PHYSICAL REVIEW D VOLUME 45, NUMBER 4 15 FEBRUARY 1992

Symplectic structures in the chirally gauged Wess-Zumino-Witten model

Tadashi Inamoto
Department of Physics, University of Tokyo, Bunkyo-ku, Tokyo 113, Japan
(Received 20 September 1991)

The algebraic structure of the chirally gauged Wess-Zumino-Witten model is studied from the Hamiltonian point of view. The consistent chiral anomaly, which is reproduced at the tree level in this model, is related to the Schwinger term of the Gauss-law algebra through descent equations constructed with phase-space differential forms. The descent equations express the effects of the consistent anomaly upon the symplectic structure of the theory, and provide the Hamiltonian analogue of the Wess-Zumino consistency condition in the Weyl gauge. We also clarify the canonical structure of the ungauged Wess-Zumino-Witten model, and the algebra associated with the global Noether symmetry is derived.

PACS number(s): 11.10.Ef, 02.40.+m, 11.30.Rd

I. INTRODUCTION

In the past decades chiral anomalies have been extensively studied by many physicists [1]. In particular, for the Hamiltonian formulation of quantized chiral gauge theories, Faddeev [2] has observed via gauge-group-cohomological analysis that the anomaly appears as an operator-valued Schwinger term ("commutator anomaly") in the Gauss-law algebra, a conjecture which has been established by perturbative calculations [3]. From physical viewpoints, the difficulties associated with the occurrence of commutator anomalies come from the fact that one cannot consistently define gauge-invariant physical states by imposing the Gauss law as a subsidiary condition. Classically, this implies that the Gauss-law constraint comes to have a second-class character in the sense of Dirac [4]. Therefore, the canonical structures of the theory drastically change owing to Schwinger terms.

On the other hand, for chiral gauge theories, it is widely known that the covariant conservation law of a chiral current acquires an anomaly as a result of the anomalous gauge variation of the chiral effective action. Such an anomaly is called a divergence (or current) anomaly. In the presence of divergence anomalies, the Ward-Takahashi identity is violated to spoil the renormalizability of the theory.

The two types of anomalies stated above are usually viewed as different manifestations of a same phenomenon, and many authors have attempted to relate commutator anomalies to divergence ones and describe these within a unified framework [5,6]. These works have shed light on the physical content of chiral anomalies.

Here we approach this problem along a somewhat different line of argument. Namely we are concerned with the chirally gauged Wess-Zumino-Witten (WZW) model in four dimensions [7]. As is well known, the chirally gauged WZW model incorporates at the classical level the anomalous nature of the corresponding chiral gauge theory. That is to say, the gauge variation of the WZW action precisely reproduces the *consistent* divergence anomaly, without going through any quantization processes. When it comes to the Hamiltonian formulation, the straightforward canonical analysis [8,9] tells us that the Poisson brackets among Gauss-law constraints give the Schwinger term, the form of which agrees (up to coboundary) with that of the Schwinger term predicted by Faddeev. Thus, for this model, one can apply a purely classical method in studying both types of anomalies, avoiding some technical complications characteristic to the quantization of field theories. This is the reason why we take up this model in the present paper.

More concretely, in the chirally gauged WZW model the WZW action is added by hand to the original (gauge-invariant) classical action. Then the total action is not invariant under gauge transformations, but develops divergence anomalies. Now one is to notice [10] that the classical action (or Lagrangian) is used to define a Lagrangian one-form on the velocity phase space—the tangent bundle over the configuration space. Consequently, the classical action defines a Lagrangian (pre)symplectic structure, which in turn gives a canonical symplectic structure on the phase space (i.e., the cotangent bundle) via Legendre transformations. Thus, the presence of the WZW term in the action requires some modification in the symplectic structure, with which the modified Poisson brackets are constructed. In this way the resulting Poisson-bracket algebra of Gauss-law constraints will be accompanied with a Schwinger term. It is hence interesting to reanalyze, along this line [11], the canonical formalism developed in Refs. [8,9], and this is the main purpose of this work.

The general framework for our study was given in Ref. [12], in which the Lagrangian (or Hamiltonian) systems with a noncanonical action of the group of symmetry transformations were shown to be equipped naturally with a certain cohomological structure. Here the terminology "noncanonical" means that the symplectic structure is not invariant under the transformation, but acquires an anomalous term. Then it was recognized that this anomalous change of the symplectic structure satisfies a one-cocycle condition, and belongs to a family of cocycles taking values in differential forms on a phase space. In fact, the *classical* symplectic-geometrical analysis of the *quantized* chiral gauge theory has been given in Refs. [13,14], however, somewhat unsatisfactory results have come out owing to quantum effects.

In this paper, we treat the chirally gauged WZW model as a system on which the group of gauge transformations acts noncanonically. The anomalous variation of the action turns out to be realized as an infinitesimal deformation of the symplectic structure in the fixed-time Hamiltonian formalism. Then it will be found that the one-cocycle condition for the deformation is just the Hamiltonian analogue of the Wess-Zumino condition [15] for the consistent anomaly. On the other hand, the one-cocycle condition is related to the algebra of Gauss-law constraints. We also study the ungauged WZW model in four dimensions [7,15], and propose the new method to derive the algebra of Noether-charge densities [16,17] corresponding to the global symmetry of the model. In addition, we briefly comment on the "Becchi-Rouet-Stora-Tyutin (BRST) structure" of the formalism.

This paper is organized as follows. We begin, in Sec. II, by giving a simple but illustrative example. A short review of the gauged nonlinear σ model without a WZW term is presented in Sec. III. We study, in Sec. IV, the chirally gauged model, and elucidate the algebraic structure of the model from the Hamiltonian point of view, emphasizing how both types of anomalies enter the descent equations. We give a new interpretation of the Wess-Zumino consistency condition in Sec. V, and also make some comments concerning the cohomological nontriviality of the Schwinger term. Section VI is devoted to studying the ungauged WZW model. We give concluding remarks in Sec. VII.

II. A SIMPLE EXAMPLE

In this section, we consider a very simple example—the motion of a charged particle in a magnetic field. Though this example has been treated also in Ref. [12], we trace the calculation more explicitly in order to make this paper self-contained.

First let us consider the Lagrangian of a free nonrelativistic particle of mass m:

$$\mathcal{L}_0=\frac{1}{2m}\dot{q}_i\dot{q}_i \ . \tag{2.1}$$

The dynamics of this system is described by the Hamiltonian

$$\mathcal{H}_0=\frac{1}{2m}p^ip^i, \quad p^i\equiv m\dot{q}_i \ , \tag{2.2}$$

with the fundamental Poisson brackets

$$\{q_i,p^j\}=\delta_i^j, \quad \{q_i,q_j\}=\{p^i,p^j\}=0 \ . \tag{2.3}$$

Once the magnetic interaction is switched on, the dynamics is governed by

$$\mathcal{H}=\frac{1}{2m}(\tilde{p}^{\,i}-A^i)^2 \ , \tag{2.4}$$

with

$$\{q_i,\tilde{p}^{\,j}\}=\delta_i^j, \quad \{q_i,q_j\}=\{\tilde{p}^{\,i},\tilde{p}^{\,j}\}=0 \ , \tag{2.5}$$

where $\tilde{p}^{\,i}\equiv m\dot{q}_i+A^i=p^i+A^i$, and A^i is an external magnetic potential.

Sternberg [18] has claimed that the Hamiltonian system (2.4) and (2.5) is alternatively described in terms of the variable p^i, not $\tilde{p}^{\,i}$, as

$$\mathcal{H}=\frac{1}{2m}p^ip^i \ , \tag{2.6}$$

$$\{q_i,p^j\}=\delta_i^j, \quad \{q_i,q_j\}=0, \quad \{p^i,p^j\}=F^{ij} \ . \tag{2.7}$$

Here F^{ij} is a magnetic field strength. Comparing Eqs. (2.6) and (2.7) with Eqs. (2.2) and (2.3), we find that the effects of the magnetic interaction appear in the modification of the Poisson brackets among the momenta, while in Eqs. (2.4) and (2.5) the modification is in the Hamiltonian. In mathematical language, modifying the Poisson brackets as above amounts to modifying the canonical symplectic structure $\Omega=dp^i\wedge dq_i$ [10] as

$$\Omega\rightarrow\Omega+dA \ , \tag{2.8}$$

where $A\equiv A^idq_i$ is a potential one-form and dA is a curvature two-form, both of which are regarded as differential forms on the phase space rather than those on the configuration space.

Now we return to the standard formalism (2.4) and (2.5), and consider the translational symmetry of the minimally coupled Lagrangian:

$$\mathcal{L}=\mathcal{L}_0+A^i\dot{q}_i \ . \tag{2.9}$$

Evidently this Lagrangian is not invariant under the translation of the coordinate,

$$q_i\rightarrow q_i+a_i \ , \tag{2.10}$$

but is accompanied with an "anomaly" linear in velocities. Indeed the Lagrangian changes under (2.10) as $\mathcal{L}\rightarrow\mathcal{L}+\Delta\mathcal{L}_a$, where

$$\Delta\mathcal{L}_a=a_i\frac{\partial A^j}{\partial q_i}\dot{q}_j \ . \tag{2.11}$$

The symplectic structure associated with the transformed Lagrangian $\mathcal{L}+\Delta\mathcal{L}_a$ is calculated as $\Omega+\omega_a$, with

$$\omega_a=\tfrac{1}{2}a_k\partial^kF^{ij}dq_i\wedge dq_j \ . \tag{2.12}$$

Then one may ask from where the anomalous change ω_a of Ω comes. Quite evidently, it originates from the second term in the Lagrangian (2.9). We proceed, however, in a slightly different (but equivalent) way. Namely we consider the system in which the symplectic structure is given by $\Omega+\Delta\Omega$ in place of Ω, where the *closed* two-form $\Delta\Omega$ is determined so that the transformation of the total symplectic structure automatically reproduces the anomalous term; i.e., $\Delta\Omega$ is required to satisfy

$$L(X_a)\Delta\Omega=\omega_a \ , \tag{2.13}$$

where X_a is a vector field defined by $i(X_a)\Omega=d(a_ip^i)$ with $i(\cdots)$ being an inner product operating on differential forms of arbitrary ranks, and $L(X_a)$ is a Lie derivative in the direction of X_a: $L(X_a)=i(X_a)d+di(X_a)$. (Note that X_a is a Hamiltonian vector field with respect to Ω, and not with respect to $\Omega+\Delta\Omega$.) In fact, the existence of $\Delta\Omega$ obeying Eq. (2.13) is guaranteed only when the following consistency condition holds:

$$L(X_a)\omega_b - L(X_b)\omega_a = 0 \ , \tag{2.14}$$

which is trivially satisfied in our case. (This is a kind of one-cocycle condition from a cohomology-theoretical point of view.) We now construct $\Delta\Omega$ explicitly by solving

$$i(X_a)\Delta\Omega = V_a + dv_a \ , \tag{2.15}$$

instead of Eq. (2.13). Here $V_a = a_i(\partial A^j/\partial q_i)dq_j$ is a one-form defined so that $dV_a = \omega_a$, and v_a is a certain zero-form. The second term on the right-hand side is needed not only to maintain full generality, but to avoid the inconsistency coming from the nilpotency of the i operator. [The one-form V_a does not satisfy $i(X_a)V_a = 0$, and hence the relation (2.15) in the absence of dv_a is inconsistent.] As expected, the solution of Eq. (2.15) compatible with Eq. (2.13) is found to be $\Delta\Omega = dA$, with $v_a = -a_i A^i$.

