Spinless Salpeter equation as a simple matrix eigenvalue problem

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(Received 6 September 1991)

We propose a new method for solving the spinless Salpeter equation. Choosing a special set of orthonormal basis functions we are able to calculate analytically the integral over the corresponding kernel as well as the matrix elements of a class of potentials. In this way the problem can be reduced to the solution of a simple matrix eigenvalue problem with explicitly known matrices, which can be solved very fast numerically. The method is demonstrated in detail for S waves using a typical interquark potential as an example.

PACS number(s): 11.10.St, 03.65.Ge, 12.40.Qq

I. INTRODUCTION

The spinless Salpeter equation represents a standard approximation to the Bethe-Salpeter equation [1]. Upon elimination of any dependence on timelike variables in a suitable manner, the Bethe-Salpeter equation reduces to the Salpeter equation [2]. Neglecting, furthermore, any reference to the spin degrees of freedom and restricting to positive-energy solutions, one arrives at the spinless Salpeter equation, which is well suited for the description of bound states consisting of scalar bosons as well as of the spin-averaged spectra of bound states consisting of fermionic constituents.

Unfortunately, all previous attempts to obtain the energy eigenvalues and corresponding wave functions of bound states from the spinless Salpeter equation rely very heavily on numerical methods [3]. Here we would like to present a technique for solving the spinless Salpeter equation which aims at the derivation of, to the utmost extent, analytical results, i.e., which tries to avoid sticking to numerical computations at a too early stage. The basic idea of the proposed procedure consists of two main steps.

(1) The solutions of the spinless Salpeter equation are approximated by a truncated expansion in terms of a complete set of orthonormal basis functions.

(2) These basis functions are chosen in such a way that the integral over the kernel representing the kinetic energy as well as the matrix elements of the interaction potential entering in the spinless Salpeter equation can be, at least in principle, evaluated analytically.

The outcome of this procedure is the formulation of the spinless Salpeter equation as a simple matrix eigenvalue problem with explicitly known matrices. The solution of this matrix equation should then pose no unusual problems.

This paper is organized as follows. Section II recalls the conversion of the spinless Salpeter equation into an equivalent integral equation [4]. Section III sketches the general procedure for treating the spinless Salpeter equation analytically. In Sec. IV, we discuss, for the sake of simplicity, only the case of states with vanishing angular momentum, whereas in Sec. V we apply our technique to the simplest potential capable to describe the strong interactions between quarks. Section VI concludes with a brief summary of the proposed method. A collection of relations needed in the previous derivations is given in the Appendix.

II. THE SPINLESS SALPETER EQUATION

The spinless Salpeter equation may be regarded as the Schrödinger equation generalized to relativistic kinematics. For the case of two particles with equal masses m interacting via a spherically symmetric potential $V(r)$, $r\equiv|\mathbf{x}|$, $\mathbf{x}$ denoting their relative coordinate, the configuration-space representation of the spinless Salpeter equation in the center-of-momentum system of the two particles reads

$$[2\sqrt{-\Delta+m^2}+V(r)]\psi(\mathbf{x})=M\psi(\mathbf{x})\ . \tag{1}$$

In the center-of-momentum system the energy eigenvalue entering the spinless Salpeter equation is nothing else but the mass M of the bound state.

For states of definite orbital angular momentum l, we define the reduced radial wave function $u(r)$ by

$$\psi(\mathbf{x})=\frac{u(r)}{r}\mathcal{Y}_{lm}(\theta,\phi)\ , \tag{2}$$

where $\mathcal{Y}_{lm}(\theta,\phi)$ are the spherical harmonics of angular momentum l and projection m. For this reduced radial wave function $u(r)$, an integral representation of the spinless Salpeter equation (1) may be found [4]:

$$\begin{aligned}[M-V(r)]u(r)&=\frac{2}{\pi}\int_0^\infty dr' G_l(mr,mr')\\&\quad\times\left[-\frac{d^2}{dr'^2}+\frac{l(l+1)}{r'^2}+m^2\right]u(r')\ ,\end{aligned} \tag{3}$$

where the kernel G_l is defined by

$$G_l(x,y)=2^l z^{l+1}\left[\frac{1}{z}\frac{\partial}{\partial z}\right]^l\frac{1}{z}[(s-z)^{l/2}K_l(\sqrt{s-z})-(s+z)^{l/2}K_l(\sqrt{s+z})], \quad (4)$$

with $s\equiv x^2+y^2$, $z\equiv 2xy$. Here K_l is the modified Bessel function of the second kind of order l [5].

