δ expansion for local gauge theories. II. Nonperturbative calculation of the anomaly in the Schwinger model

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This is the second paper in a series in which we show how to use the principles of the δ expansion to obtain nonperturbative solutions to gauge theories. The approach consists of replacing the usual minimal-coupling term $\bar{\psi}(i\partial\!\!\!/-eA\!\!\!/)\psi$ by $\bar{\psi}(i\partial\!\!\!/-eA\!\!\!/)^{\delta}\psi$ and then expanding the new theory in powers of δ. For all values of δ the theory is locally gauge invariant. Thus, local gauge invariance holds order by order in powers of δ. In this paper we show how to calculate the photon propagator and thus the anomaly in the Schwinger model (two-dimensional massless quantum electrodynamics) to first order in δ. At $\delta=1$ the exact value for the anomaly, e^2/π, is obtained.

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I. INTRODUCTION

In the first paper in this series [1] we showed how the δ expansion can be used to solve theories having a minimal covariant momentum of the form

$$\bar{\psi}(i\partial\!\!\!/-eA\!\!\!/)\psi \ . \tag{1.1}$$

The procedure is to introduce a small parameter δ into the exponent of the coupling term

$$\bar{\psi}(i\partial\!\!\!/-eA\!\!\!/)^{\delta}\psi \tag{1.2}$$

and then to solve the theory as a series in powers of δ. The expansion in δ has two advantages. First, it is nonperturbative in the coupling constant e and, second, it preserves local gauge invariance order by order in powers of δ [2].

In Ref. [1] we considered a simple model in one-dimensional space-time whose coupling has the form in (1.1). The advantage of such a model is that there is no complication coming from the γ matrices, which, in one-dimensional space-time, are completely absent. The purpose of this paper is to explain how to extend the calculational procedures developed in Ref. [1] to interaction terms such as that in (1.2) in which γ matrices are present.

Specifically, we consider the Schwinger model (massless two-dimensional quantum electrodynamics) described by the Lagrangian

$$L=-\frac{1}{4}(F_{\mu\nu})^2+\bar{\psi}(i\partial\!\!\!/-eA\!\!\!/)\psi \ . \tag{1.3}$$

We introduce the perturbation parameter δ:

$$L=-\frac{1}{4}(F_{\mu\nu})^2+M\bar{\psi}\left[\frac{i\partial\!\!\!/-eA\!\!\!/}{M}\right]^{\delta}\psi \ , \tag{1.4}$$

where M is a mass parameter that maintains the dimensional consistency of (1.4). Note that when $\delta=1$ the Lagrangian in (1.4) reduces to that in (1.3). We illustrate and describe the technical details of the δ expansion by computing the anomaly in the Schwinger model. To summarize, the general procedure is rather easy to understand but the details involve combinatoric arguments that are somewhat lengthy and elaborate in coordinate space. Thus, in this paper we limit the discussion to the determination of the first term in the δ series, namely, the coefficient of δ. The result for the anomaly is $\delta e^2/\pi$ as was shown in Ref. [2]. At $\delta=1$ this reduces to the well-known value e^2/π for the Schwinger model.

To obtain the first term in the δ expansion we follow the general rules described in Ref. [3]. We expand the Lagrangian (1.4) to first order in δ:

$$L=-\frac{1}{4}(F_{\mu\nu})^2+M\bar{\psi}\psi+\delta M\bar{\psi}\ln\left[\frac{i\partial\!\!\!/-eA\!\!\!/}{M}\right]\psi+O(\delta^2) \ . \tag{1.5}$$

Then we replace (1.5) by a provisional Lagrangian having a *polynomial* interaction term

$$L_N=-\frac{1}{4}(F_{\mu\nu})^2+M\bar{\psi}\psi+\delta M\bar{\psi}\left[\frac{i\partial\!\!\!/-eA\!\!\!/}{M}\right]^{N}\psi \ , \tag{1.6}$$

where we think of N as a positive integer. To obtain the solution to (1.5) we must solve (1.6) *for all N*. Then we differentiate with respect to N and set $N=0$ to recover

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the solution to (1.5).[1] This differentiation process of course requires that we analytically continue the solution of (1.6) off the integers. Clearly, this procedure is not rigorously justifiable. However, in all models we have studied thus far, including the gauge model described in this paper, we have always found the unambiguous and correct result. In particular, we show that to leading order in δ, the anomaly for the Schwinger model is $\delta e^2/\pi$.

This paper is organized very simply. In Sec. II we review briefly the standard diagrammatic calculation of the anomaly in the Schwinger model using weak-coupling perturbation theory. Then, in the next two sections we give a detailed calculation of the anomaly to leading order in δ and to leading order in e^2. Section III illustrates the calculation with three special cases and Sec. IV presents the complete and general calculation. Finally, in Sec. V we redo the long calculation presented in Secs. III and IV in a very brief and simple fashion: by doing the calculation in momentum space rather than in coordinate space we reduce it to just a page.

II. CONVENTIONAL CALCULATION OF THE ANOMALY IN THE SCHWINGER MODEL

In this section we review the conventional computation of the anomaly in the Schwinger model using weak-coupling perturbative methods. In Minkowski space, the Lagrangian in (1.3), or that in (1.4) evaluated at $\delta=1$, yields the following Feynman rules: $-ie\gamma^\mu$ for a fermion-fermion-boson vertex and $i(\not p)^{-1}$ for a fermion line. The anomaly is determined by computing the one-fermion-loop contribution to the photon propagator (see Fig. 1):

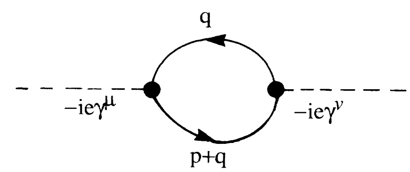

FIG. 1. The diagram from which one computes the anomaly in the weak-coupling expansion for two-dimensional massless electrodynamics.

$$\int \frac{d^dq}{(2\pi)^d}\mathrm{Tr}\left[(-1)(-ie)^2\gamma^\mu\frac{1}{\not p+\not q}\gamma^\nu\frac{i}{\not q}\right], \tag{2.1}$$

where q is the internal loop momentum. We have regulated the integral by computing it in d-dimensional space-time.

We simplify the formula in (2.1) algebraically by rationalizing the denominators,

$$-e^2\int\frac{d^dq}{(2\pi)^d}\frac{1}{(p+q)^2q^2}\mathrm{Tr}[\gamma^\mu(\not p-\not q)\gamma^\nu\not q],$$

and perform the trace

$$-e^2 2^{d/2}\int\frac{d^dq}{(2\pi)^d}\frac{(p+q)^\mu q^\nu-(p+q)\cdot q g^{\mu\nu}+(p+q)^\nu q^\mu}{(p+q)^2q^2}. \tag{2.2}$$

Next, we introduce a Feynman parameter α,

$$\frac{1}{q^2(p+q)^2}=\int_0^1 d\alpha\frac{1}{[(1-\alpha)q^2+\alpha(p+q)^2]^2},$$

perform a shift in the momentum integration variable q to eliminate terms linear in q in the denominator, and then drop terms in the numerator that are odd in q:

$$-e^2 2^{d/2}\int_0^1 d\alpha\int\frac{d^dq}{(2\pi)^d}\frac{2q^\mu q^\nu-2\alpha(1-\alpha)p^\mu p^\nu-[q^2-\alpha(1-\alpha)p^2]g^{\mu\nu}}{[q^2+\alpha(1-\alpha)p^2]^2}. \tag{2.3}$$

Using rotational symmetry in d-dimensional space we have, in general,

$$\int f(q^2)q^\mu q^\nu d^dq=\frac{g^{\mu\nu}}{d}\int f(q^2)q^2d^dq.$$

Thus, (2.3) reduces to the sum of two integrals:

$$-e^2 2^{d/2}g^{\mu\nu}\int_0^1 d\alpha\int\frac{d^dq}{(2\pi)^d}\frac{q^2\left[\frac{2}{d}-1\right]+\alpha(\alpha-1)p^2}{[q^2+\alpha(1-\alpha)p^2]^2}$$
$$+e^2 2^{d/2}p^\mu p^\nu\int_0^1 d\alpha\int\frac{d^dq}{(2\pi)^d}\frac{2\alpha(1-\alpha)}{[q^2+\alpha(1-\alpha)p^2]^2}. \tag{2.4}$$

When $d=2$ the coefficient of $p^\mu p^\nu$ [the second of the two integrals in (2.4)] is finite and this term evaluates to $e^2p^\mu p^\nu/(p^2\pi)$. The second term in the first integral in (2.4) is also finite at $d=2$. The first term in the first integral in (2.4) is infinite at $d=2$, but it contains a factor of $(2-d)/d$ which *vanishes* at $d=2$. We thus evaluate this integral keeping d arbitrary and compute the limit $d\to 2$ afterwards. Our final result for the evaluation of (2.4) is

$$\left[\frac{p^\mu p^\nu}{p^2}-g^{\mu\nu}\right]\frac{e^2}{\pi}. \tag{2.5}$$

Had we attempted to evaluate (2.1) directly in two-dimensional space-time using a momentum cutoff to regulate this integral, we would have encountered an ambiguity, namely, a logarithmically divergent integral multiplied by zero. To resolve the ambiguity, we would have imposed the requirement that the result be transverse. Since the coefficient of $p^\mu p^\nu/p^2$ is finite, the coefficient of $g^{\mu\nu}$ is then uniquely determined. The advantage of di-

[1]As noted in Ref. [1] to leading order in δ this technique bears a close resemblance to the replica method of statistical mechanics. In higher order this resemblance does not persist.

mensional regularization is that it enables us to obtain the answer in (2.5) directly without having to impose the requirement that the answer be transverse; transversality emerges naturally because dimensional regularization is consistent with gauge invariance.

Our objective in the remainder of this paper is to show how to calculate (2.5) from the Lagrangian L_N in (1.6).

III. ORDER-e^2 CONTRIBUTION TO THE ANOMALY IN THE δ EXPANSION: THREE SPECIAL CASES

In this and the next section we show how to calculate the photon two-point Green's function to order δ from the Lagrangian L_N in (1.6). In this paper we are concerned only with the coefficient of δ to order e^2.