Another consequence of the one-cocycle condition is that there exists the zero-form S_{ab} such that

$$dS_{ab} = L(X_a)V_b - L(X_b)V_a \ , \tag{2.16}$$

where we have used the commutativity of exterior derivatives with Lie derivatives, and have assumed the triviality of d-cohomology. In fact the straightforward calculation gives $dS_{ab}=0$, however, in many cases dS_{ab} may not vanish and we write dS_{ab} explicitly in the following equations not to lose the generality of our formalism.

Having obtained the closed two-form $\Delta\Omega$, the one-form V_a, and the zero-form S_{ab}, we can construct the descent equations

$$0 = d\Delta\Omega \ , \tag{2.17}$$

$$L(X_a)\Delta\Omega = dV_a \ , \tag{2.18}$$

$$L(X_a)V_b - L(X_b)V_a = dS_{ab} \ . \tag{2.19}$$

Inspired by Eq. (2.15), we also consider the descent equations in slightly modified forms:

$$0 = d\Delta\Omega \ , \tag{2.20}$$

$$L(X_a)\Delta\Omega = d(V_a + dv_a) \ , \tag{2.21}$$

$$L(X_a)(V_b + dv_b) - L(X_b)(V_a + dv_a) = d(S_{ab} + \bar{S}_{ab}) \ , \tag{2.22}$$

where $\bar{S}_{ab}$ is defined by

$$\bar{S}_{ab} = L(X_a)v_b - L(X_b)v_a = -a_i b_j F^{ij} \ . \tag{2.23}$$

As will be shown in Sec. IV, the zero-form $-(S_{ab}+\bar{S}_{ab})$ comes to play the role of a Schwinger term in the algebra of the generators of a symmetry transformation ("translation" in this case) when we pass to the description in terms of Poisson brackets. Equation (2.22) hence provides the algebraic relation between the Schwinger term and the anomaly V_a.

Before closing this section, we emphasize the analogy of this example to the chirally gauged WZW model. In both systems, the underlying symmetry is broken owing to the extra terms added to the original invariant Lagrangian. When going to the Hamiltonian formulation, the effects of these symmetry-violating terms can be implemented, at least partially, in the redefinition of the canonical symplectic structure. Thus, writing down the descent equations in the same way as above, one can clarify hidden algebraic structures of the chirally gauged WZW model, as will be extensively studied in Sec. IV.

III. HAMILTONIAN FORMULATION OF THE GAUGED NONLINEAR σ MODEL

The nonlinear σ model is in general the theory of the maps from space-time into the target manifold, which is usually taken as a group manifold of a certain compact Lie group G. The Lagrangian is given by $\mathcal{L}_0 = f^{-2}\mathrm{tr}(\partial_\mu U U^{-1}\partial^\mu U U^{-1})$, where $U(x)\in G$ and f is some constant [19]. This Lagrangian is invariant under the left (or right) translation. Let $A_\mu = A_\mu^a T^a$ be the Lie (G)-valued Yang-Mills connection, then the (left-)gauged version of the theory is described by the Lagrangian $\mathcal{L}_0 \equiv \mathcal{L}_\Sigma + \mathcal{L}_{\mathrm{YM}}$, where

$$\mathcal{L}_\Sigma = \frac{1}{f^2}\mathrm{tr}[(W_\mu + A_\mu)(W^\mu + A^\mu)] \ , \tag{3.1}$$

$$\mathcal{L}_{\mathrm{YM}} = \tfrac{1}{2}\mathrm{tr}(F_{\mu\nu}F^{\mu\nu}) \ . \tag{3.2}$$

Here $W_\mu = W_\mu^a T^a = \partial_\mu U U^{-1}$, and $F_{\mu\nu} = F_{\mu\nu}^a T^a$ is a curvature. The group of gauge transformations acts on the fields as

$$A_\mu \to A_\mu^g = g^{-1}A_\mu g + g^{-1}\partial_\mu g \ , \tag{3.3}$$

$$U \to U^g = g^{-1}U \ . \tag{3.4}$$

In order to proceed to the Hamiltonian formulation, we introduce the local coordinate ϕ^α on the group manifold [16] so that

$$W_\mu^a = \partial_\mu \phi^\alpha K_\alpha^a(\phi) \ , \tag{3.5}$$

where K_α^a is a geometrical quantity satisfying the Meurer-Cartan equation

$$\partial_\alpha K_\beta^a - \partial_\beta K_\alpha^a = f^{abc}K_\alpha^b K_\beta^c, \quad \partial_\alpha \equiv \frac{\partial}{\partial\phi^\alpha} \ , \tag{3.6}$$

with f^{abc} being the structure constant. Furthermore, we define the components K_a^α of the right-invariant Killing vector field with the properties $K_a^\alpha K_\alpha^b = \delta_a^b$, $K_a^\alpha K_\beta^a = \delta_\beta^\alpha$, and

$$K_a^\alpha \partial_\alpha K_b^\beta - K_b^\alpha \partial_\alpha K_a^\beta = -f^{abc}K_c^\beta \ . \tag{3.7}$$

In terms of the local coordinates, the Lagrangian (3.1) becomes

$$\mathcal{L}_\Sigma = -\frac{1}{2f^2}h_{\alpha\beta}\nabla_\mu\phi^\alpha\nabla^\mu\phi^\beta \ , \tag{3.8}$$

where the covariant derivative $\nabla_\mu\phi^\alpha$ and the invariant metric $h_{\alpha\beta}$ are, respectively, given by

$$\nabla_\mu \phi^\alpha = \partial_\mu \phi^\alpha + A^a_\mu K^\alpha_a \ , \tag{3.9}$$

$$h_{\alpha\beta} = K^a_\alpha K^a_\beta \ . \tag{3.10}$$

The canonical momenta are calculated as

$$P^a_\mu = -F^a_{\mu 0} \ , \tag{3.11}$$

$$\pi_\alpha = \frac{1}{f^2} h_{\alpha\beta} \nabla_0 \phi^\beta \ ; \tag{3.12}$$

then the Hamiltonian turns out to be

$$\mathcal{H}_0 = \tfrac{1}{2}(P^a_i P^a_i + \tfrac{1}{2} F^a_{ij} F^a_{ij}) + \tfrac{1}{2}\left[f^2 h^{\alpha\beta} \pi_\alpha \pi_\beta + \frac{1}{f^2} h_{\alpha\beta} \nabla_i \phi^\alpha \nabla_i \phi^\beta \right] - A^a_0 G_a \ . \tag{3.13}$$

Here $h^{\alpha\beta}$ is the inverse of $h_{\alpha\beta}$, and the coefficient of the multiplier A^a_0 is a Gauss-law function defined by

$$G_a = D_i P^a_i + K^\alpha_a \pi_\alpha, \quad D_i P^a_i \equiv \partial_i P^a_i + f^{abc} A^b_i P^c_i \ . \tag{3.14}$$

Applying Dirac's prescription for constrained systems [4], we obtain the two first-class constraints: $P^a_0 \approx 0$, $G_a \approx 0$. These constraints play the role of the generators of gauge transformations. In particular, the static gauge transformation is generated by

$$G_\xi = \int d^3x \ \xi^a(\mathbf{x}) G_a(\mathbf{x}) \ , \tag{3.15}$$

with the time-independent parameter $\xi^a(\mathbf{x})$.

As in the previous section, we adopt the symplectic-geometrical description. The phase space of our concern is an infinite-dimensional space parametrized by $A^a_i(\mathbf{x})$, $P^a_i(\mathbf{x})$, $\phi^\alpha(\mathbf{x})$, and $\pi_\alpha(\mathbf{x})$ [20]. This space is naturally endowed with a canonical symplectic structure:

$$\Omega = \int d^3x (\delta P^a_i \wedge \delta A^a_i + \delta \pi_\alpha \wedge \delta \phi^\alpha) \ , \tag{3.16}$$

where the symbol δ denotes a (functional) exterior derivative [21]. The resulting equal-time Poisson brackets are

$$\{ A^a_i(\mathbf{x}), P^b_j(\mathbf{y}) \} = \delta_{ij} \delta^{ab} \delta(\mathbf{x}-\mathbf{y}) \ , \tag{3.17}$$

$$\{ \phi^\alpha(\mathbf{x}), \pi_\beta(\mathbf{y}) \} = \delta^\alpha_\beta \delta(\mathbf{x}-\mathbf{y}) \ . \tag{3.18}$$

Since we have used the local coordinates on the group manifold, the expression of the symplectic structure has a local character from the outset. One can, however, rewrite it in terms of global geometrical objects. We define the one-form $J^a = K^a_\alpha \delta\phi^\alpha$, or, equivalently,

$$J = J^a T^a = \delta U U^{-1} \ , \tag{3.19}$$

then the σ-model part of the symplectic structure reduces to

$$\Omega_\Sigma = \int d^3x (\delta \pi_a \wedge J^a + \tfrac{1}{2} f^{abc} \pi_c J^a \wedge J^b) \ . \tag{3.20}$$

Here π_a is defined by

$$\pi_a \equiv K^\alpha_a \pi_\alpha = \frac{1}{f^2}(W^a_0 + A^a_0) \ , \tag{3.21}$$

and the Poisson brackets among the π_a's implied by Eq. (3.20) are written as

$$\{ \pi_a(\mathbf{x}), \pi_b(\mathbf{y}) \} = f^{abc} \pi_c(\mathbf{x}) \delta(\mathbf{x}-\mathbf{y}) \ . \tag{3.22}$$

The one-form J introduced above satisfies the equation of Meurer-Cartan type,

$$\delta J = J^2 \ , \tag{3.23}$$

and the inner product of the Killing vector field with J is

$$i\left[K^\alpha_a(\mathbf{x}) \frac{\delta}{\delta \phi^\alpha(\mathbf{x})} \right] J^b(\mathbf{y}) = \delta^b_a \delta(\mathbf{x}-\mathbf{y}) \ , \tag{3.24}$$

which shows that the Meurer-Cartan form J is a "dual" of the Killing vector field.

In Sec. II, the symmetry transformation was represented by the Hamiltonian vector fields for generators. In this case, the corresponding vector field X_ξ is defined by

$$i(X_\xi)\Omega = \delta G_\xi \ , \tag{3.25}$$

and we obtain

$$X_\xi = \int d^3x \left[D_i \xi^a \frac{\delta}{\delta A^a_i(\mathbf{x})} + \xi^a f^{abc} P^b_i(\mathbf{x}) \frac{\delta}{\delta P^c_i(\mathbf{x})} - \xi^a K^\alpha_a \frac{\delta}{\delta \phi^\alpha(\mathbf{x})} + \xi^a (\partial_\alpha K^\beta_a) \pi_\beta(\mathbf{x}) \frac{\delta}{\delta \pi_\alpha(\mathbf{x})} \right] , \tag{3.26}$$

which consists of the ordinary infinitesimal gauge transformations lifted to the fiber. One can easily show that the Lie-bracket algebra obeyed by the X_ξ's is isomorphic to the Lie algebra of the gauge group:

$$[X_\xi, X_\eta] = X_{[\xi,\eta]}, \quad [\xi,\eta]^a \equiv f^{abc} \xi^b \eta^c \ , \tag{3.27}$$

as expected.