Since we only consider the equal-mass case, we may eliminate on the right-hand side of Eq. (3) any dependence on the constituent mass m by rescaling the radial variable r like $x=mr$. Defining the scaled radial wave function $\tilde{u}(x):=u(x/m)=u(r)$ as well as the dimensionless bound-state mass $\tilde{M}:=M/m$ and potential $\tilde{V}(x):=V(x/m)/m=V(r)/m$, the spinless Salpeter equation (3) becomes

$$[\tilde{M}-\tilde{V}(x)]\tilde{u}(x)=\frac{2}{\pi}\int_0^\infty dy\, G_l(x,y)\times\left[-\frac{d^2}{dy^2}+\frac{l(l+1)}{y^2}+1\right]\tilde{u}(y) . \quad (5)$$

In the free case, i.e., $V(r)\equiv 0$, as well as for potentials which are less singular than the Coulomb potential, i.e., $V(r)\propto 1/r^\eta$ with $\eta<1$, the reduced radial wave function $u(r)$ behaves for small r, $r\to 0$, asymptotically like $u(r)\propto r^{l+1}$. Accordingly, we make the ansatz

$$\tilde{u}(x)=x^{l+1}w(x) . \quad (6)$$

Because of the identity

$$\left[-\frac{d^2}{dx^2}+\frac{l(l+1)}{x^2}\right]x^{l+1}w(x)=-x^{l+1}\left[\frac{d^2}{dx^2}+\frac{2(l+1)}{x}\frac{d}{dx}\right]w(x) , \quad (7)$$

Eq. (5) is equivalent to

$$[\tilde{M}-\tilde{V}(x)]x^{l+1}w(x)=\frac{2}{\pi}\int_0^\infty dy\, G_l(x,y)y^{l+1}\times\left[1-\frac{d^2}{dy^2}-\frac{2(l+1)}{y}\frac{d}{dy}\right]\times w(y) . \quad (8)$$

This (scaled) form of the spinless Salpeter equation is the starting point for our considerations.

III. STRATEGY FOR SOLUTION

In order to solve Eq. (8), we expand any function $F(x)$ we encounter into a complete orthonormal system $\{f_n(x), n=0,1,2,\dots\}$ of basis functions for $L_2(R^3)$:

$$F(x)=\sum_{n=0}^{N}F_n f_n(x) . \quad (9)$$

For finite N, the above expansion is, of course, only an approximation to the exact function $F(x)$. Because of the assumed orthonormality of the basis functions $\{f_n(x)\}$ the coefficients F_n in the above expansion may be found from

$$F_n=\int_0^\infty dx\, f_n(x)F(x) . \quad (10)$$

The main point of the present discussion is our rather sophisticated choice of the basis functions f_n which, at least in principle, allows for a thorough analytical treatment of the spinless Salpeter equation (1):

$$f_n(x):=\sqrt{2}e^{-x}L_n(2x) , \quad (11)$$

where $L_n(x)$ are the Laguerre polynomials [5]

$$L_n(x)=\sum_{k=0}^{n}\frac{(-1)^k}{k!}\begin{bmatrix}n\\k\end{bmatrix}x^k . \quad (12)$$

It is easy to convince oneself that the functions $\{f_n(x)\}$ in (11) indeed form an orthonormal set, i.e.,

$$\int_0^\infty dx\, f_n(x)f_k(x)=\delta_{nk} . \quad (13)$$

In the above spirit, the solution $w(x)$ of the spinless Salpeter equation (8) is approximated by a linear combination of the basis functions (11):

$$w(x)=\sum_{n=0}^{N}\lambda_n f_n(x) \quad (14)$$

with real coefficients λ_n.