Let us examine the term in L_N proportional to δ:

$$\delta M^{1-N}\int d^dx\, \bar{\psi}(x)(i\partial\!\!\!/-eA\!\!\!/)^N\psi(x)\ , \qquad (3.1)$$

where we regard N as a positive integer. This term gives rise to a large collection of vertices. To identify these vertices we can expand the gauge derivative term in the form of a polynomial in e^2:

$$\delta M^{1-N}\int d^dx\, \bar{\psi}i^N\partial\!\!\!/^N\psi+(\text{term of order } e)$$
$$+(\text{term of order } e^2)$$
$$+(\text{term of order } e^3)+\cdots\ .$$

Our purpose here is to examine the e^2 term in this expansion, and, using graphical methods, to calculate its contribution to the photon two-point Green's function.

The term of order e^2 is complicated for two reasons. First, the derivative operator $\partial\!\!\!/$ does not commute with the field $A(x)$. Second, the γ-matrix algebra must be taken into account. A typical e^2 term in the expansion of (3.1) has the form

$$e^2\int d^dx\, \bar{\psi}\partial\!\!\!/^{a}A\!\!\!/\partial\!\!\!/^{b}A\!\!\!/\partial\!\!\!/^{N-a-b-2}\psi\ , \qquad (3.2)$$

where a,b, $N-a-b-2\geq 0$ are all integers and the derivatives $\partial\!\!\!/$ operate forward. This term is characterized by the two integers a and b. However, we now show that the integer a is superfluous if we are only interested in graphs of order δ. We integrate by parts a times so that the derivatives $(\partial\!\!\!/)^a$ now operate *backward* on $\bar{\psi}$. Now (3.2) becomes

$$e^2(-1)^a\int d^dx\bar{\psi}\overleftarrow{\partial\!\!\!/}^{a}A\!\!\!/\partial\!\!\!/^{b}A\!\!\!/\partial\!\!\!/^{N-a-b-2}\psi\ . \qquad (3.3)$$

Now let us consider the kinds of graphs that will appear in this calculation. All graphs of order δe^2 that contribute to the anomaly have two external photon lines and one fermion loop (see Fig. 2). The graph shown arises from a vertex of the type

$$\bar{\psi}\overleftarrow{\partial\!\!\!/}^{\alpha}A\!\!\!/\partial\!\!\!/^{\beta}A\!\!\!/\partial\!\!\!/^{\gamma}\psi\ .$$

As shown, the $\partial\!\!\!/^{\alpha}$ derivatives on $\bar{\psi}$ correspond to α momentum insertions that have a direction *opposite* to the flow of the fermion loop momentum, while the $(\partial\!\!\!/)^{\gamma}$ derivatives on ψ correspond to γ momentum insertions that are in the *same* direction as the loop momentum.

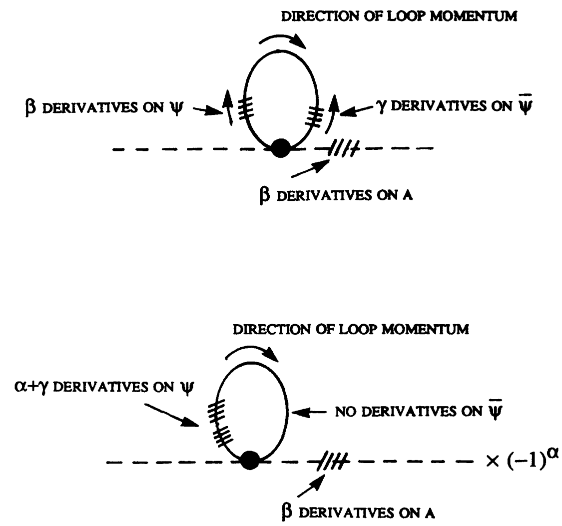

FIG. 2. A typical graph of order δe^2 for the anomaly in massless electrodynamics. Note that a graph arising from an interaction term having α derivatives on $\bar{\psi}$ and γ derivatives on ψ is equal to a graph from an interaction term having no derivatives on $\bar{\psi}$ and $\alpha+\gamma$ derivatives on ψ multiplied by $(-1)^{\alpha}$.

Thus, the graph is equal to one in which there are $\alpha+\gamma$ derivatives on the ψ field and there are no derivatives on the $\bar{\psi}$ field if the new graph is multiplied by $(-1)^{\alpha}$. Furthermore, since a trace on the γ matrices is taken around the fermion loop and traces are cyclical, we can simplify the expression in (3.3). Taking $a=\alpha$, (3.3) takes the much simpler form

$$e^2\int d^dx\, \bar{\psi}A\!\!\!/\partial\!\!\!/^{b}A\!\!\!/\partial\!\!\!/^{N-b-2}\psi\ . \qquad (3.4)$$

We emphasize that (3.3) and (3.4) are, of course, *not equal* but they give the same graphs of order δ having two external photon lines.

Thus, we can expand the original interaction Lagrangian in (3.1) as a sum of terms

$$e^2\sum_{b=0}^{N-2}\delta M^{1-N}W_b\int d^dx\, \bar{\psi}(x)A\!\!\!/(i\partial\!\!\!/)^bA\!\!\!/(i\partial\!\!\!/)^{N-b-2}\psi(x)\ , \qquad (3.5)$$

where W_b is a combinatoric weight (like a binomial coefficient) that we will now determine.

For a given value of b there are $N-b-1$ terms of the form in (3.2) having b derivatives $(\partial\!\!\!/)^b$ between $A\!\!\!/$ and $A\!\!\!/$. This is because the $N-2$ factors of $\partial\!\!\!/$ can be distributed in $N-b-1$ possible ways:

$$\bar{\psi}\partial\!\!\!/^{N-b-2}A\!\!\!/\partial\!\!\!/^{b}A\!\!\!/\psi\ ,$$
$$\bar{\psi}\partial\!\!\!/^{N-b-3}A\!\!\!/\partial^{b}A\!\!\!/\partial\!\!\!/\psi\ ,$$
$$\bar{\psi}\partial\!\!\!/^{N-b-4}A\!\!\!/\partial\!\!\!/^{b}A\!\!\!/(\partial\!\!\!/)^2\psi\ ,$$
$$\vdots$$
$$\bar{\psi}\partial\!\!\!/A\!\!\!/\partial\!\!\!/A\!\!\!/\partial\!\!\!/^{N-b-3}\psi\ ,$$
$$\bar{\psi}A\!\!\!/\partial\!\!\!/^{b}A\!\!\!/\partial\!\!\!/^{N-b-2}\psi\ .$$

Each of these terms becomes identical after integration by parts. Thus,

$$W_b = N - b - 1 \ . \tag{3.6}$$

The expression in (3.5) must be decomposed further because the term $(i\not\partial)^b$ contains derivatives that operate forward; some of these derivatives operate on $\not A$ and others operate on $\psi(x)$. We distinguish each of these cases by the integer P. Thus, we will consider separately the cases where no derivatives operate on $\not A\,(P=0)$, two derivatives operate on $\not A\,(P=2)$, four derivatives operate on $\not A\,(P=4)$, and so on. Note that we cannot have a graph in which an odd number of derivatives acts on ψ because the fermion momentum loop integral would vanish by symmetry (replacing p by $-p$ in the momentum integral shows that the integral equals its negative). Furthermore, since N is the number of γ matrices in the typical graph shown in Fig. 2, *N must be even.* If N is odd, then since the trace of a product of an odd number of γ matrices vanishes, the graph vanishes. Thus, both N and P must be even integers.

A. Special case $P=0$

The simplest special case to consider is that in which *all* derivatives in the expression $(i\not\partial)^b$ in (3.5) act on $\psi(x)$ and *no* derivatives $(P=0)$ act on $\not A$. Thus, the derivative structure of (3.5) takes the form $\int d^dx\, \bar\psi \not A \not A (i\partial)^{N-2}\psi$. However, we must also take into account the γ matrices that accompany each factor of ∂ and A. There are thus two subcases to consider: subcase *even b* is that in which an *even* number of operators $\not\partial$ lie between $\not A$ and $\not A$ and between $\not A$ and ψ in the sum in (3.5), and subcase *odd b* is that in which an *odd* number of derivative operators $\not\partial$ lie between $\not A$ and $\not A$ and between $\not A$ and ψ in the sum in (3.5). Remember that N is always even.

Subcase even b. If $P=0$ then every even-b term in the sum in (3.5) has the form

$$e^2\delta M^{1-N} W_b \int d^dx\, \bar\psi(x) A^2 (i\partial)^{N-2}\psi(x) \tag{3.7}$$

because $\not\partial^2=\partial^2$ and $\not A^2 = A^2$. Summing over all such terms is easy:

$$\sum_{\text{even } b} W_b = \sum_{k=0}^{(N/2)-1} W_{2k} = \sum_{k=0}^{(N/2)-1} N-2k-1 = (N/2)^2 \ ,$$

where we have used the expression for W_b in (3.6). Hence the total contribution from even b is

$$e^2\delta i^{N-2} M^{1-N} \frac{N^2}{4} \int d^dx\, \bar\psi(x) A^2 \partial^{N-2}\psi(x) \ . \tag{3.8}$$

Subcase odd b. Every odd-b term in the sum in (3.5) has the form

$$\delta M^{1-N} W_b \int d^dx\, \bar\psi(x) \not A \gamma_\alpha \not A \gamma_\beta \partial_\alpha \partial_\beta \partial^{N-4}\psi(x) \ .$$

Summing over all such terms is again easy:

$$\sum_{\text{odd } b} W_b = \sum_{k=1}^{(N/2)-1} W_{2k-1} = \sum_{k=1}^{(N/2)-1} N-2k = \frac{N}{2}\left[\frac{N}{2}-1\right] ,$$

where we have again used the expression for W_b in (3.6). Thus, the total contribution from odd b is

$$e^2\delta i^{N-2} M^{1-N} \frac{N}{2}\left[\frac{N}{2}-1\right] \times \int d^dx\, \bar\psi(x) \not A \gamma_\alpha \not A \gamma_\beta \partial_\alpha \partial_\beta \partial^{N-4}\psi(x) \ . \tag{3.9}$$

The expression in (3.9) may be simplified further if we recall that it is the *effective* action that will be used to evaluate diagrams of the type shown in Fig. 2. The fermion loop integral in this case is an integral whose general form can be simplified by making use of rotational symmetry:

$$\int d^dx\, \partial_\alpha \partial_\beta f(x^2) = \frac{1}{d}\delta_{\alpha\beta} \int d^dx\, \partial^2 f(x^2) \ . \tag{3.10}$$