IV. THE DESCENT EQUATIONS

A. Modification of the symplectic structure

The Lagrangian for the chirally gauged WZW model is given by $\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_{\text{WZW}}$, where the WZW *Lagrangian* can be written in the form [8,9]

$$\mathcal{L}_{\text{WZW}} = -\frac{i}{48\pi^2} \epsilon^{\mu\nu\rho\sigma} [D_{abc} A^a_\mu \partial_\nu A^b_\rho W^c_\sigma + B_{abcd} (A^a_\mu A^b_\nu A^c_\rho W^d_\sigma - \tfrac{1}{2} A^a_\mu W^b_\nu A^c_\rho W^d_\sigma - A^a_\mu W^b_\nu W^c_\rho W^d_\sigma) - i \partial_\mu \phi^\alpha \partial_\nu \phi^\beta \partial_\rho \phi^\gamma \partial_\sigma \phi^\delta \tau_{\alpha\beta\gamma\delta}] \ . \tag{4.1}$$

where $D_{abc} \equiv \mathrm{tr}(T^aT^bT^c + T^aT^cT^b)$, and $B_{abcd} \equiv \mathrm{tr}(T^aT^bT^cT^d)$. The last term comes from the usual five-dimensional integral in the WZW *action:*

$$-\frac{i}{240\pi^2}\int \mathrm{tr}[(dUU^{-1})^5] \ . \tag{4.2}$$

Here the symbol d denotes the space-time exterior derivative, and the integration is to be performed over the five-dimensional space, whose boundary is our (compactified) space-time. Note that the integrand is a closed five-form, but not exact. We have hence restricted ourselves to the specific homotopy class [22], in which one can apply a Poincaré lemma to reduce the five-dimensional integral to a four-dimensional one. Namely, the antisymmetric tensor $\tau_{\alpha\beta\gamma\delta}$ is defined so that

$$\frac{i}{240\pi^2}\mathrm{tr}[(dUU^{-1})^5] = d\left[\frac{1}{48\pi^2}\tau_{\alpha\beta\gamma\delta}\, d\phi^\alpha \wedge d\phi^\beta \wedge d\phi^\gamma \wedge d\phi^\delta\right] . \tag{4.3}$$

In analogy with the works of Sternberg [18] and others [23,24], we modify the canonical symplectic structure (3.16). Let us first introduce the canonical momenta conjugate to A_i^a and ϕ^α:

$$\tilde{P}_i^a = P_i^a + \mathcal{A}_i^a \ , \tag{4.4}$$

$$\tilde{\pi}_\alpha = \pi_\alpha + \mathcal{B}_\alpha \ , \tag{4.5}$$

where P_i^a and π_α are given by Eqs. (3.11) and (3.12), and the extra contributions from the WZW term are calculated as

$$\mathcal{A}_i^a = -\frac{i}{48\pi^2}\epsilon^{0ijk} D_{abc} A_j^b W_k^c \ , \tag{4.6}$$

$$\begin{aligned}\mathcal{B}_\alpha = &-\frac{i}{48\pi^2}\epsilon^{0ijk} K_\alpha^d [D_{abd} A_i^a \partial_j A_k^b \\ &+ B_{abcd}(A_i^a A_j^b A_k^c - A_i^a W_j^b A_k^c) \\ &- 3B_{a[bcd]} A_i^a W_j^b W_k^c] \\ &- \frac{1}{12\pi^2}\epsilon^{0ijk}\tau_{\alpha\beta\gamma\delta}\partial_i\phi^\beta \partial_j\phi^\gamma \partial_k\phi^\delta \ ,\end{aligned} \tag{4.7}$$

where the square brackets in the subscript mean antisymmetrization. We assume that the variables $\tilde{P}_i^a$ and $\tilde{\pi}_\alpha$ satisfy canonical Poisson-bracket relations, then those for the variables P_i^a and π_α develop curvaturelike terms:

$$\{P_i^a(\mathbf{x}), P_j^b(\mathbf{y})\} = \frac{\delta \mathcal{A}_j^b(\mathbf{y})}{\delta A_i^a(\mathbf{x})} - \frac{\delta \mathcal{A}_i^a(\mathbf{x})}{\delta A_j^b(\mathbf{y})} \ , \tag{4.8}$$

$$\{\pi_\alpha(\mathbf{x}), \pi_\beta(\mathbf{y})\} = \frac{\delta \mathcal{B}_\beta(\mathbf{y})}{\delta \phi^\alpha(\mathbf{x})} - \frac{\delta \mathcal{B}_\alpha(\mathbf{x})}{\delta \phi^\beta(\mathbf{y})} \ , \tag{4.9}$$

$$\{\pi_\alpha(\mathbf{x}), P_j^b(\mathbf{y})\} = \frac{\delta \mathcal{A}_j^b(\mathbf{y})}{\delta \phi^\alpha(\mathbf{x})} - \frac{\delta \mathcal{B}_\alpha(\mathbf{x})}{\delta A_j^b(\mathbf{y})} \ . \tag{4.10}$$

These terms are collected together to give a two-form on phase space:

$$\begin{aligned}\Delta\Omega \equiv &\tfrac{1}{2}\int d^3x\, d^3y \{P_i^a(\mathbf{x}), P_j^b(\mathbf{y})\}\delta A_i^a(\mathbf{x}) \wedge \delta A_j^b(\mathbf{y}) + \cdots \\ = &-\frac{i}{48\pi^2}\int \mathrm{tr}[-2W(\delta A)^2 + (AdA + dAA + A^3 + WdA + dAW - AWA - AW^2 - WAW - W^2A - W^3)J^2 \\ &+ dAJAJ + dAJWJ - JWAJA - WJAJA - AWJWJ - AJWJW - AJW^2J - W^2JWJ \\ &+ JA\delta AA + \delta AWJA + \delta AAJW + \delta AWJW + (-2dA - A^2 + AW + WA + W^2)(\delta AJ + J\delta A)] \ ,\end{aligned} \tag{4.11}$$

where we have introduced the differential forms $A = A_i dx_i$ and $W = W_i dx_i = dUU^{-1}$, on the three-dimensional space in which the Hamiltonian dynamics is described. (The exterior product is denoted simply by multiplication.) Furthermore, we assume that the functional exterior derivative anticommutes with d, i.e., $\delta A \equiv \delta A_i dx_i = -dx_i \delta A_i$.

The two-form $\Delta\Omega$ is degenerate because it does not have any vertical components. However, the combination $\Omega + \Delta\Omega$ is nondegenerate on the total phase space, which one can easily observe by writing it in the matrix form

$$\Omega + \Delta\Omega \sim \begin{bmatrix} * & -1 \\ 1 & 0 \end{bmatrix} . \tag{4.12}$$

The inverse is now expressed, in symbolic notation, as

$$(\Omega + \Delta\Omega)^{-1} = \Omega^{-1} - \Omega^{-1}(\Delta\Omega)\Omega^{-1} \ . \tag{4.13}$$

Thus, $\Omega + \Delta\Omega$ is a closed nondegenerate two-form, which we adopt as a symplectic structure. (The closedness of $\Delta\Omega$ is obvious by definition.)

A few comments are in order. A glance at the definition (4.11) shows that $\Delta\Omega$ can be expressed as $\Delta\Omega = \delta\Delta\theta$, where

$$\Delta\theta \equiv \int d^3x (\mathcal{A}_i^a \delta A_i^a + \mathcal{B}_\alpha \delta\phi^\alpha)$$
$$= -\frac{i}{48\pi^2}\int \mathrm{tr}[(WA - AW)\delta A + (AdA + dAA + A^3 - AWA - AW^2 + WAW - W^2A)J]$$
$$+ \frac{1}{12\pi^2}\int d\phi^\beta d\phi^\gamma d\phi^\delta \tau_{\alpha\beta\gamma\delta}\delta\phi^\alpha \ . \qquad (4.14)$$

Applying Stokes's theorem with an appropriate boundary condition, one can translate the last term into the *four-dimensional* expression

$$\frac{i}{48\pi^2}\int \mathrm{tr}(JW^4) + \delta\left[-\frac{1}{48\pi^2}\int \tau_{\alpha\beta\gamma\delta} d\phi^\alpha d\phi^\beta d\phi^\gamma d\phi^\delta\right] . \qquad (4.15)$$

In constructing $\Delta\Omega$ out of $\Delta\theta$, the δ-exact form does not contribute. Hence the essential role is played by the first term, which has no explicit dependence on the choice of local coordinates on the group manifold; we have eliminated the local coordinates by introducing the higher-dimensional integral. This is the characteristic feature of the theory including WZW-like terms. (Put another way, this implies the nontriviality of δ cohomology in the σ-model sector, which becomes the obstruction to the global existence of $\Delta\theta$ within three-dimensional space.) On the other hand, the two-form $\Delta\Omega$, which also has a global expression, contains no higher-dimensional terms. Such a situation will enable us to describe the chiral effective theory without any recourse to the notion of "higher dimension" [24]. That is to say, we can start with the symplectic structure in the beginning, and there is no need to know the explicit form of $\Delta\theta$. In this respect, note that the symplectic structure provides enough (physically relevant) information through, for example, the equation of motion, though it is not obtained from a three-dimensional canonical action via variational principle.

As a next comment, we point out that the effects of the WZW term cannot completely be realized in the modification of the symplectic structure in contrast with the case treated in Sec. II. The WZW term also modifies the Gauss-law function. A detailed explanation will be given in subsection C.

B. Derivation of the anomalous Schwinger term

In the presence of the WZW term, gauge invariance is explicitly broken at the classical level, and the gauge variation of the action reproduces the consistent divergence anomaly. The gauge-transformed Lagrangian is given by $\mathcal{L}+\Delta\mathcal{L}_\xi$, where

$$\Delta\mathcal{L}_\xi = \frac{i}{24\pi^2}\epsilon^{\mu\nu\rho\sigma}\,\mathrm{tr}[\xi^a T^a \partial_\mu (A_\nu \partial_\rho A_\sigma + \tfrac{1}{2} A_\nu A_\rho A_\sigma)] , \qquad (4.16)$$

up to boundary terms. The divergence anomaly can also be represented in the following way:

$$\Delta\mathcal{L}_\xi = \xi^a V_i^{ab}\dot{A}_i^b + (\partial_i \xi^a) V_i^{ab} A_0^b , \qquad (4.17)$$

again up to (spatial-)boundary terms. Here V_i^{ab} is defined by

$$V_i^{ab} \equiv \frac{i}{48\pi^2}\epsilon^{0ijk}(2D_{abc}\partial_j A_k^c + 3B_{a[bcd]}A_j^c A_k^d) . \qquad (4.18)$$

The first term on the right-hand side of Eq. (4.17) is linear in velocities, which causes the modification of the symplectic structure as briefly demonstrated in Sec. II. On the other hand, the second term is proportional to A_0^a, and is to be implemented in the modification of the Gauss law. Let us now consider the effects of the first term upon the symplectic structure of the theory.