Noting that

$$\begin{bmatrix}n\\n+1\end{bmatrix}=0 , \quad (15)$$

and introducing the shorthand notation

$$D(l;n,k)\equiv 2\sqrt{2}\frac{(-1)^k}{k!}2^k\left[(k+l+1)\begin{bmatrix}n\\k\end{bmatrix}+(k+2l+2)\begin{bmatrix}n\\k+1\end{bmatrix}\right] , \quad (16)$$

the action of the differential operator in (8) on our basis functions f_n is given by

$$x\left[1-\frac{d^2}{dx^2}-\frac{2(l+1)}{x}\frac{d}{dx}\right]f_n(x)=\sum_{k=0}^{n}D(l;n,k)x^k e^{-x} . \quad (17)$$

Accordingly, we expect to encounter in the kinetic term of the spinless Salpeter equation (8) integrals of the type

$$I_k^{(l)}(x)\equiv\int_0^\infty dy\, G_l(x,y)y^{l+k}e^{-y} . \quad (18)$$

Expanding these integrals in terms of our basis functions f_n, Eq. (11),

$$I_k^{(l)}(x) =: \sum_{m=0}^{N} c(l;k,m) f_m(x) ,$$
$$c(l;k,m) = \int_0^\infty dx\, f_m(x) I_k^{(l)}(x) , \tag{19}$$

the right-hand side of Eq. (8) becomes

$$\begin{aligned}\frac{2}{\pi}\int_0^\infty dy\, G_l(x,y) y^{l+1}\left[1-\frac{d^2}{dy^2}-\frac{2(l+1)}{y}\frac{d}{dy}\right] w(y) &= \frac{2}{\pi}\sum_{n=0}^{N}\lambda_n \int_0^\infty dy\, G_l(x,y) y^{l+1}\left[1-\frac{d^2}{dy^2}-\frac{2(l+1)}{y}\frac{d}{dy}\right] f_n(y) \\ &= \frac{2}{\pi}\sum_{n=0}^{N}\lambda_n \sum_{k=0}^{n} D(l;n,k)\int_0^\infty dy\, G_l(x,y) y^{l+k} e^{-y} \\ &= \frac{2}{\pi}\sum_{n=0}^{N}\lambda_n \sum_{k=0}^{n} D(l;n,k) I_k^{(l)}(x) \\ &= \frac{2}{\pi}\sum_{n=0}^{N}\lambda_n \sum_{k=0}^{n} D(l;n,k) \sum_{m=0}^{N} c(l;k,m) f_m(x) . \end{aligned} \tag{20}$$

Consequently, the kinetic term in the spinless Salpeter equation (8) may be represented by the operation of a "kinetic matrix" $E_{nm}^{(l)}$ acting on the vector of basis functions f_n:

$$\begin{aligned}&\frac{2}{\pi}\int_0^\infty dy\, G_l(x,y) y^{l+1}\left[1-\frac{d^2}{dy^2}-\frac{2(l+1)}{y}\frac{d}{dy}\right] w(y) \\ &\quad =: \sum_{n=0}^{N}\lambda_n \sum_{m=0}^{N} E_{nm}^{(l)} f_m(x) ,\end{aligned} \tag{21}$$

where the kinetic matrix $E_{nm}^{(l)}$ is given by

$$E_{nm}^{(l)} = \frac{2}{\pi}\sum_{k=0}^{n} D(l;n,k) c(l;k,m) . \tag{22}$$

Similarly, in order to handle the left-hand side of Eq. (8), we define a "power matrix" $P_{nm}^{(l)}$ and a "potential matrix" $V_{nm}^{(l)}$ by

$$\begin{aligned} x^{l+1} f_n(x) &=: \sum_{m=0}^{N} P_{nm}^{(l)} f_m(x) , \\ P_{nm}^{(l)} &= \int_0^\infty dx\, x^{l+1} f_n(x) f_m(x) = P_{mn}^{(l)} , \end{aligned} \tag{23}$$

and

$$\begin{aligned} \tilde{V}(x) x^{l+1} f_n(x) &=: \sum_{m=0}^{N} V_{nm}^{(l)} f_m(x) , \\ V_{nm}^{(l)} &= \int_0^\infty dx\, x^{l+1} \tilde{V}(x) f_n(x) f_m(x) = V_{mn}^{(l)} . \end{aligned} \tag{24}$$

With the help of expansion (12) of the Laguerre polynomials, we may immediately give an explicit expression for the power matrix $P_{nm}^{(l)}$:

$$P_{nm}^{(l)} = \frac{1}{2^{l+1}}\sum_{p=0}^{n}\sum_{q=0}^{m}(-1)^{p+q}\binom{n}{p}\binom{m}{q} \times \frac{(p+q+l+1)!}{p!q!} . \tag{25}$$