Hence (3.9) becomes

$$e^2\delta i^{N-2} M^{1-N} \frac{N}{2}\left[\frac{N}{2}-1\right] \times \int d^dx\, \bar\psi(x) \not A \gamma_\alpha \not A \gamma_\beta \frac{\delta_{\alpha\beta}}{d} \partial^{N-2}\psi(x) \ . \tag{3.11}$$

Finally, we make use of the γ-matrix identity

$$\not A \gamma_\alpha \not A \gamma_\alpha = (2-d) A^2 \tag{3.12}$$

to simply (3.11) to

$$e^2\delta i^{N-2} M^{1-N} \frac{2-d}{d} \frac{N}{2}\left[\frac{N}{2}-1\right] \int d^dx\, \bar\psi(x) A^2 \partial^{N-2}\psi(x) \ . \tag{3.13}$$

Combination of even-b and odd-b subcases. Combining (3.8) and (3.13) gives the effective action

$$e^2\delta i^{N-2} M^{1-N}\left[\frac{N^2}{4} + \frac{2-d}{4d} N(N-2)\right] \times \int d^dx\, \bar\psi(x) A^2 \partial^{N-2}\psi(x) \ . \tag{3.14}$$

Hence, when we evaluate the diagram in Fig. 2, which in this case has no powers of the momentum on the external photon lines, we obtain, for the amplitude [4],

$$-e^2\delta M^{1-N}\left[\frac{N^2}{2} + \frac{2-d}{2d} N(N-2)\right] g^{\mu\nu} 2^{d/2} \times \int \frac{d^dq}{(2\pi)^d} \frac{q^{N-2}}{M} \ , \tag{3.15}$$

where the factor of $2^{d/2}$ comes from computing the trace around the fermion loop. The loop momentum is q and there are $N-2$ momentum insertions as is clear from (3.14). The factor of $1/M$ comes from the fermion propagator (which is a constant), the minus sign comes from the fermion trace, and the factor of $g^{\mu\nu}$ comes from the vector interaction term $A^2 = A^\mu A_\mu$ in (3.14).

Now that we have determined the amplitude in (3.15) we must differentiate this result with respect to N and set $N=0$ (regarding N as a continuous parameter). Observe that since the even-b case gives a term proportional to N^2 this case cannot contribute to the final answer. Only the term proportional to N in the odd-b case gives a nonvanishing contribution:

$$e^2\delta\frac{2-d}{d}g^{\mu\nu}2^{d/2}\int\frac{d^dq}{(2\pi)^d}q^{-2}\ . \tag{3.16}$$

This is our final result for the contribution of the $P=0$ terms to the anomaly.

Observe the remarkable similarity between the result in (3.16) and that in (2.4). In both expressions there is an integral that is divergent when we set the dimension $d=2$ but the integral is multiplied by the vanishing quantity $d-2$. The delicate limit $d\to 2$ must be performed in order to obtain a finite and nonvanishing value for the anomaly in two-dimensional electrodynamics. We cannot perform the limit $d\to 2$ in (3.16) yet. We must first combine the $P=0$ calculation in (3.16) with the results of the calculations for $P=2, 4, 6, \ldots$. These more difficult calculations are described below.

Note also that the $P=0$ calculation is proportional to $g^{\mu\nu}$ but that the exact result in (2.5) for the anomaly contains a $p^\mu p^\nu$ term as well as a $g^{\mu\nu}$ term, in order that the result be transverse. We will see that contributions proportional to $p^\mu p^\nu$ come from the $P=2,4,6,\ldots$ calculations even though no such term occurs in the $P=0$ case.

B. Special case $P=2$

The $P=2$ case is more complicated than the $P=0$ case because when the derivative operators in the summation in (3.5) are moved to the right, exactly two derivatives must remain operating on the A field. An additional complication arises because the indices on the derivatives acting on the A field may or may not be contracted. To assist our analysis we introduce the new parameter L which counts *the number of pairs of contractions of the indices on the derivatives on the A field.*

Subcase even b. When $L=0$ we have terms of the form

$$\bar\psi A(\partial_\alpha\partial_\beta A)\partial_\alpha\partial_\beta(\partial^2)^{(N/2)-3}\psi\quad (L=0) \tag{3.17a}$$

and when $L=1$ we have terms of the form

$$\bar\psi A(\partial^2 A)(\partial^2)^{(N/2)-2}\psi\quad (L=1)\ . \tag{3.17b}$$

(In general, L takes on the integer values $L=0,1,2,\ldots,P/2$.)

The problem is now to determine the relative weights of the terms in (3.17a) and (3.17b). This problem is in fact a simple problem in probability: When terms of the form in (3.5) are simplified by allowing the derivative operators to act, it is much less likely that we will obtain a term of the $L=1$ form in (3.17b) than a term of the $L=0$ form in (3.17a). Clearly, the change of two derivative operators ∂_α having the *same* index α with both ending up differentiating the photon field A is small. Let $u_{2k,L,P}$ be the coefficient of terms of the form in (3.17) appearing in the expansion of $\not\partial^{2k}A\psi=(\partial^2)^kA\psi$. That is,

$$\begin{aligned}(\partial^2)^kA\psi = {} & A(\partial^2)^k\psi\quad (P=0\text{ term already considered})\\ & \left.\begin{aligned}&+u_{2k,0,2}(\partial_\alpha\partial_\beta A)\partial_\alpha\partial_\beta(\partial^2)^{k-2}\psi\\ &+u_{2k,1,2}(\partial^2 A)(\partial^2)^{k-1}\psi\end{aligned}\right\}(P=2\text{ terms})\\ & +(\text{additional terms with }P=4,\ 6,\ldots)\ .\end{aligned} \tag{3.18}$$

It is easy to show that

$$u_{2k,0,2}=2k(k-1) \tag{3.19a}$$

and that

$$u_{2k,1,2}=k\ . \tag{3.19b}$$

To illustrate the probabilistic nature of $u_{2k,L,P}$ we note that this coefficient is a binomial coefficient multiplied by a probability:

$$u_{2k,0,2}=\begin{bmatrix}2k\\2\end{bmatrix}\frac{2k-2}{2k-1}\ , \tag{3.20a}$$

$$u_{2k,1,2}=\begin{bmatrix}2k\\2\end{bmatrix}\frac{1}{2k-1}\ . \tag{3.20b}$$

Observe that sum of the probabilities is unity,

$$\frac{2k-2}{2k-1}+\frac{1}{2k-1}=1\ ,$$

so that

$$u_{2k,0,2}+u_{2k,1,2}=\begin{bmatrix}2k\\2\end{bmatrix}\ . \tag{3.20c}$$

Terms with $L=0$ can be replaced by terms of the $L=1$ form using the same rotational symmetry that we used in the odd-b case for $P=0$. Specifically, we may apply the identities (3.10) to the fermion momentum loop integral to replace

$$\begin{aligned}&\bar\psi A_\mu(\partial_\alpha\partial_\beta A_\mu)\partial_\alpha\partial_\beta(\partial^2)^{(N/2)-3}\psi\\ &\to\bar\psi A_\mu(\partial_\alpha\partial_\beta A_\mu)\frac{\delta_{\alpha\beta}}{d}(\partial^2)^{(N/2)-2}\psi\\ &=\frac{1}{d}\bar\psi A_\mu(\partial^2 A_\mu)(\partial^2)^{(N/2)-2}\psi\ .\end{aligned}$$

Thus, combining the $L=0$ and $L=1$ terms, we find that the $P=2$ contribution of the even-b terms in the sum in (3.5) is

$$e^2\sum_{\substack{b=2\\ \text{even } b}}^{N-2}\delta M^{1-N}W_b\int d^dx\,\bar\psi(x)A(i\not\partial)^bA(i\not\partial)^{N-b-2}\psi(x)$$

$$=i^{N-2}e^2\delta M^{1-N}\sum_{k=1}^{(N/2)-1}(N-2k-1)\left[\frac{2k(k-1)}{d}+k\right]\int d^dx\,\bar\psi(x)A_\mu(\partial^2A_\mu)(\partial^2)^{(N/2)-2}\psi(x)\ ,$$

where the γ matrices have disappeared because b is even:

$$=i^{N-2}e^2\delta M^{1-N}\frac{N}{12d}\left[\frac{N}{2}-1\right]\left[4-d+N(d-3)+\frac{N^2}{2}\right]\int d^dx\,\bar{\psi}(x)A_\mu(\partial^2 A_\mu)(\partial^2)^{(N/2)-2}\psi(x)\,. \tag{3.21}$$

To evaluate finite sums such as that in (3.21) we rewrite the summand as a sum of terms of the form $(k+a)(k+a+1)(k+a+2)\cdots(k+a+n-1)$; then we recall the simple and general identity

$$\sum_{N_1}^{N_2}(k+a)(k+a+1)(k+a+2)\cdots(k+a+n-1)$$
$$=\sum_{N_1}^{N_2}\frac{(k+a)(k+a+1)(k+a+2)\cdots(k+a+n)-(k+a-1)(k+a)(k+a+1)\cdots(k+a+n-1)}{n+1}$$
$$=\frac{(N_2+a)(N_2+a+1)(N_2+a+2)\cdots(N_2+a+n)-(N_1+a-1)(N_1+a)(N_1+a+1)\cdots(N_1+a+n-1)}{n+1}\,. \tag{3.22}$$

The sum in (3.22) is the discrete analogue of the integral

$$\int_{N_1}^{N_2}dk\,(k+a)^n=\frac{(N_2+a)^{n+1}-(N_1+a)^{n+1}}{n+1}\,.$$

Subcase odd b. When b is odd, $b=2k-1$, the sum in (3.5) contains terms of the form

$$e^2W_{2n-1}\delta M^{1-N}\int d^dx\,\bar{\psi}(x)\boldsymbol{A}(i\boldsymbol{\partial})^{2k-1}\boldsymbol{A}(i\boldsymbol{\partial})^{N-2k-1}\psi(x)$$
$$=i^{N-2}e^2W_{2n-1}\delta M^{1-N}\int d^dx\,\bar{\psi}(x)\boldsymbol{A}(\partial^2)^{k-1}\boldsymbol{\partial}\boldsymbol{A}\boldsymbol{\partial}(\partial^2)^{(N/2)-k-1}\psi(x)\,. \tag{3.23}$$