We first define V_ξ as

$$V_\xi \equiv \int d^3x\, \xi^a(\mathbf{x}) V_i^{ab}(\mathbf{x})\delta A_i^b(\mathbf{x})$$
$$= \frac{i}{48\pi^2}\int \mathrm{tr}[2\xi(\delta A dA + dA\delta A)$$
$$+ \xi(\delta A A^2 + A^2\delta A - A\delta A A)] . \qquad (4.19)$$

Being considered a one-form on the velocity phase space, this represents the anomalous infinitesimal deformation of the Lagrangian one-form, which one can obtain by applying a vertical derivative [10] to $\int d^3x\, \Delta\mathcal{L}_\xi$. Then the exterior derivative of V_ξ,

$$\delta V_\xi = \frac{i}{24\pi^2}\int \mathrm{tr}[d\xi(\delta A)^2] \equiv \omega_\xi , \qquad (4.20)$$

is to be viewed as an anomalous deformation of the symplectic structure (in which the Legendre transformation is understood). The one-cocycle condition

$$L(X_\xi)\omega_\eta - L(X_\eta)\omega_\xi - \omega_{[\xi,\eta]} = 0 \qquad (4.21)$$

can be checked explicitly. In fact the straightforward (but tedious) calculation leads us to the expected result [see Eqs. (4.28) and (4.43), as well as the Appendix]

$$L(X_\xi)\Delta\Omega = \omega_\xi , \qquad (4.22)$$

for $\Delta\Omega$ defined by Eq. (4.11), indicating that the two-form ω_ξ is a coboundary. The one-cocyclic property of ω_ξ also implies the existence of the zero-form $S_{\xi\eta}$ obeying

$$\delta S_{\xi\eta} = L(X_\xi)V_\eta - L(X_\eta)V_\xi - V_{[\xi,\eta]} . \qquad (4.23)$$

Then, after some algebra, we find

$$S_{\xi\eta} = \frac{i}{48\pi^2}\int \mathrm{tr}\{[\xi,\eta](dAA + AdA + A^3)$$
$$+ A\xi(dA)\eta - (dA)\xi A\eta\} , \qquad (4.24)$$

modulo constant terms depending only on ξ and η. Note that this expression with opposite overall sign agrees (up to the imaginary unit arising in replacing Poisson brackets by commutators) with that of the Schwinger term for the quantized chiral fermionic gauge theory, obtained by perturbative calculations [3] based on the Bjorken-Johnson-Low method [25]. It differs, however, from the Schwinger term previously predicted by Faddeev [2] by a coboundary term. We will show, later on, that the difference between Faddeev's Schwinger term and ours originates from the surface terms appearing in the integration by parts (with respect to "time") in the expression of the divergence anomaly (4.16).

Now that we have obtained a series of differential forms on the phase space, we are led to the descent equations

$$0=\delta\Delta\Omega\ , \tag{4.25}$$

$$L(X_\xi)\Delta\Omega=\delta V_\xi\ , \tag{4.26}$$

$$L(X_\xi)V_\eta-L(X_\eta)V_\xi-V_{[\xi,\eta]}=\delta S_{\xi\eta}\ . \tag{4.27}$$

In more familiar language, the first equation ensures that the Jacobi identity holds for the Poisson brackets constructed with the effective symplectic structure $\Omega+\Delta\Omega$. The second equation summarizes the effects of the divergence anomaly on the Poisson-bracket structure on our phase space. However, what the third equation means is rather obscure at present, a problem which will be discussed in Sec. V.

As pointed out in Sec. II, the fundamental relation (4.22) does not mean that the one-form V_ξ can be identified with $i(X_\xi)\Delta\Omega$. Instead, we introduce an extra term such that

$$i(X_\xi)\Delta\Omega=V_\xi+\delta v_\xi\ . \tag{4.28}$$

Thus it is natural to consider the modified descent equations

$$0=\delta\Delta\Omega\ , \tag{4.29}$$

$$L(X_\xi)\Delta\Omega=\delta(V_\xi+\delta v_\xi)\ , \tag{4.30}$$

$$L(X_\xi)(V_\eta+\delta v_\eta)-L(X_\eta)(V_\xi+\delta v_\xi)-(V+\delta v)_{[\xi,\eta]}=\delta(S_{\xi\eta}+\bar{S}_{\xi\eta})\ , \tag{4.31}$$

where $\bar{S}_{\xi\eta}$ is a coboundary defined by

$$\bar{S}_{\xi\eta}=L(X_\xi)v_\eta-L(X_\eta)v_\xi-v_{[\xi,\eta]}\ . \tag{4.32}$$

Evidently, the exact form δv_ξ in the second equation (4.30) does not contribute to the deformation of the symplectic structure. (The one-form δv_ξ represents a canonical transformation generated by v_ξ, which leaves the symplectic structure unchanged.) On the other hand, in the third equation, the Schwinger term $S_{\xi\eta}$ comes to have an additional contribution $\bar{S}_{\xi\eta}$ owing to the presence of v_ξ. In this case, the relation (4.31) admits a clear-cut interpretation [12], in contrast with the case of Eq. (4.27). The rest of this subsection is devoted to the discussion on this subject.

Since we are concerned with $\Omega+\Delta\Omega$, the effective Poisson bracket [26] between the phase-space functions g and h is represented by the Lie bracket of the vector fields $\bar{X}_g$ and $\bar{X}_h$, which are Hamiltonian with respect to $\Omega+\Delta\Omega$. Namely the relation

$$i(\bar{X}_g)(\Omega+\Delta\Omega)=\delta g \tag{4.33}$$

is required for $\bar{X}_g$, and $\bar{X}_h$ is also defined similarly. Let X_g be the Hamiltonian vector field of g with respect to Ω: $i(X_g)\Omega=\delta g$, and we seek the solution of Eq. (4.33) by setting

$$\bar{X}_g=X_g-\Delta X_g\ . \tag{4.34}$$

Inserting this into the defining equation (4.33), we obtain

$$i(\Delta X_g)(\Omega+\Delta\Omega)=i(X_g)\Delta\Omega\ , \tag{4.35}$$

which determines ΔX_g uniquely for given $\Delta\Omega$ and X_g, since $\Omega+\Delta\Omega$ is nondegenerate. With the use of Eq. (4.13), this is translated into the equivalent statements

$$i(X_g)\Delta\Omega=i(\Delta X_g)\Omega\ , \tag{4.36}$$

$$i(\Delta X_g)\Delta\Omega=0\ . \tag{4.37}$$

Being specialized to the static gauge transformation, the additional term ΔX_ξ is shown to have the form

$$\Delta X_\xi\sim\int d^3x\ \Delta X_\xi(Q)\frac{\delta}{\delta P}\ . \tag{4.38}$$

Here P and Q collectively denote "momentum" and "coordinate" variables, respectively. A straightforward consequence is that the ΔX_ξ's commute with each other:

$$[\Delta X_\xi,\Delta X_\eta]=0\ . \tag{4.39}$$

Then the Poisson brackets among the Gauss-law function(al)s are calculated as

$$\begin{aligned}\delta\{G_\xi,G_\eta\}_{\text{eff}}&=i([\bar{X}_\xi,\bar{X}_\eta])(\Omega+\Delta\Omega)\\&=-i[L(X_\xi)\Delta X_\eta-L(X_\eta)\Delta X_\xi-X_{[\xi,\eta]}](\Omega+\Delta\Omega)\\&=-L(X_\xi)(V_\eta+\delta v_\eta)+L(X_\eta)(V_\xi+\delta v_\xi)+(V+\delta v)_{[\xi,\eta]}+\delta G_{[\xi,\eta]}\\&=-\delta(S_{\xi\eta}+\bar{S}_{\xi\eta})+\delta G_{[\xi,\eta]}\ ,\end{aligned} \tag{4.40}$$

where use has been made of Eqs. (4.28) and the properties of ΔX_ξ clarified above. (The Lie derivative of vector fields is defined as $L(X)Y=[X,Y]$.) Therefore, the Schwinger term is identified with $-(S_{\xi\eta}+\bar{S}_{\xi\eta})$, modulo constant terms. Note that the zero-form $\bar{S}_{\xi\eta}$ can depend on the ϕ field, whereas $S_{\xi\eta}$ consists of only the A field. In fact, the ϕ-dependent Schwinger term disappears after the redefinition of the Gauss-law functional, as can be seen from Eq. (4.32). Namely we find

$$\bar{S}_{\xi\eta}=L(\bar{X}_\xi)v_\eta-L(\bar{X}_\eta)v_\xi-v_{[\xi,\eta]}$$
$$=\{G_\xi,v_\eta\}_{\rm eff}-\{G_\eta,v_\xi\}_{\rm eff}-v_{[\xi,\eta]}\ , \tag{4.41}$$

for the zero-form v_ξ depending only on "coordinate" variables. Combining this result with Eq. (4.40), we obtain, modulo constant,

$$\{G_\xi+v_\xi,G_\eta+v_\eta\}_{\rm eff}-(G+v)_{[\xi,\eta]}=-S_{\xi\eta}\ , \tag{4.42}$$

in which only the A-dependent (and ϕ-independent) Schwinger term remains. Now let us remember that the chirally gauged WZW model is considered to be the low-energy approximation of the corresponding chiral fermionic gauge theory, especially owing to its anomalous behavior under gauge transformations. Hence the algebra (4.42), which is isomorphic to the anomalous gauge algebra of the chiral fermionic theory, is a candidate for the "gauge algebra" in the chirally gauged WZW model, and $G_\xi+v_\xi$ is expected to be an effective Gauss-law functional. Indeed, $G_\xi+v_\xi$ with

$$v_\xi=-\frac{i}{48\pi^2}\int{\rm tr}[2\xi(dAW+WdA)$$
$$+\xi(A^2W+WA^2-AWA)$$
$$-\xi(AW^2+W^2A+WAW)-\xi W^3] \tag{4.43}$$

agrees with the Gauss-law functional computed in Refs. [8,9], when the momenta P_i^a and π_α in $G_\xi+v_\xi$ are replaced by $\tilde{P}_i^a-\mathcal{A}_i^a$ and $\tilde{\pi}_\alpha-\mathcal{B}_\alpha$, respectively. Equation (4.42) is nothing but the Gauss-law algebra derived there. [Namely, no constant terms accompany Eq. (4.42).] Here, as a consistency check, we can verify that relation (4.28) certainly holds for the above expression of v_ξ, the proof of which is outlined in the Appendix.

To conclude, the third equation of the descent equations (4.31) provides an alternative description of the anomalous gauge algebra. (Strictly speaking, the descent equation is related to the anomalous algebra multiplied by the exterior derivative, which measures field-dependent extensions of the gauge algebra.) It is also interesting to recognize that Eq. (4.31) is a consequence of the one-cocycle condition for the deformation ω_ξ of the symplectic structure. Furthermore, the deformation ω_ξ is calculated from the one-form V_ξ, and V_ξ from the divergence anomaly. In this sense the descent equations offer algebraic relations between the two types of anomalies.

