. In quantum mechanics it is well known that the asymptotic states in a Coulomb potential are not simply plane waves, but are modified by nonvanishing Coulomb phase factors, reflecting the long-range nature of the Coulomb potential. Its consequences for scattering theory were especially clarified by the work of Dollard [10], who showed that states in Hilbert space did not evolve in time asymptotically according to the free Hamiltonian, but that the true asymptotic states of the theory required the introduction of another, asymptotic Hamiltonian and, correspondingly, an appropriate modification in the definition of the S matrix. In field theory, not only are there Coulomb phases that reflect the long-range nature of the interaction, but also there is particle creation that modifies the character of the asymptotic states. In his seminal paper, Chung [11] argued that the "infrared catastrophe" could be avoided in transition amplitudes themselves if the usual Feynman-Dyson S matrix were evaluated between properly chosen *coherent states* [12,13]. This view was justified and extensively developed by Kibble [14] and, later, by Zwanziger [15], who developed a Lehmann-Symanzik-Zimmermann- (LSZ) type reduction formalism. As we shall review briefly in Sec. II and more extensively in Appendix A, these coherent states are associated with unitarily inequivalent representations of the canonical commutation relations, and so this point of view requires a modification of the traditional view of quantum field theory operating in Fock space. The justification for this departure is associated with the infinite number of soft photons created by the scattering of charged particles. This can be seen explicitly in the solution of the problem of the response of the electromagnetic field to a classical current source, in which in general it appears that an infinite number of photons are created [14,16]. However, while the asymptotic states associated with massless quanta are certainly not described by the free-particle Hamiltonian, contrary to the prevailing view [11,14,15], it is *not* necessary to leave Fock space in order to describe this situation.[1] This can be seen, in particular, in the discussion by Blanchard [17] of the Pauli-Fierz model [18] consisting of the interaction of a single, nonrelativistic, charged particle with the quantized electromagnetic field and with an unspecified, short-range potential. Although our conclusions do not depend on this revision of the usual view of coherent states, this Fock-space point of view is the one we shall develop in Sec. II. The motivation for preferring this point of view will be explained in Sec. V and Appendix A.

A kind of synthesis of the work of Chung, Dollard, Kibble, and Blanchard was developed by Kulish and Faddeev [19], for full QED, who employed the notion of a modified asymptotic Hamiltonian H_A acting in a separable but non-Fock subspace $\mathcal{H}_{\rm as}$ of physical asymptotic states. They were able to extract the generalization of the Coulomb phase factor of potential theory explicitly from the modified S matrix. This has a technical advantage over Kibble's formalism inasmuch as the S matrix acts within a single subspace of the Hilbert space rather than between inequivalent subspaces [20]. However, it is not known whether a similar construction is possible for collinear divergences.

We want to state at the outset, however, that we are not saying the more common procedure is erroneous; one can choose to regard the physical in and out spaces as non-Fock spaces and interpret S-matrix elements as transitions between unitarily inequivalent subspaces, as does Kibble [14] or, after an infinite Coulomb phase is factored out, within a common non-Fock "space of asymptotic states," as do Kulish and Faddeev [19]. However, it seems more economical and in many ways conceptually advantageous not to modify the common practice in quantum field theory and simply to work in Fock space and to regard the S matrix as operating within Fock space. To our mind it also very much facilitates the interpretation of the calculations that are actually performed in practice. However, we wish to emphasize to readers preferring the conventional view that none of our physical conclusions depend upon our reinterpretation of the underlying mathematics.

Once it is recognized that the true asymptotic states of the massless theory are not noninteracting particles as in all-massive theories, the attraction of the traditional cross-section method [1,5] is diminished, for the massless limit is bound to be problematic for the conventional Feynman-Dyson S matrix $S_{\rm FD}$. On the other hand, the less commonly used S-matrix method in the space of asymptotic states, when evaluated via the usual interaction picture with a regulator, automatically generates the proper relative weights between free-particle states involving different numbers of quanta. Transition amplitudes in this basis, unlike noninteracting particle states, have smooth limits as the particle masses tend to zero.

The bottom line on the question of summation over degenerate initial states is that while one can prepare arbitrary initial superpositions of asymptotic states, all these asymptotic states are orthogonal to the usual in and out states associated with the free-particle Hamiltonian; i.e., they lie in a different space.[2] When mass singularities are

[1]This alternate point of view (and the relevance of the work of Blanchard) was brought to our attention by D. N. Williams.

[2]This statement is true regardless of whether one adopts the conventional view of the physical Hilbert space as being a non-Fock space of coherent states with the space associated with the free Hamiltonian H_0 as Fock space $\mathcal{H}_0$ or the reverse view, advocated here, of the physical space as Fock space $\mathcal{H}_F$ with the space associated with H_0 becoming the unitarily inequivalent space of coherent states.

regulated, then all these spaces coincide and the previously orthogonal asymptotic states can be represented as sums of states of noninteracting particles. Similarly, the coherent states used in evaluating the Feynman-Dyson S matrix $S_{\rm FD}[H]$ may be represented as sums over states of noninteracting particles, but this superposition is required by the underlying asymptotic dynamics (in order to obtain matrix elements having a smooth limit as the regulators are removed) and is not to be thought of as an "ensemble average."

With regard to the second point, the uniqueness of the results, there has been remarkably little discussion in the past. This is especially surprising in view of the fact that Lee and Nauenberg, in their original paper [5], noted that there are processes in QED involving helicity flip that would appear to lead to different results than would the theory with massless electrons. Because of a collinear singularity, the total probability of an electron emitting a forward-going photon but undergoing helicity flip remains *finite* in the limit that the mass of the electron m_e tends to zero, a circumstance that led them to remark [5] that this "seems to indicate that a two-component theory of a zero mass spin-$\frac{1}{2}$ charged particle does not exist," at least not in perturbation theory.[3] If true, this would be extraordinary, since it would mean that chiral symmetry of a nonanomalous gauge theory is not restored in the limit that $m_e \to 0$. It would, moreover, suggest that a physical result depends on the regulator method, since neither giving the photon a mass nor using dimensional regularization would allow helicity flip. The theory defined by those regulators would be manifestly chirally conserving, but these are also not without paradoxes. In the case of the massive photon, it turns out that the probability of emission of a collinear longitudinal photon remains finite in the limit of zero photon mass. In the case of dimensional regularization, it turns out that the probability of emitting the extra ϵ transverse polarizations of the photon in $4+\epsilon$ dimensions does not vanish in the limit that $\epsilon \to 0$.[4] This has the appearance of a collinear anomaly, since the massless theory seems not to have properties that depend upon the method by which the singularities are regulated.[5] On the other hand, the fact remains that such effects do not occur in off-shell Green's functions, and so the possibility exists that the theory somehow has in fact a unique interpretation. Indeed, it is a corollary of our work that this apparent "anomaly" is spurious, associated with the method of calculation, and *without observable consequences.* We label as "evanescent"[6] those processes or effects that do not occur for the massless theory, but *appear* to survive in the limit that some mass (or masses) tend to zero when one evaluates the cross section via the standard Feynman-Dyson S matrix $S_{\rm FD}[H]$. In a companion paper, we discuss these pseudoanomalies in evanescent processes, using the S-matrix method to resolve this paradox concerning helicity-flip processes and the nondecoupling of longitudinal photons in massless QED [25].

Strictly speaking, the S-matrix method is based on certain unproved assumptions about the field theory, based on analogies with the solvable case of potential scattering. While those assumptions are both plausible and consistent with everything that is known about the field theory, they are assumptions nevertheless, so that explicit calculations are necessary to provide confidence in the method. We feel it is important to have settled these issues for a familiar process before applying the method in a novel context [25] where a naive application of the conventional method seems to give a different answer. The present paper establishes further confidence in the S-matrix formalism by discussing the consistency of the S-matrix method. We shall derive the equivalence between evaluating the actual S matrix in the physical Fock space and evaluating the Feynman-Dyson S matrix between unphysical coherent states [11,14], a relation that seems to have been widely assumed in practice, but not demonstrated heretofore.

We shall show that observables are, in a certain sense, independent of the particular choice of asymptotic Hamiltonian, and by both general arguments and explicit calculations, we demonstrate the equivalence between the regulated cross-section method (provided initial-state as well as final-state degeneracy is included) and the S-matrix method. Since the S matrix is derivable from the Hamiltonian of the massless theory, these results may be regarded as *justifying* the cross-section method, which deals with matrix elements having mass singularities, and as showing that summation over degenerate initial states is required for a consistent theory.

The coherent-state method has recently been extended to the treatment of collinear singularities [26]. We find that the formulation of asymptotic Hamiltonians in Ref. [26] is particularly economical and well suited to the discussion of questions of uniqueness and the correspondence between the very-high-energy and massless limits.

In non-Abelian gauge theories, the situation is especially relevant and confusing [26–30]. In a number of cases [31,32], it was shown that the Bloch-Nordsieck cancellation does not occur. However, it was later demonstrated [33] that the inclusion of initial-state degeneracy restored the cancellation of divergences. This does not establish whether physical observables always must involve such initial-state cancellations, and there seems to be a divergence of opinion on this score. Indeed, people performing radiative corrections frequently cite the inapplicabili-

[3]This paradox, its relevance to a forthcoming experiment at the DESY *ep* collider HERA, and speculations concerning its probable resolution were discussed in an earlier paper [21], but remained unresolved.

[4]These observations were made a few years ago [22], but were not published because of the suspicion, justified herein, that they were in fact unphysical artifacts of the method.

[5]Indeed, this lack of decoupling of longitudinal photons and gluons has recently been erroneously advocated as a physical effect and a new kind of "infrared anomaly" [23].

[6]This terminology has been introduced for similar kinds of ultraviolet singularities by Collins [24].

ty of the Kinoshita-Lee-Nauenberg (KLN) theorem to non-Abelian gauge theories [32,34], and there remain uncompensated logarithmic dependences on light-quark masses in these radiative corrections as traditionally calculated. It has been argued that, since the initial-state cancellation would require a very special relationship between the quark and two-particle, quark-gluon structure functions, the cancellation should not be expected to occur [32]. As a result of the work that we shall report here and subsequently [25], it is quite clear that such seemingly special relationships from the traditional point of view are natural and required.

It seems, on the contrary, that such sensitivity to light-quark masses is spurious and such cancellations are perfectly natural. While we believe that, quite generally, observables will always be independent of mass singularities and have gone considerable distance toward showing this in QED, we shall have relatively little to say in this paper about the non-Abelian case, to which we may return in later work. However, the theoretical discussion in this paper strongly suggests that there are no differences of principle between the Abelian and non-Abelian cases; the latter is simply more complicated technically. We shall indicate below why these seemingly special relationships must necessarily obtain, why there is no conflict with the superposition principle, and how the notion of partons must be revised to take them into account.

To summarize, it is the purpose of the present paper to elaborate on the S-matrix method. We extend previous results in several ways: We establish quite generally the relationship between the coherent-state method and scattering matrix in the space of asymptotic states. In so doing we also explain why standard calculations in the interaction representation in a regulated theory lead to unique, correct results in the limit that the regulator is removed. Stated another way, we argue that the *massless theory is unique,* resolving the questions originally raised in Ref. [5]. Thus massless theories have no "infrared anomalies" [23], and the perturbation expansion is well defined and independent of how the mass singularities are regulated. This will be illustrated further for the case of helicity flip in another paper [25].

Because the asymptotic Hamiltonian inevitably depends on parameters not in the original Hamiltonian and not derivable from it, this S-matrix method differs somewhat from the familiar S matrix for massive theories. To make the discussion more concrete and to support abstract claims with explicit calculations, we illustrate the concepts by reference to the classic case of the scattering of an electron in an external field and support general claims by calculating the radiative corrections to this process to lowest nontrivial order in α. We show explicitly that the S-matrix formalism not only leads to a cancellation of mass singularities, but also yields precisely the same finite, observable cross section that was previously calculated using the cross-section method [9]. In that work this was shown to give identical results regardless of whether one regulates the mass singularities via dimensional regularization or by temporarily assigning finite masses to the particles. Therefore, we have established an important extension of the classic KLN theorem [4,5] that has a number of significant practical implications for the calculation of radiative corrections.

In order to enhance our intuition, we want to understand precisely in what sense different possible choices of asymptotic Hamiltonians H_A actually describe the scattering states for the true Hamiltonian $H_{\rm QED}$. We discuss which aspects of the actual asymptotic states are determined by the Hamiltonian $H_{\rm QED}$ and show how observables are, in a sense to be described, *independent* of the choice of the particular asymptotic Hamiltonian H_A used to define the asymptotic states. Throughout the paper we pay particular attention to the relation between the choice of asymptotic Hamiltonian and the experimental resolution parameters for any given measurement. In so doing we shall indicate how one particular choice of H_A is especially natural, inasmuch as it includes all initial- and final-state degeneracies to be associated with the phase space appropriate to a given measurement. This *justifies* the intuitive interpretation commonly used [19,26] and associates an essentially unique choice of H_A with any given experimental situation. We also define a new S matrix that nicely accommodates experiments in which the initial- and final-state resolutions differ.

An outline of the paper is as follows: In Sec. II we discuss the S-matrix method in a general way, inspired by the work of Refs. [10], [17], and [19]: Much of the particular development in Sec. II is original. In particular, we derive the general coherent-state formalism and relate it to the scattering matrix in the space of actual asymptotic states. We discuss the mathematics of von Neumann space and contrast our point of view with the more conventional one in Appendix A. We also raise some questions there about some of the results in Ref. [19]. In Sec. III we apply the formalism to QED, using the method of defining the asymptotic Hamiltonian developed in Ref. [26], and discuss its intuitive interpretation. We explicitly calculate the lowest-order QED radiative corrections to the scattering of an electron in an external electromagnetic field, with particular attention not only to the cancellation of divergences, but also to showing that the finite results precisely agree with those of Ref. [9], where the cross section method was employed. In Sec. IV we return to a discussion of the S-matrix formalism, demonstrating its equivalence to the cross-section method in the case of massive particles, thereby deriving the intuitive interpretation used in Sec. III and Ref. [26]. We establish that observables are independent of the choice of H_A, but that the calculations and interpretation become especially simple for the particular choice made in Sec. III. In the process we discuss how observables may depend on the initial-state energy and angular resolutions, and we define a modified S matrix which is useful in describing situations in which the initial- and final-state resolutions differ. Some of the details of the calculations are relegated to Appendix B. Finally, in Sec. V we draw conclusions and suggest a variety of applications in which this approach is likely to provide new insights. In particular, we reflect on certain initial-state radiative correction calculations in QED that have been calculated for the

SLAC Linear Collider (SLC) and CERN e^+e^- collider LEP, and explain why they are correct for those physical situations, but would not be in the limit that the electron mass became extremely small. We also comment on certain QCD radiative corrections in deeply inelastic scattering that are sensitive to light-quark masses and that very likely are misleading and must be modified. Finally, we indicate what problems remain to be addressed and where improvements in the formalism might be made. In particular, a novel suggestion that will be explored in future work is the introduction of an "asymptotic interaction picture," as an alternative to the usual interaction picture for deriving Feynman rules and organizing calculations.

II. *S*-MATRIX METHOD AND COHERENT STATES

In this section we discuss the S-matrix method in a manner that combines the approaches of Refs. [17] and [19]. One of the goals of this section is to establish that the exact S matrix S_A between physical asymptotic states may be evaluated by evaluating the Feynman-Dyson S matrix $S_{\rm FD}$ between properly defined coherent states, as expressed in Eqs. (2.15) and (2.16) below. This is in fact the starting point of coherent-state calculations [26,27] in recent times, but we are not aware of a previous derivation in the literature. It is one thing to show that $S_{\rm FD}$ between coherent states is free of mass singularities and another to derive from first principles that this is in fact the physical transition amplitude, even though our derivation is not rigorous. The reader prepared to accept this result may wish to skip this formal discussion, referring back to the relevant equations as necessary, and proceed to the next section in which this formalism is implemented in QED for the case of the scattering of a massless electron in an external potential. However, our formal discussion is not without interest or void of results: Contrary to the conventional view of coherent states, we argue that the scattering in massless theories can be regarded as occurring in Fock space and that we need not concern ourselves with the intricacies of the larger von Neumann [35] space $\mathcal{H}_{\rm vN}$, which we discuss in Appendix A.[7] We derive the correspondence between matrix elements of the scattering operator S_A between physical, asymptotic scattering states in Fock space $\mathcal{H}_F$ and matrix elements of the Feynman-Dyson S matrix $S_{\rm FD}[H]$ in the (for us, unphysical) space of coherent states [Eq. (2.15)] that are unitarily inequivalent to Fock space. Another result is that, in the strictly massless limit, the coherent states commonly employed [27,26] lie outside the space of asymptotic scattering states [cf. Eq. (2.17) with Eq. (2.18)].

Let us first recall the situation regarding the "infrared catastrophe" associated with a massless photon interacting with a massive electron. To set the stage and to facilitate comparison, before explaining how the S matrix may be resurrected, we shall first review briefly the more familiar cross-section method. The conventional manner of dealing with these infrared problems is to give the photon a small mass, thereby temporarily replacing the long-range Coulomb potential with a short-range, exponentially damped one and restoring a mass gap to the spectrum so that propagators have isolated particle poles. In QED this enables one to use standard methods of computation, but the radiative corrections to any exclusive process, such as the elastic scattering of an electron in an external field, diverge as the photon mass m_γ tends to zero. However, the now conventional argument [1] is that any real experiment involves a detector with a finite-energy resolution ΔE, and so these exclusive processes are not observable, being physically indistinguishable from scattering accompanied by the emission of any number of sufficiently low-momentum massless photons (although, to a given order in perturbation theory, only a finite number may be emitted). When these soft bremsstrahlung cross sections are added to the radiatively corrected elastic cross section, a finite answer is obtained in the limit $m_\gamma \to 0$. However, the resulting observable cross section depends on the energy resolution ΔE and, to any finite order in perturbation theory, diverges as a power of $\ln(\Delta E)$ in the limit that $\Delta E \to 0$. In fact, if one sums up the leading logarithmic divergences in each order, one finds that the transition probability vanishes [2,16,37] as ΔE^β with some positive power $\beta \propto \alpha$; i.e., there is in fact zero probability of detecting a charged particle with a definite momentum. Thus the energy resolution of the detector enters observables in an essential way and, unlike the case of short-range interactions, cannot be removed in the definition of an observable cross section.

As reviewed in the preceding section, attempts have been made to reproduce the observable cross section by means of S-matrix methods [20]. In their influential paper, Kulish and Faddeev [19], following Dollard [10], Blanchard [17], and others, suggested that the scattering states associated with the exact Hamiltonian $H_{\rm QED}$ may be approximated by the scattering states associated with another, asymptotically correct Hamiltonian H_A. While they suggested a particular form for H_A, there is in fact a great deal of arbitrariness in its specification. The only stipulation is that all such choices of H_A must lead to the same long-range, large-time behavior as the true Hamiltonian $H_{\rm QED}$, a requirement that will be made mathematically more precise below [see Eq. (2.9)]. Unlike theories involving only massive particles, H_A involves nontrivial interactions and differs from the free-particle Hamiltonian H_0 characterizing the in and out states in theories without massless particles. In potential theory this procedure has been shown to reproduce the same results as solving the Schrödinger or Dirac equation in a Coulomb potential [20]. A similar conjecture is supposed to hold true for quantum field theory as well, and supporting evidence is provided by showing that, in the basis of asymptotic states of H_A, infrared divergences cancel, and one can obtain the correct "Coulomb phase" factors directly from H_A without solving the complete dynamical prob-

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[7] This is very much in keeping with the philosophy elaborated by Wightman [36] and the conventional view of observables.

lem [19].[8] Similarly, the extension of this method to collinear singularities removes those divergences as well [26]. In general, the definition of a particular asymptotic Hamiltonian $H_A(\Delta)$ involves a set of infrared and/or collinear parameters, denoted collectively here by Δ, that are associated with the definition of the infrared or collinear regions, and consequently, the scattering states defined from H_A also depend on these parameters.[9] This is quite different from the usual situation in which the parameters of H_0, i.e., the particle masses, are in principle determined by the parameters of the exact Hamiltonian H. It is probably more meaningful to discuss this after having some familiarity with the method, and so we defer until Sec. V the origin and implications of this difference. Since finite observable cross sections depend inextricably on experimental resolutions [5], we would anticipate that the parameters Δ may have something to do with these resolutions, and, indeed, they do.

In Sec. IV we shall show that calculations simplify considerably for the particular $H_A(\Delta)$ in which these parameters *are* identified with the experimental resolutions. For that choice the asymptotic dynamics described by $H_A(\Delta)$ may be interpreted as generating an asymptotic in state (or out state) that approximates those prepared (or measured) in the laboratory. However, we shall explicitly also show that observables are, in a sense, independent of the specific choice of $H_A(\Delta)$, and so, in principle, the theory is uniquely specified by H, as it should be.

In the following we shall give an heuristic description of the mathematical framework, with a goal toward understanding the origin of coherent states and their relationship to the asymptotic scattering states. We have not rigorously proved the assertions made in this section; rather, we are attempting to portray the circumstances that apparently *must* obtain in order to relate coherent-state calculations to the S matrix and to interpret results obtained using coherent states. As this is not a review article, we shall assume some familiarity with the concepts and results of earlier work, especially of Refs. [14] and [19].