With the above definitions, the left-hand side of Eq. (8) may be cast into the form

$$\begin{aligned}[\tilde{M}-\tilde{V}(x)] x^{l+1} w(x) &= \sum_{n=0}^{N}\lambda_n [\tilde{M}-\tilde{V}(x)] x^{l+1} f_n(x) \\ &= \sum_{n=0}^{N}\lambda_n \sum_{m=0}^{N} (\tilde{M} P_{nm}^{(l)} - V_{nm}^{(l)}) f_m(x) .\end{aligned} \tag{26}$$

Summarizing, with (26) and (21), the spinless Salpeter equation (8) reads

$$\sum_{n,m=0}^{N}\lambda_n (\tilde{M} P_{nm}^{(l)} - E_{nm}^{(l)} - V_{nm}^{(l)}) f_m(x) = 0 . \tag{27}$$

From this, one may obtain an eigenvalue equation for the coefficient vector $\lambda \equiv \{\lambda_n\}$ in the expansion (14) of the solutions $w(x)$. In self-explanatory matrix notation, this eigenvalue equation is given by

$$\tilde{M}\lambda = (P^{(l)})^{-1}[(E^{(l)})^T + V^{(l)}]\lambda . \tag{28}$$

In this way, the solution of the spinless Salpeter equation (1) can be reduced to a simple matrix eigenvalue problem. The eigenvalues of this equation are the masses M_i of the bound state under consideration. The corresponding eigenvectors $\lambda_n^{(i)}$ give the radial wave functions $u_i(r)$:

$$u_i(r) = (mr)^{l+1}\sum_{n=0}^{N}\lambda_n^{(i)} f_n(mr) . \tag{29}$$

For $N=\infty$ in the expansion (9) the above treatment would be exact. For $N<\infty$ it provides an approximation to the exact solution of increasing accuracy with increasing N, that is, with increasing size of the involved matrices.

IV. S WAVES

We illustrate the prescription given in the preceding section for the case of bound states with vanishing angular momentum, so-called S waves. In order to deal with

the eigenvalue equation (28), we need the explicit expressions of the kinetic matrix (22) and of the inverse of the power matrix (23).

A. The kinetic matrix

For a vanishing angular momentum l of the bound-state constituents, $l=0$, the spinless Salpeter equation (8) reduces to

$$[\tilde{M}-\tilde{V}(x)]xw(x) = \frac{2}{\pi}\int_0^\infty dy\, G_0(x,y)y\left[1-\frac{d^2}{dy^2}-\frac{2}{y}\frac{d}{dy}\right]w(y) \quad (30)$$

with the kernel

$$G_0(x,y)=[K_0(|x-y|)-K_0(x+y)] . \quad (31)$$

For this kernel one finds, for the integrals $I_k^{(0)}(x)$ in (18) with the help of the formulas given in the Appendix,

$$I_k^{(0)}(x)=2k!\sum_{p=0}^{[k/2]}\begin{bmatrix}k+1\\2p+1\end{bmatrix}\times\sum_{q=0}^{p}(-1)^q\begin{bmatrix}p\\q\end{bmatrix}\times\frac{x^{k-q+1}}{(2k-2q+1)!!}K_{k-q}(x) , \quad (32)$$

where

$$\left[\frac{k}{2}\right]\equiv\begin{cases}\dfrac{k}{2} & \text{for } k \text{ even} ,\\ \dfrac{k-1}{2} & \text{for } k \text{ odd} ,\end{cases} \quad (33)$$

and

$$(2n+1)!!\equiv 1\times 3\times\cdots\times(2n-1)\times(2n+1) . \quad (34)$$

Projecting out the coefficients $c(0;k,m)$ according to the second of Eqs. (19), one obtains

$$c(0;k,m)=2\sqrt{2}k!\sum_{p=0}^{[k/2]}\sum_{q=0}^{p}\sum_{r=0}^{m}(-1)^{q+r}\begin{bmatrix}k+1\\2p+1\end{bmatrix}\begin{bmatrix}p\\q\end{bmatrix}\begin{bmatrix}m\\r\end{bmatrix}\frac{2^r(r+1)(2k-2q+r+1)!}{(2k-2q+1)!!(2k-2q+2r+3)!!} . \quad (35)$$