As previously mentioned, these descent equations also give a natural framework to deal with Faddeev's Schwinger term in our formalism. Consider, for example, the following expression of the divergence anomaly:

$$-\frac{i}{24\pi^2}\epsilon^{\mu\nu\rho\sigma}{\rm tr}[\partial_\mu\xi^aT^a(A_\nu\partial_\rho A_\sigma+\tfrac{1}{2}A_\nu A_\rho A_\sigma)]$$
$$+\partial_\mu\left[\frac{i}{24\pi^2}\epsilon^{\mu\nu\rho\sigma}{\rm tr}[\xi^aT^a(A_\nu\partial_\rho A_\sigma+\tfrac{1}{2}A_\nu A_\rho A_\sigma)]\right]\ , \tag{4.44}$$

which we can obtain from Eq. (4.16) through integration by parts, keeping surface terms. Since spatial-surface terms have been systematically neglected so far, we are led to

$$(\partial_i\xi^a)V_i^{ab}A_0^b+\frac{i}{48\pi^2}\epsilon^{0ijk}D_{abc}\partial_i\xi^aA_j^b\dot{A}_k^c$$
$$+\partial_0\left[\frac{i}{24\pi^2}\epsilon^{0ijk}{\rm tr}[\xi^aT^a(A_i\partial_jA_k+\tfrac{1}{2}A_iA_jA_k)]\right]\ , \tag{4.45}$$

where the time independence of ξ^a has been used. The third term is a total divergence, and the total divergence in the Lagrangian does not cause any change in the symplectic structure. (For example, suppose that the Lagrangian $\mathcal{L}(q,\dot{q})$ is modified as $\mathcal{L}\to\mathcal{L}+[d\Gamma(q)/dt]$. The resulting change in the Lagrangian one-form is exact: $(\partial\Gamma/\partial q)dq=d\Gamma$.) Namely the third term in Eq. (4.45) contributes to V_ξ as an exact form, which in turn appears in the expression of the Schwinger term as a coboundary. Thus one will have a cohomologically equivalent Schwinger term (i.e., Faddeev's one) starting with the divergence anomaly in the absence of the surface term in Eq. (4.45). We immediately find

$$V_\xi^{\rm Faddeev}=-\frac{i}{24\pi^2}\int{\rm tr}(d\xi\, A\delta A)\ , \tag{4.46}$$

the exterior derivative of which turns out to be ω_ξ. Then, we obtain, modulo constants,

$$S_{\xi\eta}^{\rm Faddeev}=\frac{i}{24\pi^2}\int{\rm tr}[(d\xi\, d\eta-d\eta\, d\xi)A]\ , \tag{4.47}$$

for the Schwinger term defined by

$$\delta S_{\xi\eta}^{\rm Faddeev}=L(X_\xi)V_\eta^{\rm Faddeev}-L(X_\eta)V_\xi^{\rm Faddeev}-V_{[\xi,\eta]}^{\rm Faddeev}\ . \tag{4.48}$$

The corresponding Gauss-law functional is given by

$$G_\xi^{\rm Faddeev}=G_\xi+v_\xi-\frac{i}{24\pi^2}\int{\rm tr}[\xi(AdA+\tfrac{1}{2}A^3)]\ , \tag{4.49}$$

as can easily be seen from the surface term in Eq. (4.45). Moreover, subtracting the coboundary

$$-\frac{i}{144\pi^2}\left\{G_\xi,\int{\rm tr}(A^3)\right\}_{\rm eff} \tag{4.50}$$

from the above expression, one can reduce the Gauss-law functional to

$$G_\xi + v_\xi - \frac{i}{24\pi^2}\int \mathrm{tr}[\xi(dAA + AdA + A^3)] , \quad (4.51)$$

which is more familiar in the literature [3].

C. Effective Hamiltonian

We now turn our attention to the Hamiltonian for the chirally gauged WZW model. The crucial point is that the divergence anomaly contains the term proportional to A_0^a. Since the coefficient of A_0^a is a Gauss-law function, this change can be realized as a modification of the Gauss-law function defined by Eq. (3.14). Let us set the effective Hamiltonian as $\mathcal{H} = \mathcal{H}_0 + \Delta\mathcal{H}$, where $\mathcal{H}_0$ is given by Eq. (3.13), and $\Delta\mathcal{H}$ is assumed to have the form

$$\Delta\mathcal{H} = -A_0^a \Delta G_a . \quad (4.52)$$

In analogy with the case of $\Delta\Omega$, the modification of the Hamiltonian is to be subject to a relation such as (4.22). Since we have restricted ourselves, so far, to the space of connections with only spatial indices, the vector field X_ξ has been employed in Eq. (4.22) to generate static gauge transformations. The situation is, however, different when A_0^a is taken into account. Namely the generator should include an additional contribution, which is required from the gauge covariance of the equation of motion. Generally, the complete form of the generator of a gauge transformation with parameter ξ is given by [27]

$$G = \xi^{(k)} G_0 + \xi^{(k-1)} G_1 + \cdots + \xi G_k , \quad \xi^{(n)} \equiv \frac{d^n \xi}{dt^n} , \quad (4.53)$$

where G_0 is a primary first-class constraint and $G_m (1 \le m \le k)$ are the linear combinations of G_0 and secondary first-class constraints. The highest order of the time derivative, k, can be determined from the constraint structure of the theory under consideration. In our case $k = 1$, and the generator in the Yang-Mills sector has the form

$$-\int d^3x [\dot{\xi}^a P_0^a - \xi^a (D_i P_i^a + f^{abc} A_0^b P_0^c)] , \quad (4.54)$$

which produces the well-known four-dimensional gauge transformation. Thus the transformation law for A_0^a is given by $A_0^a \to A_0^a + f^{abc} A_0^b \xi^c$ for the static gauge transformation. Now the vector field with which one should be concerned is $X_\xi + X_\xi^0$, with

$$X_\xi^0 = \int d^3x \left[f^{abc} \xi^a A_0^b(\mathbf{x}) \frac{\delta}{\delta A_0^c(\mathbf{x})} \right] . \quad (4.55)$$

Here we understand that the phase space is supplemented with the coordinates A_0^a and P_0^a. Consequently, the fundamental relation to be satisfied by $\Delta\mathcal{H}$ is formulated as

$$L(X_\xi + X_\xi^0) \int d^3x \, \Delta\mathcal{H} = -\int d^3x (\partial_i \xi^a V_i^{ab} A_0^b) , \quad (4.56)$$

where the right-hand side comes from Eq. (4.17). The underlying consistency condition analogous to (4.21) can be verified explicitly.

It is easy to "solve" Eq. (4.56). Let $v_\xi = \int d^3x \, \xi^a v_a$, then the solution is given by

$$\Delta G_a = v_a , \quad (4.57)$$

as expected. Though the explicit form of v_a is rather complicated, this can be proved in an extremely simple way with the use of the alternative representation of the Schwinger term in Eq. (4.40):

$$\begin{aligned} -(S_{\xi\eta} + \bar{S}_{\xi\eta}) &= \{G_\xi, G_\eta\}_{\text{eff}} - \{G_\xi, G_\eta\} \\ &= -i(X_\xi) i(X_\eta) \Delta\Omega , \end{aligned} \quad (4.58)$$

which one can derive by recognizing

$$\{G_\xi, G_\eta\}_{\text{eff}} = i(\bar{X}_\xi) i(\bar{X}_\eta)(\Omega + \Delta\Omega) , \quad (4.59)$$

$$\{G_\xi, G_\eta\} = i(X_\xi) i(X_\eta) \Omega , \quad (4.60)$$

and Eq. (4.36). Now we calculate the left-hand side of Eq. (4.56) under (4.57). Introducing the abbreviation $\Delta G_{A_0} = \int d^3x \Delta G_a A_0^a = \int d^3x v_a A_0^a$, we obtain

$$\begin{aligned} L(X_\xi + X_\xi^0) \int d^3x \Delta\mathcal{H} &= -L(X_\xi + X_\xi^0) \Delta G_{A_0} \\ &= -L(X_\xi) \Delta G_{A_0} - \Delta G_{[A_0, \xi]} \\ &= -L(X_{A_0}) \Delta G_\xi - \bar{S}_{\xi A_0} . \end{aligned} \quad (4.61)$$

In the last line, use has been made of the relation (4.32) with the parameter η replaced by A_0. Further calculation gives us

$$\begin{aligned} (4.61) &= -i(X_{A_0}) \delta \Delta G_\xi - \bar{S}_{\xi A_0} \\ &= -i(X_{A_0})[i(X_\xi)\Delta\Omega - V_\xi] - \bar{S}_{\xi A_0} \\ &= i(X_{A_0}) V_\xi + S_{\xi A_0} , \end{aligned} \quad (4.62)$$

by virtue of Eq. (4.58). Then we are left with a simple task to derive the identity:

$$S_{\xi\eta} = \int d^3x \, \xi^a \eta^b (\partial_i V_i^{ab} + V_i^{ac} f^{cbd} A_i^d) , \quad (4.63)$$

which ensures that the final result coincides with Eq. (4.56). The effective Hamiltonian is, hence, given by

$$\mathcal{H}_{\text{eff}} = \mathcal{H}_0 - A_0^a v_a , \quad (4.64)$$

and the *unconstrained* canonical action can be written as

$$S_{\text{eff}} = \int (\theta + \Delta\theta - \mathcal{H}_{\text{eff}} dx^0) , \quad (4.65)$$

where θ is a canonical one-form:

$$\theta = \int d^3x (P_i^a \delta A_i^a + \pi_a J^a) . \quad (4.66)$$

When the constraints are taken into account, the symplectic structure is to be replaced by the one defined on a reduced phase space. Namely, the two-form $\Omega + \Delta\Omega$ is presymplectic on the larger space supplemented with A_0^a and P_0^a. Then, one arrives at the nondegenerate symplectic structure through successive symplectic-reduction processes [28], a procedure which is essentially equivalent to introducing Dirac brackets instead of Poisson brackets.

V. WESS-ZUMINO CONSISTENCY CONDITION

It is widely known that the consistent anomaly (4.16) satisfies the Wess-Zumino consistency condition [15], which means that it is a one-cocycle in the Lie-algebra cohomology. On the other hand, we have already found the analogous cohomological structure in the Hamiltonian formalism, and the question arises as to how the Wess-Zumino condition is realized within the framework of our formalism. In what follows, we show that the relation (4.27) offers partially the Hamiltonian realization of the Wess-Zumino condition.