Let H denote the exact Hamiltonian of a system involving some massless particles, such as $H_{\rm QED}$. As usual, we construct the Hilbert space $\mathcal{H}_F$ on which this acts as a Fock space.[10] In the Schrödinger picture, the states of the theory evolve according to $e^{-iHt}|\psi\rangle$, where $|\psi\rangle$ denotes some normalizable (square-integrable) wave packet in $\mathcal{H}_F$. The main difference between massive theories and theories involving massless particles is that the asymptotic behavior of the states as $t\to\mp\infty$ cannot be approximated as $e^{-iH_0t}|\psi_0\rangle$ for any choice of state $|\psi_0\rangle$, where H_0 denotes the free-particle Hamiltonian obtained from H by switching off all interactions (but keeping physical masses, if any, in H_0 [11]). This is frequently expressed in terms of the asymptotic convergence of the operator:

$$\Omega_{H,H_0}(t)\equiv e^{iHt}e^{-iH_0t}\,. \tag{2.1}$$

Then another way to state this is to say that, unlike theories with only massive particles, for no choice of state $|\psi_0^\pm\rangle$ does $\Omega_{H,H_0}(t)|\psi_0^\pm\rangle$ converge as $t\to\mp\infty$ to the Fock state $|\psi\rangle$.[12] In other words, unlike massive theories, the Møller wave operators[13]

$$\Omega^{(\pm)}_{H,H_0}\equiv \lim_{t\to\mp\infty}\Omega_{H,H_0}(t) \tag{2.2}$$

do not exist.[14] However, it is believed possible to find other asymptotic Hamiltonians $H_A(\Delta)$ to replace H_0 for which the corresponding limits *do* exist, and we shall *assume* that is the case. The basic idea in the construction of an acceptable H_A from H is that any such H_A must in-

[8]As noted at the end of Appendix A, we question the conclusion of Ref. [19] that their S matrix collapses to a Fock-space operator.

[9]In Ref. [19] this dependence on a parameter called t_0 is regarded as trivial, and even though their S matrix depends on it, it is for the most part suppressed. In Ref. [26] this dependence is represented explicitly by an infrared resolution M and an angular resolution Δ. For now, Δ simply denotes some generic way of defining what is meant by the infrared and collinear regimes.

[10]We will specifically ignore for now the possibility of unitarily inequivalent representations of the canonical commutation relations of the creation and annihilation operators. At this point we depart from the common procedure [11,14,19] of constructing the massless theory in a Hilbert space of coherent states that is unitarily inequivalent to Fock space. While the usual procedure is possible, it seems to be cumbersome and entirely unnecessary; in Appendix A we explain the relation of our procedure to that other approach.

[11]We are ignoring the possibility of bound states of H; this is an inessential complication for our purposes. If they occur, one simply projects onto the scattering states.

[12]Be warned that there is no uniform sign convention for the correspondence between the symbols + and − and the in and out states. In the literature one will frequently find the convention reversed from the one employed in this paper, which derives from the sign convention for retarded and advanced Green's functions.

[13]A pedagogical review of scattering theory in this language may be found in Chaps. 6 and 7 of Newton [38]. While this reference does not treat the massless case, his formal discussion carries over with the replacement of H_0 by H_A.

[14]By "exist," we mean as a unitary operator in $\mathcal{H}_F$. To make these statements precise, one must imagine introducing an ultraviolet cutoff, to be eventually removed by the well-known procedure of renormalization. On the other hand, we want to emphasize that these statements apply to a theory in which the mass singularities have *not* been somehow regulated, such as by temporarily giving all massless particles fictitious masses. The asymptotic convergence of the infrared regulated theory is different, as we shall discuss subsequently. The limits in which the mass regulator is removed (e.g., fictitious masses tend to zero) and the time $t\to\mp\infty$ frequently do not commute. In Appendix A we shall indicate how the situation may be regarded in the larger von Neumann space in which these operators may be formally defined.

volve precisely the same mass singularities as H, but differ in their short-range behavior. Thus, if H_A were to be regarded as the exact Hamiltonian, the associated Feynman-Dyson S matrix $S_{FD}[H_A]$ must reproduce precisely the same infrared and collinear singularities as $S_{FD}[H]$. While the existence of such an H_A has been established for potential theory [10] and for certain field-theory models [17], it is at this point an assumption about the realistic field theory for which motivation has been provided in Ref. [19] and earlier work cited therein. While this has not been proved in general to the best of our knowledge, this assumption is certainly consistent with the experience of all finite-order calculations in QED and QCD using coherent states, to which this will ultimately be shown to be related.

The replacement of H_0 by H_A is the essence of the S-matrix method. The existence of the corresponding Møller wave operators $\Omega^{(\pm)}_{H,H_A}$ (and associated S matrix S_A) as unitary operators in Fock space $\mathcal{H}_F$ is equivalent to the requirement that they be free of mass singularities. Other than the existence of these Møller operators, the properties that all such asymptotic Hamiltonians $H_A(\Delta)$ share have been only rather vaguely characterized. Their dependence on certain parameters Δ other than those in H seems to be one such property. The choices that have been made thus far [26,27] suggest that, in general, H_A is not Lorentz invariant and will not take the form of a spatial integral of a Hamiltonian density involving products of fields at the same time and place. Rather, H_A will involve fields at different times and positions, with the parameters Δ specifying the phase-space volume involved in this smearing. This does not mean that the actual theory becomes either non-Lorentz invariant or noncausal. In fact, observables in a certain sense to be explained in Sec. IV are independent of the choice of H_A. However, if one identifies Δ with experimental resolutions, certain simplifications occur. Just as there does not exist an observable cross section free of resolutions, we do not expect to be able to find an asymptotic Hamiltonian H_A (other than H itself) that does not depend on additional parameters Δ reflecting some such nonlocality.[15] As will be seen in the next section, however, the fields themselves can be taken to interact locally, so that vertices conserve energy and momentum, albeit over a restricted phase-space volume.

As a technical point, in principle, H_A may be allowed to depend explicitly on the time t, even in the Schrödinger picture. (Indeed, this may be useful, as was shown for potential theory in Refs. [10] and [19] and for field theory in Ref. [19].) In that case the evolution operator is not simply e^{-iH_At}, but rather $U_A(t)$, satisfying

$$i\frac{d}{dt}U_A(t)=H_A(t)U_A(t) \ . \tag{2.3}$$

However, in order to emphasize the formal analogy between the massive and massless cases, and because in practice we have chosen H_A to be time independent[16] (in the Schrödinger picture) as in Ref. [26], we shall assume in the following that H_A does not in fact depend explicitly on time. While this allows us to maintain conservation of energy as well as a certain economy of notation, we emphasize that few of the concepts below depend on this assumption. If e^{-iH_At} were replaced everywhere in the following by $U_A(t)$, the results would go through essentially unchanged.

To return to the discussion of asymptotic convergence, our assumption is that one can find an asymptotic Hamiltonian H_A and associated states $|\psi_A^{\pm}\rangle$, so that, for any (normalizable) state $|\psi\rangle$, its time evolution $e^{-iHt}|\psi\rangle$ approaches $e^{-iH_At}|\psi^{\pm}\rangle^A$ as $t\to\mp\infty$. More precisely, we assume the existence as (proper) unitary operators in $\mathcal{H}_F$ of the Møller wave operators[17]

$$\Omega^{(\pm)}_{H,H_A}=\lim_{t\to\mp\infty}\Omega_{H,H_A}(t) \ , \tag{2.4}$$

where

$$\Omega_{H,H_A}(t)\equiv e^{iHt}e^{-iH_At} \ . \tag{2.5}$$

The utility of these Møller operators is that they provide the correspondence between an arbitrary state $|\psi\rangle$ and the associated in and out states[18] $|\psi^{\pm}\rangle^A$, viz., $|\psi\rangle=\Omega^{(\pm)}_{H,H_A}|\psi^{\pm}\rangle^A$. Using the fact that $H\Omega^{(\pm)}_{H,H_A}=\Omega^{(\pm)}_{H,H_A}H_A$, it can be seen that, if $|n;E\rangle^H$ is an eigenstate of H with eigenvalue E, then the corresponding $|\psi_n^{\pm}(E)\rangle^A$ are the stationary-state scattering solutions of energy E associated with H_A,[19] i.e.,

$$|n;E\rangle^H=\Omega^{(\pm)}_{H,H_A}|\psi_n^{\pm}(E)\rangle^A \ . \tag{2.6}$$

The corresponding S matrix

$$S_A\equiv\Omega^{(-)\dagger}_{H,H_A}\Omega^{(+)}_{H,H_A} \tag{2.7}$$

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[15]We shall argue in Sec. V that this reflects the approximation of measurements over finite, long times and finite, large distances by an S matrix.

[16]See Eq. (3.8) below for the specific case of QED. As discussed above, $H_A=H_A(\Delta)$ also depends inextricably on certain parameters Δ, which will remain for the most part suppressed in this section, but explicitly resurrected in the next.

[17]To be mathematically precise, we will assume strong convergence to the limit.

[18]Lest there be any confusion with non-Fock asymptotic states employed in the literature [14,19], we emphasize once again that our physical Hilbert space $\mathcal{H}_F$ is a conventional Fock space, and these asymptotic states necessarily are in the same $\mathcal{H}_F$ since $\Omega^{(\pm)}_{H,H_A}$ are unitary operators within this space. Our construction differs markedly from Refs. [14] and [19], for whom the asymptotic states are non-Fock, but it coincides with that of Ref. [17] in the Pauli-Fierz model.

[19]The index n denotes any other quantum numbers, such as the momentum, necessary to specify the states. Here we are ignoring the possibility of bound states of H or H_A. If they exist, then E must be restricted to the continuous spectrum of H.

is to be associated with transitions between in states $|\phi^+\rangle^A$ and out states $|\psi^-\rangle^A$:

$$^A\langle\phi^+|\psi^-\rangle^A \equiv {}^A\langle\phi^+|S_A|\psi^+\rangle^A \ . \tag{2.8}$$

Inasmuch as the $\Omega^{(\pm)}_{H,H_A}$ exist, it follows that this S matrix S_A is a *proper* unitary operator. More physically stated, this S matrix has matrix elements in Fock space $\mathcal{H}_F$ that are free of any mass singularities. Note that S_A commutes with H_A and thus conserves energy.

Mathematically, the freedom of choice of asymptotic Hamiltonian H_A may be specified by the requirement that the operators $\Omega^{(\pm)}_{H,H_A}$ exist as unitary operators. This is clearly an equivalence relation: Two asymptotic Hamiltonians H_A and H'_A which are "asymptotically equivalent to H" in this sense, are clearly asymptotically equivalent to each other; i.e., $\Omega^{(\pm)}_{H_A,H'_A}$ also exists. This follows from[20]

$$\Omega^{(\pm)}_{H_A,H'_A} = \Omega^{(\pm)\dagger}_{H,H_A}\Omega^{(\pm)}_{H,H'_A} \ . \tag{2.9}$$

One may therefore meaningfully speak of the equivalence class of asymptotic Hamiltonians. (This does not mean that certain choices are not more convenient than others for a given set of experimental conditions.) Note that, according to this definition, the full Hamiltonian H is included in this class. Not all members of the equivalence class are relevant to a particular measurement, a point we shall elaborate on in Sec. IV.

In summary, once one recognizes that the origin of mass singularities in the usual S matrix $S_{\text{FD}}[H]$ stems from the false assumption[21] that the asymptotic states are to be identified with the free-particle eigenstates associated with H_0, one may proceed to define an S matrix S_A in a precisely analogous fashion. There is no need to modify any of the common methods of quantum field theory by which massive field theories are constructed, including the identification of the physical Hilbert space with Fock space. There are no changes in the principles underlying the field theory, only in the determination of the asymptotic states and the S matrix.[22]

As we hope the preceding discussion illustrates, the consideration of the strictly massless theory is actually quite helpful for understanding clearly what is going on. While this is all well and good, can one actually calculate anything? In particular, what is the connection with the usual perturbative solution of gauge-field theory, which is formulated in an interaction picture defined by H_0? Needless to say, this can be treacherous in the present context. The situation is very much like the manipulation of bare quantities in a renormalizable field theory, where because of Haag's theorem [40] the transformation to the interaction picture does not really exist.[23] If one is very careful, one will not make a mistake working with bare quantities, but the manipulations can only be justified by introducing an ultraviolet cutoff to render formally divergent quantities finite and carefully studying the limit as the cutoff is removed. Similarly, in order to keep things under control, it is wise to introduce at this point a regulator for the mass singularities.

Let us begin with a discussion of the options for regulating mass singularities. For QED one has the option of giving an electron an explicit mass, and the photon can be given a mass in a gauge-invariant manner as well. Even if it is compatible with gauge invariance, adding masses may change the global symmetries, such as chiral symmetry, of the theory, as we discussed in Sec. I. Moreover, giving masses to massless particles will not be so straightforward for a general gauge theory, as a bare fermion mass may be prohibited by gauge symmetries (as in the standard model), and one can never assign a bare mass to a non-Abelian gauge boson without breaking gauge invariance. Therefore, to give mass to some massless particles generally may require the introduction of some additional fictitious fields, such as Higgs fields and scalar potentials. At best, this is quite cumbersome, and one may wish to entertain other infrared regulator methods, such as dimensional regularization of infrared divergences [6,7]. Unlike other regulators, dimensional regularization leaves the form of the Hamiltonians unchanged, but modifies the evaluation of matrix elements by imagining the theory is in $4+\epsilon$ dimensions.[24] Regardless of the particular regulator employed, for economy of notation, we will denote the corresponding regulated Hamiltonians as $H^{(\epsilon)}$, $H^{(\epsilon)}_A$, and $H^{(\epsilon)}_0$, and we generically will refer to the process of removing the cutoff as $\epsilon \to 0$. We will assume that these Hamiltonians, or at least their Fock-space matrix elements, formally reduce to the correct H, H_A, and H_0 that we envisioned previously for the massless theory. It is hard to be more precise without considering a particular regulator, especially inasmuch as the regulated theories may involve additional degrees of freedom than the massless theories.[25]

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[20] Technically, this requires strong convergence of the limit in Eq. (2.4), or, for time-dependent $H_A(t)$, strong convergence of the corresponding $U_A(t)$.

[21] So far as we know, that this was the origin of the "infrared catastrophe" was first identified by Friedrichs [39].

[22] While this was also the spirit of the work by Kibble [14], he, in contrast, chose to modify the Hilbert space in which physics takes place.

[23] In fact, one can show this is mathematically precisely the same as the nonexistence of the $\Omega^{(\pm)}_{H,H_0}$ because of mass singularities [39].

[24] If dimensional regularization is also used for the ultraviolet divergences, it is important to choose the ultraviolet counterterms the same as in the massive theory. (This is necessary to define the integral $\int d^n k\, k^{-4}$, which is ill defined for all n.) That is, one must first imagine working in $4-\epsilon$ dimensions before continuing to $4+\epsilon$ dimensions.

[25] Working to lowest order in the radiative corrections, it was shown by explicit calculation in Ref. [9] that dimensional regularization and regularization by masses lead to the same observable cross section for scattering of an electron in an external field. Although this was done using the cross-section method, we will present general arguments later in this section and verify explicitly in subsequent sections that the S-matrix and cross-section methods give identical results.

Whatever the device, to be successful the regulator must correspond to a way of cutting off the long-range or collinear interactions responsible for the divergences in the usual formalism. In so doing the modified theory restores the free Hamiltonian $H_0^{(\epsilon)}$ to the same asymptotic convergence class as the true Hamiltonian $H^{(\epsilon)}$ and allows implementation of the standard formalism. It would seem that the entire preceding formal discussion might be regarded as irrelevant except that, of course, Fock-space matrix elements of the Feynman-Dyson S matrix $S_{\rm FD}[H]^{(\epsilon)}$ for the regulated theory will diverge when the cutoff ϵ is removed. In this respect the theory with a regulator of the mass singularities is *more* confusing than the massless theory, which is presumably how the "infrared catastrophe" earned its name. Traditionally, one adopts the cross-section method to make sensible predictions and to remove the cutoff. The preceeding discussion strongly suggests that these singularities are not indications of difficulties intrinsic to the theory, but are simply a reflection of the incorrect initial assumption that H_0 was in the same convergence class as H in the massless limit. The purpose of introducing regulators of mass singularities is simply to be able to make contact with the usual interaction representation. Having done that, we want to understand how to recover the sensible, finite S-matrix elements that have been discussed previously abstractly. We also wish to explore the relationship of the choice of asymptotic Hamiltonian to observable quantities and to understand why the cross-section method yields the same answer for observables as the more rigorously justifiable S-matrix method.

So let us imagine repeating the earlier discussion for the regulated theory, defining asymptotic states, transition amplitudes, etc. We assume that $H^{(\epsilon)}$, $H_A^{(\epsilon)}$, and $H_0^{(\epsilon)}$ are all properly chosen, by which we mean that the spectra of $H_0^{(\epsilon)}$ and the scattering states of $H^{(\epsilon)}$ (and $H_A^{(\epsilon)}$) coincide. An important feature of the regulated theory is that the Møller operators

$$\Omega^{(\pm)}_{H^{(\epsilon)},H_0^{(\epsilon)}}=\lim_{t\to\mp\infty}\Omega_{H^{(\epsilon)},H_0^{(\epsilon)}}(t) \qquad (2.10)$$

and

$$\Omega^{(\pm)}_{H_A^{(\epsilon)},H_0^{(\epsilon)}}=\lim_{t\to\mp\infty}\Omega_{H_A^{(\epsilon)},H_0^{(\epsilon)}}(t) \qquad (2.11)$$

now exist as proper unitary operators.[26] As a special case of the equivalence expressed by Eq. (2.9), we have

$$\Omega^{(\pm)}_{H^{(\epsilon)},H_A^{(\epsilon)}}=\Omega^{(\pm)}_{H^{(\epsilon)},H_0^{(\epsilon)}}\Omega^{(\pm)\dagger}_{H_A^{(\epsilon)},H_0^{(\epsilon)}}\ , \qquad (2.12)$$

so that the S matrix [Eq. (2.7)] may be written as

$$S_A^{(\epsilon)}=\Omega^{(-)}_{H_A^{(\epsilon)},H_0^{(\epsilon)}}S_{\rm FD}[H]^{(\epsilon)}\Omega^{(+)\dagger}_{H_A^{(\epsilon)},H_0^{(\epsilon)}}\ , \qquad (2.13)$$

where $S_{\rm FD}[H]^{(\epsilon)}$ is the usual Feynman-Dyson S matrix:

$$S_{\rm FD}[H]^{(\epsilon)}\equiv\Omega^{(-)\dagger}_{H^{(\epsilon)},H_0^{(\epsilon)}}\Omega^{(+)}_{H^{(\epsilon)},H_0^{(\epsilon)}}\ . \qquad (2.14)$$

The relation Eq. (2.13) is frequently rewritten in another form. If $|\psi\rangle$ is any Fock state, then

$$|\psi;\pm\rangle^{A,(\epsilon)}_{\rm coh}\equiv\Omega^{(\pm)\dagger}_{H_A^{(\epsilon)},H_0^{(\epsilon)}}|\psi\rangle \qquad (2.15)$$

defines the associated *coherent states* $|\psi;\pm\rangle^{A,(\epsilon)}_{\rm coh}$. Then, by definition, the Fock-space matrix elements of $S_A^{(\epsilon)}$ may be written as

$$\langle\phi|S_A^{(\epsilon)}|\psi\rangle={}^{A,(\epsilon)}_{\rm coh}\langle\phi;-|S_{\rm FD}[H^{(\epsilon)}]|\psi;+\rangle^{A,(\epsilon)}_{\rm coh}\ , \qquad (2.16)$$

for any two Fock states $|\phi\rangle,|\psi\rangle$. In other words, the Fock-space matrix elements of $S_A^{(\epsilon)}$ may be thought of as the matrix elements of the usual Feynman-Dyson S matrix $S_{\rm FD}[H^{(\epsilon)}]$ between coherent states.

Note that the coherent states relevant to the evaluation of the scattering matrix involves $\Omega^{(\pm)\dagger}_{H_A^{(\epsilon)},H_0^{(\epsilon)}}$. It will be important for the physical interpretation to recognize that these operators are the *inverses* of the operators mapping energy states of $H_0^{(\epsilon)}$ to asymptotic energy states $|\psi_n^{\pm}(E)\rangle^{A,(\epsilon)}$. To see this let $|E,n\rangle^{(\epsilon)}$ denote the usual energy eigenstates associated with $H_0^{(\epsilon)}$. Then, using $H_A^{(\epsilon)}\Omega^{(\pm)}_{H_A^{(\epsilon)},H_0^{(\epsilon)}}=\Omega^{(\pm)}_{H_A^{(\epsilon)},H_0^{(\epsilon)}}H_0^{(\epsilon)}$, their relation to the energy states associated with $H_A^{(\epsilon)}$ is

$$|\psi_n^{\pm}(E)\rangle^{A(\epsilon)}=\Omega^{(\pm)}_{H_A^{(\epsilon)},H_0^{(\epsilon)}}|E,n\rangle^{(\epsilon)}\ . \qquad (2.17)$$

These are the stationary states of $H_A^{(\epsilon)}$, which form a basis for the asymptotic scattering states. However, for finite ϵ, this simply corresponds to a unitary change of basis in Fock space $\mathcal{H}_F^{(\epsilon)}$. For a finite regulator, therefore, the two bases provide equivalent physical descriptions. Consider, as is frequently done in practice [27,26,3,33], evaluating the matrix elements of $S_A^{(\epsilon)}$ between these energy states $|E,n\rangle^{(\epsilon)}$. According to Eq. (2.16), the associated coherent states between which $S_{\rm FD}[H^{(\epsilon)}]$ is to be evaluated are given by

$$|E,n;\pm\rangle^{A,(\epsilon)}_{\rm coh}=\Omega^{(\pm)\dagger}_{H_A^{(\epsilon)},H_0^{(\epsilon)}}|E,n\rangle^{(\epsilon)}\ . \qquad (2.18)$$

Although for finite ϵ this may be regarded as another unitary transformation in $\mathcal{H}_F^{(\epsilon)}$, these coherent states are *not* to be thought of as asymptotic states or as eigenstates of $H_A^{(\epsilon)}$.[27]

Equation (2.13), or, equivalently, Eqs. (2.16) and (2.15), are useful for making the connection to ordinary perturbation theory, and we shall elaborate on this below. First, however, we must face the issue of these various re-

[26]This statement illustrates the aforementioned lack of commutivity of the limits $\epsilon\to 0$ and $t\to\infty$.