Multiplying according to (22) these expansion coefficients by

$$D(0;n,k)\equiv 2\sqrt{2}\frac{(-1)^k}{k!}2^k\left[(k+1)\begin{bmatrix}n\\k\end{bmatrix}+(k+2)\begin{bmatrix}n\\k+1\end{bmatrix}\right] , \quad (36)$$

resulting from (16), yields, for the first few entries in the kinetic matrix $E_{nm}^{(0)}$ for the case $l=0$,

$$E_{nm}^{(0)}=\frac{2}{\pi}\frac{8}{(2n+2m+3)!!}\begin{bmatrix}1 & -3 & -5 & -21 & \cdots\\ -1 & 81 & -375 & -1029 & \cdots\\ -1 & -225 & 13\,125 & -77\,175 & \cdots\\ -3 & -441 & -55125 & 3565485 & \cdots\\ \vdots & \vdots & \vdots & \vdots & \ddots\end{bmatrix} . \quad (37)$$

(Note the range of the indices n and m: $n,m=0,1,\ldots,N$.)

B. The power matrix

For $l=0$ the power matrix (23) explicitly reads

$$2P_{nm}^{(0)}\equiv\int_0^\infty dx\, xe^{-x}L_n(x)L_m(x) = \begin{cases}2n+1 & \text{for } m=n ,\\ -m & \text{for } m=n+1 ,\\ -n & \text{for } m=n-1 ,\\ 0 & \text{else}\end{cases} = \begin{bmatrix}1 & -1 & 0 & 0 & \cdots\\ -1 & 3 & -2 & 0 & \cdots\\ 0 & -2 & 5 & -3 & \cdots\\ 0 & 0 & -3 & 7 & \cdots\\ \vdots & \vdots & \vdots & \vdots & \ddots\end{bmatrix} . \quad (38)$$

The inverse of this matrix, required for the eigenvalue equation (28), depends explicitly on the size N of the involved matrices and is given by

$$(P^{(0)})_{nn}^{-1}=\sum_{k=n}^{N}\frac{2}{k+1} \quad \text{for } n=0,1,\ldots,N ,$$
$$(P^{(0)})_{nm}^{-1}=(P^{(0)})_{mn}^{-1}=(P^{(0)})_{nn}^{-1} \quad \text{for } n=0,1,\ldots,N,\ m=0,1,\ldots,n-1 . \quad (39)$$

For instance, for $N=3$ this inverse reads

$$(P^{(0)})_{4\times 4}^{-1}=\frac{1}{6}\begin{bmatrix}25 & 13 & 7 & 3\\ 13 & 13 & 7 & 3\\ 7 & 7 & 7 & 3\\ 3 & 3 & 3 & 3\end{bmatrix} . \quad (40)$$

V. A PROTOTYPE POTENTIAL

Let us apply the previously developed ideas to hadrons, considered as bound states of (constituent) quarks which are bound by the strong interaction.

We have at our disposal a generally accepted quantum field theory of the strong interaction, namely, quantum chromodynamics (QCD), a non-Abelian gauge theory for quarks and gluons. Nevertheless, because of the intrinsically nonperturbative nature of the bound-state problem, in non-Abelian gauge theories it is up to now not possible to derive the forces acting between the quarks inside a hadron from first principles; that is, QCD. Consequently, the corresponding interquark potential has to be determined phenomenologically.

The reasoning leading to the various proposed potentials $V(r)$ is rather simple. For small distances between the quarks, one expects from one-gluon exchange, by analogy to one-photon exchange in quantum electrodynamics, a Coulomb-like contribution to the potential; that is, $V(r) \propto 1/r$. For large distances, in order to be able to describe confinement, the potential has to rise to infinity. From lattice-gauge-theory computations, there are hints that this rise is a linear one; that is, $V(r) \propto r$. The most reasonable possibility to construct an interquark potential which satisfies both of the above constraints is to simply add these two contributions. This leads to the so-called funnel potential [6–8]:

$$V_F(r) = -\frac{\kappa}{r} + ar \ , \tag{41}$$

which depends on just two parameters, viz., on the Coulomb coupling constant κ and on the slope a of the linear term. This funnel-shaped potential represents the prototype of all of the proposed realistic interquark potentials. A closer inspection reveals that all phenomenologically acceptable "QCD-inspired" potentials are only variations around the funnel potential [9].