In order to investigate the Wess-Zumino condition in phase-space language, let us consider the four-dimensional gauge transformation (in the Yang-Mills sector) in the velocity phase space, which is represented by

$$X_\xi^T=\int d^4x\left[D_\mu\xi^a\frac{\delta}{\delta A_\mu^a(x)}+\frac{d}{dx^0}(D_\mu\xi^a)\frac{\delta}{\delta \dot A_\mu^a(x)}\right] . \tag{5.1}$$

When the parameter does not depend on time, the second term becomes

$$\int d^4x\, d^4y\frac{\delta(D_\mu\xi^b(\mathbf{x}))}{\delta A_\nu^a(y)}\dot A_\nu^a(y)\frac{\delta}{\delta \dot A_\mu^b(x)} . \tag{5.2}$$

The Wess-Zumino condition is now formulated as

$$L(X_\xi^T)\left[\int d^4x\,\Delta\mathcal{L}_\eta\right]-L(X_\eta^T)\left[\int d^4x\,\Delta\mathcal{L}_\xi\right]-\int d^4x\,\Delta\mathcal{L}_{[\xi,\eta]}=0 . \tag{5.3}$$

In fact, the above equation can be decomposed into two parts. Namely, when $\dot\xi=\dot\eta=0$, the terms linear in velocities on the left-hand side cancel each other, and the sum of the terms linear in A_0^a vanishes independently. First we pick up the term proportional to velocities:

$$\int d^4x\, d^4y\left[\eta^a(\mathbf{x})\frac{\delta}{\delta A_j^c(y)}[V_i^{ab}(x)D_i\xi^b(\mathbf{x})]\dot A_j^c(y)+D_i\xi^a(\mathbf{x})\left[\eta^b(\mathbf{y})\frac{\delta V_j^{bc}(y)}{\delta A_i^a(x)}-\eta^b(\mathbf{x})\frac{\delta V_i^{ba}(x)}{\delta A_j^c(y)}\right]\dot A_j^c(y)\right]-(\xi\leftrightarrow\eta)$$
$$-\int d^4x f^{dab}\xi^a(\mathbf{x})\eta^b(\mathbf{x})V_j^{dc}(x)\dot A_j^c(x) , \tag{5.4}$$

which is expressed by the eightfold integration. However, apart from $\dot A_j^c(y)$, the integrand does not contain time derivatives at all; thus, the δ function $\delta(x^0-y^0)$ arising from the (four-dimensional) functional derivative can be factored out. What remain are the sixfold spatial integration and the integration with respect to x^0. Equation (5.4) thus reads

$$\int dx^0\int d^3x\, d^3y[\cdots] , \tag{5.5}$$

where the integrand is now evaluated on the fixed-time surface. We apply the vertical derivative to the spatial-integral piece of Eq. (5.5) to get $\delta S_{\xi\eta}$ owing to the descent equation (4.27). In other words, calculating the left-hand side of the descent equation and integrating it over x^0, we arrive at Eq. (5.4) with $\dot A_j^c(y)$ replaced by $\delta A_j^c(\mathbf{y})$. Therefore, the descent equation asserts that Eq. (5.4) indeed vanishes, since

$$(5.4)=\int dx^0\frac{d}{dx^0}S_{\xi\eta}\to 0 . \tag{5.6}$$

Here we have used the time independence of the gauge parameters. Hence, generally, the velocity-dependent part of the Wess-Zumino condition is described in the Hamiltonian formalism in such a way that the coboundary of the one-form V_ξ should be an exact form, or equivalently ω_ξ should be a one-cocycle. One can also observe that the descent equation provides a Hamiltonian description of the Zumino-Stora equation [29]

$$s\omega_4^1=-d\omega_3^2 , \tag{5.7}$$

which states that the BRST transformation (denoted by s) of the consistent anomaly ω_4^1 is just a "total divergence" of the Schwinger term ω_3^2. In this context, Eq. (5.6) is equivalent to the statement

$$s\int \omega_4^1=0 , \tag{5.8}$$

in the Weyl gauge.

The term proportional to A_0^a in Eq. (5.3), which is absent when the Weyl-gauge condition is employed, also admits an interpretation in the Hamiltonian formalism; it is equivalent to the consistency condition for Eq. (4.56).

Now let us remember that the Wess-Zumino condition implies the existence of the WZW term, the gauge variation of which develops consistent anomalies. Analogously, in the Hamiltonian formalism, the existence of $\Delta\Omega$ is ensured by the one-cocycle condition. The important fact worth being mentioned here is that the WZW term necessarily includes the σ-model degrees of freedom, and it cannot be constructed with only gauge potentials. (Of course, the chiral effective action in the fermionic gauge theory produces an anomaly, but it is a nonlocal functional of background gauge potentials since the integration over fermionic degrees of freedom is implied in the effective action.) Correspondingly, it appears to be impossible to find the two-form $\Delta\Omega$ subject to Eq. (4.22) when working with only gauge potentials. However, unexpectedly the two-form

$$\Delta\Omega_A=\frac{i}{24\pi^2}\int \mathrm{tr}[A(\delta A)^2] \tag{5.9}$$

satisfies Eq. (4.22) [14]. The crucial point is that $\Delta\Omega_A$ is not a closed form [30]. Thus the effective Poisson brackets derived with $\Omega+\Delta\Omega_A$ violate the Jacobi identity; i.e., the relation

$$\{P_i^a(\mathbf{x}),P_j^b(\mathbf{y})\}_{\rm eff}=-\frac{i}{24\pi^2}\epsilon^{0ijk}D_{abc}A_k^c(\mathbf{x})\delta(\mathbf{x}-\mathbf{y}) \tag{5.10}$$

does not associate [31]. Accordingly, the algebra of the Gauss-law functions evaluated on the basis of (5.10) acquires a noncocyclic Schwinger term.

In fact, the nonclosedness of $\Delta\Omega_A$ is intimately related to the cohomological nontriviality of the Schwinger term $S_{\xi\eta}$. To see this, consider what would happen if $\Delta\Omega_A$ were closed. In such a situation, there exists the one-form $\Delta\theta_A$ so that $\Delta\Omega_A=\delta\Delta\theta_A$, with the δ cohomology in the Yang-Mills sector being trivial. Then V_ξ becomes exact modulo coboundary, which implies that the Schwinger term would be a trivial two-cocycle (up to constants).

When supplemented with the σ-model degrees of freedom, we can find the *closed* two-form $\Delta\Omega$ as in Eq. (4.11), unlike the case with only gauge potentials. Thus, on the same grounds as above, we can say that the Schwinger term $S_{\xi\eta}$ is a trivial two-cocycle. [To be precise, the one-form $\Delta\theta(\delta\Delta\theta=\Delta\Omega)$ is essentially a local object, and as a result $S_{\xi\eta}$ can be trivialized only in a local coordinate patch.] More concretely, the modified Gauss-law functional

$$(G_\xi+v_\xi)-[i(X_\xi)\Delta\theta+v_\xi]=G_\xi-i(X_\xi)\Delta\theta \tag{5.11}$$

satisfies the algebra free of Schwinger terms. The meaning of Eq. (5.11) is clear, that is, we immediately find

$$G_\xi-i(X_\xi)\Delta\theta=\int d^3x\,\xi^a(D_i\tilde{P}_i^a+K_a^\alpha\tilde{\pi}_\alpha)\ , \tag{5.12}$$

where $\tilde{P}_i^a$ and $\tilde{\pi}_\alpha$ are defined by Eqs. (4.4) and (4.5), so that the new Gauss law consists of the momenta subject to the canonical Poisson-bracket relations, leading to the ordinary Gauss-law algebra. This conclusion is not so surprising, because we can trivially construct the anomaly-free theory by subtracting the WZW term (regarded as a local polynomial of the field variables) from the Lagrangian [9].

VI. THE UNGAUGED WESS-ZUMINO-WITTEN MODEL IN FOUR DIMENSIONS

A. Formalism

The ungauged WZW model (or simply the WZW model) in four dimensions [15,7] can also be treated within the framework of our formalism. Namely the WZW model has a common feature with the chirally gauged one in the sense that the WZW term is added by hand to the original Lagrangian to drastically change the canonical structure of the theory. Then, as in the previous sections, we modify the canonical symplectic structure so as to reproduce an "anomaly" due to the WZW term. In fact, the canonical formalism of the WZW model has already been developed, concentrating on its symplectic structure [23,24] (or canonical commutation relations [32,33]) on the one hand, and on the other hand emphasizing how the algebra of Noether-charge densities is accompanied with Schwinger terms originating from the quasi-invariance of the WZW term [16,17]. Here we give a unified treatment of these subjects.

We adopt $\Omega_\Sigma+\Delta\Omega_\Sigma$ as a symplectic structure of the WZW model, where Ω_Σ is defined by Eq. (3.20) with

$$\pi_a=\frac{1}{f^2}W_0^a \tag{6.1}$$

and $\Delta\Omega_\Sigma$ is given by [23]

$$\Delta\Omega_\Sigma=-\frac{i}{48\pi^2}\int \mathrm{tr}(W^3J^2+W^2JWJ)\ . \tag{6.2}$$

One can obtain the last expression from Eq. (4.11) by setting simply $A,\delta A\to 0$. In order to convince ourselves that $\Omega_\Sigma+\Delta\Omega_\Sigma$ is the true symplectic structure, we demonstrate that the Euler-Lagrange equation evaluated with the use of $\Omega_\Sigma+\Delta\Omega_\Sigma$ indeed agrees with the equation of motion of the WZW model.

Let us start by pulling back the symplectic structure to the velocity phase space by means of the Legendre transformation (6.1):

$$\Omega_\Sigma\to(\Omega_\Sigma)_{\mathcal{L}}=\frac{1}{f^2}\int d^3x(\delta W_0^a\wedge J^a+\tfrac{1}{2}f^{abc}W_0^cJ^a\wedge J^b)\ , \tag{6.3}$$

$$\Delta\Omega_\Sigma\to(\Delta\Omega_\Sigma)_{\mathcal{L}}=\Delta\Omega_\Sigma\ . \tag{6.4}$$

In a similar manner, we define the energy density $\mathcal{E}$ as a pullback of the Hamiltonian,

$$\mathcal{H}_0=\frac{1}{2}\left[f^2\pi_a\pi_a+\frac{1}{f^2}W_i^aW_i^a\right]\ , \tag{6.5}$$

to get

$$\mathcal{E}=\frac{1}{2f^2}(W_0^aW_0^a+W_i^aW_i^a)\ . \tag{6.6}$$

Then, with the introduction of the "dynamical" vector field

$$X_{\mathcal{E}}=\int d^3x\left[\dot{\phi}^\alpha(\mathbf{x})\frac{\delta}{\delta\phi^\alpha(\mathbf{x})}+\dot{W}_0^a(\mathbf{x})\frac{\delta}{\delta W_0^a(\mathbf{x})}\right]\ , \tag{6.7}$$

the Euler-Lagrange equation becomes [10,24]

$$i(X_{\mathcal{E}})(\Omega_\Sigma+\Delta\Omega_\Sigma)_{\mathcal{L}}=-\delta\int d^3x\ \mathcal{E}\ . \tag{6.8}$$

The explicit calculation gives us the correct equation of motion:

$$\frac{1}{f^2}\partial^\mu W_\mu^a-\frac{i}{48\pi^2}\epsilon^{\mu\nu\rho\sigma}\,\mathrm{tr}(T^aW_\mu W_\nu W_\rho W_\sigma)=0\ , \tag{6.9}$$

which is obtained from the action

$$S_\Sigma=\frac{1}{f^2}\int d^4x\ \mathrm{tr}(W_\mu W^\mu)-\frac{i}{240\pi^2}\int \mathrm{tr}(W^5) \tag{6.10}$$

of the WZW model.

Here the fact worth mentioning is that the Hamiltonian formalism explained above does not use the notion of higher dimension at all. The higher dimensionality enters the formalism when we try to obtain the equation of motion through the variational principle. For example, the canonical action leading to Hamilton's equation is to be constructed with the "potential" $\theta_\Sigma+\Delta\theta_\Sigma$ of the symplectic structure, with

$$\theta_\Sigma=\int d^3x\, \pi_a J^a \ , \tag{6.11}$$

$$\Delta\theta_\Sigma=\frac{i}{48\pi^2}\int \mathrm{tr}(JW^4) \ , \tag{6.12}$$

which inevitably contains the four-dimensional term.

B. The algebra of Noether-charge densities

Having obtained the modified symplectic structure, we study its behavior under a symmetry transformation. Since there no longer exists a gauge symmetry, we are led to consider the global symmetry of the Lagrangian $\mathcal{L}_0=f^{-2}\mathrm{tr}(W_\mu W^\mu)$ and clarify the effects of the WZW term upon the symmetry properties of the model. As is well known, the Lagrangian $\mathcal{L}_0$ is invariant under the infinitesimal left translation with a global parameter ξ^a,

$$\delta_\xi\phi^\alpha=-\xi^a K_a^\alpha \ , \tag{6.13}$$

while the total Lagrangian (including the WZW term) is only quasi-invariant. It should be remembered that the quasi-invariance of the Lagrangian does not cause the deformation in the symplectic structure; however, it certainly contributes to the Schwinger term of the symmetry algebra. To see this, we perform the transformation (6.13) with the time-independent but space-dependent parameter $\xi^a(\mathbf{x})$, and investigate the response of $\Delta\Omega_\Sigma$ to this "localized" transformation [34].