[27]Readers familiar with Ref. [19] may be mystified by this statement, since the point of that paper seemed to be that the relevant states were the asymptotic states. Toward the end of Appendix A, we elaborate where we believe that reference has drawn erroneous conclusions and explain why people who evaluate $S_A^{(\epsilon)}$ between Fock states are calculating correctly. We confess to being puzzled how that prescription is to be inferred from Ref. [19], who in their Sec. V, do the opposite.

gulated quantities in the limit that the regulator is removed, $\epsilon \to 0$. The first question deals with why we have labeled our Fock space for the regulated theory with an ϵ. In general, certainly in a gauge theory, the process of regulating may change the Hilbert space or at least the physical subspace. For example, regulating the infrared divergence by adding a photon mass μ introduces a longitudinal photon as a physical degree of freedom. Even if a covariant quantization procedure had been used in the massless theory, so that the $\mathcal{H}_F$ includes an unphysical longitudinal photon, it moves from the unphysical into the physical subspace when the photon mass becomes nonzero. In the limit that $\mu \to 0$, one must demonstrate that it becomes unphysical again. This is not at all trivial; it is a simple calculation to show that the bremsstrahlung by a massless electron of a collinear, longitudinal photon does not vanish in the limit that $\mu \to 0$. More abstractly stated, certain matrix elements of singular operators such as $S_{\rm FD}[H]$ may remain finite in the limit that the regulator is removed,[28] but that does not imply they are observable. Moreover, not all the various states of the Fock space $\mathcal{H}_F^{(\epsilon)}$ defined for $\epsilon \neq 0$ remain Fock states in the limit $\epsilon \to 0$. For example, consider the relation between the asymptotic states and the eigenstates of $H_0^{(\epsilon)}$ in Eq. (2.17) or the relation between a given Fock state and its associated coherent state given in Eq. (2.15). In each case states are related via operators $\Omega^{(\pm)}_{H_A^{(\epsilon)},H_0^{(\epsilon)}}$ that are singular as $\epsilon \to 0$. As a result, it cannot remain true that the states so related both remain in Fock space in the limit; indeed, that is the inspiration for the discussion of von Neumann space $\mathcal{H}_{\rm vN}$ and the introduction of coherent states in so many of the references that we have cited. Thus not all the states lying in $\mathcal{H}_F^{(\epsilon)}$ tend to states in our original $\mathcal{H}_F$. Intuitively, it is clear that, in the limit $\epsilon \to 0$, we want the asymptotic states of the regulated theory $|\psi_n^{\pm}(E)\rangle^{A,(\epsilon)}$ to go over smoothly to the asymptotic states $|\psi_n^{\pm}(E)\rangle^{A}$ of the massless theory defined in Eq. (2.6). Inasmuch as the latter lie in the Fock space $\mathcal{H}_F$, we want the former to remain Fock states in the limit. This implies that the eigenstates of H_0, the states $|E,n\rangle^{(\epsilon)}$, do not tend to Fock states in the limit $\epsilon \to 0$. This is perfectly possible to arrange, but it is worth noting that it is the *reverse* of what is commonly done, e.g., in Refs. [14] and [19]. For us, the coherent state subspace that is unitarily inequivalent to Fock space is not the physical space, but the unphysical subspace usually associated with the Hilbert space $\mathcal{H}_0$ built up from H_0 (see Appendix A for further discussion).

From our discussion of asymptotic convergence at the outset, we know that certain of these quantities, in particular, the operators $\Omega^{(\pm)}_{H_A^{(\epsilon)},H_0^{(\epsilon)}}$, will have mass singularities in the limit. On the other hand, our assumption about the existence of H_A suggests that other quantities not involving H_0, such as $\Omega^{(\pm)}_{H^{(\epsilon)},H_A^{(\epsilon)}}$ and $S_A^{(\epsilon)}$, should be smooth in the limit $\epsilon \to 0$. This is our basic continuity assumption: *Operators that are well defined for the massless theory are approached smoothly in the limit* $\epsilon \to 0$. More precisely, if $|\psi\rangle$ is an arbitrary Fock state (independent of ϵ,) then we assume that[29]

$$\Omega^{(\pm)}_{H^{(\epsilon)},H_A^{(\epsilon)}}|\psi\rangle \to \Omega^{(\pm)}_{H,H_A}|\psi\rangle \ . \tag{2.19}$$

One might well worry that our continuity assumption on $S_A^{(\epsilon)}$ is insufficient, for it may be that we are interested in matrix elements of $S_A^{(\epsilon)}$ between states that depend on ϵ, such as the asymptotic energy states $|\psi_n^{\pm}(E)\rangle^{A,(\epsilon)}$. As stated, our convergence assumption of course does not imply that matrix elements of $S_A^{(\epsilon)}$ between states that could vary with (ϵ) converge; indeed, it can be easily seen that in general they may not. To clarify this situation requires a better understanding of the structure of the Hilbert space $\mathcal{H}_F^{(\epsilon)}$ and the way it splits up in the limit $\epsilon \to 0$. For finite ϵ the sets of states $\{|E,n\rangle^{(\epsilon)}\}$, $\{|\psi_n^{+}(E)\rangle^{A,(\epsilon)}\}$ (or $\{|\psi_n^{-}(E)\rangle^{A,(\epsilon)}\}$) each form a basis for $\mathcal{H}_F^{(\epsilon)}$. However, in the limit $\epsilon \to 0$, they become orthogonal to one another. Because the operators $\Omega^{(\pm)}_{H_A^{(\epsilon)},H_0^{(\epsilon)}}$ are singular, the asymptotic states [Eq. (2.17)], the coherent states [Eq. (2.15)], and the eigenstates of H_0 are, in the limit $\epsilon \to 0$, no longer simply related by unitary transformations in Fock space. This is normally [17,14,19] the point of departure for a discussion of the properties of von Neumann space $\mathcal{H}_{\rm vN}$ and the relation between its various unitarily inequivalent subspaces. Previous authors, with the notable exception of Blanchard [17], have regarded the Fock subspace $\mathcal{H}_F$ as unphysical and identified the physical subspace of asymptotic states as a space of coherent states unitarily inequivalent to it. However, our point of view is apparently different, as we constructed the massless theory on Fock space $\mathcal{H}_F$ from the outset, and the natural thing to do is to regard this as the physical space in the limit that $\epsilon \to 0$. Therefore, if the states between which $S_A^{(\epsilon)}$ is evaluated depend on ϵ, then, in order to obtain a finite limit in general, those states must converge to Fock states in the limit $\epsilon \to 0$. Thus, for example, so long as one adopts a procedure by which

$$\lim_{\epsilon \to 0} |\psi_n^{\pm}(E)\rangle^{A,(\epsilon)} = |\psi_n^{\pm}(E)\rangle^{A} \tag{2.20}$$

(as a well-defined limit in Fock space), then the matrix elements of $S_A^{(\epsilon)}$ will tend to the corresponding matrix elements of S_A in $\mathcal{H}_F$ for the massless theory.

[28] This is really the origin of the paradoxical "anomaly" claimed to have been found in Ref. [23]. In another paper [25] we shall deal with matters such as these more explicitly.

[29] We will assume strong convergence, although probably all that is really necessary is that $S_A^{(\epsilon)} \to S_A$ weakly. However, strong convergence of the Møller operators assures convergence of their products, such as occurs in $S_A^{(\epsilon)}$. Convergence should in principle be a derivable property, but we do not know whether it is really possible with present knowledge about field theory. One could attempt to prove it to any finite order in perturbation theory, but we have not tried. In cases that have been explored, such as the one discussed in the next section, it works.

If one is willing to assume that the thing to do is to evaluate S_A in Fock space, then one need not be concerned with the subtleties of what is going on outside of Fock space in the larger von Neumann space $\mathcal{H}_{\rm vN}$ in which $\mathcal{H}_F$ is embedded. In this larger space, previously ill-defined operators such as $\Omega^{(\pm)}_{H_A,H_0}$ in fact can be interpreted once again as a certain kind of unitary operator, an isometry between different subspaces. While it is conceptually important to understand the structure of the massless theory, in actual applications, it is irrelevant as people simply calculate matrix elements of S_A between Fock states of the regulated theory. So, in order not to get mired down in mathematical issues, we relegate to Appendix A this discussion and the connection with these opposing viewpoints.

This construction allows one to understand the equivalence between the cross-section and S-matrix methods and to resolve the quandary over the summation over degenerate initial states. For the regulated theory, these two approaches may be regarded as simply having made two different choices for the asymptotic Hamiltonians $H_A^{(\epsilon)}$ and $H_0^{(\epsilon)}$. Either choice is acceptable (in the sense of asymptotic convergence with respect to H) *so long as the mass singularities have been regulated.* The cross-section method proceeds from the choice $H_0^{(\epsilon)}$, while the S-matrix method, as shown, proceeds from $H_A^{(\epsilon)}$. Because they simply amount to a unitary change of basis, they must lead to the same observables, and therefore physical cross sections will be free of mass singularities in the limit $\epsilon\to 0$. However, the one choice $H_A^{(\epsilon)}$ leads to an S matrix $S_A^{(\epsilon)}$ that has no mass singularities in the limit $\epsilon\to 0$, while the other $H_0^{(\epsilon)}$ leads to an S matrix $S_{\rm FD}[H]^{(\epsilon)}$ that is singular in the limit $\epsilon\to 0$. It is the former that is justified by the scattering theory for the unregulated Hamiltonian, and so the cross-section method must conform thereto. In particular, it is clear that to avoid singularities in general, *the cross-section method* must *be defined to include summation over degenerate initial states.*[30] The Bloch-Nordsieck result, proved in Ref. [5] and reproduced in Sec. IV below, is that, for infrared singularities in an Abelian gauge theory, it suffices to sum only over degenerate final states. This is never true for collinear singularities and is also not true for infrared singularities in the non-Abelian theories in general [31,32].

We want to emphasize that the only reason for introducing a regulator was to make contact with the usual Feynman-Dyson S matrix, which has been given in Eq. (2.13), and to be able to evaluate quantities using the usual Feynman rules in the interaction picture, the connection to which we will now review. The standard interaction picture involves a transformation of operators and states from the Schrödinger picture of the form $\mathcal{O}_{\mathcal{S}}\to\mathcal{O}(t)\equiv e^{iH_0^{(\epsilon)}t}\mathcal{O}_{\mathcal{S}}e^{-iH_0^{(\epsilon)}t}$ and $|\psi(t)\rangle_{\mathcal{S}}\to|\tilde{\psi}(t)\rangle \equiv e^{iH_0^{(\epsilon)}t}|\psi(t)\rangle_{\mathcal{S}}$. It is a simple exercise to show that[31]

$$\Omega^{(\pm)}_{H_A^{(\epsilon)},H_0^{(\epsilon)}}=T\exp\left[-i\int_{\mp\infty}^{0}dt\,\widetilde{V_A^{(\epsilon)}(t)}\right], \qquad (2.21)$$

where the "asymptotic" interaction is given by $V_A^{(\epsilon)}\equiv H_A^{(\epsilon)}-H_0^{(\epsilon)}$ and the tilde reminds us that $V_A^{(\epsilon)}$ is to be expressed in the interaction picture. Of course, the Feynman-Dyson S matrix is, as usual,

$$S_{\rm FD}[H]^{(\epsilon)}=T\exp\left[-i\int_{-\infty}^{+\infty}dt\,\widetilde{V_I^{(\epsilon)}(t)}\right], \qquad (2.22)$$

where $V_I^{(\epsilon)}\equiv H^{(\epsilon)}-H_0^{(\epsilon)}$ is the full interaction Hamiltonian, which, in Eq. (2.22), has been transformed to the interaction picture. Thus, to evaluate Fock-space matrix elements of Eq. (2.13) in perturbation theory, one calculates $S_{\rm FD}[H]^{(\epsilon)}$ following the usual Feynman rules, while evaluating $\Omega^{(\pm)}_{H_A^{(\epsilon)},H_0^{(\epsilon)}}$ using the Feynman rules associated with vertices appropriate to the asymptotic Hamiltonian $H_A^{(\epsilon)}$.

In summary, Eq. (2.13), or, equivalently, Eqs. (2.15) and (2.16), form the starting point of all coherent-state calculations. The previous discussion has suggested that these Fock-space matrix elements smoothly go over to the corresponding matrix elements of the massless theory in the limit that the regulator is removed, and this we shall show by explicit calculation for the radiative corrections to electron scattering in an external field in the following section. Although we have shown this for specific regulators, we believe the result is quite general. In fact, it seems natural to require this of regulators, and if it were not the case for some regulator, one might be justified in labeling it as pathological. This is analogous to requiring the regulator be gauge invariant in renormalization theory; otherwise, one has to add additional counterterms so that the renormalized Green's functions satisfy the Ward-Takahashi identities required for gauge invariance. It remains to flesh out this abstract formalism in a particular example and to discuss how these matrix elements are to be related to observable cross sections. This we do for massless QED in the succeeding sections.

III. QED RADIATIVE CORRECTIONS TO ELECTRON SCATTERING

In this section we will describe the situation for massless QED, where it is possible to assign gauge-invariant bare masses to the electron and photon and thereby choose regulators quite simple and physical. However, we expect the following discussion to go through for more general regulators. We shall employ the conventional "coherent-state" method described in the preceding section.

First, we establish some notational conventions. Having indicated toward the end of the preceding section the connection with the interaction picture, in what follows, all the fields will be assumed to be given in that picture,

[30]In this regard we must register our sharp disagreement with the discussion of this point in Sec. 26 of Ref. [20].

[31]See, e.g., Ref. [38].

and we shall drop the tilde. In addition, throughout the calculations, we shall be working exclusively with the *regulated* theory in which both the electron and photon have finite masses m_e and m_γ, respectively.[32] So we shall suppress the superscript (ϵ) that we used in the preceding section to distinguish the regulated from the unregulated cases. We shall indicate explicitly whenever we wish to consider the limit in which the photon and electron masses vanish.

The interaction Hamiltonian will be written as

$$V_I(t)=V_I^{(\mathrm{QED})}(t)+V_I^{(J)}(t)\ , \tag{3.1}$$

where the usual QED interaction is

$$V_I^{(\mathrm{QED})}(t)=e\int d^3x\,:\overline{\Psi}^{(e)}(x)\gamma^\mu\Psi^{(e)}(x):A_\mu(x)\ , \tag{3.2}$$

and $V_I^{(J)}$ represents any remaining interactions, for example, the rest of the standard-model interaction Hamiltonian or, simply, an external source. The point of this separation is that we will be discussing electromagnetic radiative corrections, but will work only to first order in $V_I^{(J)}$. In this paper $V_I^{(J)}$ will simply be the interaction of an electron with a classical electromagnetic potential. We will only consider processes in which there is a finite momentum transfer exchanged with this potential, and so this will not give rise to any mass singularities. To define the asymptotic Hamiltonian H_A, it is useful to write this interaction in momentum space. Substituting the Fourier transform for the fields,[33] we find

$$V_I^{(\mathrm{QED})}(t)=e\int \widehat{d^3k_1}\widehat{d^3k_2}\sum_{l=1}^{8}h_l(\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3)\times\exp[-i(\mathcal{S}\omega)^l t]\ , \tag{3.3}$$

where $\mathcal{S}_j^l$ is the sign matrix,

$$\mathcal{S}=\begin{bmatrix}-1 & 1 & 1\\ -1 & -1 & 1\\ 1 & 1 & 1\\ 1 & -1 & 1\\ 1 & -1 & -1\\ 1 & 1 & -1\\ -1 & -1 & -1\\ -1 & 1 & -1\end{bmatrix} \tag{3.4}$$

and

$$\omega\equiv\begin{bmatrix}\omega_1\\ \omega_2\\ \omega_3\end{bmatrix}\equiv\begin{bmatrix}\omega(\mathbf{k}_1,m_e)\\ \omega(\mathbf{k}_2,m_e)\\ \omega(\mathbf{k}_3,m_\gamma)\end{bmatrix}\ . \tag{3.5}$$

Here $\{\mathbf{k}_j\}$ are the three-momenta entering the QED vertices $h^l(\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3)$ with the photon momentum $\mathbf{k}_3$ fixed by momentum conservation for each vertex l:

$$\mathbf{k}_3\equiv\mathbf{k}_3^l=-S_3^l(S_1^l\mathbf{k}_1+S_2^l\mathbf{k}_2)\ . \tag{3.6}$$

For completeness we list the eight vertices:

$$\begin{aligned}
h_1&=\sum_{\alpha_1\alpha_2\lambda}C^1 b_{\alpha_1}^\dagger(\mathbf{k}_1)b_{\alpha_2}(\mathbf{k}_2)a_\lambda(\mathbf{k}_3^1)\\
&\qquad\times\overline{U}^{\alpha_1}(\mathbf{k}_1)\not\epsilon_\lambda(\mathbf{k}_3^1)U^{\alpha_2}(\mathbf{k}_2)\ ,\\
h_2&=\sum_{\alpha_1\alpha_2\lambda}C^2 b_{\alpha_1}^\dagger(\mathbf{k}_1)d_{\alpha_2}^\dagger(\mathbf{k}_2)a_\lambda(\mathbf{k}_3^2)\\
&\qquad\times\overline{U}^{\alpha_1}(\mathbf{k}_1)\not\epsilon_\lambda(\mathbf{k}_3^2)V^{\alpha_2}(\mathbf{k}_2)\ ,\\
h_3&=\sum_{\alpha_1\alpha_2\lambda}C^3 d_{\alpha_1}(\mathbf{k}_1)b_{\alpha_2}(\mathbf{k}_2)a_\lambda(\mathbf{k}_3^3)\\
&\qquad\times\overline{V}^{\alpha_1}(\mathbf{k}_1)\not\epsilon_\lambda(\mathbf{k}_3^3)U^{\alpha_2}(\mathbf{k}_2)\ ,\\
h_4&=-\sum_{\alpha_1\alpha_2\lambda}C^4 d_{\alpha_2}^\dagger(\mathbf{k}_2)d_{\alpha_1}(\mathbf{k}_1)a_\lambda(\mathbf{k}_3^4)\\
&\qquad\times\overline{V}^{\alpha_1}(\mathbf{k}_1)\not\epsilon_\lambda(\mathbf{k}_3^4)V^{\alpha_2}(\mathbf{k}_2)\ ,\\
h_{4+l}&=h_l^\dagger(\mathbf{k}_3^l\to\mathbf{k}_3^{4+l}),\quad l=1,\dots,4\ ,
\end{aligned} \tag{3.7}$$

where we have defined $C^l\equiv[(2\pi)^3/2\omega(\mathbf{k}_3^l,m_\gamma)]^{1/2}$.

By H_0 we will of course mean the free-field theory (including mass terms for the photon and electron). We also need to identify an asymptotic Hamiltonian $H_A(\Delta)$, which, in the massless limit, contains the same mass singularities as the exact Hamiltonian H. This is the field-theory analogue of "keeping the long-range tail" of the Coulomb potential in quantum mechanics, and while there are many ways to specify H_A, we will follow the simple and intuitive method of Ref. [26]. The basic idea is to use the Lee-Nauenberg degeneracy criterion for on-mass-shell, massless particles [5]. This is particularly simple to implement in the interaction representation that we are using, since the energy associated with each virtual particle is the same as if it was on-mass shell. In time-ordered perturbation theory, the mass singularities arise from the vanishing of energy denominators obtained from the phase factors $(\mathcal{S}\omega)^l$ in Eq. (3.3). To be a good asymptotic Hamiltonian, H_A must reproduce these singularities precisely. This can be accomplished simply by defining the interaction term in $H_A(\Delta)$ to be

$$V_A^{(\mathrm{QED})}(\Delta;t)=e\int\widehat{d^3k_1}\widehat{d^3k_2}\sum_{l=1}^{8}\Theta_\Delta(\mathbf{k}_i)h_l(\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3)\times\exp[-i(\mathcal{S}\omega)^l t]\ . \tag{3.8}$$

Here we have simply inserted an unspecified function of the momentum $\Theta_\Delta(\mathbf{k}_i)$. Its purpose is to isolate the infrared and collinear contributions, and for these

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[32]In order to introduce a photon mass in a manifestly gauge-invariant way, one can introduce a fictitious scalar field in the manner of Stückelberg [41]. We imagine that we are working in the "unitary" gauge in which the scalar field has been decoupled. One can also simply insert a photon mass in the propagator and verify gauge invariance directly by showing that the $k_\mu k_\nu$ terms in the propagator make no contribution because the photon is coupled to a conserved current.