Now we recall that we are only interested in graphs of order δ and such graphs have only one vertex and *one* fermion loop. The fermion loop involves a trace and we choose to compute this trace in advance of drawing the graph; that is, we perform this trace for the interaction Lagrangian in (3.23). Of course, such a procedure would make no sense if we were going to compute a two-vertex graph, but for graphs of order δ it provides a useful and valid simplification of (3.23). Specifically, taking a trace reduces (3.23) to three distinct terms which must be considered in turn:

$$\begin{aligned}\mathrm{Tr}[\boldsymbol{A}\boldsymbol{\partial}(\partial^2)^{k-1}\boldsymbol{A}\boldsymbol{\partial}](\partial^2)^{(N/2)-k-1}=&(\mathrm{Tr}\mathbb{1})A_\alpha\partial_\alpha(\partial^2)^{k-1}A_\beta\partial_\beta(\partial^2)^{(N/2)-k-1}\quad(\text{type-I term})\\&-(\mathrm{Tr}\mathbb{1})A_\alpha\partial_\beta(\partial^2)^{k-1}A_\alpha\partial_\beta(\partial^2)^{(N/2)-k-1}\quad(\text{type-II term})\\&+(\mathrm{Tr}\mathbb{1})A_\alpha\partial_\beta(\partial^2)^{k-1}A_\beta\partial_\alpha(\partial^2)^{(N/2)-k-1}\quad(\text{type-III term})\,.\end{aligned} \tag{3.24}$$

Terms of type I. Let us consider terms of type I. After allowing the derivative operators to act to the right with the provision that exactly two derivatives must remain acting on A_β $(P=2)$, there are two possibilities that can arise: either the derivative operator ∂_α acts on A_β producing a term of the form

$$(2k-2)(\mathrm{Tr}\mathbb{1})A_\alpha(\partial_\alpha\partial_\gamma A_\beta)\partial_\gamma\partial_\beta(\partial^2)^{(N/2)-3}\quad(L=0,\ \alpha\text{ inside}) \tag{3.25a}$$

or it commutes past A_β producing terms of the form

$$(2k-2)(k-2)(\mathrm{Tr}\mathbb{1})A_\alpha(\partial_\gamma\partial_\delta A_\beta)\partial_\alpha\partial_\beta\partial_\gamma\partial_\delta(\partial^2)^{(N/2)-4}\quad(L=0,\ \alpha\text{ outside}) \tag{3.25b}$$

and

$$(k-1)(\mathrm{Tr}\mathbb{1})A_\alpha(\partial^2 A_\beta)\partial_\alpha\partial_\beta(\partial^2)^{(N/2)-3}\quad(L=1,\ \alpha\text{ outside})\,. \tag{3.25c}$$

It is of course necessary to find the correct coefficients $2k-2,(2k-2)(k-2)$, and $k-1$ of the terms in (3.25). These coefficients are determined by simple counting arguments.

We simplify the term in (3.25a) using the identity in (3.10):

$$\frac{1}{d}(2k-2)(\mathrm{Tr}\mathbb{1})A_\alpha(\partial_\alpha\partial_\beta A_\beta)(\partial^2)^{(N/2)-2}\,. \tag{3.26}$$

The term in (3.25c) is simplified using the same identity:

$$\frac{1}{d}(k-1)(\mathrm{Tr}\mathbb{1})A_\alpha(\partial^2 A_\alpha)(\partial^2)^{(N/2)-2}\,. \tag{3.27}$$

To simplify the term in (3.25b) we must use a slightly more complicated identity:

$$\int d^dx\, \partial_\alpha\partial_\beta\partial_\gamma\partial_\delta f(x^2)=\frac{1}{d(d+2)}(\delta_{\alpha\beta}\delta_{\gamma\delta}+\delta_{\alpha\gamma}\delta_{\beta\delta}+\delta_{\alpha\delta}\delta_{\beta\gamma})\int d^dx(\partial^2)^2 f(x^2)\,. \tag{3.28}$$

This identity is the *second* in an infinite sequence of identities that rely on the rotational symmetry of $f(x^2)$ in the integrand. The first such identity is given in (3.10). The Nth integral identity in the sequence is

$$\int d^dx\, \partial_{\alpha_1}\partial_{\alpha_2}\cdots\partial_{\alpha_{2N}} f(x^2)=\frac{\Gamma(d/2)}{2^N\Gamma(N+d/2)}\underbrace{(\delta_{\alpha_1\alpha_2}\cdots\delta_{\alpha_{2N-1}\alpha_{2N}}+\text{all permutations of indices})}_{(2N-1)!!\ \text{terms}}\int d^dx(\partial^2)^N f(x^2)\,. \tag{3.29}$$

Using (3.28) we simplify (3.25b) to

$$\frac{1}{d(d+2)}(2k-2)(k-2)(\mathrm{Tr}\mathbb{1})A_\alpha[(\partial^2 A_\alpha)+2(\partial_\alpha\partial_\beta A_\beta)](\partial^2)^{(N/2)-2}\,. \tag{3.30}$$

Terms of type III. The type-III term in (3.24) may be treated in the same way as the type-I term. The results are exactly the same; namely, we obtain the three expressions in (3.26), (3.27), and (3.30). (To observe the symmetry, interchange α and β and integrate by parts to make the derivatives act to the left.)

Terms of type II. As in the case of terms of type I we can expand the type-II term in (3.24) into three terms. These terms are

$$-2(k-1)(\mathrm{Tr}\mathbb{1})A_\alpha(\partial_\beta\partial_\gamma A_\alpha)\partial_\beta\partial_\gamma(\partial^2)^{(N/2)-3}\quad (L=0,\ \beta\ \text{inside})\,, \tag{3.31a}$$

$$-(k-1)(\mathrm{Tr}\mathbb{1})A_\alpha(\partial^2 A_\alpha)(\partial^2)^{(N/2)-2}\quad (L=1,\ \beta\ \text{outside})\,, \tag{3.31b}$$

$$-2(k-1)(k-2)(\mathrm{Tr}\mathbb{1})A_\alpha(\partial_\delta\partial_\gamma A_\alpha)\partial_\delta\partial_\gamma(\partial^2)^{(N/2)-3}\quad (L=0,\ \beta\ \text{outside})\,. \tag{3.31c}$$

These terms may be combined using rotational symmetry and the identity in (3.10) to

$$-\left[\frac{2(k-1)}{d}+k-1+\frac{2(k-1)(k-2)}{d}\right](\mathrm{Tr}\mathbb{1})A_\alpha(\partial^2 A_\alpha)(\partial^2)^{(N/2)-2}\,. \tag{3.32}$$

Combination of terms of types I, II, and III. Next we combine the results in (3.26), (3.27), and (3.30) and multiply by 2 (these are the type-I and type-III terms) and add the results in (3.32) (type-II terms) to obtain

$$-\frac{(k-1)(2k-2+d)}{d+2}(\mathrm{Tr}\mathbb{1})A_\alpha(\partial^2 A_\alpha)(\partial^2)^{(N/2)-2}+\frac{4(k-1)(2k-2+d)}{d(d+2)}(\mathrm{Tr}\mathbb{1})A_\alpha(\partial_\alpha\partial_\beta A_\beta)(\partial^2)^{(N/2)-2}\,. \tag{3.33}$$

Finally, we multiply the result in (3.33) by the weight $W_b=N-2k$ and sum over odd $b=2k-1$ from $k=1$ to $k=(N/2)-1$. The sum is performed using the identity in (3.22) and the result is

$$-\frac{N}{48(d+2)}(N-2)(N-4)(N+2d-2)(\mathrm{Tr}\mathbb{1})A_\alpha(\partial^2 A_\alpha)(\partial^2)^{(N/2)-2}$$
$$+\frac{N}{12d(d+2)}(N-2)(N-4)(N+2d-2)(\mathrm{Tr}\mathbb{1})A_\alpha(\partial_\alpha\partial_\beta A_\beta)(\partial^2)^{(N/2)-2}\,. \tag{3.34}$$

Combination of even-b and odd-b cases. We can now write down an effective action that produces the combined results in (3.21) and (3.34) to first order in δ:

$$i^{N-2}e^2\delta M^{1-N}\frac{N(N-2)(N^2-6N+2Nd+3d^2-2d+8)}{24d(d+2)}\int d^dx\,\bar\psi(x)A_\mu(\partial^2 A_\mu)(\partial^2)^{(N/2)-2}\psi(x)$$
$$+i^{N-2}e^2\delta M^{1-N}\frac{N(N-2)(N-4)(N+2d-2)}{12d(d+2)}\int d^4x\,\bar\psi(x)A_\mu(\partial_\mu\partial_\nu A_\nu)(\partial^2)^{(N/2)-2}\psi(x)\,. \tag{3.35}$$

The next step is to apply the operator $\partial/\partial N$ to the expression in (3.35) and then to set $N=0$. We obtain

$$\frac{e^2\delta M(3d^2-2d+8)}{12d(d+2)}\int d^dx\,\bar\psi(x)A_\mu(\partial^2 A_\mu)\partial^{-4}\psi(x)-\frac{4e^2\delta M(d-1)}{3d(d+2)}\int d^dx\,\bar\psi(x)A_\mu(\partial_\mu\partial_\nu A_\nu)\partial^{-4}\psi(x)\,. \tag{3.36}$$

The first term in (3.36) gives a graphical amplitude [4] proportional to $p^2g^{\mu\nu}$ and the second term gives an amplitude [4] proportional to $p^\mu p^\nu$ (the graphs have the form shown in Fig. 2):

$$\frac{3d^2-2d+8}{6d(d+2)}2^{d/2}e^2\delta p^2 g^{\mu\nu}\int\frac{d^dq}{(2\pi)^d}q^{-4}-\frac{8(d-1)}{3d(d+2)}2^{d/2}e^2\delta p^\mu p^\nu\int\frac{d^dq}{(2\pi)^d}q^{-4}\,. \tag{3.37}$$

Note that when $d=2$ the integrals in (3.37) are strongly infrared divergent.

C. Special case $P=4$

In preparation for the general case discussed in the next section (arbitrary P) we consider one last special case: namely, $P=4$. In this case four derivatives remain between the two A fields when the expression in (3.5) is expanded.