From (41), according to the definition given in Sec. II, the scaled funnel potential $\tilde{V}_F(x)$ reads

$$\tilde{V}_F(x) = -\frac{\kappa}{x} + \frac{a}{m^2}x \ , \tag{42}$$

which entails for the corresponding potential matrix $V_F^{(l)}$, Eq. (24),

$$V_F^{(l)} = -\kappa P^{(l-1)} + \frac{a}{m^2}P^{(l+1)} \ . \tag{43}$$

In particular, specializing again to the case $l=0$, the matrix elements of the funnel potential, taken with respect to S-wave states, are given by

$$V_F^{(0)} = -\kappa P^{(-1)} + \frac{a}{m^2}P^{(1)} \ . \tag{44}$$

Since $P^{(-1)}$ is identical to the integral on the left-hand side of the normalization condition (13) for the basis functions $\{f_n(x)\}$, we have $P^{(-1)}=1$:

$$V_F^{(0)} = -\kappa + \frac{a}{m^2}P^{(1)} \ . \tag{45}$$

The summations in (25) may be performed, giving for the $(l=1)$ power matrix $P_{nm}^{(1)}$ the result

$$\begin{aligned} P_{nn}^{(1)} &= \frac{3n(n+1)+1}{2} \quad \text{for } n=0,1,\ldots,N \ , \\ P_{n,n+1}^{(1)} &= -n(n+2)-1 \quad \text{for } n=0,1,\ldots,N-1 \ , \\ P_{n,n+2}^{(1)} &= \frac{n(n+3)+2}{4} \quad \text{for } n=0,1,\ldots,N-2 \ , \\ P_{nm}^{(1)} &= 0 \quad \text{else} \ . \end{aligned} \tag{46}$$

Explicitly, the first entries of $P^{(1)}$ read

$$P^{(1)} = \frac{1}{2}\begin{bmatrix} 1 & -2 & 1 & 0 & \cdots \\ -2 & 7 & -8 & 3 & \cdots \\ 1 & -8 & 19 & -18 & \cdots \\ 0 & 3 & -18 & 37 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} . \tag{47}$$

Shuffling together the results of Sec. IV for the kinetic matrix $E^{(0)}$ and for the inverse of the power matrix $P^{(0)}$ as well as of this section for the potential matrix of the funnel potential, which, as already mentioned, serves only for illustrative purposes, one ends up with a well-defined matrix eigenvalue equation, which represents the $(l=0)$ special case of the general matrix form (28) of the spinless Salpeter equation (1).

VI. SUMMARY

In this work we demonstrated how, by a suitable choice of basis in the vector space of wave functions, the spinless Salpeter equation may be converted to an eigenvalue equation for an analytically known matrix, which only depends on the interaction potential responsible for the dynamics of the bound state. The action of the differential operator in the kinetic term of the Salpeter equation on the basis functions is represented by a "kinetic matrix" which can be determined exactly. Since this matrix has to be calculated only once, the complexity of the required expressions, such as, for instance, Eqs. (35) and (36), is no drawback. We gave this matrix explicitly for S-wave states, i.e., $l=0$. For the case of $l \neq 0$, see Ref. [10].

As an application of the developed formalism, we considered, for simplicity, only the most naive realistic interquark potential; that is, the funnel potential. From the structure of the involved integrals, however, it is evident that the method works analytically, at least for all potentials of the type "power times exponential," i.e., for all potentials of the form

$$V(r) = \sum_n a_n r^n e^{-b_n r} \tag{48}$$

with a_n, b_n constants. Examples for the above are the funnel potential discussed in Sec. V as well as its $\exp(-r)$ improved versions [11,12].