[33]For both bosons and fermions, the normalization of the Fourier transform has been taken to be $\widehat{d^3k}\equiv d^3k/[(2\pi)^3 2\omega(\mathbf{k},m)]^{1/2}$ with $\omega(\mathbf{k},m)=(\mathbf{k}^2+m^2)^{1/2}$.

configurations, it must reproduce $V_I^{(\mathrm{QED})}$ exactly. Thus, in the massless limit, the Θ function must be equal to one for the precisely collinear $\mathbf{k}_1 \propto \mathbf{k}_2$ or infrared $\omega_3=0$ configurations. An example of the sort of function we have in mind is $\Theta(\Delta_l - |(\mathcal{S}\omega)^l|)$, where $\Theta(x)$ denotes the usual step function and Δ_l are a set of arbitrary parameters. Of course, the full asymptotic Hamiltonian is taken to be

$$H_A(\Delta) \equiv H_0 + V_A \ . \tag{3.9}$$

We have shown in the preceding section that we may take Eq. (2.13) as the starting point for calculations of the S matrix for the regulated theory and that we may restrict our attention to its matrix elements in Fock space. The Fock-space matrix elements in the regulated theory will be calculated as usual in the interaction picture, using Eqs. (2.21) and (2.22). Consider the evaluation of the Møller operators

$$\Omega_{\pm}(\Delta) \equiv \Omega^{(\pm)}_{H_A, H_0} = T \exp\left[-i \int_{\mp\infty}^{0} V_A^{(\mathrm{QED})}(\Delta; t)\, dt \right] , \tag{3.10}$$

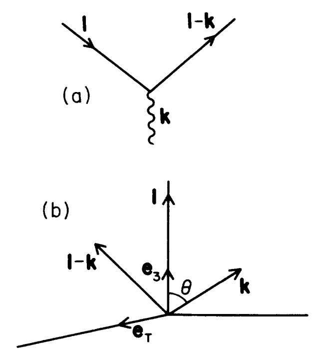

FIG. 1. Momentum assignment for the asymptotic kinematics.

where we have introduced an abbreviated notation for the Møller operators of interest to us here. Inserting the standard "adiabatic factor" $\exp(-\epsilon|t|)$ into the integrand to ensure formal convergence of the temporal integration, we obtain an expression for the nth-order term:

$$\Omega_{\pm}^{(n)}(\Delta) = \sum_{l_1 \cdots l_n} e^n V_{l_n} V_{l_{n-1}} \cdots V_{l_1} \frac{1}{(\mathcal{S}\omega)^{l_1} \pm i\epsilon} \, \frac{1}{(\mathcal{S}\omega)^{l_1} + (\mathcal{S}\omega)^{l_2} \pm 2i\epsilon} \cdots \frac{1}{(\mathcal{S}\omega)^{l_1} + (\mathcal{S}\omega)^{l_2} + \cdots + (\mathcal{S}\omega)^{l_n} \pm n i\epsilon} , \tag{3.11}$$

where the vertex V_{l_j} is defined as

$$V_{l_j} \equiv \int \widehat{d^3k}_{1_j} \widehat{d^3k}_{2_j} h_{l_j}(\mathbf{k}_{1_j}, \mathbf{k}_{2_j}, \mathbf{k}_{3_j}) \Theta_\Delta(\mathbf{k}_i) \ . \tag{3.12}$$

A. Asymptotic kinematics

At this point we introduce some kinematical notation; to be specific, consider a vertex with momentum assignments as in Fig. 1. We can write

$$\begin{aligned} \boldsymbol{l} &= (1-\chi_e^2)^{1/2} \omega(\boldsymbol{l}, m_e) \mathbf{e}_3 , \\ \boldsymbol{l} - \mathbf{k} &= (1-\chi_e^2)^{1/2} \omega(\boldsymbol{l}, m_e) \left\{ \left[1 - \cos\theta \left[\frac{z^2 - \chi_\gamma^2}{1-\chi_e^2} \right]^{1/2} \right] \mathbf{e}_3 - \sin\theta \left[\frac{z^2-\chi_\gamma^2}{1-\chi_e^2} \right]^{1/2} \mathbf{e}_T \right\} , \\ \mathbf{k} &= (z^2 - \chi_\gamma^2)^{1/2} \omega(\boldsymbol{l}, m_e)(\cos\theta \mathbf{e}_3 + \sin\theta \mathbf{e}_T) , \end{aligned} \tag{3.13}$$

where the mass fractions $\chi_{e/\gamma}$ and photon energy fraction z have been defined as

$$\chi_{e/\gamma} \equiv \frac{m_{e/\gamma}}{\omega(\boldsymbol{l}, m_e)}, \quad z \equiv \frac{\omega(\mathbf{k}, m_\gamma)}{\omega(\boldsymbol{l}, m_e)} \in [\chi_\gamma, 1-\chi_e] \ . \tag{3.14}$$

From the above expressions, we obtain

$$\omega(\boldsymbol{l} - \mathbf{k}, m_e) = \omega(\boldsymbol{l}, m_e) \{ 1 + z^2 - \chi_\gamma^2 - 2\cos\theta) [(z^2 - \chi_\gamma^2)(1-\chi_e^2)]^{1/2} \}^{1/2} \ . \tag{3.15}$$

For example, considering the vertex h_1 (which will be useful in the remainder of the paper), we have

$$\begin{aligned} v \equiv (\mathcal{S}\omega)^1 &= \omega(\boldsymbol{l} - \mathbf{k}, m_e) + \omega(\mathbf{k}, m_\gamma) - \omega(\boldsymbol{l}, m_e) \\ &= \omega(\boldsymbol{l}, m_e)(\{ 1 + z^2 - \chi_\gamma^2 - 2\cos\theta[(z^2-\chi_\gamma^2)(1-\chi_e^2)]^{1/2} \}^{1/2} + z - 1) \ . \end{aligned} \tag{3.16}$$

At this stage we will specify the structure of the domain that we previously characterized by the generic symbol Δ. As indicated previously, the kinematically singular regimes for the vertices are where the energy denominators in Eq. (3.11) may vanish in the limit that the cutoffs are removed ($m_{e/\gamma} \to 0$.) In this regard it is useful to distinguish the infrared regime, corresponding to emission of a soft photon,[34] from the collinear regimes. In general, the latter consist either of a hard photon converting to a nearly collinear electron-positron pair or of an electron or positron emitting a hard, but nearly collinear photon. In other words, the degeneracy-domain constraint that we wrote as $\Theta_\Delta(\mathbf{k}_i)$ can be specified to correspond to two complementary regimes.

(i) *Infrared (soft-photon) constraint.* This region corresponds to soft bremsstrahlung by an electron or positron and is associated with the vertices h^1, h^4 and their Hermitian conjugates h^5, h^8. For the case of interest here, scattering of an electron in an external field, the constraint may be described by the inequality

$$\chi_\gamma < z < \delta_E, \quad \theta \in [0, \pi] . \tag{3.17}$$

For the time being, we will associate a dimensionless parameter δ_E with the minimum detectable energy ΔE of an individual photon $\delta_E \equiv \Delta E / \omega(l, m_e)$. Subsequently, we shall show that δ_E may be chosen arbitrarily without changing observables, as we would expect on the basis of the general theory in the preceding section. We shall also later take up the question of whether or not, in a given experiment, the energy resolution differs between the initial state (beam) and final state (detector).

(ii) *Hard-collinear constraints.* In this category there are two different regions.

(a) Collinear photon. This region corresponds to the same vertices as the soft-photon constraint. For our situation it may be prescribed by the inequality

$$0 < \theta < \delta_\theta, \quad z \in [\delta_E, 1 - \chi_e] . \tag{3.18}$$

One may think of δ_θ as the angular resolution associated with a given measurement, but as with δ_E, observables will be independent of the choice. For a schematic view of the above regions, see Fig. 2 and Ref. [26].

(b) Collinear electron-positron pairs. This region is associated with the vertices h^2, h^3 and their conjugates h^6, h^7. To the lowest order in perturbation theory in which we are working, these do not occur and will be ignored for the remainder of this paper. In general, these vertices would be modified in a manner analogous to the preceding cases.

[34]The regime in which an electron or positron energy E becomes soft can be shown not to produce a singularity because the matrix element for that vanishes as $\sqrt{E}$. Ultimately, this difference between the massless photon and massless electron can be traced to the fact that a boson field has dimension 1, whereas a fermion field has dimension $\frac{3}{2}$.

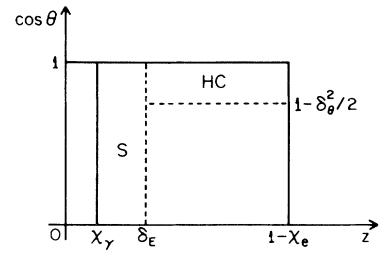

FIG. 2. Various asymptotic regions characterizing an idealized degeneracy domain: S denotes the soft region. HC denotes the hard-collinear region.

Thus, for the problem of interest, the scattering of an electron in an external field to lowest nontrivial order, we may think of specifying the asymptotic Hamiltonian by the replacement

$$\Theta_\Delta(\mathbf{k}_i) \to \Theta(\delta_E - z) + \Theta(z - \delta_E)\Theta(\delta_\theta - \theta) . \tag{3.19}$$

It is clear that the vertices here are of the standard form, but restricted to a small region of phase space where the photon is soft ($\omega_k \le \Delta E$) or where the photon is hard but nearly collinear ($\theta \le \delta_\theta$) with the electron or positron.

B. Calculation of *S*-matrix elements

Now we wish to proceed with the calculation of the lowest-order radiative corrections to electron scattering off an external source. Using the cross-section method, it has already been shown to have a smooth massless limit when the masses of the photon and electron are taken to zero [9]. Thus we already know that, following conventional methods, the properly defined observable cross section is independent of the regularization scheme applied to the mass singularities. (By properly defined we are including initial- as well as final-state degeneracy in our definition). Despite the general discussion in the preceding section, we feel it is important to verify explicitly using the *S*-matrix method that the same results are also obtained. This will give us increased confidence in the assumptions made in the abstract formalism and justify, at least in this case, the continuity assumption made in the formal treatment.

Let us determine the order e^2 corrections to the elastic amplitude. To this end one must evaluate S_A [Eq. (2.13)], using Eq. (3.11), between single electron states $|i\rangle = |e(l;i)\rangle$, $|f\rangle = |e(l';i')\rangle$. If we denote the nonradiative *S*-matrix element by $(S^{(0)})_{fi}$ (see Fig. 3) and the second-order matrix element by $(S_A^{(2)})_{fi}$, then the lowest-order radiative correction to the cross section will be

$$d\sigma^{(2)} \sim 2\,\mathrm{Re}[(S_A^{(2)})_{fi}(S^{(0)})^\dagger_{fi}] . \tag{3.20}$$

The second-order *S*-matrix element is

$$\begin{aligned}(S_A^{(2)})_{fi} = {} & \langle f|\Omega_-^{(0)}S_{\mathrm{FD}}^{(0)}\Omega_+^{(2)\dagger}|i\rangle + \langle f|\Omega_-^{(0)}S_{\mathrm{FD}}^{(1)}\Omega_+^{(1)\dagger}|i\rangle \\ & + \langle f|\Omega_-^{(2)}S_{\mathrm{FD}}^{(0)}\Omega_+^{(0)\dagger}|i\rangle + \langle f|\Omega_-^{(1)}S_{\mathrm{FD}}^{(1)}\Omega_+^{(0)\dagger}|i\rangle \\ & + \langle f|\Omega_-^{(1)}S_{\mathrm{FD}}^{(0)}\Omega_+^{(1)\dagger}|i\rangle + \langle f|\Omega_-^{(0)}S_{\mathrm{FD}}^{(2)}\Omega_+^{(0)\dagger}|i\rangle .\end{aligned} \tag{3.21}$$

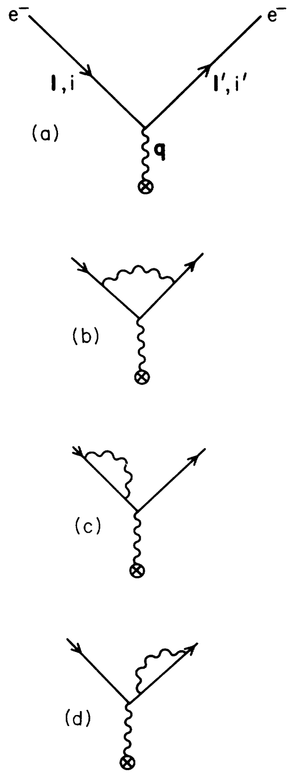

FIG. 3. Nonradiative scattering process and the virtual radiative corrections (only the singular ones are shown above).

The first two terms, involving contributions to the initial coherent state, will be denoted by $(S_A^{(2,0)})_{\rm in}$ and $(S_A^{(1,1)})_{\rm in}$, respectively. Similarly, the third and fourth terms, involving the final coherent-state contributions, will be denoted by $(S_A^{(2,0)})_f$ and $(S_A^{(1,1)})_f$, respectively. The fifth term, involving both initial and final coherent states, will be denoted by $(S_A^{(1,1)})_{\rm mix}$. Finally, the last term is the usual elastic vertex correction (since $\Omega_\pm^{(0)}=1$,) which we will write as $(S^{(0)})_{fi}\delta_V^{(2)}$. This contribution, shown in Figs. 3(b)–3(d), is totally standard and has been discussed previously [9] and will not be calculated again here. In the remainder of this section, we calculate the various other terms in Eq. (3.21).

1. Calculation of $(S_A^{(1,1)})_{in}$

We have

$$(S_A^{(1,1)})_{\rm in}=\langle e(l';i')|S_{\rm FD}^{(1)}\Omega_+^{(1)\dagger}|e(l;i)\rangle \ . \tag{3.22}$$

From Eqs. (3.11), (3.12), and the list of vertices in Eq. (3.7), we obtain the following first-order contribution to the coherent state:

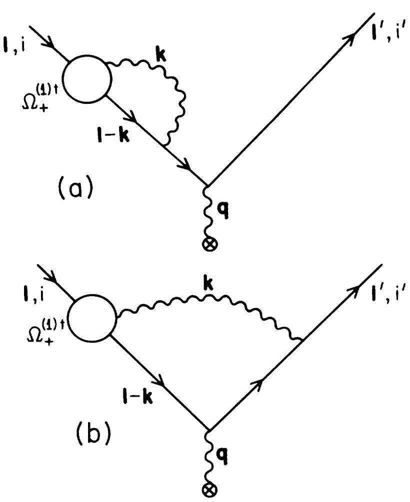

FIG. 4. First-order transformation of the initial electron state. All lines connected to the asymptotic wave operator (blob) are on mass shell. Note that the propagators are not on mass shell.

$$\begin{aligned}\Omega_+^{(1)\dagger}|e(l;i)\rangle = e\sum_{\alpha\lambda}\int \widetilde{d^3k}\,&|\gamma(\mathbf{k};\lambda)e(l-\mathbf{k};\alpha)\rangle\\ &\times \bar U^{\alpha}(l-\mathbf{k})\not\epsilon_\lambda^*(\mathbf{k})U^i(l)\\ &\times\frac{\Theta_\Delta(\mathbf{k}_i)}{2\omega(l-\mathbf{k},m_e)\nu}\ ,\end{aligned} \tag{3.23}$$

where $\Theta_\Delta(\mathbf{k}_i)$ is to be replaced as designated in Eq. (3.19). Here *and for the remainder of this paper,* we define $\widetilde{d^3k}\equiv d^3k/(2\pi)^3 2\omega(\mathbf{k},m_\gamma)$. Hence the initial single-particle state becomes an initial coherent state that can be expressed in terms of multiparticle states. The Feynman-Dyson S-matrix elements are calculated as usual with ordinary Feynman rules. The matrix element $(S_A^{(1,1)})_{\rm in}$ is depicted in Fig. 4. For the first graph [Fig. 4(a)], we can write the electronic properator as

$$\frac{N_F[\omega(\mathbf{k},m_\gamma)+\omega(l-\mathbf{k},m_e);l]}{D_F[\omega(\mathbf{k},m_\gamma)+\omega(l-\mathbf{k},m_e);l]}\ , \tag{3.24}$$

where

$$\begin{aligned}&N_F[a;\mathbf{b}]\equiv a\gamma_0-\mathbf{b}\cdot\gamma+m_e\ ,\\ &D_F[a;\mathbf{b}]\equiv a^2-\mathbf{b}^2-m_e^2\ .\end{aligned} \tag{3.25}$$

Therefore, it follows that

$$\begin{aligned}&N_F[\omega(\mathbf{k},m_\gamma)+\omega(l-\mathbf{k},m_e);l]=\not l+\nu\gamma_0+m_e\ ,\\ &D_F[\omega(\mathbf{k},m_\gamma)+\omega(l-\mathbf{k},m_e);l]=\nu[\nu+2\omega(l,m_e)]\ .\end{aligned} \tag{3.26}$$

Working similarly for the second graph [Fig. 4(b)], we obtain the S_A-matrix element:

$$\begin{aligned}(S_A^{(1,1)})_{\rm in}=-e^2\bar U^{i'}(l')\int\widetilde{d^3k}\,\Bigg\{&\not J[\Lambda(l)+\nu\gamma_0]\gamma_\mu\Lambda(l-\mathbf{k})\gamma^\mu\frac{\Theta_\Delta(\mathbf{k}_i)}{2\omega(l-\mathbf{k},m_e)\nu^2[\nu+2\omega(l,m_e)]}\\ &+\gamma_\mu[\Lambda(l-\mathbf{k})-\not q]\not J\Lambda(l-\mathbf{k})\gamma^\mu\frac{\Theta_\Delta(\mathbf{k}_i)}{2\omega(l-\mathbf{k},m_e)\nu\nu'[\nu'-2\omega(l'-\mathbf{k},m_e)]}\Bigg\}U^i(l)\ .\end{aligned} \tag{3.27}$$

In the expression above, J denotes the Fourier transform of the external current; $\Lambda(\mathbf{b})\equiv\omega(\mathbf{b},m_e)\gamma_0-\mathbf{b}\cdot\gamma+m_e$, the projector, and $\nu'\equiv\omega(\mathbf{l}'-\mathbf{k},m_e)+\omega(\mathbf{k},m_\gamma)-\omega(\mathbf{l}',m_e)$, the energy difference. (An overall energy-momentum δ function has been suppressed.) At this point we can evaluate the right-hand side of Eq. (3.27) by making the phase-space decomposition indicated by Eq. (3.19). Separating into soft and hard-collinear regions, we will define corresponding quantities $\delta_{\text{in};s}^{(1,1)}$ and $\delta_{\text{in;hc}}^{(1,1)}$ by

$$(S_A^{(1,1)})_{\text{in}}\equiv(S^{(0)})_{fi}[\delta_{\text{in};s}^{(1,1)}+\delta_{\text{in;hc}}^{(1,1)}]\ . \tag{3.28}$$

The explicit expressions for these follow.

(i) *Soft region.* Noting that $\nu\omega(\mathbf{l},m_e)\simeq\nu\omega(\mathbf{l}-\mathbf{k},m_e)\simeq(lk)$ and $\nu'\omega(\mathbf{l}'-\mathbf{k},m_e)\simeq(l'k)$, we can write

$$(S_A^{(1,1)})_{\text{in};s}=(S^{(0)})_{fi}\delta_{\text{in};s}^{(1,1)}\ , \tag{3.29}$$

with

$$(S^{(0)})_{fi}=\overline{U}^{\,i'}(\mathbf{l}')JU^{i}(\mathbf{l}) \tag{3.30}$$

and

$$\delta_{\text{in};s}^{(1,1)}=e^2\int\widetilde{d^3k}\left[\frac{-m_e^2}{(lk)^2}+\frac{(ll')}{(lk)(l'k)}\right]\Theta(\delta_E-z)\ . \tag{3.31}$$

(ii) *Hard-collinear region.* In this region we can similarly write

$$(S_A^{(1,1)})_{\text{in;hc}}=(S^{(0)})_{fi}\delta_{\text{in;hc}}^{(1,1)}\ , \tag{3.32}$$

with

$$\delta_{\text{in;hc}}^{(1,1)}=e^2\int\frac{\widetilde{d^3k}}{1-z}\left[\frac{-m_e^2}{(kl_1)^2}+\frac{1}{(kl_1)}\left[z+\frac{1}{z}-1\right]\right]\times\Theta(\delta_\theta-\theta)\Theta(z-\delta_E)\ , \tag{3.33}$$

where θ is the angle between the vectors $\mathbf{k}$ and $\mathbf{l}$. Here we used the kinematics of Sec. III A to simplify:

$$\begin{aligned}&2\omega(\mathbf{l}-\mathbf{k},m_e)\nu^2[\nu+2\omega(\mathbf{l},m_e)]\simeq4(1-z)(kl_1)^2\ ,\\&2\omega(\mathbf{l}-\mathbf{k},m_e)\nu\nu'[\nu'-2\omega(\mathbf{l}'-\mathbf{k},m_e)]\\&\qquad\simeq-4z(1-z)(kl_1)(ll')\ ,\end{aligned} \tag{3.34}$$

where the on-mass-shell four-vector l_1 is defined as $l_1\equiv(\omega(\mathbf{l}-\mathbf{k},m_e);\mathbf{l}-\mathbf{k})$.