Subcase even b. When we expand a typical even-b ($b=2k$) term in (3.5),

$$A(\partial^2)^k A(\partial^2)^{(N/2)-k-1}, \tag{3.38}$$

there are three $P=4$ terms that result:

$$A(\partial_{\alpha_1}\partial_{\alpha_2}\partial_{\alpha_3}\partial_{\alpha_4}A)(\partial^2)^{(N/2)-5}\partial_{\alpha_1}\partial_{\alpha_2}\partial_{\alpha_3}\partial_{\alpha_4}u_{2k,0,4} \quad (L=0,\ P=4), \tag{3.39a}$$

$$A(\partial_{\alpha_1}\partial_{\alpha_2}\partial^2 A)(\partial^2)^{(N/2)-4}\partial_{\alpha_1}\partial_{\alpha_2}u_{2k,1,4} \quad (L=1,\ P=4), \tag{3.39b}$$

$$A(\partial^2\partial^2 A)(\partial^2)^{(N/2)-3}u_{2k,2,4} \quad (L=2, P=4). \tag{3.39c}$$

Simple counting arguments show that the coefficients $u_{2k,L,P}$ are given by

$$\begin{aligned} u_{2k,0,4} &= \begin{bmatrix} 2k \\ 4 \end{bmatrix} \frac{2k-2}{2k-1}\,\frac{2k-4}{2k-2}\,\frac{2k-6}{2k-3}\,\frac{1}{0!}, \\ u_{2k,1,4} &= \begin{bmatrix} 2k \\ 4 \end{bmatrix} \frac{2k-2}{2k-1}\,\frac{2k-4}{2k-2}\,\frac{1}{2k-3} \begin{bmatrix} 4 \\ 2 \end{bmatrix} \frac{1}{1!}, \\ u_{2k,2,4} &= \begin{bmatrix} 2k \\ 4 \end{bmatrix} \frac{2k-2}{2k-1}\,\frac{1}{2k-2}\,\frac{1}{2k-3} \begin{bmatrix} 4 \\ 2 \end{bmatrix} \begin{bmatrix} 2 \\ 2 \end{bmatrix} \frac{1}{2!}. \end{aligned} \tag{3.40}$$

As we saw in the $P=2$ case, the coefficients $u_{2k,0,4}$ have the form of a binomial coefficient multiplied by a probability:

$$\sum_{L=0}^{2} u_{2k,L,4} = \begin{bmatrix} 2k \\ 4 \end{bmatrix}. \tag{3.41}$$

This formula is the analogue of that in (3.20c).

Next we use rotational symmetry of the fermion loop integral to simplify the formulas in (3.39). The identity in (3.28) simplifies (3.39a) to

$$A(\partial^2\partial^2 A)(\partial^2)^{(N/2)-3}\frac{2k(k-1)(k-2)(k-3)}{d(d+2)}, \tag{3.42a}$$

and the identity in (3.10) simplifies (3.39b) to

$$A(\partial^2\partial^2 A)(\partial^2)^{(N/2)-3}\frac{2k(k-1)(k-2)}{d}. \tag{3.42b}$$

The expression in (3.39c) is already in rotationally symmetric form:

$$A(\partial^2\partial^2 A)(\partial^2)^{(N/2)-3}\frac{k(k-1)}{2}. \tag{3.42c}$$

The three expressions in (3.42) must now be multiplied by the weight $W_b = W_{2k} = N-2k-1$ and summed from $k=2$ to $(N/2)-1$. We will carry out this part of the calculation later.

Subcase odd b. A typical odd-b term in (3.5) has the form

$$A(\not\partial)^{2k-1}A(\not\partial)^{N-2k-1}$$

or

$$A\not\partial(\partial^2)^{k-1}A\not\partial(\partial^2)^{(N/2)-k-1}. \tag{3.43}$$

Following the approach taken in the $P=2$ case we take the trace and identify three types of terms which we must consider in turn. These terms are listed in (3.24) and are called terms of type I, type II, and type III.

Terms of type I. If we allow the derivative operators in

$$(\mathrm{Tr}\,\mathbb{1})A_\alpha\partial_\alpha(\partial^2)^{k-1}A_\beta\partial_\beta(\partial^2)^{(N/2)-k-1}$$

to act to the right with the constraint that exactly four derivatives remain acting on A_β ($P=4$) there are exactly two possibilities that may arise. If the derivative operator ∂_α acts on A_β it produces terms of the form

$$A_\alpha(\partial_\alpha\partial_{\alpha_1}\partial_{\alpha_2}\partial_{\alpha_3}A_\beta)\partial_{\alpha_1}\partial_{\alpha_2}\partial_{\alpha_3}\partial_\beta(\partial^2)^{(N/2)-5}\begin{bmatrix} 2k-2 \\ 3 \end{bmatrix}\frac{2k-4}{2k-3}\,\frac{2k-6}{2k-4} \quad (L=0,\ \alpha \text{ inside}) \tag{3.44a}$$

and

$$A_\alpha(\partial_\alpha\partial_{\alpha_1}\partial^2 A_\beta)\partial_{\alpha_1}\partial_\beta(\partial^2)^{(N/2)-4}\begin{bmatrix} 2k-2 \\ 3 \end{bmatrix}\left[\frac{1}{2k-3}+\frac{2k-4}{2k-3}\,\frac{2}{2k-4}\right] \quad (L=1,\ \alpha \text{ inside}). \tag{3.44b}$$

On the other hand, if the derivative operator ∂_α commutes past A_β it produces terms of the form

$$A_\alpha(\partial_{\alpha_1}\partial_{\alpha_2}\partial_{\alpha_3}\partial_{\alpha_4}A_\beta)\partial_\alpha\partial_\beta\partial_{\alpha_1}\partial_{\alpha_2}\partial_{\alpha_3}\partial_{\alpha_4}(\partial^2)^{(N/2)-6}\begin{bmatrix} 2k-2 \\ 4 \end{bmatrix}\left[\frac{2k-4}{2k-3}\,\frac{2k-6}{2k-4}\,\frac{2k-8}{2k-5}\right] \quad (L=0,\ \alpha \text{ outside}), \tag{3.45a}$$

$$A_\alpha(\partial_{\alpha_1}\partial_{\alpha_2}\partial^2 A_\beta)\partial_\alpha\partial_\beta\partial_{\alpha_1}\partial_{\alpha_2}(\partial^2)^{(N/2)-5}\begin{bmatrix} 2k-2 \\ 4 \end{bmatrix}\frac{2k-4}{2k-3}\,\frac{2k-6}{2k-4}\,\frac{1}{2k-5}\begin{bmatrix} 4 \\ 2 \end{bmatrix} \quad (L=1,\ \alpha \text{ outside}), \tag{3.45b}$$

and

$$A_\alpha(\partial^2\partial^2 A_\beta)\partial_\alpha\partial_\beta(\partial^2)^{(N/2)-4}\begin{bmatrix}2k-2\\4\end{bmatrix}\frac{2k-4}{2k-3}\frac{1}{2k-4}\frac{1}{2k-5}\begin{bmatrix}4\\2\end{bmatrix}\begin{bmatrix}2\\2\end{bmatrix}\frac{1}{2!}\quad (L=2,\ \alpha \text{ outside})\ . \qquad (3.45c)$$

(Note that the α outside case has the same coefficients as the even-b case with $2k$ replaced by $2k-2$.) The probabilistic nature of the coefficients of these terms is easily demonstrated. For the α-inside terms in (3.44) the sum of the coefficients is the binomial coefficient

$$\begin{bmatrix}3k-2\\3\end{bmatrix},$$

and for the α-outside terms in (3.45) the sum of the coefficients is the binomial coefficient

$$\begin{bmatrix}2k-2\\4\end{bmatrix}.$$

Next we use rotational symmetry to simplify the α-inside expressions in (3.44). The identity in (3.28) simplifies (3.44a) to

$$\frac{4(k-1)(k-2)(k-3)}{d(d+2)}A_\alpha(\partial^2\partial_\alpha\partial_\beta A_\beta)(\partial^2)^{(N/2)-3} \qquad (3.46a)$$

and the identity in (3.10) simplifies (3.44b) to

$$\frac{2}{d}(k-1)(k-2)A_\alpha(\partial^2\partial_\alpha\partial_\beta A_\beta)(\partial^2)^{(N/2)-3}\ . \qquad (3.46b)$$

In a similar way, we simplify the α-outside expressions in (3.45). The identity in (3.29) with $N=3$ simplifies (3.45a) to

$$\frac{2(k-1)(k-2)(k-3)(k-4)}{d(d+2)(d+4)}\times[A_\alpha(\partial^2\partial^2 A_\alpha)+4A_\alpha(\partial^2\partial_\alpha\partial_\beta A_\beta)](\partial^2)^{(N/2)-3}\ . \qquad (3.47a)$$

The identity in (3.28) simplifies (3.45b) to

$$\frac{2(k-1)(k-2)(k-3)}{d(d+2)}[A_\alpha(\partial^2\partial^2 A_\alpha)+2A_\alpha(\partial^2\partial_\alpha\partial_\beta A_\beta)]\times(\partial^2)^{(N/2)-3}\ . \qquad (3.47b)$$

The identity in (3.10) simplifies (3.45c) to

$$\frac{(k-1)(k-2)}{2d}A_\alpha(\partial^2\partial^2 A_\alpha)(\partial^2)^{(N/2)-3}\ . \qquad (3.47c)$$

Terms of type III. As we observed in our discussion of the $P=2$ terms, the contribution of type-III terms is identical to the contribution of type-I terms.

Terms of type II. We can expand the type-II term in (3.24) into β-inside and β-outside terms. The two β-inside terms are

$$-A_\alpha(\partial_\beta\partial_{\alpha_1}\partial_{\alpha_2}\partial_{\alpha_3}A_\alpha)\partial_\beta\partial_{\alpha_1}\partial_{\alpha_2}\partial_{\alpha_3}(\partial^2)^{(N/2)-5}\times\frac{4}{3}(k-1)(k-2)(k-3) \qquad (3.48a)$$

and

$$-A_\alpha(\partial_\beta\partial_{\alpha_1}\partial^2 A_\alpha)\partial_\beta\partial_{\alpha_1}(\partial^2)^{(N/2)-4}2(k-1)(k-2)\ . \qquad (3.48b)$$

Using rotational symmetry these two terms become

$$-\left[\frac{4(k-1)(k-2)(k-3)}{d(d+2)}+\frac{2(k-1)(k-2)}{d}\right]\times A_\alpha(\partial^2\partial^2 A_\alpha)(\partial^2)^{(N/2)-3}\ . \qquad (3.49)$$

The three β-outside terms, after we use rotational symmetry, contribute

$$-\left[\frac{2(k-1)(k-2)(k-3)(k-4)}{d(d+2)}+\frac{2(k-1)(k-2)(k-3)}{d}+\frac{(k-1)(k-2)}{2}\right]A_\alpha(\partial^2\partial^2 A_\alpha)(\partial^2)^{(N/2)-3}\ . \qquad (3.50)$$

Observe that this result is identical to that in (3.42) for the even-b terms except that the sign is changed and k is replaced by $k-1$.