In the literature there can be found some other methods proposed for the conversion of the spinless Salpeter equation into an equivalent matrix equation by con-

TABLE I. Energy eigenvalues (in units of GeV) of the states with radial quantum number $n_r=1,2,3$ and orbital angular momentum $l=0,1$ for the funnel potential $V_F(r)=-\kappa/r+ar$ with parameters $m_c=1.45$ GeV, $\kappa=0.25$, and $a=0.18$ GeV^2. The "exact" values, taken from Table IV of Ref. [4] or Table I of Ref. [14], are given in italics. $N+1$ denotes the minimal matrix size required to reproduce these numbers by the "analytical" method of the present paper.

l	n_r = 1	2	3	N
0	3.3923	*3.9022*	4.3229	5
	3.3924		*4.2925*	7
1	*3.7345*	4.1469	4.5449	5
		4.1481	4.4972	7
			4.4963	8

structing a (at least approximate) matrix representation of the nonlocal differential operator $\sqrt{-\Delta+m^2}$ originating from the relativistic kinetic energy. Reference [13] uses a method similar to the one presented in this work, but with a different choice of the set of basis functions, namely, one which involves the generalized Laguerre polynomials. The kinetic term is treated in momentum space, where the basis functions involving generalized Laguerre polynomials become functions involving Jacobi polynomials of a somewhat complicated argument, and no attempt of an analytical computation of the encountered integral is made. Reference [14] uses a method which is closely related to the method of orthogonal collocation. The action of the square $-\Delta+m^2$ of the kinetic-energy differential operator on a set of orthogonal basis functions at some points in coordinate space, chosen in order to optimize the convergence of the numerical procedure, is determined exactly. The square root of the resulting matrix is then assumed to be an approximate matrix representation of the above square-root differential operator. For the explicit tests of the accuracy of this method, the same type of basis functions as in Ref. [13] is chosen.

The final question about the method proposed in this paper is the one about its rate of convergence with increasing size N of the involved matrices. In order to get an idea of this, we solve the spinless Salpeter equation (28) for the prototype interquark potential motivated in Sec. V, the funnel potential $V_F(r)=-\kappa/r+ar$. We adopt a standard set of values for the quark mass and potential parameters which is characteristic for the charmonium ($c\bar{c}$) system: $m_c=1.45$ GeV, $\kappa=0.25$, and $a=0.18$ GeV^2 [4,14]. The resulting energy eigenvalues of the states with radial quantum number $n_r=1,2,3$ and orbital angular momentum $l=0,1$ are given in Table I. Comparing these numbers with the "exact" values taken from Table IV of Ref. [4] or Table I of Ref. [14], we find that, as far as the matrix size necessary to achieve a given precision of the eigenvalues is concerned, our method is slightly superior to the admittedly already very fast method of orthogonal collocation developed in [14]. The reason for this is, of course, that in our approach the analytical expressions of the encountered integrals are at one's disposal, which is the main difference with all previously developed methods for solving relativistic wave equations, such as, e.g., the spinless Salpeter equation.

APPENDIX: NECESSARY FORMULAS

In this appendix we collect without any further comment all the formulas needed in Sec. IV at some intermediate steps for the computation of the kinetic matrix $E^{(0)}_{nm}$, Eq. (37), from the general expression (22). (For more details, see Ref. [10].)

Computation of the integral $I^{(0)}_k(x)$ in (32) from Eqs. (18) and (31):

$$K_0(x)=\int_0^\infty dt\frac{\cos xt}{\sqrt{1+t^2}},\quad x>0\ , \tag{A1}$$

$$\int_0^\infty dx\, x^k e^{-x}\sin xt=\frac{k!}{(1+t^2)^{(k+1)/2}}\times\sin[(k+1)\arctan t]\ , \tag{A2}$$

$$\sin[(k+1)\arctan t]=\frac{1}{(1+t^2)^{(k+1)/2}}\sum_{p=0}^{[k/2]}(-1)^p\begin{bmatrix}k+1\\2p+1\end{bmatrix}t^{2p+1}\ , \tag{A3}$$

$$\int_0^\infty dt\frac{t^{2p+1}\sin xt}{(1+t^2)^{k+3/2}}=(-1)^p\sum_{q=0}^{p}(-1)^q\begin{bmatrix}p\\q\end{bmatrix}\times\frac{x^{k-q+1}}{(2k-2q+1)!!}K_{k-q}(x)\ . \tag{A4}$$

Extraction of the expansion coefficients $c(0;k,m)$ in (35) from the result (32) for the integral $I^{(0)}_k(x)$:

$$\int_0^\infty dx\, e^{-x}x^n K_m(x)=\frac{(n+m)!(n-m)!}{(2n+1)!!}\ ,\quad m<n+1\ . \tag{A5}$$

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