2. Calculation of $(S_A^{(1,1)})_f$

Similarly, we can write

$$(S_A^{(1,1)})_f=(S^{(0)})_{fi}(\delta_{f;s}^{(1,1)}+\delta_{f;\text{hc}}^{(1,1)})\ , \tag{3.35}$$

with

$$\delta_{f;s}^{(1,1)}=e^2\int\widetilde{d^3k}\left[\frac{-m_e^2}{(l'k)^2}+\frac{(ll')}{(lk)(l'k)}\right]\Theta(\delta_E-z)\ , \tag{3.36}$$

$$\delta_{f;\text{hc}}^{(1,1)}=e^2\int\frac{\widetilde{d^3k}}{1-z}\left[\frac{-m_e^2}{(l_1'k)^2}+\frac{1}{(l_1'k)}\left[z+\frac{1}{z}-1\right]\right]\times\Theta(\delta_\theta-\theta')\Theta(z-\delta_E)\ . \tag{3.37}$$

Here θ' is the angle between the vectors $\mathbf{k}$ and $\mathbf{l}'$ and the four-vector l_1' is defined $l_1'\equiv(\omega(\mathbf{l}'-\mathbf{k},m_e);\mathbf{l}'-\mathbf{k})$. Note that $\omega(\mathbf{l},m_e)=\omega(\mathbf{l}',m_e)$, and hence the definition of z remains unchanged.

3. Calculation of $(S_A^{(1,1)})_{mix}$

Using the first-order expressions calculated previously for the soft bremsstrahlung contributions[35] to both initial and final coherent states, we may write

$$\begin{aligned}(S_A^{(1,1)})_{\text{mix}}&=-e^2\overline{U}^{\,i'}(\mathbf{l}')\int\widetilde{d^3k}\,\gamma_\mu\Lambda(\mathbf{l}'-\mathbf{k})J\Lambda(\mathbf{l}-\mathbf{k})\gamma^\mu\\&\quad\times\frac{\Theta(\delta_E-z)}{2\omega(\mathbf{l}-\mathbf{k},m_e)\nu2\omega(\mathbf{l}'-\mathbf{k},m_e)\nu'}\\&\quad\times U^i(\mathbf{l})\ .\end{aligned} \tag{3.38}$$

The corresponding diagram is shown in Fig. 5. Therefore, we can write

$$(S_A^{(1,1)})_{\text{mix}}=(S^{(0)})_{fi}\delta_{\text{mix}}^{(1,1)}\ , \tag{3.39}$$

with

$$\delta_{\text{mix}}^{(1,1)}=-e^2\int\widetilde{d^3k}\frac{(ll')}{(lk)(l'k)}\Theta(\delta_E-z)\ . \tag{3.40}$$

4. Calculation of $(S_A^{(2,0)})_{in}$ and $(S_A^{(2,0)})_f$

From Eq. (3.21) we have

$$(S_A^{(2,0)})_{\text{in}}=\langle e(\mathbf{l}';i')|S_{\text{FD}}^{(0)}\Omega_+^{(2)\dagger}|e(\mathbf{l};i)\rangle\ . \tag{3.41}$$

Calculating in a similar manner as before, the second-order initial-state contribution to the coherent state, we obtain

$$\Omega_+^{(2)\dagger}|e(\mathbf{l};i)\rangle=\sum_\alpha|e(\mathbf{l},\alpha)\rangle\overline{U}^{\,\alpha}(\mathbf{l})\Sigma_\Omega U^i(\mathbf{l})\ , \tag{3.42}$$

with

$$\Sigma_\Omega\equiv-e^2\int\widetilde{d^3k}\,\gamma_\mu\Lambda(\mathbf{l}-\mathbf{k})\gamma^\mu\times\frac{\Theta_\Delta(\mathbf{k}_i)}{2\omega(\mathbf{l},m_e)(-2i\epsilon)2\omega(\mathbf{l}-\mathbf{k},m_e)\nu}\ . \tag{3.43}$$

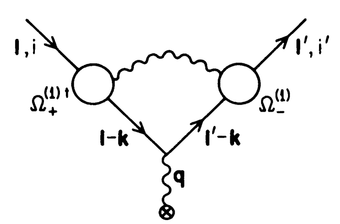

FIG. 5. Mixed initial-final state transformation.

[35]Note that this matrix element exists only in the soft region.

The diagrams corresponding to this and the final matrix element are shown in Fig. 6. Note that the denominator has a formally vanishing factor $-2i\epsilon$. This requires a certain physical interpretation. The fact that a second-order contribution to the coherent state reproduces a single-particle contribution of exactly the same on-mass-shell momentum indicates that this is a self-energy effect, as is evident from Fig. 6. This interpretation is also reinforced by the adiabatic factor $2i\epsilon$, corresponding to an interaction effect surviving in the remote past (for initial states) or the far future (for final states). This is no different from the calculation of ordinary Feynman-Dyson S-matrix elements, except this contribution is restricted to soft or collinear photons. To put it in a more familiar form, we write $v\omega(\mathbf{l}-\mathbf{k},m_e)\simeq(lk)=-\frac{1}{2}[(l-k)^2-m_e^2+i\epsilon_F]$, where ϵ_F is the Feynman displacement of the propagator poles, and the relation

$$\epsilon_F=2[\omega(\mathbf{l},m_e)-\omega(\mathbf{k},m_\gamma)]\epsilon \tag{3.44}$$

connects it to our adiabatic energy factor ϵ. We then write this as in previous cases in the form

$$(S_A^{(2,0)})_{\text{in}}=(S^{(0)})_{fi}\delta_{\text{in}}^{(2,0)} , \tag{3.45}$$

where $\delta_{\text{in}}^{(2,0)}$ is the contribution of Σ_Ω to the S-matrix element, given by

$$\delta_{\text{in}}^{(2,0)}=\frac{-e^2}{2}\int \widetilde{d^3k}\left[\frac{\partial}{\partial X(l)}\frac{\gamma_\mu\Lambda(\mathbf{l}-\mathbf{k})\gamma^\mu}{[(l-k)^2-m_e^2]}\right]_{X(l)=0}\times(1-z)\Theta_\Delta(\mathbf{k}_i) , \tag{3.46}$$

where $X(l)\equiv \not{l}-m_e$. As before, it will prove useful to divide this into soft and hard-collinear contributions:

$$(S_A^{(2,0)})_{\text{in}}=(S^{(0)})_{fi}(\delta_{\text{in};s}^{(2,0)}+\delta_{\text{in;hc}}^{(2,0)}) . \tag{3.47}$$

(i) *Soft region.* Writing $(l-k)^2-m_e^2=[X(l)-\not{k}][X(l)-\not{k}+2m_e]$ and evaluating the derivative, we obtain

$$\delta_{\text{in};s}^{(2,0)}=\frac{e^2}{2}\int\widetilde{d^3k}\left[\frac{m_e^2}{(lk)^2}-\frac{1}{(lk)}\right]\Theta(\delta_E-z)$$
$$\simeq\frac{e^2}{2}\int\widetilde{d^3k}\frac{m_e^2}{(lk)^2}\Theta(\delta_E-z) , \tag{3.48}$$

since the second term does not contribute to the soft limit.

(ii) *Hard-collinear region.* Working similarly, we find

$$\delta_{\text{in;hc}}^{(2,0)}=\frac{e^2}{2}\int\frac{\widetilde{d^3k}}{1-z}\left[\frac{m_e^2}{(kl_1)^2}-\frac{(1-z)}{(kl_1)}\right]\times\Theta(\delta_\theta-\theta)\Theta(z-\delta_E)$$
$$=\frac{e^2}{2}\int\frac{\widetilde{d^3k}}{1-z}\left[\frac{m_e^2}{(kl_1)^2}-\frac{z}{(kl_1)}\right]\times\Theta(\delta_\theta-\theta)\Theta(z-\delta_E) . \tag{3.49}$$

Obviously, $(S_A^{(2,0)})_f$ is given by expressions identical to those above with l_1 replaced by l_1'.

C. Summary of results

Summarizing the results of all these calculations, we may put together the various contributions:

$$S_{fi}^{(2)}=S_{fi}^{(0)}(\delta_{\text{in};s}^{(2,0)}+\delta_{\text{in};s}^{(1,1)}+\delta_{f;s}^{(2,0)}+\delta_{f;s}^{(1,1)}+\delta_{\text{mix}}^{(1,1)}+\delta_{\text{in;hc}}^{(2,0)}+\delta_{\text{in;hc}}^{(1,1)}+\delta_{f;\text{hc}}^{(2,0)}+\delta_{f;\text{hc}}^{(1,1)}+\delta_V^{(2)}) , \tag{3.50}$$

where we have included the usual vertex correction in the last term. Denoting the sum inside the parentheses simply by $\delta^{(2)}$, we have

$$\delta^{(2)}=e^2\int\widetilde{d^3k}\left[-\frac{m_e^2}{2(lk)^2}-\frac{m_e^2}{2(l'k)^2}+\frac{(ll')}{(lk)(l'k)}\right]\times\Theta(\delta_E-z)$$
$$+e^2\int\frac{\widetilde{d^3k}}{1-z}\left[-\frac{m_e^2}{(kl_1)^2}+\frac{1}{(kl_1)}\left[z+\frac{2}{z}-2\right]\right]\times\Theta(\delta_\theta-\theta)\Theta(z-\delta_E)+\delta_V^{(2)} . \tag{3.51}$$

In this form it is easy to compare with the results obtained via the cross-section method [9]. The first term is exactly equal to half the soft bremsstrahlung contribution, denoted $\frac{1}{2}\delta_{\text{BS}}$, while the second term equals the sum of hard collinear bremsstrahlung, denoted δ_{BH}. Hence

$$\delta^{(2)}=\tfrac{1}{2}\delta_{\text{BS}}+\delta_{\text{BH}}+\delta_V^{(2)} . \tag{3.52}$$

The lowest-order radiative correction to the cross section [Eq. (3.20)] is therefore

$$\frac{d\sigma^{(2)}}{d\Omega}=2\delta^{(2)}\frac{d\sigma^{(0)}}{d\Omega}=(\delta_{\text{BS}}+2\delta_{\text{BH}}+2\delta_V^{(2)})\frac{d\sigma^{(0)}}{d\Omega} . \tag{3.53}$$

This is precisely the same cross section obtained in Ref.

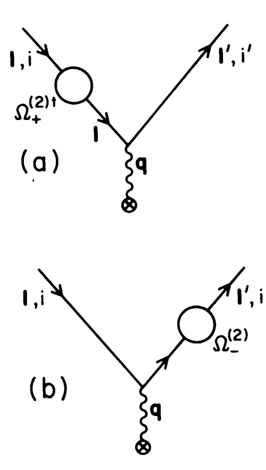

FIG. 6. Asymptotic self-energy transformations of the (a) initial and (b) final states.

[9], thereby establishing the identity of the S-matrix method with the cross-section method in this case. But Ref. [9] established that this formula has a smooth limit as the photon and electron masses tend to zero, and so no further work is required here. Moreover, since Ref. [9] also showed that the result was the same regardless of whether one used dimensional regularization or mass cutoffs, we know this result contains no anomalies. The present discussion implies that this result will also obtain for fixed masses in the limit of very high energies; i.e., the result is entirely uniform.

IV. OBSERVABLES AND THE "EXPERIMENTAL" ASYMPTOTIC HAMILTONIAN

Having illustrated the S-matrix formalism in the case of the scattering of an electron in an external field, we wish to reflect on the method more generally and explain its relationship to the traditional cross-section method. To some extent this section picks up where Sec. II leaves off, with reference to the explicit calculation of the previous section. Heretofore, we have intuitively interpreted the parameters in the asymptotic Hamiltonian as the degree of physical degeneracy appropriate to a particular experiment. In this section we among other things derive that interpretation, showing that the preceding calculation is correct only for that particular choice. In the process we show that observables are, in a sense, independent of the particular choice of $H_A(\Delta)$. We answer other questions such as, what if the initial- and final-state resolutions differ? To this end we introduce a new S matrix that involves different asymptotic Hamiltonians for the initial and final states. We find that the "infrared catastrophe" is independent of the initial-state energy resolution, so long as that is more precise than the final-state energy resolution. At first sight this result appears counterintuitive, until one understands that, in a given measurement, the determination that a soft photon has not been emitted is set by the worst rather than the best resolution in the problem. We also show that, in this special case, one does not need to sum over degenerate initial Fock states, just as in the Bloch-Nordsieck prescription. However, this is never true of the collinear singularities associated with the incoming particles, for which initial-state summation is crucial. As a result, observables remain functions of the initial-state angular resolution (or, equivalently, the transverse-momentum resolution).

We shall show that this intuitive picture is justified. The concept employed by Bloch and Nordsieck [1] and by Lee and Nauenberg [5] is that, for massless particles, there are always indistinguishable final states to be associated with a given measurement. Such states will be said to lie within a certain *degeneracy domain* for a given experiment. For example, one cannot differentiate between a single electron of a certain momentum and multiparticle states having the same quantum numbers lying within a certain region of phase space that cannot be distinguished by the measurement. The usual examples are that one cannot distinguish between an electron together with a sufficiently soft photon or a sufficiently collinear electron-photon pair. We shall show that, for multiparticle states lying within the degeneracy domain, all matrix elements of S_A vanish. This means that for states that are experimentally indistinguishable, the scattering operator S_A is completely characterized by its matrix elements between particles with momenta that are all distinguishable. It is important to understand that this applies to massive as well as massless particles; it just depends on the relation between the particles' masses and the actual resolutions in a given situation.

To begin, suppose that all particles, both electrons and photons, were massive. One may still develop the theory in terms of a set of asymptotic Hamiltonians $H_A(\Delta)$ such as the ones considered previously, and for a certain range of kinematic parameters, there will be nontrivial solutions of the constraints provided by the Θ functions. For the massive theory, the previously singular Møller operators $\Omega_\pm \equiv \Omega^{(\pm)}_{H_A,H_0}$ are well-defined unitary transformations in Fock space, and the formalism may be interpreted simply as calculating in a basis different from the more customary Fock states having a definite number of particles. One can therefore obtain physical predictions in a manner similar to the usual cross-section method. In fact, for any given reaction at finite energy, one can, in principle, choose the parameters Δ sufficiently small so that, for the allowed range of phase space, $\Omega_\pm = 1$, in which case this becomes *identical* to the usual cross-section formalism. As we shall discuss, it seems to us that the formalism, at least the cross-section method, is inherently ambiguous when it comes to initial-state degeneracy.

When the photon or electron becomes massless, the equivalence to the standard formalism breaks down. Because of the infrared behavior of H_A, the $\Omega_\pm \equiv \Omega^{(\pm)}_{H_A,H_0}$ cease to be operators in Fock space. This is true regardless of whether one interprets the physical space as Fock space $\mathcal{H}_F$, as we have done here, or the free-particle states in Fock space $\mathcal{H}_0$, as was done in Ref. [19]. In either case the operators $\Omega_\pm$ map eigenstates of H_0 to eigenstates of $H_A(\Delta)$, states which more nearly approximate the actual asymptotic states of the theory, but which depend on some parameters Δ. Although it is believed the singular behavior accurately reflects the infrared structure of $H_{\rm QED}$, the asymptotic Hamiltonian $H_A(\Delta)$ depends on parameters Δ that are not specified by $H_{\rm QED}$, but which reflect the fact that observable cross sections depend inextricably on experimental parameters. This naturally raises a number of questions concerning (1) the uniqueness of the form of the $H_A(\Delta)$, (2) the meaning of the parameters in $H_A(\Delta)$ and their relation to experimental parameters, and (3) the relationship of the S-matrix method to the cross-section method.

With respect to the first point, the asymptotic Hamiltonian H_A certainly is not unique, at least not if the only requirement is the existence of the corresponding Møller wave operators $\Omega^{(\pm)}_{H,H_A}$. We recall that in the case of QED the form of H_A used in Ref. [19] is not derived from $H_{\rm QED}$. Indeed, the precise form of H_A used in Ref. [19] differs from the ones used, for example, in Ref. [26]. However, it is argued that either choice is plausibly correct because matrix elements in the respective asymptotic-state bases do not contain infrared divergences to all orders in perturbation theory. Similarly, when both the electron and photon are massless, so that collinear divergences also appear, it has been argued [26] that the choice of $H_A(\Delta)$ that we have employed leads to matrix elements free of collinear singularities as well. We have seen explicitly in the case of the scattering of an electron in an external field that working with the $S_A(\Delta)$ associated with the asymptotic Hamiltonian $H_A(\Delta)$ does indeed produce nonsingular matrix elements, at least to lowest nontrivial order. One can easily see that this result is independent of the identification of Δ with any experimental parameters. It is clear that the *crucial ingredient for the absence of such mass singularities is that H_A reproduce the same infrared and collinear singularity structure as $H_{\rm QED}$*. This is the universal feature of acceptable asymptotic Hamiltonians having the property that the matrix elements so generated are free of mass singularities. In other respects there is considerable arbitrariness in how $H_A(\Delta)$ is specified.

What then are we to make of our demonstration in the foregoing section that, if we identify the parameters in a particular $H_A(\Delta)$ with the experimental resolutions, we obtain the same answer as the usual cross-section method? For the time being, we will continue to imagine that initial- and final-state resolutions are the same. Subsequently, we will generalize to the more realistic case when they differ. The parameters generically denoted by Δ define the degenerate phase space for a given experiment, the relevant soft and nearly collinear regimes. Within this degeneracy domain, a particular experiment cannot distinguish between multiparticle states differing by a number of sufficiently soft or sufficiently collinear quanta. Correspondingly, all matrix elements of $S_A[\Delta]$ between states having quanta within the degenerate phase space vanish. We have demonstrated this by explicit calculation, to lowest nontrivial order, in Appendix B. A heuristic way to see this is in the asymptotic interaction picture, discussed in Appendix C, in which the interaction operator is $V_I = H - H_A(\Delta)$. In this picture all vertices vanish for the emission or absorption of quanta within the degeneracy domain. For example, consider the vertices associated with H_A restricted by the Θ function as in Eq. (3.19). V_I involves a Θ function with reversed arguments. Thus, if Θ_Δ is given as in Eq. (3.19), then $1-\Theta_\Delta = \Theta(z-\delta_E)\Theta(\theta-\delta_\theta)$. Such vertices in this picture emit only hard, noncollinear photons. Similar arguments apply to other vertices involving collinear electron-positron pairs. Thus $S_A[\Delta]$ is nonzero only between states having quanta all of whose momenta lie outside the degeneracy domain. For example, in the case treated in Sec. III, scattering of an electron in an external field is completely characterized by the single-particle matrix elements that we calculated. Since the inelastic matrix elements vanish and the elastic matrix elements of S_A reproduce the results of the cross-section method involving elastic and inelastic matrix elements of $S_{\rm FD}$, our identification of the parameters Δ in $H_A(\Delta)$ with the experimental resolution is justified.

But what if we had chosen $H_A(\Delta)$ with parameters Δ not equal to the experimental resolution? The foregoing discussion applies to both massless quanta and to particles having a small but finite mass. As we remarked earlier, for massive particles, choosing Δ sufficiently small removes any possibility of degeneracy. Thus, for sufficiently small Δ, $S_A[\Delta]$ reduces to $S_{\rm FD}$. If this choice of Δ is small compared to the experimental resolutions, then it is clear that the cross-section method must be used to construct the observable cross section. This observation generalizes as follows: If the choice of asymptotic Hamiltonian $H_A(\Delta)$ involves parameters Δ that are smaller than the *experimental* degeneracy, one must add to the elastic cross section, the inelastic contributions of soft and collinear quanta that the measurement cannot resolve. This is proved by explicit calculations in Appendix B for the scattering of an electron in an external field. More precisely stated, if Δ^e denotes the experimental resolution, and $\Delta < \Delta^e$, then

$$\begin{aligned} d\sigma_{\rm obs}(\Delta^e) &= |\langle l'|S_A[\Delta^e]|l\rangle|^2 \\ &= |\langle l'|S_A[\Delta]|l\rangle|^2 + \sum_{i,f}{}' |\langle f|S_A[\Delta]|i\rangle|^2 , \end{aligned} \tag{4.1}$$

where the second term represents a sum over those inelastic initial and final states that cannot be experimentally distinguished from the elastic contribution. The necessity to include initial-state degeneracy differs from the Bloch-Nordsieck [1] result for Abelian theories. However, it is well known to be required for collinear singularities [5,6] and for non-Abelian theories in amplitudes involving nonsinglet initial states [31,32].

The equality displayed in Eq. (4.1) requires a specific relative weighting among degenerate initial states, viz., the same phase space normalizations that apply to final states. While this relation is an indisputable mathematical fact, it carries the paradoxical implication that initial-state degeneracy is to be associated with a certain relative weight between, say, an incoming single electron of definite energy and an electron of much lower energy accompanied by a hard but nearly collinear photon. This conflicts with the intuitive notion of an electron beam as well as the idea that one may prepare arbitrary linear combinations of states in Hilbert space. A complete resolution of this paradox requires a more careful analysis of the measurement process. While we have not carried out such a study, we believe it would show that the S matrix $\langle e(l')|S_A[\Delta^e]|e(l)\rangle$ involving experimental parameters Δ^e is an approximation to the long-time, large-distance correlation functions. The corrections to these transition amplitudes presumably vanish in the limit that $\Delta^e \to 0$. Thus one cannot so easily separate the measurement from the dynamical system characterized by H, which is, of course, free of such parameters. For the massless theory, the measured states are approximately states described by the in and out states associated with asymptotic Hamiltonians $H_A(\Delta)$. However, it is not *a priori* clear which asymptotic Hamiltonian is relevant to a particular measurement. We would turn the preceding discussion around and specify that H_A for which S_A annihilates multiparticle states that are experimentally indistinguishable.