$p^\mu p^\nu$ term. We can look ahead to the final answer and determine the origin of its structure. The final answer contains one term proportional to $p^\mu p^\nu$ and another proportional to $g^{\mu\nu}$. The $p^\mu p^\nu$ term arises from an effective Lagrangian containing an interaction of the form $A_\alpha(\partial_\alpha\partial_\beta\partial^2 A_\beta)$ and the $g^{\mu\nu}$ term arises from an interaction of the form $A_\alpha(\partial^2\partial^2 A_\beta)$. Hence the $p^\mu p^\nu$ term comes only from the type-I and type-III terms in the odd-b case. Thus, we must collect all the $p^\mu p^\nu$-producing terms in (3.46) and (3.47) and multiply by 2:

$$\left[\frac{16(k-1)(k-2)(k-3)}{d(d+2)}+\frac{4(k-1)(k-2)}{d}+\frac{16(k-1)(k-2)(k-3)(k-4)}{d(d+2)(d+4)}\right]A_\alpha(\partial^2\partial_\alpha\partial_\beta A_\beta)(\partial^2)^{(N/2)-3}\ . \qquad (3.51)$$

We must then multiply by the weight $W_b=N-2k$ and sum over odd $b=2k-1$ from $k=1$ to $k=(N/2)-1$. To perform this sum we use the identity in (3.22). The result is

$$\frac{N(N-2)(N-4)(N-6)(5d^2-18d+8-12N+6Nd+2N^2)}{120d(d+2)(d+4)}A_\alpha(\partial_\alpha\partial_\beta\partial^2 A_\beta)(\partial^2)^{(N/2)-3} . \quad (3.52)$$

It is easy now to insert the expression in (3.52) into (3.5) to construct the effective order-δ action:

$$\frac{N(N-2)(N-4)(N-6)(5d^2-18d+8-12N+6Nd+2N^2)}{120d(d+2)(d+4)}e^2\delta M^{1-N}i^{N-2}\int d^dx\,\bar\psi(x)A_\alpha(\partial_\alpha\partial_\beta\partial^2 A_\beta)(\partial^2)^{(N/2)-3}\psi(x) . \quad (3.53)$$

Finally, we differentiate with respect to N and set $N=0$:

$$\frac{2(5d^2-18d+8)}{5d(d+2)(d+4)}e^2\delta M\int d^dx\,\bar\psi(x)A_\alpha(\partial_\alpha\partial_\beta\partial^2 A_\beta)(\partial^2)^{-3}\psi(x) . \quad (3.54)$$

This action produces a two-photon amplitude from a graph of the form shown in Fig. 2 [4]:

$$\frac{4(5d^2-18d+8)}{5d(d+2)(d+4)}e^2\delta 2^{d/2}p^2p^\mu p^\nu\int\frac{d^dq}{(2\pi)^d}q^{-6} . \quad (3.55)$$

$g^{\mu\nu}$ *term*. The $g^{\mu\nu}$ comes from both even-b and odd-b terms. The contribution from even-b terms is taken from (3.42),

$$\sum_{k=2}^{(N/2)-1}\left[\frac{2k(k-1)(k-2)(k-3)}{d(d+2)}+\frac{2k(k-1)(k-2)}{d}+\frac{k(k-1)}{2}\right](N-2k-1)\boldsymbol{A}\partial^2\partial^2\boldsymbol{A}(\partial^2)^{(N/2)-3} , \quad (3.56)$$

and the contribution from odd-b terms is taken from (3.47) (multiplied by two to account for terms of type I and type III, (3.49), and (3.50):

$$\sum_{k=1}^{(N/2)-1}\left[\frac{4(k-1)(k-2)(k-3)(k-4)}{d(d+2)(d+4)}-\frac{(k-1)(k-2)}{d}-\frac{2(k-1)(k-2)(k-3)(k-4)}{d(d+2)}\right.$$
$$\left.-\frac{2(k-1)(k-2)(k-3)}{d}-\frac{(k-1)(k-2)}{2}\right](N-2k)A_\alpha(\partial^2\partial^2 A_\alpha)(\partial^2)^{(N/2)-3} . \quad (3.57)$$

We now evaluate the sums in (3.56) and (3.57) using the identity in (3.22) and combine the results to obtain a form for the effective action to order δ:

$$\frac{N(N-2)(N-4)}{240d(d+2)(d+4)}(N^3+3dN^2-12N^2+5d^2N-12dN+60N+5d^3+4d-144)$$
$$\times e^2\delta M^{1-N}i^{N-2}\int d^dx\,\bar\psi(x)A_\mu(\partial^2\partial^2A_\mu)(\partial^2)^{(N/2)-3}\psi(x) . \quad (3.58)$$

Finally, we differentiate with respect to N and set $N=0$:

$$-\frac{5d^3+4d-144}{30d(d+2)(d+4)}e^2\delta M\int d^dx\,\bar\psi(x)A_\mu(\partial^2\partial^2A_\mu)(\partial^2)^{-3}\psi(x) . \quad (3.59)$$

This action produces a two-photon amplitude from a graph of the form shown in Fig. 2 [4]:

$$-\frac{5d^3+4d-144}{15d(d+2)(d+4)}e^2\delta 2^{d/2}p^4g^{\mu\nu}\int\frac{d^dq}{(2\pi)^d}q^{-6} . \quad (3.60)$$

D. Summary

Here is a summary of the results derived in this section taken from (3.16), (3.37), (3.55), and (3.60). Terms containing $g^{\mu\nu}$:

$$g^{\mu\nu}e^2\delta 2^{d/2}\left[\frac{2-d}{d}\int\frac{d^dq}{(2\pi)^d}q^{-2} \qquad (P=0)\right.$$
$$+\frac{3d^2-2d+8}{6d(d+2)}p^2\int\frac{d^dq}{(2\pi)^d}q^{-4} \qquad (P=2)$$
$$+\frac{144-4d-5d^d}{15d(d+2)(d+4)}p^4\int\frac{d^dq}{(2\pi)^d}q^{-6} \qquad (P=4)$$
$$\left.+\cdots\right] . \quad (3.61)$$

Terms containing $p^\mu p^\nu$:

$$\frac{p^\mu p^\nu}{p^2} e^2\delta 2^{d/2}\left[\frac{8(1-d)}{3d(d+2)}p^2\int\frac{d^dq}{(2\pi)^d}q^{-4} \qquad (P=2)\right.$$
$$+\frac{4(5d^2-18d+8)}{5d(d+2)(d+4)}p^4\int\frac{d^dq}{(2\pi)^d}q^{-6} \qquad (P=4)$$
$$\left.+\cdots\right] . \tag{3.62}$$

Observe that while the scale mass M appears in the Lagrangian in (1.4), it does not appear in the formulas (3.61) and (3.62). This convenient cancellation occurs because the fermion propagator is the constant $1/M$ and the vertex amplitude is proportional to M^{1-N}, which becomes M at $N=0$.

Note also that subsequent integrals in (3.61) and (3.62) are increasingly infrared divergent. It will be necessary to perform a summation over P (under the integral) before attempting to evaluate these integrals. We will see in the next section that this summation gives an integrand that is no longer infrared divergent.

IV. ORDER-e^2 CONTRIBUTION TO THE ANOMALY IN THE δ EXPANSION

In this section we show how to find the general term in the series in (3.61) and (3.62) for arbitrary values of P. It is easier to derive the coefficient of $p^\mu p^\nu$ so we consider this case first.

A. $p^\mu p^\nu$ terms

From our experience with the special cases considered in Sec. III it is clear that $p^\mu p^\nu$ terms come only from terms of type I and type III for the case of odd b. Recall that a typical odd-b term in (3.5) has the form

$$A(\bar{\partial})^{2k-1}A(\bar{\partial})^{N-2k-1}$$

or

$$A\bar{\partial}(\partial^2)^{k-1}A\bar{\partial}(\partial^2)^{(N/2)-k-1} .$$

When the trace is taken we obtain a type-I term of the form [see Eq. (3.24)]

$$(\mathrm{Tr}\mathbb{1})A_\alpha\partial_\alpha(\partial^2)^{k-1}A_\beta\partial_\beta(\partial^2)^{(N/2)-k-1} . \tag{4.1}$$

Now, when we allow all derivatives to act to the right we obtain two types of terms: α-inside terms (where the derivative with index α acts on A_β and remains between the two A fields) and α-outside terms [where the derivative with index α commutes past A_β and acts on $\psi(x)$].