Let X_ξ be the vector field [35] generating the transformation, i.e.,

$$i(X_\xi)\Omega_\Sigma=\delta G_\xi \ , \tag{6.14}$$

with

$$G_\xi=\int d^3x\, \xi^a(\mathbf{x})\pi_a(\mathbf{x}) \ . \tag{6.15}$$

Then the calculation results in

$$L(X_\xi)\Delta\Omega_\Sigma=\omega_\xi \ , \tag{6.16}$$

where

$$\omega_\xi=\frac{i}{48\pi^2}\int \mathrm{tr}[d\xi(2J^2W^2+W^2J^2+JWJW+WJWJ+WJ^2W)] \ . \tag{6.17}$$

Obviously, the deformation ω_ξ is a one-cocycle by definition. Along the same line as in Sec. IV, we can derive ω_ξ by studying how the Lagrangian $\mathcal{L}$ changes under the localized transformation. Namely, when $\dot{\xi}=0$, we find

$$\mathcal{L}\to\mathcal{L}+\frac{1}{f^2}W_i^a\partial_i\xi^a+\partial_0\left[\frac{i}{48\pi^2}\epsilon^{0ijk}\mathrm{tr}(\xi W_i W_j W_k)\right]+\frac{i}{48\pi^2}\epsilon^{0ijk}\mathrm{tr}[\partial_i\xi(W_0W_jW_k-W_jW_0W_k+W_jW_kW_0)] \ , \tag{6.18}$$

up to spatial-boundary terms. Thus the "anomalous" deformation V_ξ of the Lagrangian (or canonical) one-form is given by

$$V_\xi=-\frac{i}{48\pi^2}\int \mathrm{tr}[d\xi(JW^2-WJW+W^2J)] \ , \tag{6.19}$$

modulo exact form δv_ξ, with

$$v_\xi=\frac{i}{48\pi^2}\int \mathrm{tr}(\xi W^3) \ . \tag{6.20}$$

Here we can prove the identity

$$i(X_\xi)\Delta\Omega_\Sigma=V_\xi+\delta v_\xi \ , \tag{6.21}$$

and consequently $\delta V_\xi=\omega_\xi$, as expected. On the other hand, the remaining piece in the deformation of the Lagrangian $f^{-2}W_i^a\partial_i\xi^a$ indicates the noninvariance of the Hamiltonian (6.5) under the localized transformation, and is not important in the present story.

The one-cocyclic property of ω_ξ enables us to write down the descent equation, and $S_{\xi\eta}$ defined by

$$\delta S_{\xi\eta}=L(X_\xi)V_\eta-L(X_\eta)V_\xi-V_{[\xi,\eta]} \tag{6.22}$$

plays the role of the Schwinger term (with opposite sign) of our interest. The explicit calculation shows

$$S_{\xi\eta}=-\frac{i}{48\pi^2}\int \mathrm{tr}[(d\xi\, d\eta-d\eta\, d\xi)W] \ . \tag{6.23}$$

We introduce $I_a^0(\mathbf{x})$ such that $G_\xi+v_\xi=\int d^3x\, \xi^a(\mathbf{x})I_a^0(\mathbf{x})$, that is,

$$I_a^0=\pi_a+\frac{i}{48\pi^2}\epsilon^{0ijk}\mathrm{tr}(T^aW_iW_jW_k) \ , \tag{6.24}$$

and the current algebra implied by the descent equation can be written as

$$\{I_a^0(\mathbf{x}),I_b^0(\mathbf{y})\}_{\mathrm{eff}}=f^{abc}I_c^0(\mathbf{x})\delta(\mathbf{x}-\mathbf{y})-\frac{i}{24\pi^2}\epsilon^{0ijk}D_{abc}\partial_i W_j^c(\mathbf{x})\partial_k\delta(\mathbf{x}-\mathbf{y}) \ . \tag{6.25}$$

Here the function $I_a^0(\mathbf{x})$ is interpreted as a Noether-charge density associated with the global transformation (6.13) [16,17]. Indeed, the equation of motion (6.9) reads $\partial_\mu I_a^\mu=0$, where

$$I_a^\mu=-\frac{1}{f^2}W_a^\mu+\frac{i}{48\pi^2}\epsilon^{\mu\nu\rho\sigma}\,\mathrm{tr}(T^aW_\nu W_\rho W_\sigma)\ , \tag{6.26}$$

which turns out to be the Noether current. The second term on the right-hand side comes from the WZW term. In comparing this result with that of Refs. [16,17], one should understand that the effects of the WZW term appear also in the noncanonical Poisson-bracket relations obeyed by the momenta (6.1). Namely π_a is to be replaced by $K_a^\alpha(\tilde{\pi}_\alpha-\mathcal{B}_\alpha)$ with $\mathcal{B}_\alpha$ defined by Eq. (4.7) in the absence of gauge fields.

C. Comments on the BRST structure of the formalism

Faddeev's original prediction for the Schwinger term was based on the observation that the anomalous Gauss-law algebra is to be subject to the Jacobi identity. As was shown in Ref. [33] with the use of the BRST technique [29], the associativity can also be a guiding principle in determining the possible extensions of the current algebra in the WZW model. The classical counterpart of the current algebra was found to be

$$\{\pi_a(\mathbf{x}),\pi_b(\mathbf{y})\}_{\mathrm{eff}}=f^{abc}\pi_c(\mathbf{x})\delta(\mathbf{x}-\mathbf{y})-\frac{i}{48\pi^2}\epsilon^{0ijk}\,\mathrm{tr}([T^a,T^b]W_iW_jW_k-W_iW_jT^aW_kT^b+W_iW_jT^bW_kT^a)\delta(\mathbf{x}-\mathbf{y})\ , \tag{6.27}$$

which coincides with the noncanonical Poisson brackets for the π_a's implied by Eqs. (3.20) and (6.2). In the present formalism, the two-cocyclic property of the extension is merely a consequence of the associativity of the underlying Poisson brackets—the closed nature of the symplectic structure. Therefore, seeking the BRST-closed two-form to get a two-cocycle is in some sense equivalent, in the context of our formalism, to seeking a δ-closed two-form $\Delta\Omega_\Sigma$. Indeed, the relation $\delta\Delta\theta_\Sigma=\Delta\Omega_\Sigma$ has the structure of a *dimensional descent* equation characteristic to the BRST formulation of the theory of anomalies. More precisely, one can observe

$$\delta W=-dJ+JW+WJ,\quad dJ\equiv dx_i\partial_iJ=-\partial_iJdx_i\ , \tag{6.28}$$

so that $(d+\delta)(J+W)=(J+W)^2$. Then, expanding $(d+\delta)\mathrm{tr}[(J+W)^5]=0$ in powers of J and picking up the terms of order J^2, we obtain

$$\delta\,\mathrm{tr}(JW^4)=-d\,\mathrm{tr}(W^3J^2+W^2JWJ)\ , \tag{6.29}$$

which is nothing but the statement $\delta\Delta\theta_\Sigma=\Delta\Omega_\Sigma$. Such a construction is the four-dimensional generalization of the technique proposed in Ref. [24].

The BRST procedure has also been applied to the gauged WZW model [36], now treating gauge potentials as external variables. It is also possible to interpret the results within the framework of our formalism. Let δ_Σ be the exterior derivative operating only on the σ-model degrees of freedom; then we find

$$\delta_\Sigma\hat{J}=-\hat{J}^2\ , \tag{6.30}$$

$$\delta_\Sigma A^U=-d\hat{J}-\hat{J}A^U-A^U\hat{J}\ , \tag{6.31}$$

where $\hat{J}\equiv U^{-1}JU$ and $A^U\equiv U^{-1}AU+U^{-1}dU$. Applying again the BRST method, we obtain a three-dimensional δ_Σ-closed form as

$$\Delta\Omega_\Sigma=\frac{i}{48\pi^2}\int\mathrm{tr}\{\hat{J}^2[A^UdA^U+dA^UA^U+(A^U)^3]+\hat{J}A^U\hat{J}dA^U\}=\tfrac{1}{2}S_{\hat{J}\hat{J}}(A^U)\ , \tag{6.32}$$

where we have used the notation in Eq. (4.24). This expression agrees with the $J\wedge J$ part of the two-form $\Delta\Omega$ defined by Eq. (4.11). When the gauge fields acquire dynamics, one should use the exterior derivative δ rather than δ_Σ. The resulting expression for the two-form $\Delta\Omega$ includes the corrections consisting of the differential forms of the types $\delta A\wedge\delta A$ or $\delta A\wedge J$.

VII. CONCLUSIONS AND REMARKS

We have applied the classical symplectic-geometrical analysis to the chirally gauged WZW model, which implements the anomalies at the classical level. The consistent anomaly affects the canonical structure of the theory in such a way that the symplectic structure is deformed along gauge orbits. Quite analogously to the Lagrangian formulation, the extra two-form $\Delta\Omega$ is added by hand to the canonical symplectic structure to reproduce the deformation. The existence of $\Delta\Omega$ is a consequence of the one-cocyclic property of the deformation, which is also a foundation for establishing the algebraic relation between the divergence anomaly and the Schwinger term.

On the other hand, the one-cocyclic property is found to be a Hamiltonian realization of the Wess-Zumino condition in the Weyl gauge. Though our analysis has been performed in the context of the WZW model, such an interpretation will be valid even for the underlying chiral fermionic gauge theory (in the Weyl gauge). Namely the Wess-Zumino condition is formulated in terms of the gauge fields only, and the σ-model degrees of freedom are introduced in the process of finding the "solution" to the condition. Similarly, in the Hamiltonian formulation of the chiral gauge theory, one can start with the one-cocycle condition, now regarding it as a first principle. The σ-model fields are introduced in order to construct the *closed* two-form $\Delta\Omega$ subject to Eq. (4.22).

When it comes to the ungauged model, we have been able to obtain the algebra of Noether-charge densities by treating a localized transformation. The algebra derived here is considered to be more essential than Eq. (6.27) in elucidating the symmetry properties of the model.

Our formalism is applicable to the higher-dimensional chirally gauged (or ungauged) model, and will reveal the hidden algebraic structure analogous to that of the four-dimensional theories.

At the end we point out the interesting subjects which deserve further investigations. In addition to the descent equations of our main concern, we have established another algebraic formula connecting the divergence anomaly with the Schwinger term, i.e., Eq. (4.63). Employing a four-dimensional description, we can rewrite it in the form

$$-S_{\xi\eta}=\int d^3y\,\eta^b(y)\int dx^0\left[\frac{\delta}{\delta A_0^b(y)}-\left[D_i(y)\frac{\delta}{\delta \dot{A}_i(y)}\right]^b\right]\Delta L_\xi(x^0)\ , \tag{7.1}$$

where $\Delta L^\xi(x^0)=\int d^3x\Delta\mathcal{L}_\xi$ is a consistent anomaly integrated over the three-dimensional space. This expression has been obtained in Refs. [6] in different lines of arguments, and is expected to give some hints to the physical understanding of chiral gauge theories also within the framework of our formalism. Next, we call one's attention to the Chern-Simons-Yang-Mills theories. When the Chern-Simons term is added to the Yang-Mills Lagrangian, the canonical momenta change their forms and the Poisson brackets among them acquire the additional terms, which turn out to be Abelian functional curvatures in the space of static Yang-Mills connections in the Weyl gauge [37]. Moreover, the curvature term can be regarded as a presymplectic structure on the space of connections (not on the phase space) to give a natural geometrical setting [38] for covariant anomalies [39]. It is interesting to interpret these works in phase-space language, which problem remains to be studied and the results will be reported elsewhere.