Suppose one had chosen an asymptotic Hamiltonian $H_A(\Delta)$ corresponding to a resolution larger than the actual experimental resolution. It is clear that, to realize the same observable cross section $d\sigma_{\text{obs}}$ as in Eq. (4.1), one must *subtract* those inelastic contributions corresponding to quanta that can in fact be distinguished, but that have been included in the "elastic" term by the choice of $H_A(\Delta)$. Thus one obtains a similar expression as Eq. (4.1), but with a minus sign before the inelastic term. This shows that two asymptotic Hamiltonians may be mathematically equivalent in the sense discussed previously, but may actually describe very different physical situations.

What if, as is usually the case, the initial- and final-state resolutions differ? In that case it is useful to introduce a modification of $S_A[\Delta]$ to accommodate such circumstances. The S matrix S_A, given in Eq. (2.7) or (2.8), corresponds to transitions between in and out states defined with respect to a particular choice of asymptotic Hamiltonian H_A. Nothing prevents one from using a different asymptotic Hamiltonian for the out states than for the in states; that is, one may define

$$S_{A'A} \equiv \Omega^{(-)\dagger}_{H,H'_A} \Omega^{(+)}_{H,H_A} \tag{4.2}$$

or

$${}^{A'}\langle \phi^+ | \psi^- \rangle^A \equiv {}^{A'}\langle \phi^+ | S_{A'A} | \psi^+ \rangle^A , \tag{4.3}$$

where H'_A is associated with out states and H_A with in states. Mathematically, this amounts to a trivial change of basis in the Hilbert space since

$$\Omega^{(-)\dagger}_{H,H'_A} = \Omega^{(-)}_{H'_A,H_A} \Omega^{(-)\dagger}_{H,H_A} . \tag{4.4}$$

Physically, of course, we have in mind a situation in which the final-state resolution (generically denoted by Δ_f) differs from the initial-state resolution (denoted by Δ_i), suggesting that we associate $H'_A \equiv H_A(\Delta_f)$ with final states and $H_A \equiv H_A(\Delta_i)$ with initial states and denote $S_{A'A}$ as $S_{A'A}[\Delta_f,\Delta_i]$. One may relate $S_{A'A}$ to our previous S_A by using Eq. (4.3), to wit,

$$S_{A'A}[\Delta_f,\Delta_i] = \Omega^{(-)}_{H'_A H_A} S_A[\Delta_i] . \tag{4.5}$$

It is straightforward to generalize the preceding discussion to show that multiparticle matrix elements of $S_{A'A}[\Delta_f,\Delta_i]$ vanish whenever final- or initial-state quanta lie within their respective degeneracy domains, i.e., when states involve quanta that are indistinguishable by a measurement characterized by final-state resolution Δ_f and initial-state resolution Δ_i. To see this one may apply the asymptotic interaction picture argument that we previously gave for $S_A[\Delta]$ separately to $\Omega^{(+)}_{H,H_A}$ and $\Omega^{(-)}_{H,H'_A}$; i.e., one may employ different interaction pictures in each case to establish the result. One may also verify it by explicit calculation in the usual interaction picture as shown in Appendix B. For massive particles (or a regulated theory), one may write

$$S_{A'A}[\Delta_f,\Delta_i] = \Omega^{(-)}_{H'_A,H_0} S_{\text{FD}}[H] \Omega^{(+)\dagger}_{H_A,H_0} . \tag{4.6}$$

This is the most natural form for calculating the matrix elements of $S_{A'A}$ using the usual Feynman rules of the interaction picture. For example, for scattering in an external field, one is instructed to evaluate the Feynman-Dyson S matrix between the in-coherent state $|i^+\rangle = \Omega^{(+)\dagger}_{H_A,H_0}|e(l)\rangle$, associated with initial-state resolution Δ_i, and the out-coherent state $|f^-\rangle = \Omega^{(-)\dagger}_{H'_A,H_0}|e(l')\rangle$, associated with final-state resolution Δ_f.

Thus far, we have avoided the rather vexing question of how, for a given experiment, one is to infer the initial-state resolution Δ_i. In analogy with detector resolutions, we want to identify the uncertainty with which the energy and direction of each particle may be known. This is not the same as the classical, statistical distribution of momenta of all the particles in a beam bunch, since, in principle, each particle's momentum is definite. Rather, one should entertain the notion of the extent to which, given the accelerator, a single particle, such as an electron, can be distinguished from multiparticle states, such as electron plus a soft photon or an electron plus a collinear, hard photon. Another way to think about this is to view the accelerator as a measurement of a particle's momentum to a certain accuracy. To better understand the symmetry between initial- and final-state resolutions, it may be conceptually helpful to restrict one's attention for a moment to theories invariant under time reversal. Then, formally,

$$\langle f | S_{A'A}[\Delta_f,\Delta_i] | i \rangle = \langle -i | S_{AA'}[\Delta_i,\Delta_f]^\dagger | -f \rangle^* . \tag{4.7}$$

Thus the initial-state resolution becomes the final-state resolution for the time-reversed process, for which one is invited to imagine the accelerator as the detector.

For Abelian theories, as remarked earlier, so long as the initial-state energy resolution ΔE_i is less than the detector resolution, the normal situation, then the matrix element will in fact be independent of ΔE_i. However, this is not true for collinear singularities, and the matrix element is not independent of the initial-state angular resolution $\delta\theta_i$ or, alternatively, the initial-state transverse-momentum resolution. For an electron beam, for example, $\delta\theta_i$ will be the angle below which the accelerator cannot differentiate a single electron from an electron accompanied by a collinear photon or a photon from a collinear electron-positron pair, and so it behooves us to understand better what this refers to. One would naively think one could distinguish the single electron from the electron-photon pair simply by observing the photon or, since it has lower energy, by observing the electron bend at a larger angle in a magnetic field. But one cannot preclude the possibility that the photon is absorbed by the electron in the process of further interaction, and so observations on the final state are irrelevant. Indeed, looking back at the contribution of the initial coherent state to the result in the preceding section or, for that matter, recalling the analogous calculation using the cross-section method [5], it is just this probability of absorption in addition to interaction that contributes to the cancellation of the mass singularity in the virtual, vertex correction.

In the final formula for the observable cross section [Eq. (3.53)], the lower limit on the angle θ is set by the initial-state angular resolution $\delta_\theta^{\text{in}}$. As mentioned in Sec. II, this S-matrix or coherent-state formalism may be applied to massive particles as well as to massless particles, a point on which we shall elaborate further in our companion paper [25]. But for massive particles, the near vanishing of the energy denominator [Eq. (3.16)] is governed not simply by the angular resolution $\delta_\theta^{\text{in}}$ or the corresponding transverse-momentum resolution $\delta_{p\perp}$, but rather by the so-called transverse mass $m_\perp=(p_\perp^2+m^2)^{1/2}$ or corresponding angle $m_\perp/E=(\delta_\theta^{\text{in}2}+\chi^2)^{1/2}$, where, as before, $\chi\equiv m/E$. This is the quantity that sets the scale of the cutoff on the collinear singularity. Thus, if $\delta_\theta^{\text{in}}\gg\chi$, then the mass may be neglected, whereas if $\delta_\theta^{\text{in}}\ll\chi$, it is the mass rather than the resolution that is the important parameter. In this latter case, treating the particle as a single particle and neglecting the experimental resolutions is a valid approximation.

In a given experimental situation, what is this initial-state angular resolution $\delta_\theta^{\text{in}}$? One would think that a classical source could have arbitrarily precise resolution, given that there is an infinite amount of time to prepare the beam. In such a case, the initial-state resolution would be set by quantum limits on the precision of a measurement because of the uncertainty principle. For example, the time that it takes for an electron to travel from the final focus (FF) of the accelerator to the intersection point sets a limit on the electron energy resolution $\Delta E\hbar/T$, and within this energy band, you could not tell the difference between an electron and an electron plus a soft photon. In fact, for a sufficiently energetic electron and a hard photon, one can easily show from Eq. (3.16) that a lower limit on ΔE already imposes a lower limit on the degree of collinearity of the electron-photon pair. Similarly, the knowledge that the particle lies within a certain region in the transverse plane sets a lower limit on the transverse-momentum resolution. Typically, such limits are extremely small, even for the highest-energy electron accelerators, so that there is no question that it would be the electron mass rather than these limits that would cut off collinear singularities. However, it seems that, even though there is in principle an infinite time to prepare an electron beam, the characteristics of the accelerator itself rather than the geometry of the interaction region are the determining factors in limiting the initial-state angular and energy resolutions. We shall discuss this from a couple of different points of view, but nevertheless, it still turns out that, in realistic situations, it is the electron mass rather than the angular resolution that is the primary regulator of collinear singularities.

As the beam approaches the intersection point, it passes through a final focus in which the momenta of the particles are precisely controlled so that they converge to a tight spot at the intersection point (IP) some meters away. At a symmetric point along its orbit, the mean amplitude r and direction r' of the beam may be inferred from the transverse emittance ϵ and the amplitude function for transverse (betatron) oscillations β, viz.,

$$r=\sqrt{\beta\epsilon},\quad r'=\sqrt{\epsilon/\beta}\ . \tag{4.8}$$

In fact, in electron accelerators, these are very different in the horizontal and vertical directions, which we will take into account below. For the sake of discussion, however, let us suppose for a moment that the beam is azimuthally symmetric around the beam axis. After the final focus, the beam converges through a drift region[36] to the IP where its transverse amplitude is minimum and where it normally collides with a similarly focused beam in the opposite direction. Our interest is in how well differentiated any given electron is from a nearly degenerate multiparticle state, such as a nearly collinear electron-photon pair. In the drift region, the angular divergence r' of the beam is constant and defines the angle of the cone within which particles of the beam pass as they emerge from the FF until they arrive at the IP. Quantities such as r and r' describe the classical statistical distribution of the particles in the beam, each one of which has a definite momentum. The angular resolution we seek is, however, not r', but the typical uncertainty with which an electron's momentum is defined. For this it is important to understand that transverse oscillations are excited by the quantum fluctuations in the emission of bremsstrahlung by a relativistic electron [42]. Such bremsstrahlung forms a "searchlight," peaking at an angle χ_e with respect to the initial direction of the electron. The electron's recoil, this inevitable jitter in the electron's direction due to random emission of photons, is the mechanism that drives the betatron oscillations and creates uncertainty in the electron's direction. Thus we anticipate that, in an optimally designed accelerator, the initial-state angular resolution $\delta_\theta^{\text{in}}$ will be of the order of χ_e, although, since the probability of bremsstrahlung is rather small, the resolution may be smaller than this on the average. As we have seen, the relative size of $\delta_\theta^{\text{in}}$ and χ_e is all important in determining whether initial-state degeneracy or coherence is important in practice, and so we need to be more precise than this order-of-magnitude estimate. We wish to deduce $\delta_\theta^{\text{in}}$ directly from accelerator parameters just as δ_θ^f is deduced from the resolution properties of the detector. Because initial-state degeneracy is an unfamiliar concept and because we are exploring unfamiliar territory, we shall present two estimates of $\delta_\theta^{\text{in}}$. In the first we deduce an easily understandable geometrical upper limit for a given accelerator; in the second we will infer a more precise estimate based on the parameters of the final focus.

This geometrical upper limit may be deduced simply as follows: Imagine a particle emerging from the final focus on the beam axis, but with some transverse momentum. If the characteristic uncertainty in the transverse momentum carried the particle outside the cone formed by the final focus, the particle would lie outside the transverse

[36]In practice, the beam is usually surrounded by a detector where, in principle, there is little or no external field affecting its motion.

area at the IP. To put it another way, an electron emitting a photon just as it emerged from the final focus may not arrive at the IP if its change in direction is too great. Moreover, if this electron or photon were not obscured by the other electrons in the beam bunch, then presumably this state could be distinguished from a single-particle state. So it appears that $\delta_\theta^{\rm in} \le r({\rm IP})/L$, where r(IP) represents the characteristic size $\sqrt{\epsilon\beta({\rm IP})}$ of the beam at the IP and L is the drift distance from the FF to the IP. These quantities are readily available [43]. However, we must face the fact that an electron beam is far more elliptical rather than cylindrical, the amplitude in the plane of the orbit being much larger than vertically. What amplitude should be used in estimating $\delta_\theta^{\rm in}$? Imagine the extreme in which the vertical amplitude were very small so that the beam profile were nearly planar. We have argued that ultimately the electron's resolution is related to the emission of bremsstrahlung, and we know that this searchlight is azimuthally symmetric about the electron's direction of motion at an angle of order χ_e. If such an emission angle were large compared to the angle defined by the vertical amplitude, then it would be very infrequent that the emitted photon or recoiling electron would lie in the plane of the beam. Thus we would argue that for the highly skewed beam profile one ought to use the smallest dimension, the vertical height y, in estimating $\delta_\theta^{\rm in}$. At the SLC, for example, where the beam is very narrowly focused, this corresponds to about 2–3 μm divided by about 3 m, giving $y/L=10^{-6}$. This is to be compared with $\chi_e=m_e/E\approx 10^{-5}$. Thus it is the mass rather than the angle that is relevant and cuts off the collinear singularity.[37] At LEP the beam radius and drift distance are larger, so that we find $y/L=12$ μm/3.5 m$=3.4\times 10^{-6}$, and so the angle remains less than χ_e. The final-state angular resolution is typically much worse, with δ_θ^f typically on the order of 10^{-4} or more.

Our second, somewhat less intuitive, estimate of the $\delta_\theta^{\rm in}$ is based on the properties of the final focusing of the beam just before the intersection region. The momentum of the beam is precisely defined by the FF, and correspondingly, the amplitude of the beam reaches a maximum in the FF, where the angular dispersion is minimum, $r'_{\rm min}=\sqrt{\epsilon/\beta_{\rm max}({\rm FF})}$. If the typical uncertainty in direction were not smaller than this, the FF would not work as designed. On the other hand, the uncertainty cannot be smaller than this, since this is the highest-momentum resolution in the accelerator. Thus we suggest that this minimum of $r'_{\rm min}$ in the FF corresponds to the initial-state angular resolution $\delta_\theta^{\rm in}$ that we seek.

By way of consistency, one can show that this value is certainly less than our previously estimated upper limit. Clearly, the maximum amplitude $\beta_{\rm max}$(FF) inside the FF is greater than its value at the face of the FF when the beam enters the drift region. Using the free-space equations, one can relate β at the face to β(IP) at the intersection point. To a good approximation, at the face, $\beta\approx L^2/\beta({\rm IP})$. Therefore,

$$r'_{\rm min}\le\left[\frac{\epsilon\beta({\rm IP})}{L^2}\right]^{1/2}=\frac{r({\rm IP})}{L}\ , \tag{4.9}$$

which we recognize as our geometrical upper limit. In the normal case that $\beta_{\rm max}$(FF) is very different in the horizontal and vertical directions, once again, we would argue that it is legitimate to use the larger of the two in estimating the resolution. While the order of magnitude is important, in many applications, it is sufficient to know the initial-state resolution approximately, because the resolution appears in a logarithm multiplied by the fine-structure constant. However, evanescent processes are directly proportional to the ratio $\chi_e/\delta_\theta^{\rm in}$, and so it is important to determine these quantities precisely.

It may seem paradoxical that the initial-state angular resolution could be deduced to be much smaller than χ_e, the angle at which bremsstrahlung is maximum and the ultimate cause of imprecision in the knowledge of the direction of the electrons. The answer, we believe, is that, because the fine-structure constant is small, the probability of bremsstrahlung remains small, so that, on average, the direction may be relatively better defined than the recoil angle.

In the design of future linear accelerators with electrons of energy on the order of 1 TeV, $\chi_e=m_e/E$ $\approx 0.5\times 10^{-6}$. However, to obtain the necessary luminosity, the radius r(IP) of the beam spot at the intersection point falls to 50 nm or so, so that the angular definition is reduced by at least as much as m_e/E. Thus the same conclusion will continue to hold for future machines, a dramatic illustration that the *high-energy limit is not necessarily the same as the massless limit.*

V. DISCUSSION AND CONCLUSIONS

In this section we wish to reflect further on the implications of our formalism, on remaining questions, and on potential future developments.

Although it would be unconventional in the case that a particle's mass really is nonzero, one may choose an asymptotic Hamiltonian H_A other than H_0, since H_A and H_0 are in the same equivalence class. What is more, this could even be useful in situations where the energy is very high and/or the experimental resolutions are large compared to the mass. An example in QED is the helicity-flip process discussed in Ref. [25]. One appealing aspect of our formalism is that both the massive and massless theory are in Fock space, and so no conceptual modification of the usual way of thinking is required. In such cases the particle's mass is physical and not just a regulator for the massless theory, and the calculations, for example, in Sec. III, should be viewed in a different light. Then whether a mass is experimentally relevant can rather easily be decided. The massive theory will approximate the massless theory so long as the mass fractions χ_e,χ_γ are smaller than the corresponding experimental resolutions $\{\Delta E,\delta\theta\}$.

Besides being able to treat the high-energy and mass-

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[37] By way of contrast, the angular divergence of the beam, not to be confused with the initial-state angular resolution, is on the order of 200 μrad, an order of magnitude *larger* than χ_e.

less limits in a uniform manner, there are other advantages of the S-matrix method over the conventional cross-section method: (a) The experimental degeneracy of states, necessary to achieve finite transition rates, is taken into account at the S-matrix level and not at the cross-section level (as an incoherent sum of cross sections within a Lee-Nauenberg degenerate set [5]). This is especially advantageous in the discussion of evanescent processes [25]. (b) The formalism naturally accounts for degeneracy both in the initial as well as the final state. In this sense initial- and final-state resolutions are treated on the same footing. The calculations in this paper, in the context of a well-studied example, establish that the S-matrix method is reliable and equivalent to the traditional cross-section method, at least where the latter method has been applied. The treatment of the collinear singularities in this example illustrates the general result that initial-state degeneracy *must* be taken into account in order to establish that observables are nonsingular as a particle's mass tends to zero. This is, of course, not new [5].[38] Although we have not proved the equivalence between the S-matrix formalism and the cross-section method in general, our results make it at least plausible that they will lead to identical predictions for observables, provided initial-state degeneracy has been properly taken into account. This has far-reaching implications: The KLN results [4,5] concerning the cancellation of mass singularities in observables undoubtedly can be generalized to show that properly defined observables have a finite, smooth limit as any mass tends to zero. This limit is uniquely described by the corresponding massless theory and is independent of the method introduced to regulate the mass singularities at intermediate stages of the calculation.

In particular, as indicated in the preceding section, the radiative corrections which have been performed at SLC and LEP to take into account initial-state bremsstrahlung and, in particular, to extract the Z^0 mass and width, should be numerically reliable. In a hypothetical situation in which the weak interactions were unbroken and the electron, neutrino, and vector bosons were all massless, one would be forced to take the initial-state resolution into account. This might be relevant to early universe calculations, where the temperature sets the scale of the soft and collinear cutoff.

As another related application, the naive calculation of the background in the search for right-handed charged currents at HERA, first discussed in Ref. [21], does in fact survive for the experimental resolutions.[39] This is relevant also for radiative corrections [44,45] to the polarization asymmetries as SLC and LEP. These were quantitatively small corrections in any case, but our work shows that, for realistic experimental resolutions, these calculations are in fact approximately correct, effectively being of order α and not of order $\alpha m_e^2/s$. These matters will be discussed further elsewhere [25,46].

One consequence of our work is that the apparent anomalies in massless QED that have been attributed to such mass singularities [23] are spurious, a feature of the method of calculation in terms of Feynman-Dyson S-matrix elements, but absent from observables, calculated either in the S-matrix S_A discussed herein or in a properly defined observable cross section. These pseudoanomalies will be discussed further in our companion paper [25].

Of course, for QCD, the discussion is further complicated by the fact that there are no asymptotic quark and gluon states because of confinement. We are speaking here of the perturbative contributions used for the discussion of the "hard processes" factorized from wave functions in the formalism of perturbative QCD. Thus the closest one may come to a description in terms of massless quarks and gluons is probably the replacement of the usual parton distributions by coherent-state distributions corresponding to an asymptotic Hamiltonian corresponding to a characteristic energy and transverse-momentum uncertainty on the order of $\Lambda_{\rm QCD}$. This resolves the paradox concerning the failure of the KLN theorem noted in Ref. [32]. Our results also invalidate the suggestion of Ioffe and co-workers [23] that protons acquire a longitudinal gluon structure function.