α-inside terms. When $P=0$ there are no α-inside terms, when $P=2$ there is one such term corresponding to $L=0$ [see (3.25a)], and when $P=4$ there are two such terms corresponding to $L=0$ and $L=1$ [see (3.44a) and (3.44b)]. In general there are $P/2$ such terms corresponding to $L=0$, $L=1,\ldots,L=(P-2)/2$. The coefficients of these terms fit a relatively simple and recognizable pattern. For example, for $P=6$ we have

$$\frac{(2k-2)(2k-4)(2k-6)}{5!}\left[(2k-8)(2k-10) \qquad (L=0\ \text{term})\right.$$
$$+\begin{bmatrix}5\\2\end{bmatrix}(2k-8) \qquad (L=1\ \text{term})$$
$$\left.+\begin{bmatrix}5\\2\end{bmatrix}\begin{bmatrix}3\\2\end{bmatrix}\frac{1}{2!} \qquad (L=2\ \text{term})\right] . \tag{4.2a}$$

For $P=8$ we have

$$\frac{(2k-2)(2k-4)(2k-6)(2k-8)}{7!}\left[(2k-10)(2k-12)(2k-14) \quad (L=0\ \text{term})\right.$$
$$+\begin{bmatrix}7\\2\end{bmatrix}(2k-10)(2k-12) \qquad (L=1\ \text{term})$$
$$+\begin{bmatrix}7\\2\end{bmatrix}\begin{bmatrix}5\\2\end{bmatrix}\frac{1}{2!}(2k-10) \qquad (L=2\ \text{term})$$
$$\left.+\begin{bmatrix}7\\2\end{bmatrix}\begin{bmatrix}5\\2\end{bmatrix}\begin{bmatrix}3\\2\end{bmatrix}\frac{1}{3!} \qquad (L=3\ \text{term})\right] \tag{4.2b}$$

and for $P=10$ we have

$$\frac{(2k-2)(2k-4)(2k-6)(2k-8)(2k-10)}{9!}\Bigg[(2k-12)(2k-14)(2k-16)(2k-18)\quad (L=10\ \text{term})$$
$$+\begin{bmatrix}9\\2\end{bmatrix}(2k-12)(2k-14)(2k-16)\quad (L=1\ \text{term})$$
$$+\begin{bmatrix}9\\2\end{bmatrix}\begin{bmatrix}7\\2\end{bmatrix}\frac{1}{2!}(2k-12)(2k-14)\quad (L=2\ \text{term})$$
$$+\begin{bmatrix}9\\2\end{bmatrix}\begin{bmatrix}7\\2\end{bmatrix}\begin{bmatrix}5\\2\end{bmatrix}\frac{1}{3!}(2k-12)\quad (L=3\ \text{term})$$
$$+\begin{bmatrix}9\\2\end{bmatrix}\begin{bmatrix}7\\2\end{bmatrix}\begin{bmatrix}5\\2\end{bmatrix}\begin{bmatrix}3\\2\end{bmatrix}\frac{1}{4!}\quad (L=4\ \text{term})\Bigg]. \tag{4.2c}$$

The pattern is now clear; the coefficient of a term for a particular value of P, $2k-1$, and L has the form

$$u(2k-1,L,P)=\frac{2^{P-L-1}}{(P-1)!L!}\times\prod_{n=1}^{P-L-1}(k-n)\prod_{l=1}^{L}\begin{bmatrix}P+1-2l\\2\end{bmatrix}. \tag{4.3}$$

This formula reproduces correctly all of the odd-b results we have seen so far; namely, (3.25a) for $P=2$, (3.44a) for $P=4$, and (4.2) for $P=6$, 8, and 10.

The formula in (4.3) can be generalized to include even b as well as odd b [5]:

$$u(2k-a,L,P)=\frac{2^{P-a-L}}{(P-a)!L!}\times\prod_{n=a}^{P-L-1}(k-n)\prod_{l=1}^{L}\begin{bmatrix}P+2-a-2l\\2\end{bmatrix}, \tag{4.4}$$

where $a=0$ for the case of even b and $a=1$ for the case of odd b. Now, when $a=0$, (4.4) gives the even-b results in (3.7) for $P=0$, in (3.19) for $P=2$, and in (3.40) for $P=4$.

α-outside terms. Next, we examine the α-outside terms. When $P=0$ there is one α-outside term [see (3.9)], when $P=2$ there are two α-outside terms [(3.25b) for $L=0$ and (3.25c) for $L=1$], and when $P=4$ there are three α-outside terms [(3.45a) for $L=0$, (3.45b) for $L=1$, and (3.45c) for $L=2$]. Once again, the coefficients for these terms fit a simple pattern:

$$u(2k-1,L,P)=\frac{2^{P-L}}{P!L!}\prod_{n=1}^{P-L}(k-n)\prod_{l=1}^{L}\begin{bmatrix}P+2-2l\\2\end{bmatrix}. \tag{4.5}$$

Imposition of rotational symmetry. Next we apply the identity in (3.29) to the α-inside terms. Rotational symmetry reduces the coefficient in (4.3) to

$$\frac{2^{P-L-1}}{(P-1)!L!}\prod_{l=1}^{L}\begin{bmatrix}P+1-2l\\2\end{bmatrix}\left[\prod_{n-1}^{P-L-1}(k-n)\right]\times\frac{(P-2L-1)!!}{\prod_{n=1}^{P/2-L}[d+2(n-1)]}, \tag{4.6}$$

where $L=0,1,2,\ldots,(P/2)-1$.

Applying the identity in (3.29) to the α-outside terms changes the coefficient in (4.5) to

$$\frac{2^{P-L}}{P!L!}\left[\prod_{n=1}^{P-L}(k-n)\right]\prod_{l=1}^{L}\begin{bmatrix}P+2-2l\\2\end{bmatrix}\times\frac{(P+1-2L)!!-(P-1-2L)!!}{\prod_{n=1}^{(P/2)-L+1}[d+2(n-1)]}, \tag{4.7}$$

where $L=0,1,\ldots,(P/2)-1$.

Summation on k and L. Just as we did in the special examples considered in Sec. III we multiply the expressions in (4.6) and (4.7) by the weight $W_b=N-2k$ and sum over the odd $b=2k-1$ from $k=(P+2)/2$ to $k=(N/2)-1$. To perform this summation we use the elementary identity in (3.22). We also sum on L from 0 to $(P/2)-1$ [6]. We must multiply these sums by 2 because we have identical contributions coming from terms of type I and terms of type III, as we saw in Sec. III. We spare the reader the algebra in performing these sums and merely present the final result, which is a sum over P:

$$-4e^2N\frac{p_\mu p_\nu}{p^2}\int\frac{d^2q}{(2\pi)^2}q^N\left[\frac{p^2}{6q^4}+\frac{p^4}{15q^6}+\frac{3p^6}{140q^8}+\frac{2p^8}{315q^{10}}+\cdots+\frac{\frac{P}{2}\left[\left[\frac{P}{2}\right]!\right]^2p^P}{(P+1)!q^{2+P}}+\cdots\right], \tag{4.8}$$

where we have set $d=2$. For large N the first few integrals in this sum are strongly ultraviolet divergent. However, recall that we must differentiate with respect to N and set $N=0$. Performing this operation *before* attempting to integrate removes all ultraviolet divergences and gives the result

$$-4e^2\frac{p_\mu p_\nu}{p^2}\int\frac{d^2q}{(2\pi)^2}\sum_{\substack{P=2\\ \text{even } P}}^{\infty}\frac{\frac{P}{2}[(P/2)!]^2p^P}{(P+1)!q^{2+P}} . \tag{4.9}$$

If we set $d=2$ in (3.62) we obtain the first two terms in this sum corresponding to the special cases $P=2$ and $P=4$.

Of course, each term in (4.9) gives an integral that is infrared divergent. However, if we perform the sum over P first, we obtain a convergent integral. Taking $P=2n$ and $z=|p|/|q|$ we have the identity

$$\sum_{n=0}^{\infty}\frac{z^{2n}n(n!)^2}{(2n+1)!}=\frac{1}{2}z\frac{d}{dz}\left[\frac{2}{-iz}\frac{\ln\left[\frac{z}{2i}+\sqrt{1-z^2/4}\right]}{\sqrt{1-z^2/4}}\right] . \tag{4.10}$$

Thus, substituting (4.10) into (4.9) gives

$$-4e^2\frac{p_\mu p_\nu}{p^2}\int_0^\infty\frac{dz}{4\pi}\frac{d}{dz}\left[\frac{2}{-iz}\frac{\ln\left[\frac{z}{2i}+\sqrt{1-z^2/4}\right]}{\sqrt{1-z^2/4}}\right], \tag{4.11}$$

which can be evaluated trivially:

$$-\frac{e^2}{\pi}\frac{p_\mu p_\nu}{p^2}\frac{2}{-iz}\frac{\ln\left[\frac{z}{2i}+\sqrt{1-z^2/4}\right]}{\sqrt{1-z^2/4}}\Bigg|_0^\infty=\frac{e^2p_\mu p_\nu}{\pi p^2} . \tag{4.12}$$

This answer agrees, of course, with the correct result in (2.5).

B. $g^{\mu\nu}$ terms

As we have seen from the special examples considered in Sec. III, there are three sources of $g^{\mu\nu}$ terms: the even-b terms, part of the type-I and type-III odd-b terms, and the type-II odd-b terms.

The even-b case is given in (4.4) with $a=0$. Imposing rotational symmetry [see (3.29)] now gives

$$\frac{2^{P-L}}{P!L!}\prod_{l=1}^{L}\left[\begin{matrix}P+2-2l\\ 2\end{matrix}\right]\times\prod_{n=0}^{P-L-1}(k-n)\frac{(P-2L-1)!!}{\prod_{n=1}^{(P/2)-L}(d+2n-2)} . \tag{4.13}$$

The corresponding formula for odd-b terms of type I and type III is

$$\frac{2^{P-L}}{P!L!}\prod_{l=1}^{L}\left[\begin{matrix}P+2-2l\\ 2\end{matrix}\right]\times\prod_{n=1}^{P-L}(k-n)\frac{(P-1-2L)!!}{\prod_{n=1}^{(P/2)-L+1}(d+2n-2)} \tag{4.14}$$

and the corresponding formula for odd-b terms of type II is

$$\frac{-2^{P-L}}{P!L!}\prod_{l=1}^{L}\left[\begin{matrix}P+2-2l\\ 2\end{matrix}\right]\prod_{n=1}^{P-L}(k-n)\frac{(P-1-2L)!!}{\prod_{n=1}^{(P/2)-L}(d+2n-2)} \tag{4.15}$$

for the α-outside terms and the result in (4.15) times $[(P/2)-L]/(k-P+L)$ for the α-inside terms.

As before, we multiply the even-b terms by the weight $N-2k-1$ and the odd-b terms by the weight $N-2k$ and sum over k, making use of the identity in (3.22). We sum over L using the identity in Ref. [6]. In the final result we must set $d=2$, differentiate with respect to N, and set $N=0$. The resulting series is *identical* to that in (4.9) except that the opposite sign occurs:

$$+4e^2g_{\mu\nu}\int\frac{d^2q}{(2\pi)^2}\sum_{\substack{P=2\\ \text{even } P}}^{\infty}\frac{\frac{P}{2}\left[\left[\frac{P}{2}\right]!\right]^2p^P}{(P+1)!q^{2+P}} . \tag{4.16}$$

Indeed, if we set $d=2$ in (3.61) we obtain the first two (nonzero) terms in this series.

Summing the series in (4.16) as we did above and performing the integration over q we get

$$\frac{-e^2}{\pi}g_{\mu\nu} . \tag{4.17}$$

This answer agrees with the correct result in (2.5).