ACKNOWLEDGMENTS

This work was partially supported by Grant-in-Aid for Scientific Research No. 03-0097 from the Ministry of Education, Science and Culture of Japan, and is also supported by Japan Society for the Promotion of Science.

APPENDIX

In this appendix, we give a simplified proof of Eq. (4.22), which can be traced more easily than the direct computation of $L(X_\xi)\Delta\Omega$. First of all, we write the two-form $\Delta\Omega$ without using the differential-geometric notation:

$$\begin{aligned}\Delta\Omega=&\frac{i}{48\pi^2}\int d^3x\,\epsilon^{0ijk}D_{abc}W_k^c\delta A_i^a\delta A_j^b+\frac{i}{48\pi^2}\int d^3x\,\epsilon^{0ijk}[3B_{[abc]d}(A_i^c+W_i^c)\partial_jA_k^d+B_{[ab]cde}A_i^cA_j^dA_k^e\\&\qquad+(B_{[ab]cde}+B_{[ae]dbc}+B_{[eb]dac})A_i^cA_j^dW_k^e\\&\qquad-2B_{[abcde]}(3A_i^c+W_i^c)W_j^dW_k^e]J^aJ^b\\&+\frac{i}{48\pi^2}\int d^3x\,\epsilon^{0ijk}[-2D_{abc}\partial_jA_k^c-3B_{a[bcd]}A_j^cA_k^d\\&\qquad+3B_{b[acd]}(2A_j^c+W_j^c)W_k^d]J^a\delta A_i^b\ .\end{aligned} \tag{A1}$$

Next, we calculate $i(X_\xi)\Delta\Omega-V_\xi$ to obtain

$$\begin{aligned}i(X_\xi)\Delta\Omega-V_\xi=&\frac{i}{48\pi^2}\int d^3x\,\epsilon^{0ijk}[-2D_{abc}D_j\xi^aW_k^c-3\xi^aB_{b[acd]}(2A_j^c+W_j^c)W_k^d]\delta A_i^b\\&-\frac{i}{24\pi^2}\int d^3x\,\epsilon^{0ijk}\xi^a[3B_{[abc]d}(A_i^c+W_i^c)\partial_jA_k^d+(B_{[ab]cde}+B_{[ae]dbc}+B_{[eb]dac})A_i^cA_j^dW_k^e\\&\qquad-2B_{[abcde]}(3A_i^c+W_i^c)W_j^dW_k^e]J^b\\&-\frac{i}{48\pi^2}\int d^3x\,\epsilon^{0ijk}\{-3B_{b[acd]}\partial_i\xi^aA_j^cA_k^d+D_i\xi^a[-2D_{abc}\partial_jA_k^c+3B_{a[bcd]}(2A_j^c+W_j^c)W_k^d]\}J^b\ ,\end{aligned} \tag{A2}$$

which agrees with the exterior derivative of v_ξ defined by Eq. (4.43):

$$v_\xi=-\frac{i}{48\pi^2}\int d^3x\,\epsilon^{0ijk}\xi^a(2D_{abc}\partial_iA_j^bW_k^c+3B_{[abc]d}A_i^bA_j^cW_k^d-3B_{a[bcd]}A_i^bW_j^cW_k^d-B_{a[bcd]}W_i^bW_j^cW_k^d)\ . \tag{A3}$$

Thus we have verified Eq. (4.28), and consequently we find

$$L(X_\xi)\Delta\Omega=\delta(V_\xi+\delta v_\xi)=\omega_\xi\ , \tag{A4}$$

which completes the proof.

[1] R. Jackiw, in *Current Algebra and Anomalies,* edited by S. B. Treiman, R. Jackiw, B. Zumino, and E. Witten (World Scientific, Singapore, 1985), and references therein.

[2] L. D. Faddeev, Phys. Lett. **145B**, 81 (1984); see also L. D. Faddeev and S. L. Shatashvilli, Theor. Math. Phys. **60**, 770 (1984); J. Michelson, Commun. Math. Phys. **97**, 361 (1985).

[3] S.-G. Jo, Nucl. Phys. **B259**, 616 (1985); K. Kobayashi, K. Seo, and A. Sugamoto, *ibid.* **B273**, 607 (1986).

[4] P. A. M. Dirac, *Lectures on Quantum Mechanics* (Yeshiva University Press, New York, 1964).

[5] K. Fujikawa, Phys. Lett. B **171**, 424 (1986); D. S. Hwang, Nucl. Phys. **B286**, 231 (1987); A. Yu. Alekseev, Ya. Madaichik, L. D. Faddeev, and S. L. Shatashvilli, Theor. Math. Phys. **73**, 1149 (1987); P. Mitra, Phys. Lett. B **188**, 111 (1988); **214**, 105 (1988); Phys. Rev. Lett. **60**, 265 (1988); **61**, 2005(E) (1988); I. V. Kolokolov and A. S. Yelkhovsky, Nucl. Phys. **B312**, 419 (1989); S. Hosono, Phys. Rev. D **39**, 1743 (1989); Y.-Z. Zhang, J. Phys. A **21**, L1055 (1988); Phys. Rev. Lett. **62**, 2221 (1989); J. Phys. A **22**, L371 (1989).

[6] T. D. Kieu, Phys. Lett. B **223**, 72 (1989); C. Schwiebert, *ibid.* **241**, 223 (1990); **242**, 415 (1990).

[7] E. Witten, Nucl. Phys. **B223**, 422 (1983).

[8] R. Percacci and R. Rajaraman, Phys. Lett. B **184**, 256 (1988); Int. J. Mod. Phys. A **4**, 4177 (1989).

[9] T. Fujiwara, S. Hosono, and S. Kitakado, Mod. Phys. Lett. A **3**, 1585 (1988); T. Fujiwara and S. Kitakado, Z. Phys. C **43**, 201 (1989).

[10] G. Marmo, E. J. Saletan, A. Simoni, and B. Vitale, *Dynamical Systems* (Wiley Interscience, New York, 1985).

[11] The relation between the symplectic structure and chiral anomalies has been investigated in the context of the Berry phase. See, for example, S. Iida and H. Kuratsuji, Phys. Lett. B **184**, 242 (1987); H. Kuratsuji and S. Iida, Phys. Rev. D **37**, 441 (1988); E. Gozzi and W. D. Thacker, *ibid.* **35**, 2398 (1987); E. Gozzi, D. Rohrlich, and W. D. Thacker, *ibid.* **42**, 2752 (1990).

[12] J. F. Cariñena and L. A. Ibort, J. Math. Phys. **29**, 541 (1988).

[13] W.-B. Yeung and H.-L. Yu, Phys. Rev. D **36**, 3066 (1987); T. Inamoto, *ibid.* **39**, 3179 (1989).

[14] T. Inamoto, Prog. Theor. Phys. **82**, 408 (1989).

[15] J. Wess and B. Zumino, Phys. Lett. **37B**, 95 (1971).

[16] A. C. Davis, J. A. Gracy, and A. J. Macfarlane, Phys. Lett. B **194**, 415 (1987).

[17] J. A. de Azcárraga, J. M. Izquierdo, and A. J. Macfarlane, Ann. Phys. (N.Y.) **202**, 1 (1990).

[18] S. Sternberg, Proc. Natl. Acad. Sci. U.S.A. **74**, 5235 (1977).

[19] Basically, we follow the notation of Refs. [8] unless otherwise stated.

[20] We consider such an infinite-dimensional space only at the formal level. More rigorous treatments are not discussed here.

[21] C. Crnković and E. Witten, in *300 Years of Gravitation,* edited by S. W. Hawking and W. Israel (Cambridge University Press, Cambridge, England, 1988).

[22] A. M. Polyakov and P. B. Wiegmann, Phys. Lett. **131B**, 121 (1983); **141B**, 223 (1984).

[23] T. R. Ramadas, Commun. Math. Phys. **93**, 355 (1984).

[24] I. Bakas and D. McMullan, Phys. Lett. B **189**, 141 (1987).

[25] J. Bjorken, Phys. Rev. **148**, 1467 (1966); K. Johnson and F. Low, Prog. Theor. Phys. B **14**, 269 (1969).

[26] The Poisson brackets constructed with $\Omega+\Delta\Omega$ are nothing but the ones defined by Eqs. (4.8)–(4.10), now regarding P_i^a and π_α as fundamental variables.

[27] L. Castellani, Ann. Phys. (N.Y.) **143**, 357 (1982).

[28] L. D. Faddeev and R. Jackiw, Phys. Rev. Lett. **60**, 1692 (1988).

[29] B. Zumino, in *Relativity, Groups and Topology II,* Proceedings of the Les Houches Summer School, Les Houches, France, 1983, edited by B. S. DeWitt and R. Stora, Les Houches Summer School Proceedings Vol. 40 (North-Holland, Amsterdam, 1984); R. Stora, in *Progress in Gauge Field Theory,* Proceedings of the Cargese Summer Institute, Cargese, France, 1983, edited by G. 't Hooft, A. Singer, and R. Stora, NATO Advanced Study Institutes, Series B: Physics, Vol. 85 (Plenum, New York, 1987).

[30] Note that the fundamental relation (4.22) does not require $\delta\Delta\Omega=0$, but only the gauge invariance of $\delta\Delta\Omega$ is required: $L(X_\xi)\delta\Delta\Omega=0$.

[31] Remarkably, this algebra agrees with that of non-Abelian electric fields in the chiral gauge theory. See, for example, A. Niemi and G. Semenoff, Phys. Rev. Lett. **55**, 927 (1985); **56**, 1019 (1986); S.-G. Jo, Phys. Lett. **163B**, 353 (1985); B. Banerjee, Phys. Rev. Lett. **61**, 514 (1988); S. Ghosh and R. Banerjee, Z. Phys. C **41**, 121 (1988).

[32] S. G. Rajeev, Phys. Rev. D **29**, 2944 (1984).

[33] T. Fujiwara, Phys. Lett. **152B**, 103 (1985).

[34] This procedure is somewhat similar to the standard method to compute Noether currents. Namely, transforming the Lagrangian with a global parameter ξ, and then one obtains a Noether current as a coefficient of $\partial_\mu\xi$.

[35] Hereafter we use the same symbols as in the preceding sections to avoid notational complications.

[36] T. Fujiwara, S. Kitakado, and T. Nonoyama, Phys. Lett. **155B**, 91 (1985).

[37] G. V. Dunne and C. A. Trugenberger, Ann. Phys. (N.Y.) **204**, 281 (1990).

[38] D. Bao and V. P. Nair, Commun. Math. Phys. **101**, 437 (1985).

[39] W. Bardeen and B. Zumino, Nucl. Phys. **B244**, 421 (1984).