There are many other potential applications of our general results, some of which we will treat in future publications. One natural application of the preceding is to QCD, where it has been suggested [34,32] that there are very large corrections proportional to $\ln(Q^2/m^2)$ associated with gluon bremsstrahlung from light quarks. Assuming our results extend to non-Abelian theories—and we see no reason why they should not—these results must be incorrect. The mean transverse momentum of a parton in a hadron is set by the scale of confinement $\Lambda_{\rm QCD}$, which, although not well known, is on the order of 100–200 MeV. Thus light quarks (and gluons) appear in parton-model calculations as beams of partons whose transverse momentum is large compared to their mass, quite unlike the case of electron bremsstrahlung discussed above. For u and d quarks, for which $\Lambda_{\rm QCD} \gg m$, the mass singularity will be cutoff by $\Lambda_{\rm QCD}$ rather than m. So the large $\ln(m)$ terms in Ref. [34] are spurious, a point made earlier in Ref. [32]. However, in Ref. [32] it is remarked that in order for the KLN theorem to hold in deeply inelastic scattering, there would have to be a very special relation between the single-particle structure function for a quark, say, and the two-particle structure function for a quark plus a gluon.[40] We have seen for the

[38]However, just because a process may not involve a collinear divergence, it does not follow that initial-state degeneracy can be ignored because there are finite effects, such as the helicity-flip process to be discussed in detail in our companion paper [25], which require initial-state degeneracy to avoid faulty conclusions about observables.

[39]This contradicts our conjecture made in Ref. [21], although it remains true that, strictly in the limit $m_e \to 0$, it would be suppressed and become of order $\alpha m_e^2/s$.

[40]" . . . such a cancellation would require a miracle for which we cannot see any reason" [32].

case of QED that the asymptotic states of the massless theory are not the usual eigenstates of the free Hamiltonian H_0, but of a modified Hamiltonian H_A, and the same would be true for the non-Abelian case. For any given measurement, these describe the flux of incoming and outgoing particles, and when the initial resolution is larger than the mass, these remain the appropriate states. Correspondingly, the correct basis for describing the parton states in a hadron are not the quark eigenstates of H_0, but the eigenstates of $H_A(\Delta)$, where we would expect the characteristic energy and transverse-momentum resolution to be of order $\Lambda_{\rm QCD}$. It follows that, for quarks whose masses are small compared to $\Lambda_{\rm QCD}$, the situation more nearly resembles the massless rather than the massive case. When expressed in terms of states with definite numbers of particles, the single- and multiple-particle states enter in precisely the right combination to remove mass singularities in S-matrix elements; i.e., the "miracle" referred to in the preceding footnote becomes natural. Alternatively stated, the "in states" which are appropriate to a perturbative evaluation of "hard-scattering processes" are not the single-parton states (whose matrix elements involve mass singularities), but rather coherent states (whose matrix elements are free of mass singularities) associated with a resolution on order $\Lambda_{\rm QCD}$. Although we have not proved that such a relation exists in QCD, we believe it must obtain, since the quark masses ought to play no role whatsoever in determining the short-distance results. From our point of view, it is better to think of the incoming quark parton not as a single-particle state in perturbative QCD, but rather as a coherent state with quark quantum numbers. A modified formalism then should allow a consistent treatment of higher-twist phenomenology of hard scattering, but we have not attempted to develop that application.

Other topics for future development are more formal. It would be theoretically satisfying if a formalism could be developed in which there were no need for regulators of mass singularities to be introduced. It would be of practical significance as well, since it would reduce the need for extreme accuracy on calculating terms having singularities that cancel when added. This seems very likely to be possible; we saw in our calculations of S_A for the regulated theory that when we combined the integrands of the various Feynman diagrams occurring in a given order, no mass singularities remained.[41] Thus it would seem to be a matter of organizing the calculation more efficiently at the outset, perhaps by developing a different set of Feynman rules. An attractive but as yet not fully developed possibility in this regard would be to use an alternative to the interaction picture, replacing H_0 everywhere with H_A. We call this the "asymptotic interaction picture" and anticipate returning to it in future work. We shall offer some preliminary considerations in Appendix C, indicating why this might be a mass-singularity-free formalism for the asymptotic S matrix.

Other potential formal developments include a more manifestly covariant formalism, such as might be provided by light-cone or infinite-momentum-frame techniques [47]. Moreover, although we do not expect to encounter any new matters of principle, we have not fully explored the complications that arise in generalizing this formalism to non-Abelian theories.

We wish to reflect on the fact that the finite S-matrix elements in theories with massless particles depend on parameters, generically labeled here by the parameters Δ in our asymptotic Hamiltonians $H_A(\Delta)$. Correspondingly, as we have remarked at the outset and elaborated in Sec. IV, all observable cross sections depend inextricably on experimental parameters. Unlike familiar Feynman-Dyson S-matrix elements, there do not exist transition amplitudes involving only the parameters of the Hamiltonian H determinable, in principle, in the idealized limit of infinitely precise measurements. This does not mean that the dynamics specified by H is necessarily incomplete, but it does mean that it cannot be isolated so easily from the measurement process. One must specify precisely the characteristics of the measurement, such as the finite time over which the measurements are made and the finite extent over which the measurement is carried out. This stands to reason, since a soft photon corresponds to a quantum of nearly infinite wavelength, and so the notion of its localization or its association with either the initial- or final-state detector becomes meaningless, as one of the pioneers of this subject has emphasized [48]. In field-theoretic language, this suggests that one is measuring certain finite-time, finite-length correlation functions associated with gauge-invariant sources. What then is the meaning of the S matrix S_A, which is defined by discussing transitions between the infinitely distant past and the infinitely far future? Our conjecture is that the S-matrix method, embodied in S_A, is a technique for isolating the leading dependence on the measurement parameters, up to terms that vanish in the idealized limit that the experimental resolutions tend to zero. It would be quite illuminating to demonstrate this explicitly, but we have not attempted to do so.

A related aspect of the differences between theories with mass singularities and those without concerns the analytic properties of the Green's functions. It is well known [20] that in QED the ordinary Green's functions do not have simple poles at the electron mass, but rather a branch point characteristic of the threshold for inelastic processes. And Kulish and Faddeev also emphasized that the "relativistic concept of a charged particle does not exist" [19].

Finally, we wish to reflect on the meaning of a massless electron in QED. Over the years there has often been speculation that such a theory makes no sense, at least not in perturbation theory. Occasionally, people have argued that there are nonperturbative effects that rescue it and lead to a very different physical spectrum than the perturbative one [49]. While it is certainly possible that massless QED has a nonperturbative, strong-coupling phase [50], we see no reason to question the existence of the usual perturbative, weak-coupling solution. We know from the renormalization group that the β function is

[41] This contrasts with the cross-section method where cancellations occur only after adding together incoherent cross sections.

infrared-free, and so there certainly does not exist a massless charged particle, defined as the interaction of a fermion with a static electric field (photon of infinite wavelength). Stated otherwise, the vacuum polarization completely screens the charge of the fermion. Thus there is no correspondence (in the sense of Bohr) of the quantum field theory with the massless limit of a charged particle in classical electrodynamics, which certainly seems fraught with inconsistencies. Nevertheless, there is no indication of problems in a perturbative solution of the quantum field theory defined by the usual Feynman rules, provided only that one does not attempt to define a finite, on-shell renormalized charge. However, the usual Feynman rules defined for example in minimal subtraction can be used. Of course, as emphasized throughout this paper, the usual Feynman-Dyson S-matrix elements do not exist, but we understand that this is no indication of an underlying inconsistency, but rather a reflection of the incorrect choice of asymptotic Hamiltonian in H_0. In short, if one is careful, it seems that massless QED makes perfect sense, and massless perturbation theory is internally consistent.

Note added in proof. N. Nakanishi has kindly called our attention to an early paper by Morota [51] in which S_A was first defined and, indeed, in which the asymptotic interaction picture of Appendix C was employed.

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APPENDIX A

Here we contrast our view of the physical Hilbert space as Fock space with the conventional view in terms of coherent states: In the text we have built up field theories of massless particles in Fock space $\mathcal{H}_F$ in precisely the same manner that theories without massless quanta are defined. This represents a departure from what has become common practice over the past 20 years, wherein the physical states associated with massless quanta, such as photons, are taken to be non-Fock coherent states. We will argue here that either approach is permissible and that physical predictions are the same in either case.[42]

We have described how the Møller operators $\Omega^{(\pm)}_{H,H_0}$ in Eq. (2.2) do not exist as unitary operators in $\mathcal{H}_F$. However, $\mathcal{H}_F$ may be embedded in a larger space, the infinite tensor product space of von Neumann [35] $\mathcal{H}_{\rm vN}$. This space has been described elsewhere [35,14,19], and these discussions will not be duplicated here. Suffice it to say that von Neumann dealt with the meaning of operators (and their products) of the form $\exp(iF)$, with F a formally Hermitian but singular operator, such as $H_{\rm QED}$. Even though matrix elements of F may be singular, $\exp(iF)$ can be made well defined. Essentially, von Neumann's trick is to interpret matrix elements with factors of the form $\exp(i\infty)$ as zero and to show that this can be made consistent with precise notions of convergence and continuity. In our context examples of such formally unitary operators are those having mass singularities such as the $\Omega^{(\pm)}_{H,H_0}$ and $S_{\rm FD}$. Similarly, for any Fock state $|\psi\rangle$, the corresponding states

$$|\psi_0^\pm\rangle \equiv \Omega^{(\pm)\dagger}_{H,H_0}|\psi\rangle \tag{A1}$$

exist not in Fock space $\mathcal{H}_F$, but in the larger Hilbert space $\mathcal{H}_{\rm vN}$ of von Neumann. In fact, they are orthogonal [with von Neumann's convention that $\exp(i\infty)=0$] to all of $\mathcal{H}_F$, a property necessary for consistency.

The conventional construction [14,19] is the reverse of ours: The free-particle subspace associated with H_0 is built up as a Fock space $\mathcal{H}_0$, while the space $\Omega^{(\pm)}_{H,H_0}\mathcal{H}_0$ of physical states and the space $\mathcal{H}_A$ of asymptotic states [19] involve coherent states orthogonal to $\mathcal{H}_0$. In contrast, we have associated the Fock space $\mathcal{H}_F$ with the physical space, while the unphysical $\Omega^{(\pm)\dagger}_{H,H_0}\mathcal{H}_F$ must be thought of as a space of non-Fock, "coherent" states. The two viewpoints simply correspond to choosing different, unitarily inequivalent representations of the canonical commutation relations [17].

We may describe the use of non-Fock spaces further as follows: Consider the Eq. (A1). There is no question that the Møller operators $\Omega^{(\pm)\dagger}_{H,H_0}$ do not exist in $\mathcal{H}_F$; however, to interpret this relation, one has a choice. Conventionally, the in or out states $|\psi_0^\pm\rangle$ are regarded as Fock states, and so the space associated with $\Omega^{(\pm)}_{H,H_0}|\psi_0^\pm\rangle$ is regarded as the non-Fock space of physical states and $\Omega^{(\pm)}_{H_A,H_0}|\psi_0^\pm\rangle$ is the non-Fock space of asymptotic states [19]. We, on the contrary, choose the physical space as Fock space, so that the $|\psi_0^\pm\rangle$ defined in Eq. (A1) become non-Fock. In this approach the free-field Hamiltonian is defined on a non-Fock space, but since H_0 plays no role in the physics associated with H, we find this point of view attractive. Another advantage of this viewpoint is that the asymptotic states lie in the same physical Fock space $\mathcal{H}_F$ as the

[42]This is not entirely original with us, as the possibility was noted by Blanchard [17], and the equivalence is expressed in Sec. V of that reference.

physical states, because the Møller operators $\Omega^{(\pm)}_{H,H_A}$ exist for any acceptable asymptotic Hamiltonian H_A. In the more conventional view [19], the space of asymptotic states (at least for QED) is a denumerable subspace of the larger, nonseparable von Neumann space $\mathcal{H}_{\rm vN}$.

Less mathematically speaking, one might well ask whether one choice is preferable on physical grounds. We would argue that our choice is the more natural and easily interpretable. Let us recall the motivation [14] for introducing coherent states in the first place: Kibble considered the response of the quantized electromagnetic field to a classical electromagnetic current. He showed that, if the current carries a nonzero charge, then the no-photon state evolves in general to a coherent state with an infinity of photons. This is all well and good and has been interpreted as evidence that the physical states with their cloud of photons created by charged particles lie outside Fock space. However, our point is that the no-photon state is itself an unphysical state in a world of nonzero charge. The correct asymptotic states in the presence of an external $J^{\rm cl}_\mu$ current may be obtained quite analogously to the manner in which H_A has been obtained from $H_{\rm QED}$ in the text. In the presence of a nonzero charge, asymptotic states would always involve soft photons and would be defined by retaining the low-frequency components of the current in the asymptotic Hamiltonian H_A. Explicitly, we may simply take the asymptotic interaction of the form

$$V_A=\int_{k_\mu<\epsilon}\frac{d^4k}{(2\pi)^4}J_\mu(k)^{\rm cl}A^\mu(k)\ . \tag{A2}$$

With either choice a consistent formalism for transition amplitudes can be developed because *observables always involve square-integrable amplitudes,* free of mass singularities (and, as we have argued, free of ambiguities). There are conceptual advantages of identifying the physical space with Fock space. For one thing, massless and massive particles can be treated on an equal footing. Moreover, if one wishes, one may use an asymptotic Hamiltonian H_A with nontrivial interactions for treating massive particles as well. This is not just aesthetically attractive; it enables us to answer the question when a particle's mass is relevant or irrelevant for a given measurement. In other words, in a certain experimental situation, is the massive theory well approximated by the massless limit? This is crucial for the sort of processes discussed in Ref. [25]. Aside from the uniformity of treatment of massless and massive particles, there are conceptual advantages for massless particles as well. Physical transition amplitudes exist between states within the same space instead of as mappings between unitarily inequivalent subspaces [14].[43] Moreover, there is no question that the familiar mathematics of Fock space is simpler than the subtleties of von Neumann space. In fact, in principle, it is unnecessary to consider the larger von Neumann space $\mathcal{H}_{\rm vN}$ in which $\mathcal{H}_F$ is embedded, a result that is philosophically attractive [36]. In practice, however, one may be forced to face up to $\mathcal{H}_{\rm vN}$ when one attempts to develop Feynman rules in the usual interaction representation. However, even this is not compelling, since there is no problem interpreting the evolution operator $\exp(-iH_0t)$ in Fock space. The niceties arise when one discusses the limits as $t\to\pm\infty$. With massive theories the full Hamiltonian H may be expressed as a Hamiltonian taking the form of H_0, but with in or out creation-annihilation operators replacing the original creation-annihilation operators. These two sets of creation-annihilation operators are unitarily equivalent, provided an ultraviolet cutoff is present. However, for massless theories, the unitary equivalence breaks down because of mass singularities, requiring the discussion of coherent states and von Neumann space.

Just as with uv divergences, if the mass singularities are regulated, then the unitary equivalence is resurrected, and perturbation theory may be developed as usual. This is illustrated in the text by our discussion of the regulated theory and the limit $\epsilon\to0$. We have remarked that, in the regulated theory, all the states in Hilbert space may be regarded as Fock states. However, the usual Feynman-Dyson S-matrix elements become singular as the regulator is removed. This singular behavior is reflected by the fact that the limit

$$\Omega^{(\pm)}_{H_A,H_0}=\lim_{\epsilon\to0}\Omega^{(\pm)}_{H_A^{(\epsilon)},H_0^{(\epsilon)}} \tag{A3}$$

does not exist as an operator in Fock space, but only as an isometry in von Neumann space. Therefore, the limit of the asymptotic states of Eq. (2.17) and the coherent states of Eq. (2.18) cannot be in the same space as the limit of the eigenstates $|E,n\rangle^{(\epsilon)}$ of $H_0^{(\epsilon)}$. While it is conventional to insist that the last remain in Fock space, our approach is to require the asymptotic states $|\psi_n^\pm(E)\rangle^{A(\epsilon)}$ to remain in Fock space $\mathcal{H}_F$ in the limit that $\epsilon\to0$. Note that this implies that the coherent states arising in the limit $\epsilon\to0$ in Eq. (2.18) are neither in Fock space nor in the space of the free-particle Hamiltonian to which the $|E,n\rangle^{(\epsilon)}$ tend.

This concludes our brief discussion of some of the mathematical niceties associated with coherent states, inequivalent representations of the canonical commutation relations, and von Neumann space. We wish to add a few remarks concerning some points of disagreement with Ref. [19]. For purposes of this discussion, we shall follow their notation and treatment of the canonical communation relations and asymptotic space. First of all, it is clear that their $W(t)$ and $W^\dagger(t)$ lead to inequivalent asymptotic spaces.[44] The true asymptotic states are in $W^\dagger(t)\mathcal{H}_F$, whereas it is $W(t)$ that occurs in the definition

[43] By explicitly extracting a Coulomb "phase factor" Φ, Kulish and Faddeev [19] finesse this aspect of Kibble's construction and succeed in having transitions occur within a single space of asymptotic states. It is not clear whether it is possible to extend it to collinear singularities or how to generalize their technique beyond QED.

[44] $W(t)$ is like our $\exp(iH_At)$, but with a Coulomb phase $\exp[i\Phi(t)]$ factored out.

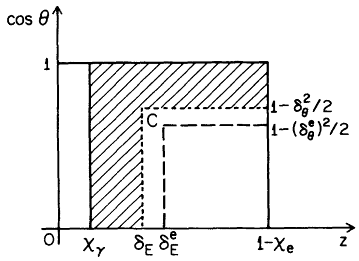

FIG. 7. Arbitrary vs experimental degeneracy domain. The shaded region denotes the arbitrary domain. Region C denotes its complement up to the experimental one.

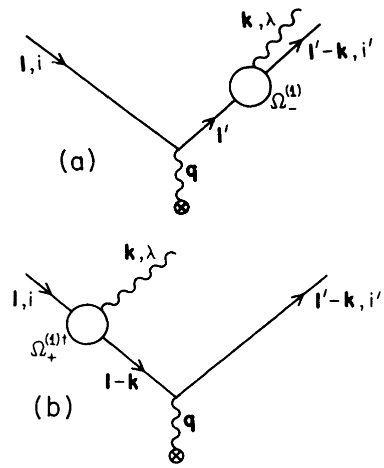

FIG. 8. First-order asymptotic transformation of the two-particle final state.

of their asymptotic operator $U_{\rm as}(t)$. They argue that, in fact, $W(t)$ can, in the limit $t\rightarrow\mp\infty$, be replaced by the identity operator in the modified S matrix, so that their equivalent of S_A reduces to a Fock-space operator. We believe this is incorrect; this is a very singular limit. As evidence of just how delicate the situation is, we note that, at the same time, they argue that the asymptotic states in $W^{\dagger}(t)\mathcal{H}_F$ do not, in the limit $t\rightarrow\mp\infty$, reduce to Fock states, but remain orthogonal thereto. We believe that this is inconsistent. In fact, in their application to scattering in an external field, they show that a Chung coherent state is equivalent to the state $\exp(R_f)b_i^{\dagger}|0\rangle$ and not to the asymptotic state $\exp(-R_f)b_i^{\dagger}|0\rangle$ as they claim. This sign difference is absolutely essential, showing that what was really calculated was $S_{\rm FD}[H]$ between coherent states, as we have derived, and not between asymptotic states, as they have claimed.

APPENDIX B

In this appendix we shall prove that there is a continuous transition between the S-matrix and cross-section formalisms, depending on the choice of the parameters in the asymptotic Hamiltonian.

Denoting the S_A matrix by $S_A(\Delta)\equiv\Omega_-(\Delta)S_{\rm FD}\Omega_+^{\dagger}(\Delta)$, we may prove the equality

$$\begin{aligned}
&2\,{\rm Re}\{\langle e(\boldsymbol{l}\,';i')|S_A^{(2)}(\Delta)|e(\boldsymbol{l};i)\rangle[\langle e(\boldsymbol{l}\,';i')|S_A^{(0)}|e(\boldsymbol{l};i)\rangle]^{\dagger}\}+\sum_{\lambda}\int^{\Delta_s^e}d\Phi|\langle e(\boldsymbol{l}\,'-\mathbf{k};i')\gamma(\mathbf{k};\lambda)|S_A^{(1)}(\Delta_s)|e(\boldsymbol{l};i)\rangle|^2\\
&\quad+\sum_{\lambda}\int^{\Delta_{\rm hc}^e}d\Phi_F|\langle e(\boldsymbol{l}\,'-\mathbf{k};i')\gamma(\mathbf{k};\lambda)|S_A^{(1)}(\Delta_{\rm hc})|e(\boldsymbol{l};i)\rangle|^2+\sum_{\lambda}\int^{\Delta_{\rm hc}^e}d\Phi_I|\langle e(\boldsymbol{l}\,';i')|S_A^{(1)}(\Delta_{\rm hc})|\gamma(\mathbf{k};\lambda)e(\boldsymbol{l}-\mathbf{k};i)\rangle|^2\\
&\qquad=2\,{\rm Re}\{\langle e(\boldsymbol{l}\,';i')|S_A^{(2)}(\Delta^e)|e(\boldsymbol{l};i)\rangle[\langle e(\boldsymbol{l}\,';i')|S_A^{(0)}|e(\boldsymbol{l};i)\rangle]^{\dagger}\}\ . \qquad \text{(B1)}
\end{aligned}$$

In the above $d\Phi_{F/I}$ is the photon phase space of either the initial or final degenerate Fock state, and the decomposition of the asymptotic phase space into soft and hard-collinear regions is denoted by $\Delta=\Delta_s\cup\Delta_{\rm hc}$ and $\Delta^e=\Delta_s^e\cup\Delta_{\rm hc}^e$ for the arbitrary and experimental degeneracy domains, respectively (see Fig. 7).