V. MOMENTUM-SPACE CALCULATION OF THE PHOTON PROPAGATOR

The calculation presented in Secs. III and IV is easy in principle but technically complicated because the structure in (3.2) contains both γ matrices and derivatives. Since the derivatives do not commute with $A(x)$ and the

γ matrices are also noncommuting objects, the calculation required to find the photon propagator determined by (3.2) as a function of N is rather complicated.

The combinatorics for this problem can be simplified enormously if we rework this problem in momentum space. Now the derivative operator ∂_μ is replaced by the commuting object p_μ. We describe the calculation in this section.

We begin by rewriting the interaction term in (3.2) in momentum space. We substitute

$$\psi(x)=\int \frac{d^dq}{(2\pi)^d}\psi(q)e^{iqx} \tag{5.1a}$$

for the Fermi field,

$$\boldsymbol{A}(x)=\int \frac{d^dk}{(2\pi)^d}\boldsymbol{A}(k)e^{ikx}\ , \tag{5.1b}$$

$$\boldsymbol{A}(x)=\int \frac{d^dl}{(2\pi)^d}\boldsymbol{A}(l)e^{ilx}\ , \tag{5.1c}$$

for the two photon fields, and

$$\bar{\psi}(x)=\int \frac{d^ds}{(2\pi)^d}\bar{\psi}(s)e^{-isx}\ . \tag{5.1d}$$

Inserting (5.1) into (3.2) and keeping the factors of δ and M^{1-N} in (3.1) gives an interaction term of the form

$$\delta e^2M^{1-N}\sum_{a,b}\int\int\int \frac{d^dq}{(2\pi)^d}\frac{d^dk}{(2\pi)^d}\frac{d^dl}{(2\pi)^d}\bar{\psi}(q+k+l)(\not{q}+\not{k}+\not{l})^a\boldsymbol{A}(l)(\not{q}+\not{k})^b\boldsymbol{A}(k)\not{q}^{\,N-a-b-2}\psi(q)\ . \tag{5.2}$$

This interaction term is represented by the four-point vertex shown in Fig. 3. The amplitude for this vertex factor is

$$-\delta e^2M^{1-N}(\not{q}+\not{k}+\not{l})^a\gamma^\mu(\not{q}+\not{k})^b\gamma^\nu\not{q}^{\,N-a-b-2}\ . \tag{5.3}$$

To construct the photon propagator $\Pi^{\mu\nu}(p)$ we connect the two fermion legs on the vertex in Fig. 3 together to make the graph shown in Fig. 4. Recall that the amplitude for the fermion propagator is $1/M$. Thus, the order-δ contribution to the photon propagator $\Pi^{\mu\nu}(p)$ is

$$\Pi^{\mu\nu}(p)=\delta M^{1-N}\frac{e^2}{M}\sum_{a,b}\int\frac{d^dq}{(2\pi)^d}\mathrm{Tr}\gamma^\mu(\not{q}+\not{p})^b\gamma^\nu\not{q}^{\,N-b-2}\ . \tag{5.4}$$

To obtain (5.4) we set $q+k+l=q$ and integrate over q. There are two ways to construct such a graph, one for which $k=-l=p$ and another for which $-k=l=p$; let us now consider just the first. The closed fermion loop is associated with a negative trace. Under the trace sign a structure such as

$$\mathrm{Tr}\not{q}^{\,a}\gamma^\mu(\not{q}+\not{p})^b\gamma^\nu\not{q}^{\,N-a-b-2}$$

simplifies to

$$\mathrm{Tr}\gamma^\mu(\not{q}+\not{p})^b\gamma^\nu\not{q}^{\,N-b-2}\ .$$

Note that the total number of γ matrices must be even or else the trace vanishes. Thus, N is even as we concluded at the beginning of Sec. III and we write $N=2n$. There are now two cases to consider: b even ($b=2k$) and b odd ($b=2k+1$).

Even b. When $N=2n$ and $b=2k$ the expression in (5.4) collapses immediately:

$$\begin{aligned}\Pi^{\mu\nu}_{\rm even}(p)&=\delta M^{1-N}\frac{e^2}{M}\sum_{k=0}^{n-1}\sum_{a=0}^{N-2k-2}\int\frac{d^dq}{(2\pi)^d}(q+p)^{2k}q^{2n-2k-2}\mathrm{Tr}(\gamma^\mu\gamma^\nu)\\&=\delta M^{-N}e^2g^{\mu\nu}2^{d/2}\sum_{k=0}^{n-1}\int\frac{d^dq}{(2\pi)^d}(q+p)^{2k}q^{2n-2k-2}(2n-2k-1)\ .\end{aligned} \tag{5.5}$$

Next we sum over k using the identities

$$\sum_{k=0}^{n-1}x^k=\frac{1-x^n}{1-x} \tag{5.6a}$$

and

$$\sum_{k=0}^{n-1}kx^k=\frac{x}{(1-x)^2}[1+(n-1)x^n-nx^{n-1}]\ . \tag{5.6b}$$

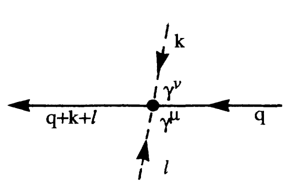

FIG. 3. The four-point vertex represented by the interaction term in (5.2).

Note that each sum in (5.6) vanishes at $N=2n=0$ and that

$$\frac{\partial}{\partial N}\sum_{k=0}^{n-1}x^k\bigg|_{N=0}=-\frac{1}{2}\frac{\ln x}{1-x} \tag{5.7a}$$

and

$$\frac{\partial}{\partial N}\sum_{k=0}^{n-1}kx^k\bigg|_{N=0}=-\frac{1}{2}\frac{x}{(1-x)^2}\ln x-\frac{1}{2(1-x)}\ . \tag{5.7b}$$

Thus, after differentiating with respect to N and setting $N=0$, we have

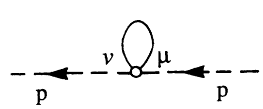

FIG. 4. The graph constructed from the vertex in Fig. 3 representing the photon propagator.

$$\Pi^{\mu\nu}_{\text{even}}(p)=\frac{1}{2}\delta e^2 g^{\mu\nu}2^{d/2}\int\frac{d^dq}{(2\pi)^d}\frac{q^2+(q+p)^2}{[q^2-(q+p)^2]^2}\ln\frac{(q+p)^2}{q^2}+\delta e^2 g^{\mu\nu}2^{d/2}\int\frac{d^dq}{(2\pi)^d}\frac{1}{q^2-(q+p)^2}\;. \tag{5.8}$$

Hence, as we saw in Secs. III and IV, the even-b case only produces terms proportional to $g^{\mu\nu}$.

Odd b. Substituting $N=2n$ and $b=2k+1$ into (5.4) gives

$$\Pi^{\mu\nu}_{\text{odd}}(p)=e^2\delta M^{-N}\sum_{k=0}^{n-1}\sum_{a=0}^{2n-2k-3}\int\frac{d^dq}{(2\pi)^d}(q+p)^{2k}q^{2n-2k-4}\mathrm{Tr}\gamma^\mu(\not{q}+\not{p})\gamma^\nu\not{q}\;.$$

Evaluating the trace and summing over a gives

$$\Pi^{\mu\nu}_{\text{odd}}(p)=e^2 2^{d/2}\delta M^{-N}\sum_{k=0}^{n-1}(2n-2k-2)\int\frac{d^dq}{(2\pi)^d}(q+p)^{2k}q^{2n-2k-4}[(q^\mu+p^\mu)q^\nu+q^\mu(q^\nu+p^\nu)-g^{\mu\nu}(q^2+q\cdot p)]\;. \tag{5.9}$$

Finally, we sum on k, compute $\partial/\partial N$, and set $N=2n=0$:

$$\Pi^{\mu\nu}_{\text{odd}}(p)=\delta e^2 2^{d/2}\int\frac{d^dq}{(2\pi)^d}\left[\frac{1}{[q^2-(p+q)^2]^2}\ln\frac{(p+q)^2}{q^2}+\frac{1}{q^2[q^2-(p+q)^2]}\right]\times\left[(p+q)^\mu q^\nu+q^\mu(p+q)^\nu-g^{\mu\nu}(q^2+q\cdot p)\right]\;. \tag{5.10}$$

Combination of odd-b and even-b terms. We combine $\Pi^{\mu\nu}_{\text{even}}(p)$ in (5.8) and $\Pi^{\mu\nu}_{\text{odd}}(p)$ in (5.10) to obtain $\Pi^{\mu\nu}(p)$:

$$\Pi^{\mu\nu}(p)=\delta e^2 2^{d/2}\int\frac{d^dq}{(2\pi)^d}\frac{\ln[(q+p)^2/q^2]}{[q^2-(q+p)^2]^2}[\tfrac{1}{2}g^{\mu\nu}p^2+(p+q)^\mu q^\nu+q^\mu(p+q)^\nu]+\delta e^2 2^{d/2}\int\frac{d^dq}{(2\pi)^d}\frac{(p+q)^\mu q^\nu+q^\mu(p+q)^\nu-g^{\mu\nu}q\cdot p}{q^2[q^2-(p+q)^2]}\;. \tag{5.11}$$

Contribution of the second graph. Finally, we must add to the result in (5.11) the contribution from the second graph of the form in Fig. 4 coming from the vertex in Fig. 3. The amplitude for this graph is obtained simply from that in (5.11) by replacing p with $-p$ and interchanging μ with ν. If we make a shift of the integration variable of the form $q\to q+p$ in the new amplitude and add it to (5.11) we obtain the result in (2.2) multiplied by δ, which is easily evaluated to give the result (2.5). Again, we have successfully evaluated the photon propagator and the anomaly.

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[4] Observe that the coefficient in (3.15) is twice that in (3.14). An action of the form $\delta\int A^2\bar{\psi}\psi$ has a vertex of the form 2δ.

[5] The probabilisitic nature of the coefficients $u(2k-a,L,P)$ in (4.4) is easily demonstrated. Summing over all L for fixed P gives a binomial coefficient:

$$\sum_L u(2k-a,L,P)=\binom{2k-2a}{P-a}\quad(a=0,1)\;.$$

Therefore, we can express the coefficients u as probabilities multiplied by the binomial coefficient $\binom{2k-2a}{P-a}$.

[6] To perform the summation over L we use the identity

$$\sum_{L=0}^{M}(-1)^L\frac{(2M-L)!}{(M-L)!L!(2M-L+2)(D+M-L)!}=\frac{(-1)^D M!(M+1-D)!}{2(2M+1)!}\;.$$