To prove the above statement in the soft region,[45] let us compute the matrix element

$$\begin{aligned}
(S_A^{(1)})_{fi}&=\langle e(\boldsymbol{l}\,'-\mathbf{k};i')\gamma(\mathbf{k};\lambda)|[\Omega_-(\Delta_s)S_{\rm FD}\Omega_+^{\dagger}(\Delta_s)]^{(1)}|e(\boldsymbol{l};i)\rangle\\
&=\langle e(\boldsymbol{l}\,'-\mathbf{k};i')\gamma(\mathbf{k};\lambda)|\Omega_-^{(1)}(\Delta_s)S_{\rm FD}^{(0)}|e(\boldsymbol{l};i)\rangle+\langle e(\boldsymbol{l}\,'-\mathbf{k};i')\gamma(\mathbf{k};\lambda)|S_{\rm FD}^{(0)}\Omega_+^{(1)\dagger}(\Delta_s)|e(\boldsymbol{l};i)\rangle\\
&\quad+\langle e(\boldsymbol{l}\,'-\mathbf{k};i')\gamma(\mathbf{k};\lambda)|S^{(1)}|e(\boldsymbol{l};i)\rangle\equiv S^{(1,0)}+S^{(0,1)}+S^{(1)}\ .
\end{aligned}$$

Computing the first term, we find

$$S^{(1,0)}=-e\sum_{\alpha}\langle e(\boldsymbol{l}\,';\alpha)|S_{\rm FD}^{(0)}|e(\boldsymbol{l};i)\rangle\overline{U}^{\,i'}(\boldsymbol{l}\,'-\mathbf{k})\not{\epsilon}_{\lambda}^{*}(\mathbf{k})U^{\alpha}(\boldsymbol{l}\,')\frac{\Theta(\Delta-\nu')}{2\omega(\boldsymbol{l}\,',m_e)\nu'}\ .$$

This term corresponds to the diagram of Fig. 8(a). Similarly,

$$S^{(0,1)}=e\sum_{\alpha}\langle e(\boldsymbol{l}\,'-\mathbf{k};i')|S^{(0)}|e(\boldsymbol{l}-\mathbf{k};\alpha)\rangle\overline{U}^{\,\alpha}(\boldsymbol{l}-\mathbf{k})\not{\epsilon}_{\lambda}^{*}(\mathbf{k})U^{i}(\boldsymbol{l})\frac{\Theta(\Delta-\nu)}{2\omega(\boldsymbol{l}-\mathbf{k},m_e)\nu}\ .$$

[45]The hard-collinear region can be treated in a completely analogous manner.

The corresponding diagram is shown in Fig. 8(b). Finally, the bremsstrahlung matrix element, represented by the Feynman graphs of Fig. 9, is

$$S^{(1)}=e\bar{U}^{\,i'}(\mathbf{l}\,'-\mathbf{k})\not{\epsilon}^{*}_{\lambda}(\mathbf{k})\frac{N_F[\omega(\mathbf{l}\,',m_e)+\nu';\mathbf{l}\,']}{D_F[\omega(\mathbf{l}\,',m_e)+\nu';\mathbf{l}\,']}\boldsymbol{J}U^{i}(\mathbf{l})+e\bar{U}^{\,i'}(\mathbf{l}\,'-\mathbf{k})\boldsymbol{J}\frac{\not{l}-\not{k}+m_e}{(l-k)^2-m_e^2}\not{\epsilon}^{*}_{\lambda}(\mathbf{k})U^{i}(\mathbf{l})\ .$$

Note that

$$D_F[\omega(\mathbf{l}\,',m_e)+\nu';\mathbf{l}\,']=\nu'[\nu'+2\omega(\mathbf{l},m_e)]$$

and

$$(l-k)^2-m_e^2=\nu[\nu-2\omega(\mathbf{l}-\mathbf{k},m_e)]\ .$$

We may perform the Taylor expansions

$$\frac{1}{D_F[\omega(\mathbf{l}\,',m_e)+\nu';\mathbf{l}\,']}=\frac{1}{2\omega(\mathbf{l},m_e)\nu'}-\frac{1}{[2\omega(\mathbf{l},m_e)]^2}\sum_{k=0}^{\infty}(-1)^k\chi_{\nu'}^{k}\ ,$$

$$\frac{1}{(l-k)^2-m_e^2}=-\frac{1}{2\omega(\mathbf{l}-\mathbf{k},m_e)\nu}-\frac{1}{[2\omega(\mathbf{l}-\mathbf{k},m_e)]^2}\sum_{k=0}^{\infty}\chi_{\nu}^{k}\ ,$$

with $\chi_{\nu'}\equiv\nu'/2\omega(\mathbf{l},m_e)$ and $\chi_{\nu}\equiv\nu/2\omega(\mathbf{l}-\mathbf{k},m_e)$.

Therefore, the asymptotic S-matrix element in the soft region will be

$$\begin{aligned}(S_A^{(1)})_{fi}=e\bar{U}^{\,i'}(\mathbf{l}\,'-\mathbf{k})\Bigg\{&\not{\epsilon}^{*}_{\lambda}(\mathbf{k})(\not{l}+m_e)\boldsymbol{J}\frac{\Theta(\nu'-\Delta_s)}{2\omega(\mathbf{l},m_e)\nu'}-\boldsymbol{J}(\not{l}-\not{k}+m_e)\not{\epsilon}^{*}_{\lambda}(\mathbf{k})\frac{\Theta(\nu-\Delta_s)}{2\omega(\mathbf{l}-\mathbf{k},m_e)\nu}\\&+\boldsymbol{J}\gamma_0\not{\epsilon}^{*}_{\lambda}(\mathbf{k})\frac{\Theta(\Delta_s-\nu)}{2\omega(\mathbf{l}-\mathbf{k},m_e)}-\not{\epsilon}^{*}_{\lambda}(\mathbf{k})[\not{l}\,'+m_e-2\omega(\mathbf{l},m_e)\gamma_0]\boldsymbol{J}\frac{1}{[2\omega(\mathbf{l},m_e)]^2}\sum_{k=0}^{\infty}(-1)^k\chi_{\nu'}^{k}\\&-\boldsymbol{J}(\not{l}-\not{k}+m_e)\not{\epsilon}^{*}_{\lambda}(\mathbf{k})\frac{1}{[2\omega(\mathbf{l}-\mathbf{k},m_e)]^2}\sum_{k=0}^{\infty}\chi_{\nu}^{k}\Bigg\}U^{i}(\mathbf{l})\ .\end{aligned}$$

Note that the arguments of the Θ functions that multiply singular denominators in this region of the degenerate domain have their arguments reversed, as a result of a partial cancellation between the diagrams of Figs. 8 and 9, respectively. Therefore, when individual diagrams are combined, the total S_A-matrix element is completely regular in the masses *even without the use of an explicit set of regulators.* The only contribution of the above quantity in the soft region is

$$(S_A^{(1)})_{fi}=e\bar{U}^{\,i'}(\mathbf{l}\,')\boldsymbol{J}U^{i}(\mathbf{l})\left[\frac{(\epsilon^{*}_{\lambda}l')}{(l'k)}-\frac{(\epsilon^{*}_{\lambda}l)}{(lk)}\right]\Theta(z-\delta_E)\ . \tag{B2}$$

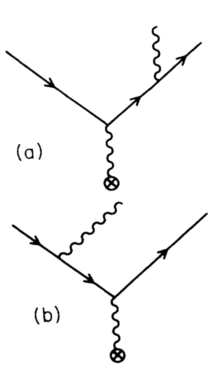

FIG. 9. Final-state bremsstrahlung in the soft region.

Therefore, the contribution to the cross section will be

$$\begin{aligned}\int_{\chi_\gamma}^{\delta_E^e}\widetilde{d^3k}\,|(S_A^{(1)})_{fi}|^2&\\=|(S^{(0)})_{fi}|^2&\\\times e^2\int_{\delta_E}^{\delta_E^e}\widetilde{d^3k}&\left[-\frac{m_e^2}{(lk)^2}-\frac{m_e^2}{(l'k)^2}+\frac{2(ll')}{(lk)(l'k)}\right]\ .\end{aligned} \tag{B3}$$

We see immediately that this contribution shifts the arbitrary parameter δ_E of the interference term on the left-hand side (LHS) of Eq. (B1) to the correct experimental value of the corresponding terms on the RHS. A similar proof can be given for the hard-collinear region. We note that, had we chosen the asymptotic Hamiltonian to coincide with the one describing exactly the experimental asymptotic states, which in terms of asymptotic phase space would mean that we would identify the arbitrary degeneracy domain Δ with the experimental one, Δ^e, we would not need the inelastic terms appearing in Eq. (B1). Hence, in a sense, the transformation of the Fock states into coherent states can be fine-tuned according to the values of the degeneracy domain. When we tune the coherent states onto the experimental ones, the cross-section formalism (inelastic approach) goes over continuously to the S-matrix formalism (elastic approach).

Although we have only proved it in lowest nontrivial order, it is a plausible conjecture that, with the particular asymptotic Hamiltonian chosen to simulate the experimental resolutions, there are no contributions from $\Omega_\pm^\dagger$ acting on multiparticle Fock states to all orders. This shows that all asymptotic Hamiltonians lead to the same observables, but choosing H_A with a particular measurement in mind enormously simplifies both the interpretation of the associated asymptotic states and the complexity of the calculations that must be performed.

One can actually show that Eq. (B1) as well as the complete symmetry between initial and final states, even in the soft region, follows from the unitarity of the asymptotic S matrix. One could argue against the symmetrical status of Eq. (B1) on the ground that it only contains a final-state soft photon. We show that one of the corollaries of unitarity is the fact that the treatment of the soft region is indeed completely symmetrical, but only the usual Bloch-Nordsieck soft bremsstrahlung survives. This establishes the physical intuition that a soft photon has really no direction.

Proceeding with the proof of the above statements, we shall restrain the discussion to the soft region. Given that the process is factorizable, the hard-collinear region may be treated in an analogous manner. By construction,

$$S_A^\dagger(\Delta_s)S_A(\Delta_s)=1 \ . \tag{B4}$$

The soft degenerate Hilbert subspace of an electron of four-momentum l is spanned by a complete set of Fock states defined as

$$\{|n\rangle\}=\left\{\prod_{i=0}^{n}|\gamma(\mathbf{k}_i)\rangle\otimes\left|e\left[\mathbf{l}-\sum_{i=0}^{n}\mathbf{k}_i\right]\right\rangle\right\} \ .$$

Completeness in the soft region may be written as

$$\sum_{n=0}^{\infty}\prod_{i=0}^{n}\int^{\Delta_s^e}\widetilde{d^3k_i}|n\rangle\langle n|\equiv\int_{n=0}^{\infty}|n\rangle\langle n|=1 \ . \tag{B5}$$

In the above we used the notation $|n=0\rangle\equiv|e(l')\rangle$ and $\widetilde{d^3k_0}\equiv\widetilde{d^3l'}$. Note that we have taken the boundary of the degenerate soft subspace Δ_s^e to coincide with the experimental energy resolution, since this is to be thought of as a completeness relation containing those states that are experimentally indistinguishable. Also, in Eq. (B5) we have not included states containing more than one electron, since those states will not contribute in this order.

From Eqs. (B4) and (B5) follows the identity

$$\left\langle e(l)\left|S_A^\dagger(\Delta s)\int_{n=0}^{\infty}\right|n\right\rangle\langle n|S_A(\Delta_s)|e(l)\rangle=\left\langle e(l)\left|S_A^\dagger(\Delta_s')\int_{n=0}^{\infty}\right|n\right\rangle\langle n|S_A(\Delta_s')|e(l)\rangle \quad \forall\Delta_s,\Delta_s'\le\Delta_s^e \ .$$

Making a perturbative expansion of S_A and keeping the second-order terms from the above identity, we have

$$\begin{aligned}\int_{n=0}^{\infty}2\,\mathrm{Re}\{\langle n|S_A^{(2)}(\Delta_s)|e(l)\rangle[\langle n|S_A^{(0)}(\Delta_s)|e(l)\rangle]^\dagger\}+\int_{n=0}^{\infty}|\langle n|S_A^{(1)}(\Delta_s)|e(l)\rangle|^2&\\=\int_{n=0}^{\infty}2\,\mathrm{Re}\{\langle n|S_A^{(2)}(\Delta_s')|e(l)\rangle[\langle n|S_A^{(0)}(\Delta_s')|e(l)\rangle]^\dagger\}+\int_{n=0}^{\infty}|\langle n|S_A^{(1)}(\Delta_s')|e(l)\rangle|^2 \ .&\end{aligned} \tag{B6}$$

Let us now see what the degenerate states contributing in Eq. (B6) are. Concentrating first on the interference term, we can see that $S_A^{(2)}$ contains four soft "virtual" integrations (i.e., integrations with an upper limit Δ_s) and six particle operators. On the other hand, the second factor of the interference term survives only for $|n\rangle=|n=0\rangle\equiv|e(l')\rangle$. Hence we obtain three δ functions and one soft "virtual" integration survives.

Looking at the square term, we remark that $S_A^{(1)}$ contains two soft "virtual" integrations and three particle operators. The degenerate Fock state contributing is $|n\rangle=|n=1\rangle$. That means that the total number of extra particle operators[46] is four; hence two δ functions are produced. Consequently, no soft "virtual" integrations survive. Nevertheless, a "real" integration over the degenerate phase space is introduced from the completeness of the degenerate subspace.[47] Noting that the matrix element of $S_A^{(1)}$ is regular, as was explicitly shown earlier in this appendix,[48] we conclude that Eq. (B6) leads to the identity

$$\begin{aligned}2\,\mathrm{Re}\{\langle e(l')|S_A^{(2)}(\Delta_s)|e(l)\rangle[\langle e(l')|S_A^{(0)}(\Delta_s)|e(l)\rangle]^\dagger\}+\int_{\Delta_s}^{\Delta_s^e}\widetilde{d^3k}|\langle 1|S_A^{(1)}(\Delta_s)|e(l)\rangle|^2&\\=2\,\mathrm{Re}\{\langle e(l')|S_A^{(2)}(\Delta_s')|e(l)\rangle[\langle e(l')|S_A^{(0)}(\Delta_s')|e(l)\rangle]^\dagger\}+\int_{\Delta_s'}^{\Delta_s^e}\widetilde{d^3k}|\langle 1|S_A^{(1)}(\Delta_s')|e(l)\rangle|^2 \quad \forall\Delta_s,\Delta_s'\le\Delta_s^e \ .&\end{aligned} \tag{B7}$$

Choosing $\Delta_s'=\Delta_s^e$, we recover the soft part of Eq. (B1).

An interesting corollary follows from Eq. (B6) if we apply it to a two-particle initial state $|i\rangle=|e(\mathbf{l}-\mathbf{k})\gamma(\mathbf{k})\rangle\equiv|e\gamma(l)\rangle$. If we now concentrate on the interference term, we see that only the state $|n\rangle=|1\rangle$ survives. The matrix element $\langle i|S_A^{(2)}(\Delta_s)|1\rangle$ contains four integrations and six plus two additional particle operators; therefore, four δ functions are introduced and no soft "virtual" integration survives. For the square term we find, reasoning as before, that the only state contributing is the $|n\rangle=|n=0\rangle$ one. On the other hand, all matrix elements are regular; i.e., they contain Θ functions with inverted arguments. Therefore, integrating Eq. (B6) on the degenerate final space, we obtain the equality[49]

[46]"Extra" meaning in addition to the two electronic operators necessary to produce zeroth-order electron scattering.

[47]To be precise, an overall integration $\widetilde{d^3l'}$ also remains for all the terms in Eq. (B6). This is canceled by the overall energy-momentum conservation characterizing the zeroth-order process.

[48]This was shown to result from the reversal of the argument of the Θ function.

[49]Note that the $n=1$ integration is cancelled by the *disconnected* matrix element $\langle 1|S_A^{(0)}(\Delta_s)|i\rangle$.

$$\int_{\Delta_s}^{\Delta_s^e}\widetilde{d^3k}\left[2\,\mathrm{Re}\left[\int_{n=1}\langle 1|S_A^{(2)}(\Delta_s)|i\rangle[\langle 1|S_A^{(0)}(\Delta_s)|i\rangle]^\dagger\right]+|\langle 0|S_A^{(1)}(\Delta_s)|i\rangle|^2\right]$$

$$=\int_{\Delta_s'}^{\Delta_s^e}\widetilde{d^3k}\left[2\,\mathrm{Re}\left[\int_{n=1}\langle 1|S_A^{(2)}(\Delta_s')|i\rangle[\langle 1|S_A^{(0)}(\Delta_s')|i\rangle]^\dagger\right]+|\langle 0|S_A^{(1)}(\Delta_s')|i\rangle|^2\right]. \quad \text{(B8)}$$

Hence, choosing $\Delta_s'=0$, we verify that the sum of the inelastic contributions, which should in principle be included to restore the symmetrical status of Eq. (B1) in the soft region, is equal to zero. In other words, the usual Bloch-Nordsieck mechanism is recovered in the soft region, but this mechanism is proven here to be indeed *symmetric* as far as initial- and final-state degeneracies are concerned.

APPENDIX C: ASYMPTOTIC INTERACTION PICTURE

Let us define an "asymptotic interaction picture" by transforming by the unitary operator e^{iH_At} from the Schrödinger picture. Schrödinger picture operators $\mathcal{O}_{\mathcal{S}}$ become $\widehat{\mathcal{O}(t)}\equiv e^{iH_At}\mathcal{O}_{\mathcal{S}}e^{-iH_At}$ and Schrödinger states $|\psi(t)\rangle$ go over to $|\hat{\psi}(t)\rangle\equiv e^{iH_At}|\psi(t)\rangle$. It is straightforward to show that

$$i\frac{d}{dt}\Omega^\dagger_{H,H_A}(t)=\widehat{V_I(t)}\Omega^\dagger_{H,H_A}(t)\ , \quad \text{(C1)}$$

where the interaction is defined as $\widehat{V_I(t)}\equiv e^{iH_At}V_Ie^{-iH_At}$ and $V_I\equiv H-H_A$. If H_A were replaced by H_0 here, one would recognize this as the familiar evolution equation in the interaction picture. Note that, by the choice of H_A, $H-H_A$ involves vertices that specifically *exclude* infrared or collinear quanta; that is, these vertices precisely vanish for those phase-space configurations that potentially give rise to mass singularities. The solution to Eq. (C1) is, as usual,

$$\Omega^\dagger_{H,H_A}(t)=T\exp\left[-i\int_0^t dt'\widehat{V_I(t')}\right]. \quad \text{(C2)}$$

Lest there by any confusion, this operator has no mass singularities. They could potentially arise in the limit as the time $t\to\mp\infty$; i.e., the operators

$$\Omega^{(\pm)}_{H,H_A}=T\exp\left[-i\int_{\mp\infty}^0 dt\ \widehat{V_I(t)}\right],$$
$$S_A=T\exp\left[-i\int_{-\infty}^{+\infty}dt\ \widehat{V_I(t)}\right] \quad \text{(C3)}$$

might have mass singularities. By our assumption about asymptotic convergence, such operators do not have mass singularities when evaluated between states of $\mathcal{H}_A$. It is the purpose of this discussion to argue that not just the matrix elements in $\mathcal{H}_A$, but also the matrix elements in $\mathcal{H}_F$ are free of mass singularities.[50]

Imagine evaluating Fock-space matrix elements of $\Omega^{(\pm)}_{H,H_A}$. In the asymptotic interaction picture, the propagators take the form of the exact propagators in a theory with Hamiltonian H_A. Whereas the vertices associated with $V_I=H-H_A$ involve no infrared or collinear quanta, the propagators do. This is a great advantage of the asymptotic interaction picture, inasmuch as all mass singularities are isolated in the propagators, and we know that these singularities will show up only when one attempts to go on mass shell. Fock-space matrix elements involving external on-shell particles manifest mass singularities when an internal propagator goes on mass shell, but these regions of phase space are specifically *excluded* by the vertices coming from V_I. So the potential singularities coming from propagators multiply vertices that vanish precisely where those singularities would occur. Thus the proper vertices [one-particle irreducible (1PI), truncated n-point functions] of the theory and matrix elements of operators such as $\Omega^{(\pm)}_{H,H_A}$ that only involve noninfrared and noncollinear particles have no mass singularities at all.

In this argument we have finessed a subtle complication here, inasmuch as the relation between n-point functions and properly normalized matrix elements involve on-mass-shell "wave-function" renormalization constants. But in the massless theory, the exact propagator and the propagator evolving by the asymptotic dynamics H_A do not have a simple particle pole, but rather only a branch point for the continuum of states that, in the ordinary interaction picture, consist of a particle together with any number of soft or collinear quanta. Thus the very definition of wave-function renormalization is problematic, although it can be dealt with [14] and one can even develop an LSZ-type reduction formalism [15]. This is best addressed in the context of regularization of mass singularities, similar to the discussion in Sec. II, but we note that the wave-function renormalization is gauge dependent, and one might even find gauges in which it does not have mass singularities. So we think these complications are technical rather than fundamental.

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[50]It is plausible that if an operator is a unitary operator in one equivalence class, it changes occupation numbers by a finite amount and so is unitary in all the equivalence classes in the $\mathcal{H}_{\mathrm{vN}}$. This seems to be conjectured by Barton [40], and if it could be proved, our subsequent argument could be omitted.

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